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Attractors and time averages for random maps

by

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ABSTRACT. – Considering random noise in finite dimensional parameterized families of diffeomorphisms of a compact finite dimensional boundaryless manifold M, we show the existence of time averages for almost every orbit of each point of M, imposing mild conditions on the families; see Section 2.4. Moreover these averages are given by a finite number of physical absolutely continuous stationary probability measures.

We use this result to deduce that situations with infinitely many sinks and Hénon-like attractors are not stable under random perturbations, e.g., Newhouse's and Colli's phenomena in the generic unfolding of a quadratic homoclinic tangency by a one-parameter family of diffeomorphisms. © 2000 Éditions scientifiques et médicales Elsevier SAS

Key words: Random perturbations, Time averages, Physical probabilities, Homoclinic bifurcations

RÉSUMÉ. – On considère un bruit aléatoire dans des familles paramétrées de dimension finie de difféomorphismes d'une variété compacte sans bord, M, de dimension finie, et on montre, sous certains conditions pas très fortes sur ces familles, l'existence de moyennes temporelles (asymptotiques) pour presque toute orbite de chaque point de M (*voir* Section 2.4). Ces moyennes sont données par un nombre fini de mesures de probabilité stationnaires physiques absolument continues.

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On utilise ce résultat pour déduire que les situations de coexistence d'une infinité de puists et d'attracteurs de type Hénon ne sont pas stables par des perturbations aléatoires ; par exemple, les phénomènes de Newhouse et de Colli dans le dédoublement générique d'une tangence homoclinique quadratique par une famille de difféomorphismes à un paramètre. © 2000 Éditions scientifiques et médicales Elsevier SAS

Mots Clés: Perturbations aléatoires, Moyennes temporelles, Probabilités physiques, Bifurcations homocliniques

1. INTRODUCTION

Newhouse proved in [16–18] that many surface diffeomorphisms have infinitely many attracting periodic orbits (sinks), a serious blow to early hopes that generic systems might have only finitely many attractors. Indeed, see [18] and also [22], arbitrarily close to any C^2 diffeomorphism on a surface M with a homoclinic tangency there exist open subsets of $\mathrm{Diff}^2(M)$ whose generic elements have infinitely many sinks or sources.

This result was extended to arbitrary dimensions by Palis and Viana in [23], see also [25] and [11]. Diffeomorphisms with infinitely many coexisting hyperbolic attractors were constructed by Gambaudo and Tresser in [10]. Colli showed in [7] that diffeomorphisms displaying infinitely many Hénon-like strange attractors are dense in some open subsets of $\mathrm{Diff}^\infty(M)$, if $\dim M = 2$. Even more recently, Bonatti and Díaz in [4] showed that coexistence of infinitely many sinks or sources is generic in some open subsets of $\mathrm{Diff}^1(M)$, if $\dim M \geqslant 3$.

However, apart from these existence results, diffeomorphisms with infinitely many attractors or repellers are still a mystery. Results of [14,8, 5] show that maps which cannot be approximated by others with infinitely many sinks or sources have properties of partial hyperbolicity. In this case the dynamics of these maps can be understood to some degree, see e.g., [3,24,12,6,1]. It would be nice to know that systems with infinitely many sinks or sources are negligible from the measure theoretical point of view. Indeed, it has been conjectured that such systems correspond to zero Lebesgue measure in parameter space for generic families (finite number of parameters) of maps, see [29] and [22]. Nevertheless this is not yet know.

Here we show that this phenomenon of coexistence of infinitely many sinks or sources can indeed be discarded in the setting of maps endowed with random noise. We prove that (Theorem 1) *every diffeomorphism of a compact finite dimensional boundaryless manifold M under absolutely continuous random perturbations along a parameterized family has only finitely many physical measures whose basins cover Lebesgue-a.e. point of M.*

In the context of the generic unfolding of quadratic homoclinic tangencies by uniparametric arcs of surface diffeomorphisms, where the coexistence phenomenon of infinitely many attractors was first shown to occur, we prove (Theorem 2) a result similar to the previous one concerning points whose perturbed orbits visit a neighborhood of the tangency infinitely often with positive probability, which we call *recurrent points.*

This result is a corollary of the former since we show the random parametric perturbations applied on the recurrent points to be *absolutely continuous* as well. For an uniparametric arc to satisfy this property in a surface a quadratic homoclinic tangency is used: the mixture of expanding and contracting directions near a homoclinic tangency point, in a neighborhood of it in the manifold for every diffeomorphism close to the one exhibiting the tangency, is what permits us to get absolute continuity even when only a single parameter is at hand.

We conclude (Section 14) that *there cannot be infinitely many attractors (or physical measures) whose orbits (respectively, supports) pass near a quadratic homoclinic tangency point or its generic unfolding under random parametric perturbations (i.e., random errors in the parameters)—in this sense, diffeomorphisms with infinitely many attractors are not stable under random perturbations.*

These results can be seen from the perspective of a broad program proposed by J. Palis in [20]. In particular, he conjectured that systems with finitely many attractors are dense in the space of all systems. Moreover, these attractors should have nice statistical properties, including existence of physical measures supported on them, and stochastic stability under small random noise—see, e.g., [31].

Fornaess and Sibony in [9] have shown a result similar to Theorem 1 to hold in the context of random perturbations of rational functions. The precise form of the statement of this theorem and of some definitions was inspired on Theorem 1.1 of theirs.

Relevant setting and all definitions are in Sections 2 and 3 along with the precise statement of the result, including the kind of noise to be used

and some examples. A summary of the steps of the proof is given in Section 4, where we also sketch the contents of Sections 5 through 9. In Section 10 we apply our results to perturbations of an example of Bowen. This provides a good insight into the meaning of these results.

Relevant settings, definitions and the statement of Theorem 2 are in Section 11. Its proof in Sections 12 and 13.

Several questions arise in this context of systems with random noise and the simple methods used in this work to derive Theorems 1 and 2 should be generalized and extended. Some of those questions are presented in the last section (Section 15) of this paper.

2. SOME NOTATIONS, DEFINITIONS AND THE MAIN THEOREM

Throughout this paper M will signify a compact boundaryless manifold with finite dimension, m will be some normalized ($m(M) = 1$) Riemannian volume form on M and $d_M : M \times M \to \mathbb{R}$ a distance given by some Riemannian structure on M, fixed once and for all. When not otherwise mentioned, absolute continuity will be taken with respect to the probability m.

The random perturbations to be considered will act on the dynamics of diffeomorphisms of a parameterized family given by the C^1 function $f : M \times B^n \to M$, where $B^n = \{y \in \mathbb{R}^n : \|y\|_2 < 1\}$ is the unit ball of $\mathbb{R}^n$, $n \geqslant 1$, $\|\cdot\|_2$ is the Euclidean norm and the map $f_t : M \to M$, $x \in M \mapsto f(x,t)$ is a diffeomorphism for every $t \in B^n$.

2.1. Perturbations around a parameter

Let us fix $a \in B^n$ and take $\varepsilon > 0$ such that the closed ε-neighborhood of a be contained in B^n, $\overline{B}^n(a,\varepsilon) \subset B^n$. We define the *perturbation space around a of size ε* to be

$$\Delta = \Delta_\varepsilon(a) = \overline{B}^n(a,\varepsilon)^{\mathbb{N}} = \{\underline{t} = (t_j)_{j=1}^\infty : \|t_j - a\|_2 \leqslant \varepsilon,\ j \geqslant 1\}$$

with the product topology, which is equivalent to the topology induced by the metric $d(\underline{t},\underline{s}) = \sum_{j=1}^\infty 2^{-n} \cdot \|t_j - s_j\|_2$, $\underline{t},\underline{s} \in \Delta$, and the measure ν^∞ given by the product of the normalized Lebesgue volume measure ν over each $\overline{B}^n(a,\varepsilon)$. For sets $A_1, \ldots, A_k$ of the Borel family in $\overline{B}^n(a,\varepsilon)$ we have $\nu^\infty(A_1 \times \cdots \times A_k \times \overline{B}^n(a,\varepsilon)^{\mathbb{N}}) = \nu(A_1)\cdots\nu(A_k)$ and if

$A \subset \overline{B}^n(a,\varepsilon)$ then $\nu(A) = |\overline{B}^n(a,\varepsilon)|^{-1} \cdot |A|$, where $|A|$ will mean the Lebesgue volume measure of A.

Now we define the *perturbed iterates* of f by

$$f_{\underline{t}}^k(z) = f^k(z,\underline{t}) = f_{t_k} \circ \cdots \circ f_{t_1}(z), \quad z \in M,\ \underline{t} \in \Delta$$

and state the useful convention that $f^0(z,\underline{t}) = z$ and

$$f_V^k(U) = f^k(U,V) = \{f_{\underline{t}}^k(z)\colon \underline{t} \in V,\ z \in U\}, \quad U \subset M,\ V \subset \Delta$$

for every $k \geqslant 1$. We emphasize a very often used property in what follows.

PROPERTY 2.1. – *For every fixed $k \geqslant 1$ it holds that*

(1) $(z,t_1,\ldots,t_k) \in M \times \overline{B}^n(a,\varepsilon) \times \overset{k}{\cdots} \times \overline{B}^n(a,\varepsilon) \mapsto f^k(z,t_1,\ldots,t_k) = f_{t_k} \circ \cdots \circ f_{t_1}(z) \in M$ *is differentiable*;

(2) $(z,\underline{t}) \in M \times \Delta \mapsto f^k(z,\underline{t}) \in M$ *is continuous (with the product topology)*;

(3) $z \in M \mapsto f^k(z,t_1,\ldots,t_k) \in M$ *is a diffeomorphism for every* $t_1,\ldots,t_k \in \overline{B}^n(a,\varepsilon)$.

Given $\underline{t} \in \Delta$ and $z \in M$ we will call $\{f_{\underline{t}}^k(z)\}_{n=1}^{\infty}$ the *$\underline{t}$-orbit* of z and many times write $\mathcal{O}(z,\underline{t})$.

In this way, perturbations are implemented by a random choice of parameters of a parameterized family of diffeomorphisms at each iteration, the choice being made in a ε-neighborhood of a fixed parameter according to a uniform probability. Such choices are represented by a vector $\underline{t}$ in Δ, an infinite product of intervals, and the greater or lesser importance of the set of perturbations taken into account will be evaluated by the measure ν^{∞}.

This kind of random iteration will be referred to as *parametric noise*. With the settings given above, the family of diffeomorphisms acting on M with parametric noise of level ε around f_a will be written $\mathcal{F}_{a,\varepsilon} = \{f_t\colon t \in \overline{B}^n(a,\varepsilon)\}$. To simplify writing the factors of Δ we set $T = \overline{B}^n(a,\varepsilon)$ from now on, so that $\Delta = T^{\mathbb{N}}$.

2.2. Stationary probabilities

We can define a shift operator $S\colon M \times \Delta \to M \times \Delta$, $(z,\underline{t}) \mapsto (f_{t_1}(z), \sigma(\underline{t}))$, where σ is the left shift on sequences of Δ: $\sigma(\underline{t}) = \underline{s}$ with $\underline{t} = (t_1,t_2,t_3,\ldots)$ and $\underline{s} = (t_2,t_3,t_4,\ldots)$. By the definition of S and Property 2.1(2) we deduce that S is continuous.

A probability measure μ in M is said a *stationary probability* if the measure $\mu \times \nu^{\infty}$ is S-invariant:

$$\mu \times \nu^{\infty}\big(S^{-1}A\big) = \mu \times \nu^{\infty}(A), \quad \text{for every Borel subset } A \text{ of } M \times \Delta. \tag{1}$$

This is equivalent to say that μ satisfies the following identity

$$\iint \varphi\big(f(z,t)\big)\, d\mu(z)\, d\nu(t) = \int \varphi(z)\, d\mu(z), \quad \forall \varphi \in C^0(M). \tag{2}$$

In fact, writing (1) for $A = U \times \Delta$, where U is a Borel subset of M, we have

$$\begin{aligned}
\mu \times \nu^{\infty}\big(S^{-1}(U \times \Delta)\big) &= \mu \times \nu^{\infty}\Big(\bigcup_{s\in T} f_s^{-1}(U) \times \{s\} \times \Delta\Big) \\
&= \mu \times \nu\Big(\bigcup_{s\in T} f_s^{-1}(U) \times \{s\}\Big) \times \nu^{\infty}(\Delta) \\
&= \iint 1_U\big(f(x,s)\big)\, d\mu(x)\, d\nu(s)
\end{aligned} \tag{3}$$

which is equal to $\mu \times \nu^{\infty}(U \times \Delta) = \mu(U)$, that is,

$$\iint 1_U\big(f(x,s)\big)\, d\mu(x)\, d\nu(s) = \mu(U) = \int 1_U(x)\, d\mu(x),$$

where 1_U is such that $1_U(x) = 1$ if $x \in U$ and $1_U(x) = 0$ otherwise. Then (2) holds for every $\varphi \in L^1_{\mu}(M, \mathbb{R}) \supset C(M, \mathbb{R})$, because simple functions are dense in L^1_{μ} and the relation (2) is linear.

Conversely, if (2) holds for every $\varphi \in C^0(M, \mathbb{R})$, then it holds for every element of $L^1_{\mu}(M, \mathbb{R})$ because μ and ν are Borel measures and $f : M \times B \to M$ is continuous (so that the left hand side of (2) gives a regular measure over M). In particular, it holds for $\varphi = 1_U$, and (3) is equal to $\int 1_U(x)\, d\mu(x) = \mu(U) = \mu \times \nu^{\infty}(U \times \Delta)$ proving that (2) implies $\mu \times \nu^{\infty}(S^{-1}(U \times \Delta)) = \mu \times \nu^{\infty}(U \times \Delta)$. Now we see that, if $V \subset \Delta$ is also a Borel subset,

$$\begin{aligned}
\mu \times \nu^{\infty}\big(S^{-1}(U \times V)\big) &= \mu \times \nu^{\infty}\Big(\bigcup_{s\in T} f_s^{-1}(U) \times \{s\} \times V\Big) \\
&= \mu \times \nu\Big(\bigcup_{s\in T} f_s^{-1}(U) \times \{s\}\Big) \times \nu^{\infty}(V)
\end{aligned}$$

$$= \iint 1_U(f(x,s))\, d\mu(x)\, d\nu(s) \times \nu^\infty(V)$$

$$= \int 1_U(x)\, d\mu(x) \times \nu^\infty(V) = \mu \times \nu^\infty(U \times V)$$

proving the equivalence between (2) and (1).

2.3. Ergodicity, generic points, ergodic basin

In the same way we have defined a stationary probability, by utilizing the shift S, we will say that μ is a *stationary ergodic* probability measure if $\mu \times \nu^\infty$ is S-ergodic.

In this situation, Birkhoff's ergodic theorem ensures that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi\big(S^j(x,\underline{t})\big) = \int \psi\, d(\mu \times \nu^\infty)$$

for $\mu \times \nu^\infty$-a.e.$(x,\underline{t}) \in M \times \Delta$ and for every $\psi \in C^0(M \times \Delta, \mathbb{R})$. In particular, putting $\psi = \varphi \circ \pi$, with $\varphi \in C^0(M,\mathbb{R})$ and $\pi : M \times \Delta \to M$ the projection on the first factor, we obtain $\psi(S^j(x,\underline{t})) = \varphi(f^j(x,\underline{t}))$, $j = 0, 1, 2, \ldots$ and $\int \psi\, d(\mu \times \nu^\infty) = \int \varphi\, d\mu$, thus for every continuous $\varphi : M \to \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f^j(x,\underline{t})\big) = \int \varphi\, d\mu,$$

$$\text{for } \mu \times \nu^\infty\text{-a.e. } (x,\underline{t}) \in M \times \Delta. \tag{4}$$

We now remark that, because $\mu \times \nu^\infty$ is a product measure, we have the following property. Let X be the set of $(x,\underline{t})$ that satisfy (4) for every continuous function $\varphi : M \to \mathbb{R}$. If $\{\varphi_n\}_{n=1}^\infty$ is a denumerable and dense sequence in $C^0(M,\mathbb{R})$ and X_n the set of those points $(x,\underline{t}) \in M \times \Delta$ that satisfy (4) for φ_n, $n \geqslant 1$, then it is easy to see (cf. [15, Chapter II.6]) that $X = \bigcap_{n\geqslant 1} X_n$ is a set of $\mu \times \nu^\infty$-measure 1. Let us consider now $X(x) = \{\underline{t} \in \Delta\colon\ (x,\underline{t}) \in X\}$, the section of X through $x \in M$. Then we have $\nu^\infty(X(x)) = 1$ for μ-a.e. $x \in M$. Indeed, by Fubini's theorem, $\mu \times \nu^\infty(X) = \int \nu^\infty(X(x))\, d\mu(x) = 1$ with $0 \leqslant \nu^\infty(X(x)) \leqslant 1$ for every $x \in M$. Hence, the last identity implies the statement, because μ is a probability measure.

The points x that satisfy $\nu^\infty(X(x)) = 1$, that is, for which the limit in (4) exists and equals $\int \varphi\, d\mu$ for ν^∞-a.e. $\underline{t} \in \Delta$ and every continuous $\varphi : M \to \mathbb{R}$, will be called *$\mu$-generic points*. The set of μ-generic points,

when μ is stationary and ergodic, will be the *ergodic basin* of μ and will be written $E(\mu)$.

To complete this setting of terms and symbols, those ergodic stationary probability measures μ whose basin has positive volume, $m(E(\mu)) > 0$, will be called *physical measures* of the perturbed system. We also convention to write $f^k(x, \nu^\infty)$ for the push-forward of ν^∞ by $f^k(x, \cdot)$, that is $f^k(x, \nu^\infty)\varphi = \int \varphi(f^k(x, \underline{t}))\, d\nu^\infty(\underline{t})$ for every $k \geqslant 1$, $x \in M$ and $\varphi \in C^0(M, \mathbb{R})$ by definition.

2.4. Statement of the results

THEOREM 1. – *Let $f : M \to M$ be a diffeomorphism of class C^r, $r \geqslant 1$, of a compact connected boundaryless manifold M of finite dimension. If $f = f_a$ is a member of a parametric family under parametric noise of level $\varepsilon > 0$, as in Section 2.1, that satisfies the hypothesis: there are $K \in \mathbb{N}$ and $\xi_0 > 0$ such that, for all $k \geqslant K$ and $x \in M$*

A) $f^k(x, \Delta) \supset B(f^k(x), \xi_0)$;

B) $f^k(x, \nu^\infty) \ll m$;

then there is a finite number of probability measures $\mu_1, \ldots, \mu_l$ in M with the properties

1. *$\mu_1, \ldots, \mu_l$ are physical absolutely continuous probability measures;*
2. *$\operatorname{supp} \mu_i \cap \operatorname{supp} \mu_j = \emptyset$ for all $1 \leqslant i < j \leqslant l$;*
3. *for all $x \in M$ there are open sets $V_1 = V_1(x), \ldots, V_l = V_l(x) \subset \Delta$ such that*
   (a) *$V_i \cap V_j = \emptyset$, $1 \leqslant i < j \leqslant l$;*
   (b) *$\nu^\infty(\Delta \setminus (V_1 \cup \cdots \cup V_l)) = 0$;*
   (c) *for all $1 \leqslant i \leqslant l$ and ν^∞-a.e. $\underline{t} \in V_i$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x, \underline{t})) = \int \varphi\, d\mu_i, \quad \text{for every } \varphi \in C(M, \mathbb{R}).$$

   *Moreover, the sets $V_1(x), \ldots, V_l(x)$ depend continuously on $x \in M$ with respect to the distance $d_\nu(A, B) = \nu^\infty(A \triangle B)$ between ν^∞-*mod 0 *subsets of Δ.*

The theorem assures the existence of a finite number of physical probability measures with respect to the perturbed system $\mathcal{F}_{a,\varepsilon}$, as defined in the previous subsections, which describe the asymptotics of the Birkhoff averages of almost every perturbed orbit of every point of M. Section 10 gives perhaps a clearer meaning for this result.

The conditions on the noise are about "how much spread" suffer the orbits under perturbation when compared with those without perturbation. They demand that the perturbations "scatter" the orbits in an "uniform" way around the nonperturbed ones, at least from some iterates onward, and ask for negligible perturbations (of ν^∞ measure zero) to produce negligible effects: the result of such perturbations should only be a set of m measure zero.

These hypothesis try to translate the intuitive idea of random perturbations not having "privileged direction or size", causing deviations from the ideal orbit that will "fill" a full neighborhood of that orbit and "ignoring" sets of perturbations of zero probability. In the light of this, parametric noise satisfying conditions A) and B) may aptly be referred to as *physical parametric noise.*

Example 1. – Let $M = \mathbb{T}^n$ be the n-torus, $n \geqslant 1$, and $f_0 : \mathbb{T}^n \to \mathbb{T}^n$ a C^r-diffeomorphism, $r \geqslant 1$. Since $\mathbb{T}^n$ is parallelizable, $T\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$, we can find n globally orthonormal (hence nonvanishing) vector fields in $\mathcal{X}^r(M)$. For instance, through the identification $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$ via the natural projection, we may take $X_1(x) = e_1 = (1, 0, \ldots, 0), X_2(x) = e_2 = (0, 1, \ldots, 0), \ldots, X_n(x) = e_n = (0, 0, \ldots, 1)$ for all $x \in \mathbb{T}^n$.

We construct a family of differentiable maps defining $f : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n$ by

$$(x, t) \in \mathbb{T}^n \times \mathbb{R}^n \mapsto f_0(x) + t_1 X_1\big(f_0(x)\big) + \cdots + t_n X_n\big(f_0(x)\big) \bmod \mathbb{Z}^n,$$

or equivalently by $f_t(x) = f(x, t_1, \ldots, t_n) = f_0(x) + (t_1, \ldots, t_n) \bmod \mathbb{Z}^n$.

We note that since $\|t\|_2 < \varepsilon$ implies $\|f_t - f_0\|_{C^r} < \varepsilon$ for every $\varepsilon > 0$ and $\mathrm{Diff}^r(\mathbb{T}^n)$ is open in $C^r(\mathbb{T}^n, \mathbb{T}^n)$ (cf. [21, Chapter I]), there is $\varepsilon_0 > 0$ such that the restriction $f_| : \mathbb{T}^n \times B^n(0, \varepsilon_0) \to \mathbb{T}^n$ is a C^r-family of C^r-diffeomorphisms of $\mathbb{T}^n$.

It is not difficult to see that f satisfies hypothesis A) and B) of Theorem 1 for $K = 1$ and for every family $\mathcal{F}_{a,\varepsilon} = \{f_t\colon \|t - a\|_2 < \varepsilon\}$ such that $\overline{B}^n(a, \varepsilon) \subset B^n(0, \varepsilon_0)$. We may say, in the light of this, that this specific kind of random parametric perturbation is an *absolutely continuous random perturbation.*

Theorem 1 follows and we see that *any random absolutely continuous perturbation of a diffeomorphism of the torus (or of any parallelizable manifold) is such that Birkhoff averages exist for almost every orbit of every point of the torus. Moreover, their values are defined by a*

finite number of absolutely continuous physical stationary probability measures.

Remark 2.1. – Example 1 shows that given any diffeomorphism f of a parallelizable manifold we may easily embed f in a suitable parameterized family of diffeomorphisms satisfying hypothesis A) and B).

Example 2. – We now construct an absolutely continuous random perturbation around any given diffeomorphism $f \in \mathrm{Diff}^r(M)$, $r \geqslant 1$, of every compact finite dimensional boundaryless manifold M, assuming M to be endowed with some Riemannian metric. It is most likely that this kind of construction can be carried out with $n = \dim(M)$ or $n+1$ parameters.

We start by taking a finite number of coordinate charts $\{\psi_i : B(0,3) \to M\}_{i=1}^l$ such that $\{\psi_i(B(0,3))\}_{i=1}^l$ is an open cover of M and $\{\psi_i(B(0,1))\}_{i=1}^l$ also (this is a standard construction, cf. [21, Section 1.2]). In each of those charts we define $n = \dim(M)$ orthonormal vector fields $\widetilde{X}_{i1}, \ldots, \widetilde{X}_{in} : B(0,3) \to T_{\psi_i(B(0,3))}M$ and extend them to the whole of M with the help of bump functions. This may be done in such a way that the extensions X_{ij} are null outside $\psi_i(B(0,2))$ and coincide with $\widetilde{X}_{ij}$ in $\psi_i(\overline{B(0,1)})$, $i = 1, \ldots, l$; $j = 1, \ldots, n$. We then see that

- At every $x \in M$ there is some $1 \leqslant i \leqslant l$ such that $X_{i1}(x), \ldots, X_{in}(x)$ is an orthonormal basis for $T_x M$—and likewise for $X_{i1}, \ldots, X_{in}$—because $\{\psi_i(B(0,1))\}_{i=1}^l$ was an open cover of M.

Finally we define the following parameterized family

$$F : \left(\mathbb{R}^n\right)^l \to C^r(M, M),$$

$$F\left((u_{ij})_{\substack{i=1,\ldots,l \\ j=1,\ldots,n}}\right)(x) = \Phi\left(f(x), \sum_{i=1}^{l}\sum_{j=1}^{n} u_{ij} \cdot X_{ij},\ 1\right),$$

where $\Phi : TM \times \mathbb{R} \to M$ is the geodesic flow associated to the given Riemannian metric. Then for some $\varepsilon_0 > 0$ we get a finite dimensional parameterized family of diffeomorphisms $F_| : B^{n \cdot l}(0, \varepsilon_0) \to \mathrm{Diff}^r(M)$ satisfying conditions A) and B) of Theorem 1 for $K = 1$ and some $\xi_0 > 0$, and for every family $\mathcal{F}_{a,\varepsilon} = \{F_t : \|t - a\|_2 < \varepsilon\}$ where $a \in \overline{B}^{n \cdot l}(a, \varepsilon) \subset B^{n \cdot l}(0, \varepsilon_0)$.

Example 3. – In the context of *random perturbation of rational functions*, as in [9], hypothesis A) and B) are immediate.

Indeed, let $R : \overline{\mathbb{C}} \times W \to \overline{\mathbb{C}}$ be analytic, where $W \subset \mathbb{C}$ is open an connected, $z \mapsto R(z, c)$ is rational for all $c \in W$ and $c \in W \mapsto R(z, c)$ is nonconstant for every $z \in \overline{\mathbb{C}}$ (i.e., R is a *generic family of*

rational functions). Then it is easy to get a $\xi = \xi(c_0, \varepsilon) > 0$ such that $R(z, B(c_0, \varepsilon)) \supset B(R(z, c_0), \xi)$ for all $z \in \overline{\mathbb{C}}$, whenever $B(c_0, \varepsilon) \subset W$, by compactness of $\overline{\mathbb{C}}$ and because analytic nonconstant functions are open. Moreover, if λ is Lebesgue measure normalized and restricted to $B(c_0, \varepsilon)$, then $R(z, \lambda) \ll$ Lebesgue on $\mathbb{C}$. Hence we get A) and B) with $K = 1$.

Theorem 1 then proves something more than Theorem 0.1 of [9]: we get physical measures whose support contains neighborhoods of the attracting cycles of R_{c_0} and which give the time averages of almost every orbit of each point of the Riemann sphere.

Example 4. – Let $f : M \times T \to M$ be a parameterized family of diffeomorphisms as in Section 2 such that for some $a \in T$ the diffeomorphism f_a is transitive. Let us suppose further that for some $\varepsilon > 0$ the parametric noise of level ε around f_a, $\mathcal{F}_{a,\varepsilon}$, satisfies hypothesis A) and B). Hence Theorem 1 holds and let μ_i be one of the physical absolutely continuous probabilities given by the theorem.

Since f_a is transitive, there is a residual set $\mathcal{R}$ in M whose points $x_0 \in \mathcal{R}$ give dense f_a-orbits: $\overline{\{f_a^k(x_0)\}_{k=0}^{\infty}} = M$. Moreover, the c-invariance of $\operatorname{supp} \mu_i$ (v. Section 3, Definition 3.1) and hypothesis A) imply that $\operatorname{int}(\operatorname{supp} \mu_i) \neq \emptyset$, and thus there is $x_0 \in (\mathcal{R} \cap \operatorname{int}(\operatorname{supp} \mu_i))$.

We deduce that

$$\operatorname{supp} \mu_i \supset \overline{\{f^k(x_0, \Delta)\}_{k=1}^{\infty}} \supset \overline{\{f_a^k(x_0)\}_{k=0}^{\infty}} = M$$

and so there is only one physical absolutely continuous probability in M, whose support is the whole of M.

In particular, *every diffeomorphism of the torus* $\mathbb{T}^n$ $(n \geqslant 1)$ *with a dense orbit, under absolutely continuous noise of arbitrary level* $\varepsilon > 0$, *has a single physical absolutely continuous probability whose support is* M (*and likewise if* M *is any parallelizable compact boundaryless manifold*).

In Section 11 we shall see that certain arcs (uniparametric families) of diffeomorphisms of class C^r $(r \geqslant 3)$ generically unfolding a quadratic homoclinic tangency satisfy both conditions of Theorem 1, restricted to a neighborhood of the point of homoclinic tangency. For more specifics, check the abovementioned section. We will then have

THEOREM 2. – *There are open sets of arcs* (*in the* C^3 *topology*) $\{f_t\}_{t \in]-1,1[}$ *of diffeomorphisms of class* C^3 *of a compact boundaryless surface generically unfolding a quadratic homoclinic tangency at* f_0 *such that, in a neighborhood* $\mathcal{Q}$ *of a point of homoclinic tangency and for all* f_{t_0} *sufficiently near* f_0 *under parametric noise of sufficiently small level*

$0 < \varepsilon < \varepsilon_0$, there are a finite number of probability measures $\mu_1, \ldots, \mu_h$ in $\mathcal{Q}$ that satisfy the conditions 1) *and* 2) *and also* 3) *of Theorem* 1, *for points $x \in M$ whose orbits $\mathcal{O}(x, \underline{t})$ have an infinite number of iterates in $\mathcal{Q}$ with respect to a ν^∞ positive measure set of perturbations.*

This result, combined with Newhouse's phenomenon, shows that the infinity of periodic hyperbolic attractors (sinks) that coexist in a neighborhood of a point of homoclinic tangency, for "many" parameter values near the bifurcation parameter, cannot "survive" the random parametric perturbation. Moreover it must subsist, at most, a finite number of *analytic continuations under random perturbation* of a sink. Section 14 will specify this conclusions and extend the result in a simple manner to Colli's phenomenon, where the infinity of hyperbolic periodic attractors is replaced by an infinity of Hénon-like strange attractors.

Now we will concentrate on the proof of Theorem 1.

3. INVARIANT DOMAINS

Let μ be a stationary probability measure with respect to a parametric perturbation of noise level $\varepsilon > 0$ around f_a. Then $\operatorname{supp}\mu$ is S-invariant: $S(\operatorname{supp}(\mu \times \nu^\infty)) \subset \operatorname{supp}(\mu \times \nu^\infty)$.

Let us observe that since $\operatorname{supp}(\mu) = \operatorname{supp}(\mu) \times \Delta$ we have for all $(x, \underline{t}) \in \operatorname{supp}(\mu) \times \Delta$ that $f^k(x, \underline{t}) \in \operatorname{supp}\mu$, for all $k \geqslant 1$. That is, $\operatorname{supp}\mu$ is completely invariant according to

DEFINITION 3.1. – *A part C of M is said* completely invariant *or* c-invariant *if $f^k(x, \underline{t}) \in C$ for all $x \in C$, $\underline{t} \in \Delta$ and $k \geqslant 1$.*

With the purpose of showing the existence of the kind of stationary probability measures stated in Theorem 1 and to better understand the dynamics of the points in their support as well, we make a series of definitions.

DEFINITION 3.2. – *An* invariant domain under an ε-perturbation with respect to the family f around the parameter $a \in I$ *will be a finite collection $\mathcal{U}_0, \ldots, \mathcal{U}_{r-1}$ of pairwise* separated open sets, *that is, $i \neq j \Rightarrow \overline{\mathcal{U}_i} \cap \overline{\mathcal{U}_j} = \emptyset$, such that $f^k(\mathcal{U}_0, \Delta) \subseteq \mathcal{U}_{k \bmod r}$ for all $k \geqslant 1$, and it will be written $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$. The number $r \in \mathbb{N}$ above will be referred to as the* period *of the invariant domain.*

Let us observe that the open set $\mathcal{U}_0$ has a privileged role in the above definitions.

DEFINITION 3.3. – *An invariant domain that also satisfies*

$$f^k(\mathcal{U}_i, \Delta) \subseteq \mathcal{U}_{(k+i) \bmod r}, \quad \forall k \geqslant 1 \tag{5}$$

whatever $i \in \{0, \ldots, r-1\}$ *will be a* symmetrically invariant *domain or* s-invariant *domain.*

This kind of domains will be at the heart of the arguments within next sections and the proof of their existence and finite number is the key to every other result in this paper.

Remark 3.1. – Since the f_t are diffeomorphisms for all $t \in T$, we see that if the collection $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ is s-invariant, then $\overline{\mathcal{D}} = (\overline{\mathcal{U}}_0, \ldots, \overline{\mathcal{U}}_{r-1})$ also satisfies (5) and conversely: if the closure $\overline{\mathcal{D}} = (\overline{\mathcal{U}}_0, \ldots, \overline{\mathcal{U}}_{r-1})$ satisfies (5) with $\mathcal{U}_0, \ldots, \mathcal{U}_{r-1}$ pairwise disjoint open sets, then $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ is an s-invariant domain.

3.1. Partial order and minimality

Let $\mathcal{D}$ be the family of s-invariant domains. We define the following partial order relation between its elements.

Let $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ and $D' = (\mathcal{U}'_0, \ldots, \mathcal{U}'_{r'-1})$ be elements of $\mathcal{D}$.

First, $D = D'$ if there are $i, i' \in \mathbb{N}$ such that $\mathcal{U}_{(i+k) \bmod r} = \mathcal{U}'_{(i'+k) \bmod r'}$, $\forall k \geqslant 1$ which implies $r = r'$, because the open sets that form each invariant domain are pairwise disjoint.

We say $D \prec D'$ if there are $i, i' \in \mathbb{N}$ such that $\mathcal{U}_{i \bmod r} \subseteq \mathcal{U}'_{i' \bmod r'}$ but $\mathcal{U}_{i \bmod r} \neq \mathcal{U}'_{i' \bmod r'}$, and $\mathcal{U}_{(i+k) \bmod r} \subseteq \mathcal{U}'_{(i'+k) \bmod r'}$ for all $k \geqslant 1$ (see Fig. 1 for an example with $r = 3$ and $r' = 6$).

We write $D \preceq D'$ if, and only if, $D = D'$ or $D \prec D'$.

Clearly $(\mathcal{D}, \preceq)$ is now a partially ordered set.

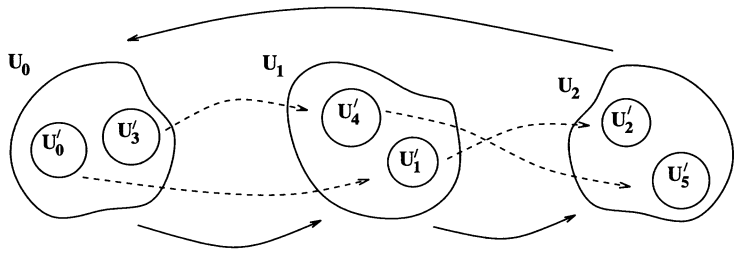

Fig. 1. Domains D,D' with $D' \prec D$.

DEFINITION 3.4. – *A* **minimal invariant domain** *is a domain $D \in \mathcal{D}$ which is minimal with respect to the partial order $\preceq$ just defined.*

Minimal domains will be represented by the letter $\mathcal{M}$ throughout this text.

4. A TOUR OF THE PROOF

With the notions given in previous sections we can now divide the proof of Theorem 1 in the following steps:

(1) To show that $\mathcal{D}$ has some minimal invariant domain and that any invariant domain contains some minimal one (Section 6.1).

(2) To show that minimal invariant domains are pairwise disjoint (Section 6.2).

By now we can already deduce the number of minimals is finite. In fact, a minimal invariant domain $\mathcal{M}$ is completely invariant and by hypothesis A) of Theorem 1 we see that every open set of the finite collection forming $\mathcal{M}$ contains a ball of radius $\geqslant \xi_0 > 0$. The compactness of M and step (2) above ensure there can only be a finite number of such open sets and thus a finite number of minimals also.

(3) Every minimal domain is *randomly transitive* or *r-transitive*, this notion will be specified in Section 6.3.

(4) The orbits of every point $z \in \mathcal{M}$ under noise generate a stationary probability measure μ which is absolutely continuous (Section 7.1).

From (3) and (4) we deduce that there exists an absolutely continuous stationary probability μ in the closure of each minimal $\mathcal{M}$ (since $\mathcal{M}$ contains every orbit of $z \in \mathcal{M}$) whose support is the closure of $\mathcal{M}$ (by the c-invariance of the support and item (3): $\operatorname{supp} \mu = \overline{\mathcal{M}}$.

(5) Every stationary absolutely continuous probability measure μ supported on a minimal domain $\mathcal{M}$ is ergodic and its ergodic basin $E(\mu)$ contains the whole of $\mathcal{M}$: $E(\mu) \supset \overline{\mathcal{M}}$ (Section 7.2).

Being ergodic, absolutely continuous and supported on the whole of $\mathcal{M}$, this probability μ is physical, since the minimal invariant domain is a collection of open sets. Consequently since for every such measure $E(\mu) \supset \overline{\mathcal{M}}$ holds, this is the only stationary ergodic absolutely continuous probability measure supported on $\mathcal{M}$. It will be referred to as the *characteristic probability* of the minimal $\mathcal{M}$.

(6) Every stationary probability measure is supported on some s-invariant domain (Section 8).

This crucial step gives the converse of the property deduced from step (5). Moreover, combining with the results of the previous steps we will deduce from step (6) that

(7) Every stationary probability measure is a finite convex linear combination of characteristic probabilities (Section 8).

(8) Finally, in Section 9, we will use items (4) and (7) to deduce that ν^∞-a.e. perturbation $\underline{t} \in \Delta$ is such that $\mathcal{O}(z, \underline{t})$ eventually falls into some minimal $\mathcal{M}$. The perturbations sending z into different minimals form the partition of item (3) of Theorem 1. Since $\mathcal{M}$ supports a characteristic measure which is physical, we further derive that Birkhoff averages exist for $\mathcal{O}(z, \underline{t})$ and satisfy (4).

5. FUNDAMENTAL LEMMAS

The measure theoretical lemma that follows will be used frequently within the arguments of this and next sections.

LEMMA 5.1. – *Given $V \subset \Delta$ with $\nu^\infty(V) > 0$, we define for fixed $\underline{\theta} \in \Delta$ and $k \geqslant 1$*

$$V(\underline{\theta}, k) = \{\underline{\omega} \in V\colon\ \omega_1 = \theta_1, \ldots, \omega_k = \theta_k\}$$

the k-section of V along $\underline{\theta}$. Then we have

$$\nu^\infty\big(\sigma^k V(\underline{\theta}, k)\big) \to 1 \quad \textit{when } k \to \infty$$

for ν^∞-a.e. $\underline{\theta} \in V$, where $\sigma : \Delta \to \Delta$ is the left shift on sequences: $\sigma(\underline{\psi}) = \underline{\varphi}$ with $\varphi_n = \psi_{n+1}$, $n = 1, 2, 3 \ldots$.

Note. – From now on we will say that a vector $\underline{\theta}$ satisfying the above limit with respect to a set $V \subset \Delta$ is *V-generic*.

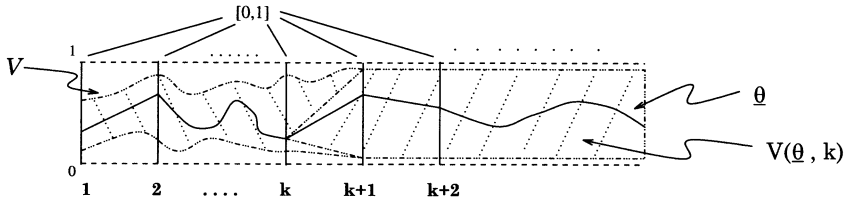

Fig. 2. Representation of the infinite product of the interval [0, 1], a vector $\underline{\theta}$ and the sets V and $V(\underline{\theta}, k)$.

Proof. – We may assume, for definiteness, that $\Delta = [0,1]^{\mathbb{N}}$ with ν the Lebesgue measure in $[0,1]$ so that ν^∞ is a probability in Δ. Let $V \subset \Delta$ be such that $\nu^\infty(V) > 0$.

If $\mathcal{B}$ is the Borel σ-algebra in $[0,1]$ and

$$\mathcal{B}_k = \overbrace{\mathcal{B} \times \cdots \times \mathcal{B}}^{k} \times [0,1]^{\mathbb{N}}, \quad k \geqslant 1,$$

then $\mathcal{A} = \sigma(\bigcup_{k=1}^\infty \mathcal{B}_k)$ is the σ-algebra of Δ over which ν^∞ is defined, the σ-algebra generated by all $\mathcal{B}_k$. For every $f \in L^1(\Delta, \mathcal{A}, \nu^\infty)$ and each $k \geqslant 1$ the map $A \in \mathcal{B}_k \mapsto \int_A f\, d\nu^\infty$ defines a finite measure on $(\Delta, \mathcal{B}_k, \nu^\infty)$, which clearly is absolutely continuous with respect to the measure $A \in \mathcal{B}_k \mapsto \nu^\infty(A)$ (the restriction of ν^∞ to $\mathcal{B}_k$). By Radon–Nikodym's theorem there is $E(f|\mathcal{B}_k) \in L^1(\Delta, \mathcal{B}_k, \nu^\infty)$, the *conditional expectation of f with respect to the σ-algebra $\mathcal{B}_k$*, such that

$$\int_A E(f|\mathcal{B}_k)\, d\nu^\infty = \int_A f\, d\nu^\infty, \quad \forall A \in \mathcal{B}_k \tag{6}$$

and this function is unique with this property in $L^1(\Delta, \mathcal{B}_k, \nu^\infty)$.

Let $X_k = E(f|\mathcal{B}_k)$, $k = 1, 2, \ldots$. We are going to see that $\{X_k\}_{k=1}^\infty$ is a martingale with respect to the sequence $\{\mathcal{B}_k\}_{k=1}^\infty$ of σ-algebras.

Indeed, because $\mathcal{B}_k \subset \mathcal{B}_{k+1}$ we have $\int_A E(f|\mathcal{B}_{k+1})\, d\nu^\infty = \int_A f\, d\nu^\infty$ for all $A \in \mathcal{B}_k$ and by (6) and uniqueness of conditional expectation

$$E(X_{k+1}|\mathcal{B}_k) = E\big(E(f|\mathcal{B}_{k+1})|\mathcal{B}_k\big) = E(f|\mathcal{B}_k) = X_k \quad \nu^\infty\text{-a.e.}$$

By the martingale convergence theorem (cf. [19] for simple definitions and proofs), the sequence $\{X_k\}_{k=1}^\infty$ has a ν^∞-a.e. limit that we shall write $X \in L^0(\Delta, \mathcal{A})$.

By (6) and because $f \in L^1(\Delta, \mathcal{A}, \nu^\infty)$ we have, assuming $f \geqslant 0$, that $X_k \geqslant 0$ ν^∞-a.e., $k \geqslant 1$, and consequently $X \geqslant 0$ ν^∞-a.e. Moreover

$$\int |X_k|\, d\nu^\infty = \int X_k\, d\nu^\infty = \int f\, d\nu^\infty = \int |f|\, d\nu^\infty$$

and so $X \in L^1(\Delta, \mathcal{A}, \nu^\infty)$ by dominated convergence and $\int |X|\, d\nu^\infty = \int X\, d\nu^\infty$ gives $\int f\, d\nu^\infty$. Furthermore, if $A \in \mathcal{B}_k$ then $\int_A X_j\, d\nu^\infty = \int_A f\, d\nu^\infty$ for all $j \geqslant k$ and from this we get $\int_A X\, d\nu^\infty = \int_A f\, d\nu^\infty$ for all $A \in \mathcal{B}_k$ and $k \geqslant 1$.

By the absolute continuity of the integral of a L^1-function and by definition of $\mathcal{A}$, for every $\varepsilon > 0$ and $A \in \mathcal{A}$ there are $\delta > 0$, $k \geqslant 1$ and $B \in \mathcal{B}_k$ such that $\nu^\infty(A \triangle B) < \delta$, $\int_{A \triangle B} |X| \, d\nu^\infty < \varepsilon$ and $\int_{A \triangle B} |f| \, d\nu^\infty < \varepsilon$. Now we have, in succession

$$\left| \int_A f \, d\nu^\infty - \int_B X_k \, d\nu^\infty \right| = \left| \int_A f \, d\nu^\infty - \int_B f \, d\nu^\infty \right| \leqslant \int_{A \triangle B} |f| \, d\nu^\infty < \varepsilon,$$

$$\left| \int_A X \, d\nu^\infty - \int_B X_k \, d\nu^\infty \right| = \left| \int_A X \, d\nu^\infty - \int_B X \, d\nu^\infty \right| \leqslant \int_{A \triangle B} |X| \, d\nu^\infty < \varepsilon,$$

and from this we get $|\int_A X \, d\nu^\infty - \int_A f \, d\nu^\infty| \leqslant 2\varepsilon$ with $\varepsilon > 0$ arbitrary.

We conclude that $\int_A X \, d\nu^\infty = \int_A f \, d\nu^\infty$, $\forall A \in \mathcal{A}$ and so $X = f\nu^\infty$-a.e.

In particular if $f = 1_V$ we have $X_k \to 1_V \nu^\infty$-a.e. and $\int_B E(1_V | \mathcal{B}_k) \, d\nu^\infty = \int_B 1_V \, d\nu^\infty$ equals $\nu^\infty(V \cap B)$ by definition of conditional expectation. But $\nu^\infty(V \cap B) = \int_B 1_V \, d\nu^\infty$ also equals $\int_B \nu_k^\infty(V(\underline{\theta}, k)) \, d\nu^k(\underline{\theta})$ for every $B \in \mathcal{B}_k$ and $k \geqslant 1$ by Fubini's theorem, where $\nu_k^\infty(A) = \nu^\infty(\sigma^k A)$ and $\nu^k(A) = \nu^k(\pi_k(A))$, $A \in \mathcal{A}$ with $\pi_k : \Delta \to [0,1]^k$ the natural projection $\underline{\theta} = (\theta_i)_{i=1}^\infty \mapsto (\theta_1, \ldots, \theta_k)$. That is $\nu_k^\infty(V(\underline{\theta}, k)) = E(1_V | \mathcal{B}_k) = X_k$ ν^∞-a.e. $\underline{t} \in \Delta$, and the proof is complete. □

This lemma will be utilized essentially in the following way. Let V, W be subsets of Δ with ν^∞-positive measure and $\underline{t}$ a V-generic vector. Then there is $k_0 \in \mathbb{N}$ such that

$$k \geqslant k_0 \Rightarrow \nu^\infty\big(W \cap \sigma^k V(\underline{t}, k)\big) > 0. \tag{7}$$

Since $\nu^\infty(V(\underline{t}, k)) = 0$ for all $\underline{t} \in \Delta$ and $k \geqslant 1$ we may wonder whether we may use (7) in arguments proving some ν^∞-a.e. result. The answer is in the following

LEMMA 5.2. – *Let $V, W \subset \Delta$ be such that $\nu^\infty(V), \nu^\infty(W) > 0$. Then for ν^∞-a.e. $\underline{t} \in V$ there is a $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$ and every $\eta > 0$*

$$\nu^\infty\big\{\underline{s} \in V : d(\underline{s}, \underline{t}) < \eta \text{ and } \sigma^k \underline{s} \in W\big\} > 0.$$

Hence we may not have (7) but we know we can choose with positive probability a vector in V arbitrarily close to $\underline{t}$ whose kth shift is in W. This will be enough for our purposes.

Proof. – Let $V \subset \Delta$ be such that $\nu^\infty(V) > 0$. For every $n \geqslant 1$ and $j \geqslant 1$ let $K_{n,j}$ be a compact set inside V such that $\nu^\infty(V \setminus K_{n,j}) < (n \cdot 2^j)^{-1}$

and $E(1_V|\mathcal{B}_j)_{|K_{n,j}}$ is continuous—we are using Luzin's theorem (v. [13, Chapter IV, Section 21]). Then $C_n = \bigcap_{j\geqslant 1} K_{n,j}$ is a compact subset of V, $\nu^\infty(V \setminus C_n) \leqslant n^{-1}$ and $E(1_V|\mathcal{B}_j)_{|C_n}$ is continuous for every $j, n \geqslant 1$.

We have $V = \bigcup_{n\geqslant 1} C_n$, ν^∞ mod 0 and so ν^∞-a.e. $\underline{t} \in V$ is in some C_n, $n \geqslant 1$. Moreover ν^∞-a.e. $\underline{t} \in V$ is a ν^∞-density point of some C_n and we may suppose $\nu^\infty(C_n) > 0$ for all $n \geqslant 1$ (otherwise we consider only $n \geqslant n_0$ for some big $n_0 \in \mathbb{N}$).

From now on we suppose $\underline{t}$ is V-generic and a ν^∞-density point of some C_n with $\nu^\infty(C_n) > 0$. We let $W \subset \Delta$ be such that $\nu^\infty(W) > 0$, set $\delta = \frac{1}{4}\nu^\infty(W) > 0$ and let $k_0 \in \mathbb{N}$ be such that $\nu^\infty(\sigma^k V(\underline{t}, k)) \geqslant 1 - \delta$, for every $k \geqslant k_0$ by Lemma 5.1. By the choice of $\underline{t}$ and C_n we have $\nu^\infty(B(\underline{t}, \eta) \cap C_n) > 0$ for all $\eta > 0$ and for some $\eta_0 > 0$ we have further that, fixing $k \geqslant k_0$,

$$d(\underline{s}, \underline{t}) < \eta_0, \underline{s} \in C_n \Rightarrow \nu^\infty\big(\sigma^k V(\underline{s}, k)\big) \geqslant 1 - 2\delta$$

by the continuity of $E(1_V|\mathcal{B}_k)_{|C_n}$ at $\underline{t}$. Therefore we deduce that

$$d(\underline{s}, \underline{t}) < \eta_0, \underline{s} \in C_n \Rightarrow \nu^\infty\big(W \cap \sigma^k V(\underline{s}, k)\big) \geqslant 2\delta > 0$$

and so, for any $\eta > 0$, we have

$$\begin{aligned}
&\nu^\infty\{\underline{s} \in V\colon d(\underline{s}, \underline{t}) < \eta \text{ and } \sigma^k \underline{s} \in W\} \\
&\quad \geqslant \nu^\infty\{\underline{s} \in V\colon d(\underline{s}, \underline{t}) < \eta_1 = \min\{\eta_0, \eta\} \text{ and } \sigma^k \underline{s} \in W\} \\
&\quad \geqslant \int_{B(\underline{t},\eta_1)\cap C_n} 1_W(\sigma^k \underline{s})\, d\nu^\infty(\underline{s}) \\
&\quad = \int_{B(\underline{t},\eta_1)\cap C_n} \int_{\sigma^k V(\underline{s},k)} 1_W(\underline{u})\, d\nu^\infty(\underline{u})\, d\nu^k(\underline{s}) \\
&\quad = \int_{B(\underline{t},\eta_1)\cap C_n} \nu^\infty\big(W \cap \sigma^k V(\underline{s}, k)\big)\, d\nu^k(\underline{s}) \\
&\quad \geqslant 2\delta \cdot \nu^k\big(B(\underline{t}, \eta_1) \cap C_n\big) \geqslant 2\delta \cdot \nu^\infty\big(B(\underline{t}, \eta_1) \cap C_n\big) > 0
\end{aligned}$$

where we have used Fubini's theorem and ν^k is as before in Lemma 5.1. □

In Section 13 a slight generalization of Lemma 5.1 will be needed.

DEFINITION 5.1. – *Given $V \subset \Delta$ and $\underline{t}, \underline{s} \in \Delta$ we define a* double section *through $\underline{t}$ and $\underline{s}$ at $k \geqslant 1$ by $V(\underline{t}, k, \underline{s}) = \{\underline{\varphi} \in V\colon \varphi_1 = t_1, \ldots, \varphi_k = t_k$ and $\varphi_{k+2} = s_1, \varphi_{k+3} = s_2, \ldots\}$.*

LEMMA 5.3. – *Let $V \subset \Delta$ be such that $\nu^\infty(V) > 0$. Then for ν^∞-a.e. $\underline{t} \in V$ and for every $0 < \gamma, \delta < 1$ there exists $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$ there is a set $W_k \subset V$ with the properties*

1. $\underline{t} \in W_k$;
2. $\nu^\infty(W_k) > 0$;
3. $\nu(p_{k+1} W_k(\underline{t}, k, \underline{s})) \geqslant 1 - \delta$ *for ν^∞-a.e. $\underline{t} \in W_k$ and $\underline{s}$ in a subset of Δ with ν^∞-measure $\geqslant 1 - \gamma$,*

where $p_k : \Delta \to B$ is the projection on the kth coordinate.

Proof. – (An application of Lemma 5.1 and Fubini's theorem.)

Defining $V_n = \{\underline{t} \in V\colon\ \nu^\infty(\sigma^k V(\underline{t}, k)) \geqslant 1 - \delta \cdot (1 - \gamma),\ \forall k \geqslant n\}$ we have $V_n \subset V_{n+1}$ and Lemma 5.1 says $V = \bigcup_{n \geqslant 1} V_n$, ν^∞ mod 0. We set $k_0 \in \mathbb{N}$ such that $\nu^\infty(V_k) \geqslant \frac{4}{5}\nu^\infty(V)$ for every $k \geqslant k_0$. Definition 5.1 and Fubini's theorem imply

$$1 - \delta \cdot (1 - \gamma) \leqslant \nu^\infty\big(\sigma^k V(\underline{t}, k)\big) = \int \nu\big[p_{k+1} V(\underline{t}, k, \underline{s})\big]\, d\nu^\infty(\underline{s})$$

for every $\underline{t} \in V_{k_0}$ and $k \geqslant k_0$. We define now for each $\underline{t} \in V_{k_0}$ and $k \geqslant k_0$ the set

$$W_k(\underline{t}) = \big\{\underline{s} \in \Delta\colon\ \nu\big[p_{k+1} V(\underline{t}, k, \underline{s})\big] \geqslant 1 - \delta\big\}$$

and by the last inequality we see that $\nu^\infty(W_k(\underline{t})) \geqslant 1 - \gamma$. Then defining for $k \geqslant k_0$

$$W_k = \bigcup \big\{V(\underline{t}, k, \underline{s})\colon\ \underline{t} \in V_{k_0} \text{ and } \underline{s} \in W_k(\underline{t})\big\},$$

we get

$$\begin{aligned}
\nu^\infty(W_k) &= \int_{V_{k_0}} \int_{W_k(\underline{t})} \nu\big[p_{k+1} V(\underline{t}, k, \underline{s})\big]\, d\nu^\infty(\underline{s})\, d\nu^k(\underline{t}) \\
&\geqslant \int_{V_{k_0}} (1 - \delta) \cdot \nu^\infty\big(W_k(\underline{t})\big)\, d\nu^k(\underline{t}) \\
&\geqslant (1 - \delta) \cdot (1 - \gamma) \cdot \nu^k(V_{k_0}) \\
&\geqslant (1 - \delta)(1 - \gamma) \cdot \nu^\infty(V_{k_0}) \\
&\geqslant (1 - \delta)(1 - \gamma) \cdot \frac{4}{5} \cdot \nu^\infty(V) > 0.
\end{aligned}$$

We finally note that ν^∞-a.e. $\underline{t} \in V$ is in every V_k for sufficiently big k. □

The following notions will be extremely useful. They are mere adaptations of the usual notions of ω-limit to the context of random parametric perturbations.

DEFINITION 5.2. – *We take z to be some point in M, U some subset of M, $\underline{t}$ some vector in Δ and define*

$$\begin{aligned}
\omega(z,\underline{t}) = \big\{ & w \in M\colon \exists n_1 < n_2 < \cdots \text{ in } \mathbb{N} \\
& \text{such that } f_{\underline{t}}^{n_j}(z) \to w \text{ when } j \to \infty \big\} \\
& \text{(the usual definition of } \omega\text{-limit for the orbit } \mathcal{O}(z,\underline{t})); \\
\omega(U,\underline{t}) = \big\{ & w \in M\colon \exists \{u_j\}_{j=1}^{\infty} \subset M \, \exists n_1 < n_2 < \cdots \text{ in } \mathbb{N} \\
& \text{such that } f_{\underline{t}}^{n_j}(u_j) \to w \text{ when } j \to \infty \big\} \\
& \text{(the } \omega\text{-limit of a set under a perturbation vector } \underline{t}); \\
\omega(z,\Delta) = \big\{ & w \in M\colon \exists \{\underline{\theta}^{(j)}\}_{j=1}^{\infty} \subset \Delta \, \exists n_1 < n_2 < \cdots \text{ in } \mathbb{N} \\
& \text{such that } f_{\underline{\theta}^{(j)}}^{n_j}(z) \to w \text{ when } j \to \infty \big\} \\
& \text{(the } \omega\text{-limit of a point under every perturbation}); \\
\omega(U,\Delta) = \big\{ & w \in M\colon \exists \{u_j\}_{j=1}^{\infty} \subset M \, \exists \{\underline{\theta}^{(j)}\}_{j=1}^{\infty} \subset \Delta \, \exists n_1 < n_2 < \cdots \text{ in } \mathbb{N} \\
& \text{such that } f_{\underline{\theta}^{(j)}}^{n_j}(u_j) \to w \text{ when } j \to \infty \big\} \\
& \text{(the same as before with respect to a set)}.
\end{aligned}$$

LEMMA 5.4. – *Let us suppose U to be a subset of M whose orbits, under a positive ν^{∞}-measure set $V \subset \Delta$ of perturbations, go through a finite family of pairwise separated open sets $A_0, \ldots, A_{l-1}$ in a cyclic way, that is*

$$f_V^k(U) \subset A_{k \bmod l}, \quad \forall k \geqslant 1 \tag{8}$$

(example: the set $\mathcal{U}_0$ of an invariant domain $D \in \mathcal{D}$ with respect to $\mathcal{U}_0, \ldots, \mathcal{U}_{r-1}$).

Then the set $\omega(U,\underline{\theta})$ of accumulation points of the orbit of U under a V-generic perturbation $\underline{\theta} \in V$ is such that $\omega(U,\underline{\theta}) \subset \overline{A}_0 \cup \cdots \cup \overline{A}_{l-1}$ and if $z \in \omega(U,\underline{\theta}) \cap \overline{A}_i$ with $0 \leqslant i \leqslant l-1$ and $\underline{\psi} \in \Delta$, then $f_{\underline{\psi}}^k(z) \in \overline{A}_{(i+k) \bmod l}$ for all $k \geqslant 1$.

LEMMA 5.5. – *If in the last lemma we had $V = \Delta$ then the set $\omega(U,\Delta)$, besides having orbits that go in a cyclic way through the $\overline{A}_i$, $i = 0, \ldots, l-1$, under* any *perturbation, would also be invariant under* every *perturbation: $f_{\underline{\psi}}^k(z) \in \omega(U,\Delta)$ for all $k \geqslant 1$, for all $z \in \omega(U,\Delta)$ and for all $\underline{\psi} \in \Delta$.*

These lemmas essentially state that whenever we look at limits of *generic* perturbations we find a point whose perturbed orbit does *not* depend on the perturbation chosen, in the sense that it is carried cyclically through some specified family of sets. This property is the key idea behind the construction of s-invariant domains in Lemma 5.6

Proof of Lemma 5.4. – Let us fix $\underline{\psi} \in \Delta$ and $z \in \omega(U, \underline{\theta})$ with $\nu^\infty(\sigma^j V(, \underline{\theta}, j)) \to 1$ when $j \to \infty$.

Then there are sequences $\{u_j\}_{j=1}^\infty \subset U$ and $\{n_j\}_{j=1}^\infty \subset \mathbb{N}$ with $n_1 < n_2 < \cdots$ such that $z_j = f^{n_j}(u_j, \underline{\theta}) \to z$ when $j \to \infty$. It is clear that $z \in \overline{A}_0 \cup \cdots \cup \overline{A}_{l-1}$.

Let us now fix $k \in \mathbb{N}$ and assume $z \in \overline{A_i}$ for some $i \in \{0, \ldots, l-1\}$. We want to show that $f^k(z, \underline{\psi}) \in \overline{A}_{(i+k) \bmod l}$.

Once k is fixed, Property 2.1 implies that, for given $\delta > 0$, there are $\gamma, \upsilon > 0$ such that

$$d(\underline{\psi}, \underline{\varphi}) < \gamma,\ \underline{\varphi} \in \Delta \Rightarrow d_M\big(f^k(z, \underline{\psi}), f^k(z, \underline{\varphi})\big) < \delta;$$

$$d_M(z_1, z_2) < \upsilon,\ z_1, z_2 \in M,\ \underline{\varphi} \in \Delta \Rightarrow d_M\big(f^k(z_1, \underline{\varphi}), f^k(z_2, \underline{\varphi})\big) < \delta. \quad (9)$$

By Lemma 5.2 and the convergence of $\{z_j\}_{j=1}^\infty$, making $W = B(\underline{\psi}, \gamma/2)$ we may choose a sufficiently big $j \in \mathbb{N}$ such that $d_M(f^{n_j}(u_j, \underline{\theta}), z) < \upsilon/2$ and a sufficiently small $\eta > 0$ such that, with positive probability, there can be found $\underline{\varphi} \in V$ with

$$d(\underline{\varphi}, \underline{\theta}) < \eta,\ d\big(\sigma^{n_j}\underline{\varphi}, \underline{\psi}\big) < \gamma/2 \quad \text{and also} \quad d_M\big(f^{n_j}(u_j, \underline{\varphi}), z\big) < \upsilon. \quad (10)$$

Hence, by the choice of γ and υ we will have that:

$$\begin{aligned} &d_M\big(f^k(z, \underline{\psi}), f^k\big(z_j, \sigma^{n_j}\underline{\varphi}\big)\big) \\ &\quad \leqslant d_M\big(f^k(z, \underline{\psi}), f^k\big(z, \sigma^{n_j}\underline{\varphi}\big)\big) + d_M\big(f^k\big(z, \sigma^{n_j}\underline{\varphi}\big), f^k\big(z_j, \sigma^{n_j}\underline{\varphi}\big)\big) \\ &\quad \leqslant \delta + \delta = 2\delta. \end{aligned}$$

But we can take $\upsilon > 0$ so small that besides (10) and we get

$$d_M(w, z) < \upsilon,\ w \in A_0 \cup \cdots \cup A_{l-1} \Rightarrow z \in A_i. \quad (11)$$

With this we have $z_j \in A_i$ and also $f^k(z_j, \sigma^{n_j}\underline{\varphi}) \in A_{(i+k) \bmod l}$ by the hypothesis (8), with $\delta > 0$ arbitrary, and the lemma follows immediately. □

Proof of Lemma 5.5. – Let us take $z \in \omega(U, \Delta)$ and suppose $z \in \overline{A_i}$ for some $i \in \{0, \ldots, l-1\}$. We fix $k \geqslant 1$ and $\underline{\psi} \in \Delta$.

Then there are $\{\underline{\theta}^{(j)}\}_{j=1}^{\infty} \subset \Delta$, $\{u_j\}_{j=1}^{\infty} \subset U$ and $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ with $n_1 < n_2 < \cdots$ in such a way that $z_j = f^{n_j}(u_j, \underline{\theta}^{(j)}) \to z$ when $j \to \infty$.

For $\delta > 0$ let us take $\upsilon > 0$ as in (9), and υ so small that (11) holds. Moreover, let $j_0 \in \mathbb{N}$ be such that $j \geqslant j_0 \Rightarrow d_M(f^{n_j}(u_j, \underline{\theta}^{(j)}), z) < \upsilon$.

We now have

$$d_M\big(f^k(z, \underline{\psi}), f^k(z_j, \underline{\psi})\big) = d_M\big(f^k(z, \underline{\psi}), f^{k+n_j}\big(u_j, \tilde{\underline{\theta}}^{(j)}\big)\big) \leqslant \delta$$

for $j \geqslant j_0$, where $\tilde{\underline{\theta}}^{(j)} = (\theta_1^{(j)}, \ldots, \theta_{n_j}^{(j)}, \psi_1, \ldots, \psi_k, \psi_{k+1}, \ldots) \in \Delta$. But $\delta > 0$ is arbitrary, thus we get that

$$f^{k+n_j}\big(u_j, \tilde{\underline{\theta}}^{(j)}\big) \to f_{\underline{\psi}}^k(z) \quad \text{when } j \to \infty.$$

Now we see that for $f^k(z, \underline{\psi})$ there exist $\{\tilde{\underline{\theta}}^{(j)}\}_{j=1}^{\infty} \subset \Delta$, $\{u_j\}_{j=1}^{\infty} \subset U$ and $\{k + n_j\}_{j+1}^{\infty} \subset \mathbb{N}$ with $k + n_1 < k + n_2 < \cdots$ in such a way that $f^k(z, \underline{\psi}) \in \omega(U, \Delta)$. □

We state the following lemma (which should be a corollary of the previous two) with a slight abuse of language: we say an invariant domain $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ contains (is contained by) a set C if $\mathcal{U}_0 \cup \cdots \cup \mathcal{U}_{r-1} \supset C$ (respectively, $C \supset \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_{r-1}$).

LEMMA 5.6. – *If C is a c-invariant set contained in some domain $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ invariant with respect to a system $\mathcal{F}_{a,\varepsilon}$ under parametric noise satisfying hypothesis* A) *of Theorem* 1, *then it contains some s-invariant domain.*

Proof. – Let C and D be as stated and let us consider $X = \omega(C, \Delta)$ (cf. Definition 5.2).

By Lemma 5.5 we know that $X \subseteq \overline{C} \subset \overline{\mathcal{U}}_0 \cup \cdots \cup \overline{\mathcal{U}}_{r-1}$ is a c-invariant set whose points are carried cyclically through the $\overline{\mathcal{U}}_i$, $i = 0, 1, \ldots, r-1$.

By hypothesis A) of Theorem 1 it holds that $\operatorname{int}(X) \neq \emptyset$. Thus the collection $\widetilde{D} = (\mathcal{U}_0 \cap \operatorname{int}(X), \ldots, \mathcal{U}_{r-1} \cap \operatorname{int}(X))$ is a member of $\mathcal{D}$, an s-invariant domain.

Indeed, since the f_t are diffeomorphisms for all $t \in B$, the interior of X must be sent into the interior of X. But, by Lemma 5.5, the orbits of points of X must respect the cyclic order of the $\mathcal{U}_i$, $i = 0, \ldots, r-1$.

We conclude that X contains an s-invariant domain in its interior (the open sets forming $\widetilde{D}$ are pairwise separated by construction). Since $X \subset \overline{C}$, we have the same for C. □

DEFINITION 5.3. – *Let $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ be an s-invariant domain ($D \in \mathcal{D}$) and $z \in M$. We define $G(z) = G_D(z) = \{\underline{t} \in \Delta\colon\ \exists n \in \mathbb{N}$ such that $f^n_{\underline{t}}(z) \in D\}$ and $H(z) = H_D(z) = \Delta \setminus G(z)$, the perturbation vectors that will send z into D and those that never do so, respectively.*

LEMMA 5.7. – *Let us suppose that $z \in M$ is such that $\nu^\infty(H_D(z)) > 0$ for some $D \in \mathcal{D}$ and $\underline{t}$ is a H-generic vector ($H = H(z) = H_D(z)$).*
Then $H(w) = H_D(w) = \Delta$ for every $w \in \omega(z, \underline{t})$.

This lemma assures that those points whose perturbed orbits never fall in some invariant domain D for *many* (ν^∞-positive measure) perturbations have ω-limit points (under generic perturbations) which are *never* sent into the same domain D by every perturbation. This is another "independence of perturbation" property for the orbits of ω-limit points.

Proof. – Let us fix a H-generic perturbation vector $\underline{t}$ and $w \in \omega(z, \underline{t})$.
By contradiction, let us suppose there are $\underline{s} \in \Delta$ and $n \in \mathbb{N}$ such that $f^n_{\underline{s}}(w) \in D$. Then there must be a neighborhood U_w of w in M and a neighborhood $V_{\underline{s}}$ of $\underline{s}$ in Δ such that $f^n(U_w \times V_{\underline{s}}) \subseteq D$ by the continuity of $f^n : M \times \Delta \to M$ (by Property 2.1).
But $w \in \omega(z, \underline{t})$ and $\underline{t}$ is H-generic, thus there are $k \in \mathbb{N}$ and $\underline{\theta} \in H$ very close to $\underline{t}$, with positive probability, such that $f^k(z, \underline{t}) \in U_w$ and $\sigma^k\underline{\theta} \in V_{\underline{s}}$ by Lemma 5.2, since $\nu^\infty(H) > 0$. Therefore $f^{k+n}(z, \underline{\theta}) \in D$ contradicting $\underline{\theta} \in H$. □

LEMMA 5.8. – *Let z be a point of M and V a subset of Δ, with $\nu^\infty(V) > 0$, such that for ν^∞-a.e. vector $\underline{t} \in V$ and every $w \in \omega(z, \underline{t})$ there is $\underline{s} \in \Delta$ ($\underline{s} = \underline{s}(\underline{t}, w)$) such that the orbit $\mathcal{O}(w, \underline{s})$ eventually falls in some minimal invariant domain*:

$$\exists \mathcal{M} = \mathcal{M}(\underline{s}) \text{ minimal } \exists n = n(\underline{s}) \in \mathbb{N}\colon\ f^n_{\underline{s}}(w) \in \mathcal{M}.$$

Then we will have a $\preceq$-minimal domain $\mathcal{M}$, a set $W \subseteq V$, with $\nu^\infty(W) > 0$, and a $m \in \mathbb{N}$ such that $f^m_{\underline{\theta}}(z) \in \mathcal{M}$ for every $\underline{\theta}$ in W.

Let us observe that the hypothesis does not prevent the point from being sent into different invariant domains by different perturbations, but the lemma ensures there will be a positive measure set of perturbation vectors sending the point *into the same invariant domain*! In other words, the system under parametric noise cannot be unstable to the extent of sending a given point into completely different places by nearby perturbations.

Proof. – As in the proof of Lemma 5.2 let us fix $\delta > 0$ and a compact C contained in V such that $\nu^\infty(V \setminus C) < \delta$ and $E(1_V|\mathcal{B}_j)_{|C}$ is continuous for every $j \geqslant 1$. We may assume $\nu^\infty(C) > 0$.

Now we take $\underline{t} \in C$ such that $\underline{t}$ is both V-generic and a ν^∞-density point of C.

Let w be a point in $\omega(z, \underline{t})$ and $\{n_j\}_{j=1}^\infty \subseteq \mathbb{N}$ a sequence $n_1 < n_2 < \cdots$ such that $f_{\underline{t}}^{n_j}(z) \to w$ when $j \to \infty$. We will fix, from the hypothesis, a minimal domain $\mathcal{M}$, an integer $k \in \mathbb{N}$ and a perturbation vector $\underline{\theta} \in \Delta$ such that $f^k(w, \underline{\theta}) \in \mathcal{M}$.

Since $\mathcal{M}$ is open and $f^k : M \times \Delta \to M$ is continuous (see Property 2.1), there are neighborhoods U_w of w in M and $U_{\underline{\theta}}$ of $\underline{\theta}$ in Δ such that $f^k(U_w \times U_{\underline{\theta}}) \subseteq \mathcal{M}$.

By the choice of w and $\underline{t}$ there is $m \in \mathbb{N}$ with the property

$$j \geqslant m \Rightarrow \big(f_{\underline{t}}^{n_j}(z) \in U_w \text{ and } \beta = \nu^\infty\big(\sigma^{n_j} V(\underline{t}, n_j) \cap U_{\underline{\theta}}\big) > 0\big).$$

Because $E(1_V|\mathcal{B}_j)_{|C}$ is continuous, there is $\rho > 0$ such that

$$\underline{s} \in B(\underline{t}, \rho) \cap C \Rightarrow \big|E(1_V|\mathcal{B}_{n_m})(\underline{s}) - E(1_V|\mathcal{B}_{n_m})(\underline{t})\big| < \frac{\beta}{2}$$

and $f^{n_m}(z, B(\underline{t}, \rho)) \subseteq U_w$ by the continuity of $f^{n_m} : M \times \Delta \to M$.

Then we have $\nu^\infty(\sigma^{n_m} V(\underline{s}, n_m) \cap U_{\underline{\theta}}) \geqslant \beta/2 > 0$ for every $\underline{s} \in B(\underline{t}, \rho) \cap C$ because

$$\begin{aligned}
&\big|\nu^\infty\big(\sigma^{n_m} V(\underline{t}, n_m) \cap U_{\underline{\theta}}\big) - \nu^\infty\big(\sigma^{n_m} V(\underline{s}, n_m) \cap U_{\underline{\theta}}\big)\big| \\
&\quad = \nu^\infty\big[\big(U_{\underline{\theta}} \cap \sigma^{n_m} V(\underline{t}, n_m)\big) \Delta \big(U_{\underline{\theta}} \cap \sigma^{n_m} V(\underline{s}, n_m)\big)\big] \\
&\quad = \nu^\infty\big[U_{\underline{\theta}} \cap \big(\sigma^{n_m} V(\underline{t}, n_m) \Delta \sigma^{n_m} V(\underline{s}, n_m)\big)\big] \\
&\quad \leqslant \nu^\infty\big(\sigma^{n_m} V(\underline{t}, n_m) \Delta \sigma^{n_m} V(\underline{s}, n_m)\big) \\
&\quad = \big|\nu^\infty\big(\sigma^{n_m} V(\underline{t}, n_m)\big) - \nu^\infty\big(\sigma^{n_m} V(\underline{s}, n_m)\big)\big| < \frac{\beta}{2}.
\end{aligned}$$

It follows that

$$W = \bigcup_{\underline{s} \in B(\underline{t}, \rho) \cap C} \big\{(s_1, \ldots, s_{n_m}, u_1, u_2, \ldots) : \underline{u} \in \big(\sigma^{n_m} V(\underline{s}, n_m) \cap U_{\underline{\theta}}\big)\big\}$$

is a subset of V such that

$$\nu^{\infty}(W) = \int_{B(\underline{t},\rho)\cap C} \nu^{\infty}\big(\sigma^{n_m} V(\underline{s}, n_m) \cap U_{\underline{\theta}}\big)\, d\nu^{n_m}(\underline{s})$$
$$\geqslant \frac{\beta}{2} \cdot \nu^{\infty}\big(B(\underline{t}, \rho) \cap C\big) > 0$$

because $\underline{t}$ is a ν^{∞}-density point of C. Moreover

$$f^{n_m}(z, W) \subseteq f^{n_m}\big(z, \big(B(\underline{t}, \rho) \cap C\big)\big) \subseteq f^{n_m}\big(z, B(\underline{t}, \rho)\big) \subseteq U_w$$

and $f^k(U_w \times U_{\underline{\theta}}) \subseteq \mathcal{M}$ with $\sigma^{n_m} W \subseteq U_{\underline{\theta}}$. Thus $f^{n_m+k}(z, W) \subseteq \mathcal{M}$, completing the proof of the lemma. □

6. FINITE NUMBER OF MINIMAL INVARIANT DOMAINS

Two basic properties of the members of $\mathcal{D}$ are the following direct consequences of hypothesis A) of Theorem 1 and Definitions 3.2 and 3.3.

PROPERTY 6.1. – *Any s-invariant domain $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ is such that every open set $\mathcal{U}_i$ contains some ball of radius $\xi_0 > 0$, $i = 0, \ldots, r-1$. Consequently, each open set has a volume (m measure) greater than some constant $l_0 > 0$.*

PROPERTY 6.2. – *The period of any invariant domain $D \in \mathcal{D}$ is bounded from above by a constant $T_p \in \mathbb{N}$ dependent on l_0 ($T_p \leqslant 1/l_0$).*

6.1. Minimals exist

We start by showing that Zorn's lemma can be applied to the partially ordered set $(\mathcal{D}, \preceq)$ of completely and symmetrically invariant domains of M. Having established this, we conclude that there are minimal invariant domains in M.

Let $\mathcal{C}$ be a $\preceq$-chain in $(\mathcal{D}, \preceq)$, that is, if $D, D' \in \mathcal{C}$ then either $D \preceq D'$ or $D' \preceq D$. By Property 6.2, the domains of $\mathcal{C}$ have a finite number of distinct periods. So if $\rho : \mathcal{C} \to \mathbb{N}$ is the map that associates to each $D \in \mathcal{C}$ its period $\rho(D) \in \mathbb{N}$, then $\rho(\mathcal{C}) = \{r_1, \ldots, r_l\}$ and $\mathcal{C} = \bigcup_{j=1}^{l} \rho^{-1}\{r_j\}$. We need to find a lower bound for $\mathcal{C}$ in $(\mathcal{D}, \preceq)$. We can suppose that $\mathcal{C}$ does not have a minimum, otherwise we would have nothing to prove. Now we establish

CLAIM 6.1. – *There is a $j_0 \in \{1, \ldots, l\}$ such that the subchain $\mathcal{S} = \rho^{-1}\{r_{j_0}\}$ does not have a lower bound in $\mathcal{C}$. Moreover $\mathcal{S}$* precedes *every element of $\mathcal{C}$: for all $D \in \mathcal{C}$ there is a $D' \in \mathcal{S}$ such that $D' \preceq D$.*

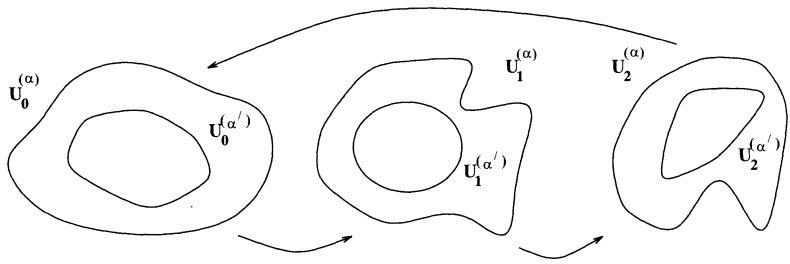

Fig. 3. $D_\alpha \prec D_{\alpha'}$ with $D_\alpha, D_{\alpha'}$ in a subchain of period three after suitable arrangement of indexes.

Indeed, if every subchain of constant period $\mathcal{S}_j = \rho^{-1}\{r_j\}$ had a lower bound $D_j \in \mathcal{C}$ for $j = 1, \ldots, l$, then the minimum of the subchain $\mathcal{S}' = \{D_1, \ldots, D_l\} \subseteq \mathcal{C}$ (which always exists because $\mathcal{S}'$ is finite) would be a minimum for $\mathcal{C}$, in contradiction to the supposition we started with. So there is some $\mathcal{S} = \mathcal{S}_{j_0}$ without a lower bound in $\mathcal{C}$.

Now for the second part of the claim. Let us suppose, by contradiction, that there is a $\widetilde{D} \in \mathcal{C}$ such that $D \not\preceq \widetilde{D}$ for every $D \in \mathcal{S}$. But we are within a chain, thus $\widetilde{D} \prec D$ for all $D \in \mathcal{S}$, that is, $\widetilde{D}$ would be a lower bound for $\mathcal{S}$ in $\mathcal{C}$, and this contradiction proves the claim.

Now we just need to show that $\mathcal{S}$ has some lower bound in $(\mathcal{D}, \preceq)$ in order to get a lower bound for $\mathcal{C}$.

To do that, let us first observe that $\mathcal{S}$ is made by nested invariant domains of equal period, all symmetrically invariant. Thus we can always write $D \in \mathcal{S}$ as $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ and, for any other $D' = (\mathcal{U}'_0, \ldots, \mathcal{U}'_{r-1})$, we can never have two different $\mathcal{U}'_i, \mathcal{U}'_j$ intersect the same $\mathcal{U}_k$, $i, j, k \in \{0, \ldots, r-1\}$ and $i \neq j$ (see Fig. 3 for a representation of $\mathcal{S}$ with period three).

Hence we can rearrange the lower indexes of the open sets that form the domains of $\mathcal{S}$ in order to obtain $\mathcal{S} = \{D_\alpha\}_{\alpha \in \mathcal{A}}$ with $D_\alpha = (\mathcal{U}_0^{(\alpha)}, \ldots, \mathcal{U}_{r-1}^{(\alpha)})$ for $\alpha \in \mathcal{A}$, $\mathcal{A}$ some set of indexes, and satisfying the following property

$$D_\alpha \preceq D_{\alpha'} \Leftrightarrow \left(\mathcal{U}_i^{(\alpha)} \subseteq \mathcal{U}_i^{(\alpha')},\ i = 0, \ldots, r-1\right);$$

for all $\alpha, \alpha' \in \mathcal{A}$.

We can now consider the intersections $\widetilde{\mathcal{U}}_i = \bigcap_{\alpha \in \mathcal{A}} \mathcal{U}_i^{(\alpha)}$, $i = 0, \ldots, r-1$, and observe that, because each D_α is s-invariant, the family $(\widetilde{U}_0, \ldots, \widetilde{U}_{r-1})$ satisfies

$$f^k(\widetilde{\mathcal{U}}_i, \Delta) \subseteq \widetilde{\mathcal{U}}_{(k+i) \bmod r}, \quad \forall k \geqslant 1 \ \ \forall i = 0, \ldots, r-1 \tag{12}$$

and since fixing $\alpha_0 \in \mathcal{A}$ we have $\widetilde{\mathcal{U}}_i \subset \mathcal{U}_i^{(\alpha_0)}$ for $i = 0, \ldots, r-1$, the $\widetilde{\mathcal{U}}_0, \ldots, \widetilde{\mathcal{U}}_{r-1}$ are pairwise separated, because the $\mathcal{U}_0^{(\alpha_0)}, \ldots, \mathcal{U}_{r-1}^{(\alpha_0)}$ already were pairwise separated.

Finally, hypothesis A) of Theorem 1 and (12) ensure that every $\widetilde{\mathcal{U}}_i$ has nonempty interior ($i = 0, \ldots, r-1$). Since the f_t are diffeomorphisms for $t \in B$, hence open maps, we deduce that $\widetilde{D} = (\mathrm{int}(\widetilde{\mathcal{U}}_0), \ldots, \mathrm{int}(\widetilde{\mathcal{U}}_{r-1}))$ is an s-invariant domain of $\mathcal{D}$ which clearly is a lower bound for the subchain $\mathcal{S}$. Consequently we got a lower bound for the chain $\mathcal{C}$ we started with and proved that Zorn's lemma can be applied to $(\mathcal{D}, \preceq)$.

Moreover, it is easy to see that *each member of $\mathcal{D}$ contains a minimal domain.*

In fact, let us now fix $D_0 \in \mathcal{D}$ and consider the partially ordered set $(\mathcal{D}_{D_0}, \preceq)$, where $\mathcal{D}_{D_0} = \{D \in \mathcal{D}\colon\ D \preceq D_0\}$. Since it can be shown that each chain of $(\mathcal{D}_{D_0}, \preceq)$ has a lower bound in $\mathcal{D}_{D_0}$, in the same way we did before, there must be some minimal domain in $(\mathcal{D}_{D_0}, \preceq)$ which, by the definition of $\mathcal{D}_{D_0}$, is also a minimal domain of $(\mathcal{D}, \preceq)$.

We conclude that each domain in $\mathcal{D}$ contains a minimal domain of $(\mathcal{D}, \preceq)$.

6.2. Minimals are pairwise disjoint

Let us now observe that, because each open set of the collection that forms an invariant domain has a volume (Riemannian measure m on M) of at least $l_0 > 0$ by Property 6.1, to prove there is a finite number of $\preceq$-minimals we need only show they are pairwise disjoint.

Let $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ and $D' = (\mathcal{U}'_0, \ldots, \mathcal{U}'_{r'-1})$ be two minimals of $(\mathcal{D}, \preceq)$ whose open sets have some intersection, $\mathcal{U}_i \cap \mathcal{U}'_j \neq \emptyset$ say, for some $i \in \{0, \ldots, r-1\}$ and $j \in \{0, \ldots, r'-1\}$.

Because both D and D' are s-invariant, we have for all $k \geqslant 1$

$$f_\Delta^k(\mathcal{U}_i \cap \mathcal{U}'_j) \subset f_\Delta^k(\mathcal{U}_i) \subset \mathcal{U}_{(i+k) \bmod r} \quad \text{and}$$
$$f_\Delta^k(\mathcal{U}_i \cap \mathcal{U}'_j) \subset f_\Delta^k(\mathcal{U}'_j) \subset \mathcal{U}'_{(j+k) \bmod r'}$$

and thus $f_\Delta^k(\mathcal{U}_i \cap \mathcal{U}'_j) \subset \mathcal{U}_{(i+k) \bmod r} \cap \mathcal{U}'_{(j+k) \bmod r'}$. Therefore if we define

$$\widehat{D} = (\mathcal{U}_i \cap \mathcal{U}'_j, \mathcal{U}_{(i+1) \bmod r} \cap \mathcal{U}'_{(j+1) \bmod r'}, \ldots,$$
$$\mathcal{U}_{(i+[r,r']-1) \bmod r} \cap \mathcal{U}'_{(j+[r,r']-1) \bmod r'})$$

we will get $\hat{D} \in \mathcal{D}$ (here $[r, r']$ is the least common multiple of r and r').

The invariance property is clear. Let us check that the open sets forming $\hat{D}$ are pairwise *separated*. Indeed, if we had

$$\overline{\mathcal{U}_{(i+k_1) \bmod r} \cap \mathcal{U}'_{(j+k_1) \bmod r'}} \cap \overline{\mathcal{U}_{(i+k_2) \bmod r} \cap \mathcal{U}'_{(j+k_2) \bmod r'}} \neq \emptyset$$

with $0 \leqslant k_1 < k_2 \leqslant [r, r'] - 1$ then, in particular,

$$\overline{\mathcal{U}}_{(i+k_1) \bmod r} \cap \overline{\mathcal{U}}_{(j+k_2) \bmod r} \neq \emptyset \quad \text{and}$$
$$\overline{\mathcal{U}}'_{(i+k_1) \bmod r'} \cap \overline{\mathcal{U}}'_{(j+k_2) \bmod r'} \neq \emptyset.$$

However by Definitions 3.2 and 3.3 we conclude that $k_1 \equiv k_2 \pmod{r}$ and $k_1 \equiv k_2 \pmod{r'}$ with $0 \leqslant k_1 < k_2 \leqslant [r, r'] - 1$, contradicting the Chinese Remainder Theorem.

We have now $\hat{D} \preceq D$ and $\hat{D} \preceq D'$, so the minimality of both D and D' implies $D = \hat{D} = D'$. We have shown that if two $\preceq$-minimals intersect then they are equal. Consequently, we have that they are pairwise disjoint and, as mentioned above, we conclude there is a finite number of minimals in $(\mathcal{D}, \preceq)$.

6.3. Minimals are transitive

The following is an expression of the dynamical indivisibility of minimal invariant domains.

LEMMA 6.1. – *Every minimal invariant domain $\mathcal{M} = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ is transitive in the following sense. For every $z \in \mathcal{M}$ (meaning $z \in \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_{r-1}$) the sequence $\{f_\Delta^n(z)\}_{n=1}^\infty$ is dense in $\mathcal{M}$.*

We will say that minimal invariant domains are *randomly transitive* or *r-transitive* when referring to this kind of transitiveness.

Proof. – In fact, let $\mathcal{M} = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1}) \in \mathcal{D}$ be a minimal and let us take some point $z \in \mathcal{U}_i$ with $i \in \{0, \ldots, r-1\}$ and $X = \omega(z, \Delta)$ (cf. Definition 5.2).

By Lemma 5.5, we have $X \subseteq \overline{\mathcal{M}} = \overline{\mathcal{U}}_0 \cup \cdots \cup \overline{\mathcal{U}}_{r-1}$, X is c-invariant and goes cyclically through the $\overline{\mathcal{U}}_0, \ldots, \overline{\mathcal{U}}_{r-1}$, under every perturbation vector of Δ. Besides, by Lemma 5.6 there is $D \in \mathcal{D}$ such that $D \subset X$. So $D \preceq \mathcal{M}$, in contradiction with the $\preceq$-minimality of $\mathcal{M}$.

Hence it must be that $\mathcal{M} = D$ and then $\{f_\Delta^n(z)\}_{n=1}^\infty$ is dense in $\mathcal{U}_0 \cup \cdots \cup \mathcal{U}_{r-1}$, as stated. □

Given a minimal $\mathcal{M} = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$, since it is s-invariant, it will also be invariant with respect to f_t for every $t \in T$, because the vector $(t, t, t, \ldots)$ is in Δ.

This means we have $f_t^k(\mathcal{U}_i) \subset \mathcal{U}_{(i+k) \bmod r}$ for all $k \geqslant 1$ and $i = 0, \ldots, r-1$.

However, we cannot state any kind of indivisibility for this domain with respect to f_t because the domain was originally a minimal domain, but with noise. The perturbations around the system f_a may have mixed, in a single collection of open sets, several attractors indivisible with respect to f_t, but that under random choices of parameters were indistinguishable. We cannot proceed further in this because we made no hypothesis about the dynamics of the f_t without noise.

7. STATIONARY PROBABILITY MEASURES

7.1. Existence and absolute continuity

Let z be a point of M. The formalization of the dynamics under noise by means of the operator S enables us to naturally associate a probability measure to the orbits of the system: the push-forward of ν^∞ from Δ to M via the map f^k given by $f^k(z, \nu^\infty)$, $k \geqslant 1$. We have defined this as the probability which integrates continuous functions $\varphi : M \to \mathbb{R}$ as

$$f^k(z, \nu^\infty)\varphi = \left[f^k(z, \cdot)_* \nu^\infty\right]\varphi = \int \varphi\left(f^k(z, \underline{t})\right) d\nu^\infty(\underline{t}), \quad k \geqslant 1.$$

These probabilities are not stationary in general, but if we consider their averages

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} f^j(z, \nu^\infty), \quad n = 1, 2, \ldots, \tag{13}$$

we obtain a sequence of probability measures in M which, by compactness of the space $\mathbb{P}(M)$ of probabilities measures over M with the weak topology, has some limit point $\mu_\infty = \lim_{i \to \infty} \frac{1}{n_i} \sum_{j=1}^{n_i} f^j(z, \nu^\infty)$. This means the integral of a continuous $\varphi : M \to \mathbb{R}$ with respect to μ_∞ is given by

$$\mu_\infty(\varphi) = \lim_{i \to \infty} \frac{1}{n_i} \sum_{j=1}^{n_i} \int \varphi\left(f^j(z, \underline{t})\right) d\nu^\infty(\underline{t}).$$

This accumulation point is a stationary probability. In fact,

$$\iint \varphi \circ f(w, s)\, d\mu_\infty(w)\, d\nu(s)$$

$$= \int \left[\lim_{i\to\infty} \frac{1}{n_i} \sum_{j=1}^{n_i} \int (\varphi \circ f_s)\big(f^j(z,\underline{t})\big)\, d\nu^\infty(\underline{t}) \right] d\nu(s)$$

and

$$\begin{aligned}
&\int \left[\frac{1}{n_i} \sum_{j=1}^{n_i} \int (\varphi \circ f_s)\big(f^j(z,\underline{t})\big)\, d\nu^\infty(\underline{t}) \right] d\nu(s) \\
&= \frac{1}{n_i} \sum_{j=1}^{n_i} \int\int \varphi\big(f_s \circ f_{t_j} \circ \cdots \circ f_{t_1}(z)\big)\, d\nu^\infty(\underline{t})\, d\nu(s) \\
&= \frac{1}{n_i} \sum_{j=1}^{n_i} \int \varphi\big(f_{\underline{t}}^{j+1}(z)\big)\, d\nu^\infty(\underline{t}) \\
&= \frac{1}{n_i} \sum_{j=1}^{n_i} \int \varphi\big(f_{\underline{t}}^{j}(z)\big)\, d\nu^\infty(\underline{t}) \\
&\quad + \frac{1}{n_i} \left[\int \varphi\big(f_{\underline{t}}^{n_i+1}(z)\big)\, d\nu^\infty(\underline{t}) - \int \varphi\big(f_{\underline{t}}(z)\big)\, d\nu^\infty(\underline{t}) \right]
\end{aligned}$$

for $i \geqslant 1$. Since $\sup_{w\in M} |\varphi(w)| = \|\varphi\|$ is finite, the second term of the last expression converges to zero when $i \to \infty$, while the first term gives the integral of φ with respect to μ_∞, that is

$$\begin{aligned}
\int \varphi\, d\mu_\infty &= \lim_{i\to\infty} \int \left[\frac{1}{n_i} \sum_{j=1}^{n_i} \int (\varphi \circ f_s)\big(f^j(z,\underline{t})\big)\, d\nu^\infty(\underline{t}) \right] d\nu(s) \\
&= \int \left[\lim_{i\to\infty} \frac{1}{n_i} \sum_{j=1}^{n_i} \int (\varphi \circ f_s)\big(f^j(z,\underline{t})\big)\, d\nu^\infty(\underline{t}) \right] d\nu(s) \\
&= \int\int \varphi \circ f_s(w)\, d\mu_\infty(w)\, d\nu(s),
\end{aligned}$$

where we have used the dominated convergence theorem to exchange the limit and the integral signs. In addition, because $C^0(M,\mathbb{R})$ is dense in $L^1(M,\mu_\infty)$ with the L^1-norm, we see the last identity holds for every μ_∞-integrable $\varphi : M \to \mathbb{R}$.

Moreover, if E is any Borel subset of M we can write

$$\begin{aligned}
\mu_\infty(E) &= \int 1_E\, d\mu_\infty = \int\int 1_E\big(f_t(x)\big)\, d\mu_\infty(x)\, d\nu(t) \\
&= \int\int\int 1_E\big(f_{t_2} \circ f_{t_1}(x)\big)\, d\mu_\infty(x)\, d\nu(t_1)\, d\nu(t_2) \\
&\ \ \vdots
\end{aligned}$$

$$= \int\int 1_E\big(f_{\underline{t}}^k(x)\big)\, d\mu_\infty(x)\, d\nu^\infty(\underline{t})$$
$$= \int\int 1_E\big(f_{\underline{t}}^k(x)\big)\, d\nu^\infty(\underline{t})\, d\mu_\infty(x)$$
$$= \int f^k(x,\nu^\infty)(E)\, d\mu_\infty(x)$$

for $k = 1, 2, 3\ldots$. Hypothesis B) of Theorem 1 guarantees that $f^k(x,\nu^\infty) \ll m$ for $k \geqslant K$. Thus $\mu_\infty(E) = 0$ whenever $m(E) = 0$. We have just proved

LEMMA 7.1. – *Given $z \in M$, any accumulation point of the averages* (13) *is a stationary absolutely continuous probability measure over M.*

Let us remark that $\mu_\infty = \mu_\infty(z)$ depends on $z \in M$ and the accumulation point of the averages (13) may not be unique.

7.2. Ergodicity and characteristic probabilities

Let us suppose $z \in D$ for some $D \in \mathcal{D}$. Then it is clear that $\operatorname{supp}\mu_\infty \subset \overline{D}$, whatever accumulation point of the averages (13) we choose. Moreover, by Remark 3.1 we have that $\overline{\mathcal{D}} = (\overline{\mathcal{U}}_0, \ldots, \overline{\mathcal{U}}_{r-1})$ satisfies (5) also. Thus if $w \in \operatorname{supp}\mu_\infty$, we get by hypothesis A) that $f^k(w,\Delta) \supset B(f_{t_0}^k(w), \xi_0)$ for all $k \geqslant K$, and by the invariance of the support we conclude $\operatorname{supp}\mu \cap D \neq \emptyset$ because $\operatorname{int}(\partial D) = \operatorname{int}(\partial\mathcal{U}_0 \cup \cdots \cup \partial\mathcal{U}_{r-1}) = \emptyset$. In addition, if z belonged to a minimal $\mathcal{M} \in \mathcal{D}$, then the invariance of the support, the fact that $\operatorname{supp}\mu \cap D \neq \emptyset$ and the r-transitiveness of $\mathcal{M}$ (given by Lemma 6.1) together imply $\operatorname{supp}\mu = \overline{\mathcal{M}}$.

LEMMA 7.2. – *If $\mathcal{M} \in \mathcal{D}$ is a minimal invariant domain and μ a stationary absolutely continuous probability measure with* $\operatorname{supp}\mu = \overline{\mathcal{M}}$, *then*

$$\varphi(x) = \int \varphi\big(f_t(x)\big)\, d\nu(t), \quad \mu\text{-a.e. } x \Rightarrow \varphi \text{ is } \mu\text{-a.e. constant}$$

for every bounded measurable function $\varphi : M \to \mathbb{R}$.

Proof. – What we want to prove is equivalent to the following for every Borel set E:

$$1_E(x) = \int 1_E\big(f_t(x)\big)\, d\nu(t), \quad \mu\text{-a.e. } x \Rightarrow \mu(E) = 0 \text{ or } 1. \tag{14}$$

Let E be a Borel set that satisfies the left hand side of (14) and let us suppose that $\mu(E) > 0$. Since $\mu \ll m$ we have $m(E) > 0$ and thus there is a closed $F \subseteq E$ such that $m(E \setminus F) = 0 = \mu(E \setminus F)$. Moreover the following holds μ-a.e.

$$1_F(x) = 1_E(x) = \int 1_E\big(f_t(x)\big)\, d\nu(t)$$
$$= \iint 1_E\big(f_t(f_s(x))\big)\, d\nu(s)\, d\nu(t). \tag{15}$$

In fact, let N be the set of those points x which do satisfy the left hand side identity of (14). Then $\mu(N) = 1$ and also $\mu \times \nu^\infty(N \times \Delta) = 1$. Since μ is stationary we have $\mu \times \nu^\infty(S^{-1}(N \times \Delta)) = 1$, that is (cf. Section 2.2) $\int \mu(f_s^{-1}(N))\, d\nu(s) = 1 \Leftrightarrow \mu(f_s^{-1}(N)) = 1$ for ν^∞-a.e. $s \in T$. Moreover the set

$$N_s = \left\{ x \in M\colon\ 1_E\big(f_s(x)\big) = \int 1_E\big(f_t(f_s(x))\big)\, d\nu(t) \right\}$$

is equal to $(f_s)^{-1}(N)$ for all $s \in T$. Therefore $\mu(N_s) = 1$ for ν-a.e. $s \in T$. This means $1_E(f_s(x)) = \int 1_E(f_t(f_s(x)))\, d\nu(t)$, for ν-a.e. $s \in T$ and μ-a.e. x.

In particular we get (15) when integrating both sides with respect to s.

Likewise we can have (15) with any number of compositions, that is

$$1_F(x) = 1_E(x) = \int 1_E\big(f^k(x, \underline{t})\big)\, d\nu^\infty(\underline{t}), \quad \mu\text{-a.e. } x,\ k = 1, 2, \ldots,$$

and we can write

$$\int 1_E\big(f^k(x, \underline{t})\big)\, d\nu^\infty(\underline{t}) = f^k(x, \nu^\infty)(E) = f^k(x, \nu^\infty)(F)$$
$$\text{for } k \geqslant K \tag{16}$$

by hypothesis B) of Theorem 1. From the last two identities we arrive at

$$1_F(x) = \int 1_F\big(f^k(z, \underline{t})\big)\, d\nu^\infty(\underline{t}), \quad \mu\text{-a.e. } x.$$

This identity implies that for μ_∞-a.e. $x \in F$ we have $f^k(x, \underline{t}) \in F$ for ν^∞-a.e. $\underline{t} \in \Delta$ and $k \geqslant K$. However, since $\underline{t} \in \Delta \mapsto f^k(x, \underline{t}) \in M$ is continuous for every fixed $k \geqslant 1$ and $f^k(x, \underline{t}) \in F$ for a dense set of vectors $\underline{t}$ in Δ (because ν^∞-a.e. implies density in Δ), we deduce that $f^k(x, \underline{t}) \in F$ for *all* $\underline{t} \in \Delta$ (because F is closed) and $k \geqslant K$. Then, if we

define $\mathcal{U}_i' = \mathcal{U}_i \cap \bigcup_{x\in F, k\geqslant K} \operatorname{int}(f^k(x,\Delta))$, $i = 0,\dots,r-1$, we see that the $\mathcal{U}_i' \subset F$ are open, nonempty (by hypothesis A) and because $\operatorname{supp}\mu = \overline{\mathcal{M}}$ and $\operatorname{int}(\overline{\mathcal{M}}) = \mathcal{M} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_{r-1}$) and so $D = (\mathcal{U}_0',\dots,\mathcal{U}_{r-1}')$ is an s-invariant domain.

In fact, fixing $y \in \mathcal{U}_i'$ for some $0 \leqslant i \leqslant r-1$, $\underline{s} \in \Delta$ and $n \geqslant 1$, there are $k \geqslant K$ and $\delta > 0$ such that $B(y,\delta) \subset f^k(x,\Delta)$ and $f_{\underline{s}}^n(B(y,\delta)) \subset f^{k+n}(x,\Delta) \subset \mathcal{U}_{i+n \bmod r}$ by definition of $\mathcal{U}_i'$. Hence $f_{\underline{s}}^n(y) \in \operatorname{int}(f^{k+n}(x,\Delta)) \cap \mathcal{U}_{i+n \bmod r}$ after Property 2.1(3).

We have built an s-invariant domain $D \in \mathcal{D}$ such that $D \preceq \mathcal{M}$. The minimality of $\mathcal{M}$ gives $D = \mathcal{M}$ and hence $F \supseteq \overline{\mathcal{M}}$, that is, $\mu(E) = \mu(F) \geqslant \mu(\mathcal{M}) = 1$. □

Lemma 7.2 implies that μ_∞ is ergodic, that is, $\mu_\infty \times \nu^\infty$ is S-ergodic. (For ease of writing we make $\mu = \mu_\infty$ in the following discussion.)

Indeed, let us assume that $\psi : M \times \Delta \to \mathbb{R}$ is an S-invariant bounded measurable function: $\psi(S(z,\underline{t})) = \psi(z,\underline{t})$, $\mu\times\nu^\infty$-a.e. $(z,\underline{t}) \in M \times \Delta$.

For each $k \geqslant 0$ we define

$$\psi_k(x,t_1,\dots,t_k) = \int \psi(x,t_1,t_2,\dots,t_k,t_{k+1},\dots)\,d\nu(t_{k+1})\,d\nu(t_{k+2})\dots$$

and we have, by the invariance of ψ,

$$\begin{aligned}\psi_0(x) &= \int \psi(x,t_1,t_2,\dots)\,d\nu(t_1)\,d\nu(t_2)\dots \\ &= \int \psi\big(f_{t_1}(x),t_2,t_3,\dots\big)\,d\nu(t_2)\,d\nu(t_3)\dots d\nu(t_1) \\ &= \int \psi_0\big(f_{t_1}(x)\big)\,d\nu(t_1), \quad \mu\text{-a.e. } x \in M.\end{aligned}$$

Therefore, by Lemma 7.2, we conclude that ψ_0 is μ-a.e. constant. In general, for $k \geqslant 1$,

$$\begin{aligned}\psi_k(x,t_1,\dots,t_k) &= \int \psi(x,t_1,t_2,\dots,t_k,t_{k+1},\dots)\,d\nu(t_{k+1})\,d\nu(t_{k+2})\dots \\ &= \int \psi\big(f_{t_1}(x),t_2,\dots,t_k,t_{k+1},\dots\big)\,d\nu(t_{k+1})\,d\nu(t_{k+2})\dots \\ &= \psi_{k-1}\big(f_{t_1}(x),t_2,\dots,t_k\big), \quad \mu\times\nu^k\text{-a.e. } (x,t_1,\dots,t_k).\end{aligned}$$

We then have $\psi_1 \equiv \psi_0$, $\mu\times\nu$-a.e.; $\psi_2 \equiv \psi_1$, $\mu\times\nu^2$-a.e.;... and so, by induction

$$\psi_k \equiv \psi_0 \equiv \text{constant}, \quad \mu\times\nu^k\text{-a.e., for every } k \geqslant 1.$$

However if we identify $\psi_k(x, \underline{t})$ with $\psi_k(x, t_1, \ldots, t_k)$, then ψ_k coincides with $E(\psi|\mathcal{B}_k)$, $\mu \times \nu^\infty$-a.e. and we have seen in Lemma 5.1 that $E(\psi|\mathcal{B}_k) \to \psi$, $\mu \times \nu^\infty$-a.e., when $k \to \infty$. Hence we have also $\psi \equiv$ constant, $\mu \times \nu^\infty$-a.e., and conclude that $\mu \times \nu^\infty$ is S-ergodic.

Ergodicity, Birkhoff's theorem and the absolute continuity imply that $\mu = \mu_\infty$ is physical. Indeed, $\operatorname{supp}\mu = \overline{\mathcal{M}} = E(\mu)$ μ-a.e. because $\mu(E(\mu)) = 1$ by Birkhoff's ergodic theorem. So, if $E = E(\mu)$, then $1_E(x) = \int 1_E(f(x,t))\, d\nu(t)$ for μ-a.e. x because E is invariant. Hence, recalling the proof of Lemma 7.2, we get $E = E(\mu) \supset \overline{\mathcal{M}}$ and $m(E(\mu)) \geqslant m(\mathcal{M}) > 0$ ($\mathcal{M}$ is a collection of open sets).

We easily deduce that any two physical probability measures μ_1, μ_2 whose support is $\overline{\mathcal{M}}$ must be equal. Indeed, since both $E(\mu_1)$ and $E(\mu_2)$ contain $\overline{\mathcal{M}}$, the time averages of every continuous $\varphi : M \to \mathbb{R}$ on the orbits of some $x \in \mathcal{M}$ must equal both $\int \varphi\, d\mu_1$ and $\int \varphi\, d\mu_2$.

The above arguments prove the existence of a *characteristic measure* for each minimal invariant domain.

PROPOSITION 7.3. – *Given a minimal $\mathcal{M} \in \mathcal{D}$ there is only one physical absolutely continuous probability measure whose support is contained in $\overline{\mathcal{M}}$. Moreover, every $x \in \overline{\mathcal{M}}$ is in the ergodic basin of this characteristic measure.*

8. DECOMPOSITION OF STATIONARY PROBABILITIES

Let μ be a stationary probability. Then $\operatorname{supp}\mu$ is a c-invariant set. By hypothesis A) of Theorem 1 we deduce that $\operatorname{int}(\operatorname{supp}\mu) \neq \emptyset$.

Let $C_1, C_2, \ldots$ be the connected components of $\operatorname{int}(\operatorname{supp}\mu)$: it is an at most countable family of connected sets and $\operatorname{int}(\operatorname{supp}\mu) = \bigcup_{i \geqslant 1} C_i$.

Since f_t is a diffeomorphism for every $t \in T$, thus a continuous open map, we deduce that each $f_t(C_i)$ is a connected open set contained in $\operatorname{supp}\mu$, by the c-invariance. Hence there is some $j = j(i,t)$ such that $f_t(C_i) \subset C_j$ by openness and connectedness.

In particular, by the same reasoning, we see that every point in C_i is sent by f_t in the interior of $\operatorname{supp}\mu$ for all $t \in T$ and $i \geqslant 1$.

We show that $j = j(i,t)$ does not depend on $t \in T$.

By contradiction, let us suppose there are $i \geqslant 1$, t_0 and t_1 in B such that $j_0 = j(i,t_0) \neq j(i,t_1) = j_1$ and let us fix $x \in C_i$. We take a continuous curve $\gamma : [0,1] \to T$ with endpoints t_0 and t_1 in B: $\gamma(0) = t_0$

and $\gamma(1) = t_1$. We know that

$$f(x, \gamma(s)) \in \operatorname{int}(\operatorname{supp}\mu) = \bigcup_{i \geqslant 1} C_i \quad \text{for all } s \in [0, 1],$$

but since $f(x, \gamma(0)) = f(x, t_0) \in C_{j_0}$ and $f(x, \gamma(1)) = f(x, t_1) \in C_{j_1}$ with C_{j_0}, C_{j_1} distinct connected components of $\operatorname{int}(\operatorname{supp}\mu)$, we conclude there is $\bar{s} \in]0, 1[$ such that

$$f(x, \gamma(\bar{s})) \in \partial C_{j_0} \subset \partial(\operatorname{supp}\mu) = \operatorname{supp}\mu \setminus (\operatorname{int}(\operatorname{supp}\mu)),$$

a contradiction. So every C_i is sent into some $C_{j(i)}$ by any f_t and the permutation $i \mapsto j(i)$ does not depend on $t \in T$.

We remark, in particular, that if for $x \in C_i$ we have $f^k(x, \underline{t}) \in C_j$ for some $j, k \geqslant 1$ and $\underline{t} \in \Delta$, then $f^k(x, \Delta) \subset C_j$.

Since $\mu \times \nu^\infty(C_i \times \Delta) > 0$ $(i \geqslant 1)$ Poincaré's recurrence theorem guarantees that $\mu \times \nu^\infty$-a.e. pair $(x, \underline{t}) \in C_i \times \Delta$ is ω-recurrent with regard to the action of S. By last remark, we see that $f^k(C_i, \Delta)$ returns to C_i infinitely often, for every fixed i. Hence, again by hypothesis (A), each C_i contains a ξ_0-ball. Thus, because M is compact, the pairwise disjoint family $C_1, C_2, \ldots$ must be finite and so $\operatorname{int}(\operatorname{supp}\mu) = C_1 \dot{\cup} \cdots \dot{\cup} C_l$ (a disjoint union).

The open sets $C_1, \ldots, C_l$ may not be pairwise separated. However, the following reflexive and symmetric relation $C_i \sim C_j \Leftrightarrow \overline{C_i} \cap \overline{C_j} \neq \emptyset$, $(1 \leqslant i, j \leqslant l)$ *generates* a unique equivalence relation $\simeq$ such that, if $\widetilde{C}_1, \ldots, \widetilde{C}_q$ are the $\simeq$-equivalence classes, then $W_1 = \bigcup \widetilde{C}_1, \ldots, W_q = \bigcup \widetilde{C}_q$ are pairwise separated open sets. Moreover, these sets are interchanged by any f_t $(t \in T)$ in the same way the $C_1, \ldots, C_l$ were, that is, the permutation of their indexes by the action of f_t does not depend on t.

The permutation of the indexes of the $W_1, \ldots, W_q$ has a finite number of cycles which are a finite collection of pairwise separated open sets satisfying Definition 3.3. We have proved

PROPOSITION 8.1. – *Every stationary measure μ is such that the interior of its support is made of a finite number of s-invariant domains.*

Remark 8.1. – If μ were ergodic, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} 1_{W_i}(f_{\underline{t}}^j(x)) = \mu(W_i) > 0$$

for $\mu \times \nu^{\infty}$-a.e. $(x, \underline{t}) \in M \times \Delta$ and $1 \leqslant i \leqslant q$. So almost every point of $W_1 \cup \cdots \cup W_q$ returns to W_i infinitely many times. In this case the **interior of** $\operatorname{supp} \mu$ **is made of a single s-invariant domain**.

Let now $\mathcal{M}_1, \ldots, \mathcal{M}_h$ be *all* the minimal domains inside the s-invariant domains given by Proposition 8.1 (recall Section 6.1). Provisionally we assume the following

LEMMA 8.2. – *The normalized restriction of a stationary measure to a c-invariant set is a stationary probability.*

Let the normalized restrictions be $\mu_{\mathcal{M}_i}(A) = \frac{1}{\mu(\mathcal{M}_i)} \cdot \mu(A \cap \mathcal{M}_i)$, $i = 1, \ldots, h$, where A is any Borel set and $\mu(\mathcal{M}_i) > 0$ (because $\mathcal{M}_i$ is a collection of open sets inside $\operatorname{int}(\operatorname{supp} \mu)$). By Proposition 7.3, $\mu_{\mathcal{M}_i}$ must be the characteristic probability of $\mathcal{M}_i$, $i = 1, \ldots, h$.

Remark 8.2. – This means the characteristic probability of each $\mathcal{M}_i$ must give zero mass to the border $\partial \mathcal{M}_i$, since it coincides with its normalized restriction to the interior of $\mathcal{M}_i$.

To see that these probabilities are enough to define μ, we consider $\lambda = \mu - \mu(\mathcal{M}_1) \cdot \mu_{\mathcal{M}_1} - \cdots - \mu(\mathcal{M}_h) \cdot \mu_{\mathcal{M}_h}$. If $\lambda \not\equiv 0$, then λ is a stationary measure (of course, being stationary is an additive property) whose support is nonempty. By Proposition 8.1 and by Section 6.1 we have some minimal domain $\mathcal{M}$ in $\operatorname{supp} \lambda$ with $\lambda(\mathcal{M}) > 0$. But $\operatorname{supp} \lambda \subset \operatorname{supp} \mu \setminus (\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_h)$ and the $\mathcal{M}_1, \ldots, \mathcal{M}_h$ are the only minimals in $\operatorname{supp} \mu$. We have reached a contradiction, so $\lambda \equiv 0$ and we have proved (apart Lemma 8.2)

PROPOSITION 8.3. – *Every stationary probability is a linear finite and convex combination of characteristic probabilities.*

Let us note that these arguments show that $\operatorname{supp} \mu = \operatorname{supp} \mu_{\mathcal{M}_1} \dot{\cup} \cdots \dot{\cup} \operatorname{supp} \mu_{\mathcal{M}_h}$ and consequently $\mu(\mathcal{M}_1) + \cdots + \mu(\mathcal{M}_h) = 1$, that is, the linear combination above is indeed convex.

To end this section we prove the remaining lemma.

Proof of Lemma 8.2. – Let μ be a stationary measure and C a c-invariant set.

We remark that we know every point of C stays in C, but we do not know whether points in the complement $\operatorname{supp} \mu \setminus C$ enter in C by the action of f_t.

First, we show $D = \operatorname{supp} \mu \setminus C$ to be *almost* completely invariant.

In fact, we may assume $\mu(D) > 0$ (otherwise $C = \operatorname{supp}\mu$, μ-mod 0) and write

$$0 < \mu(D) = \int 1_D(x)\, d\mu(x) = \iint 1_D\big(f(x,t)\big)\, d\mu(x)\, d\nu(t)$$

because μ is S-invariant. By the invariance of C, $x \in C \Rightarrow f(x,t) \in C \Rightarrow 1_D(f(x,t)) = 0$ for every $t \in T$ and so

$$\iint 1_D\big(f(x,t)\big)\, d\mu(x)\, d\nu(t) = \iint_D 1_D\big(f(x,t)\big)\, d\mu(x)\, d\nu(t).$$

Defining $D_1(t) = \{x \in D\colon\ f(x,t) \in D\}$ and $D_2(t) = \{x \in D\colon\ f(x,t) \notin D\}$ for $t \in T$, we have

$$\mu(D) = \int \int_{D_1(t)\cup D_2(t)} 1_D\big(f(x,t)\big)\, d\mu(x)\, d\nu(t) = \int \mu\big(D_1(t)\big)\, d\nu(t) > 0,$$

where $\mu(D_1(t)) \leqslant \mu(D)$ for every $t \in T$. Thus $\mu(D_1(t)) = \mu(D)$ for ν-a.e. t, that is, $f(x,t) \in D$ for $\mu \times \nu$-a.e. $(x,t) \in D \times T$. In other words, points outside C almost never enter in C.

Now we know that $1_C(x) = 1_C(f(x,t))$ for $\mu \times \nu$-a.e. pair (x,t). Hence,

$$\begin{aligned}\int \varphi \cdot 1_C\, d\mu &= \iint \varphi\big(f_t(x)\big) \cdot 1_C\big(f_t(x)\big)\, d\mu(x)\, d\nu(t)\\ &= \iint \varphi\big(f_t(x)\big) \cdot 1_C(x)\, d\mu(x)\, d\nu(t)\end{aligned}$$

for any $\varphi \in C^0(M, \mathbb{R})$, that is, the restriction of μ to C is stationary. □

9. TIME AVERAGES AND MINIMAL DOMAINS

What remains to be done is essentially to fit together previous results. Indeed, Sections 6 and 7 prove items 1 and 2 in the statement of Theorem 1. To achieve the decomposition of item 3 we are going to show that every point $z \in M$ is sent into some minimal domain by ν^∞-a.e. perturbation of Δ and the ν^∞-mod 0 partition of Δ obtained by this property satisfies 3(a), 3(b) and 3(c), since we already know that m-a.e. point inside a minimal belongs to the respective ergodic basin.

Let $z \in M$ and let μ be a stationary probability given by some accumulation point of the averages (13). By Proposition 8.3 we know μ

decomposes in the following way

$$\mu = \alpha_1 \cdot \mu_1 + \cdots + \alpha_h \cdot \mu_h \tag{17}$$

where $0 < \alpha_1, \ldots, \alpha_h < 1$, $\alpha_1 + \cdots + \alpha_h = 1$ and $\mu_1, \ldots, \mu_h$ are the characteristic probabilities of the minimals $\mathcal{M}_1, \ldots, \mathcal{M}_h$, respectively.

Decomposition (17) and the construction of μ ensure there is, for every $i = 1, \ldots, h$, a set $V_i \subset \Delta$ with $\nu^\infty(V_i) > 0$ such that there is $k \in \mathbb{N}$ satisfying $f^k(z, \underline{s}) \in \mathcal{M}_i$ for every $\underline{s} \in V_i$.

Indeed, $\mu(\mathcal{M}_i) > 0$ implies there exist open sets $U \subset \overline{U} \subset V \subseteq \mathcal{M}_i$ such that $\mu(U) > 0$ and so $\varphi \in C(M, \mathbb{R})$ with $0 \leqslant \varphi \leqslant 1$, $\operatorname{supp}\varphi \subseteq V$ and $\varphi_{|U} \equiv 1$ satisfies $\mu(\varphi) = \lim_{i\to\infty} \frac{1}{n_i} \sum_{j=1}^{n_i} \int \varphi(f^j(z, \underline{t}))\, d\nu^\infty(\underline{t}) > 0$. Then we have, for some $j \in \mathbb{N}$:

$$\nu^\infty\{\underline{t} \in \Delta\colon\ f^j(z, \underline{t}) \in \mathcal{M}_i\} \geqslant \int \varphi\big(f^j(z, \underline{t})\big)\, d\nu^\infty(\underline{t}) > 0.$$

Now we claim the sets V_i occupy the entire space Δ or equivalently (cf. Definition 5.3)

PROPOSITION 9.1. – *For every $z \in M$ we have $G_{\mathcal{M}_1}(z) \cup \cdots \cup G_{\mathcal{M}_l}(z) = \Delta$, ν^∞-mod 0 and $G_{\mathcal{M}_i}(z) \cap G_{\mathcal{M}_j}(z) = \emptyset$ for every pair $1 \leqslant i < j \leqslant l$ where $\mathcal{M}_1, \ldots, \mathcal{M}_l$ are* all *the minimal invariant domain of $\mathcal{D}$.*

Proof. – By contradiction, let us suppose there is $V \subset \Delta$ with $\nu^\infty(V) > 0$ such that $\nu^\infty(V \cap G_{\mathcal{M}_i}(z)) = 0$, $i = 1, \ldots, l$ (or $V \subset \bigcap_{i=1}^{l} H_{\mathcal{M}_i}(z)$, ν^∞-mod 0).

Let $\underline{t}$ be a V-generic vector and let $w \in \omega(z, \underline{t})$. By Lemma 5.7 we have $\bigcap_{i=1}^{l} H_{\mathcal{M}_i}(w) = \Delta$, ν^∞-mod 0, that is, the orbit of w under almost every perturbation never falls in $\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l$. Consequently any stationary probability obtained from the orbits of w as in Section 7.1 will admit a (nontrivial) decomposition (according to Proposition 8.3) $\mu = \beta_1 \cdot \tilde{\mu}_1 + \cdots + \beta_{\tilde{h}} \cdot \tilde{\mu}_h$ such that $0 \leqslant \beta_1, \ldots, \beta_h \leqslant 1$, $\beta_1 + \cdots + \beta_h = 1$ and each $\tilde{\mu}_i$ is the characteristic probability of $\widetilde{\mathcal{M}}_i$, $i = 1, \ldots, h$, where each of the $\widetilde{\mathcal{M}}_1, \ldots, \widetilde{\mathcal{M}}_h$ is distinct from $\mathcal{M}_1, \ldots, \mathcal{M}_l$.

This contradict the supposition that the $\mathcal{M}_1, \ldots, \mathcal{M}_l$ are *all* the minimal invariant domains of $\mathcal{D}$ and so such a set V cannot exist. □

We now easily derive the continuous dependence of the sets $V_i(x)$ from $x \in M$ with respect to the distance between ν^∞-mod 0 sets $A, B \subset \Delta$ given by $d_\nu(A, B) = \nu^\infty(A \triangle B)$.

We fix $x \in M$ and note that each $V_i(x)$ can be written as

$$V_i(x) = \bigcup_{k=1}^{\infty} V_{i,k}(x) \quad \text{where } V_{i,k}(x) = \{\underline{t} \in \Delta\colon\ f^k(x,\underline{t}) \in \mathcal{M}_i\},$$
$$k \geqslant 1, \tag{18}$$

are open and $V_{i,k}(x) \subseteq V_{i,k+1}(x)$ for all $k \geqslant 1$ by the complete invariance of $\mathcal{M}_i$, $i = 1, \ldots, l$. This implies that for some $\delta > 0$ we can find $k_0 \in \mathbb{N}$ such that $\nu^\infty(V_i(x) \setminus V_{i,k_0}(x)) \leqslant \delta$ for all $1 \leqslant i \leqslant l$.

On the one hand, by the finiteness of k_0, Property 2.1 and the openness of the domains that form $\mathcal{M}_i$, we get the existence of $\gamma > 0$ with the property $V_{i,k_0}(y) \supseteq V_{i,k_0}(x)$ for all $y \in B(x,\gamma)$. Hence $\nu^\infty(V_i(y)) \geqslant \nu^\infty(V_{i,k_0}(y)) \geqslant \nu^\infty(V_{i,k_0}(x)) \geqslant \nu^\infty(V_i(x)) - \delta$ whenever $d_M(y,x) < \gamma$ and for every $i = 1, \ldots, l$.

On the other hand

$$\begin{aligned}
\nu^\infty\big(V_i(y)\big) &= 1 - \nu^\infty\big(V_1(y)\big) - \cdots - \nu^\infty\big(V_{i-1}(y)\big) \\
&\quad - \nu^\infty\big(V_{i+1}(y)\big) - \cdots - \nu^\infty\big(V_h(y)\big) \\
&\leqslant 1 - \nu^\infty\big(V_1(x)\big) - \cdots - \nu^\infty\big(V_{i-1}(x)\big) \\
&\quad - \nu^\infty\big(V_{i+1}(x)\big) - \cdots - \nu^\infty\big(V_h(x)\big) + (h-1)\cdot\delta \\
&= \nu^\infty\big(V_i(x)\big) + (h-1)\cdot\delta
\end{aligned}$$

for all $1 \leqslant i \leqslant l$ and continuity follows.

We are left to show item 3(c) of Theorem 1 holds with respect to this decomposition.

Let us fix $1 \leqslant i \leqslant l$ such that $\nu^\infty(V_i) > 0$.

We note that (18), the openness of the $\mathcal{M}_i$ and the continuity Property 2.1(1) imply the $V_i(z)$ to be open subsets of Δ, that is, for every $\underline{t} \in V_i(z)$ there are $k \in \mathbb{N}$ and $\rho > 0$ such that $f^k(z, B(\underline{t},\rho)) \subset \mathcal{M}_i$ and so $V_i(z) \supset B(\underline{t},\rho)$. According to Section 7.2 we have

$$\mathcal{M}_i \subset E(\mu_i) \quad \text{and thus} \quad \{\underline{s} \in V_i(z)\colon\ f^k(z,\underline{s}) \in E(\mu_i)\} \supset B(\underline{t},\rho). \tag{19}$$

This means that every $\underline{s}$ in $B(\underline{t},\rho) \subset V_i = V_i(z)$ is such that $w = f^k(z,\underline{s}) \in E(\mu_i)$, that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f^j(w,\underline{u})\big) = \int \varphi \, d\mu_i \quad \text{for every } \varphi \in C^0(M,\mathbb{R})$$
$$\text{and } \nu^\infty\text{-a.e. } \underline{u} \in \Delta.$$

Since time averages do not depend on any finite number of iterates, item 3(c) of Theorem 1 follows and the proof of Theorem 1 is complete.

Remark 9.1. – We note that *diffeomorphism* in the arguments and definitions of Sections 2 through 9 may be replaced throughout by *continuous open map*. This means Theorem 1 is a result of *continuous Ergodic Theory* and not specific of *differentiable Ergodic Theory*: a C^0-continuous family of continuous open maps $f_t : M \to M$, $t \in B$, would suffice.

Remark 9.2. – The conclusions of Theorem 1 can be obtained with weaker hypothesis instead of the stated A) and B).

Indeed, it is very easy to see that the integer N may depend on x in the statement of A). Thus it can be replaced by

A′) There is $\xi_0 > 0$ such that for all $x \in M$ there exists $N = N(x) \in \mathbb{N}$ satisfying $f^k(x, \Delta) \supset B\big(f^k(x), \xi_0\big)$ for all $k \geqslant N$.

Moreover, B) can be weakened so that the absolute continuity of a stationary probability μ still holds by allowing $f^k(x, \nu^\infty) \ll m$ for *some* $k \geqslant 1$. If this k does not depend on $x \in M$, then we can still write (16) and proceed to prove Lemma 7.2.

Other weakenings of B) are possible, one such will be of use following Section 11 dealing with random parametric perturbations near homoclinic bifurcations.

10. BOWEN'S EXAMPLE

This is the answer to a question raised by C. Bonatti. This example captures the meaning of Theorem 1: even if a given *deterministic* (*noiseless*) system is devoid of physical measures (its Birkhoff averages do not exist almost everywhere) we may nevertheless get a finite number of physical probabilities describing the asymptotics of almost every orbit just by *adding* a small amount of random noise.

Example 5. – *Bowen's example* (see [28] for the not very clear reason for the name) is a folklore example showing that Birkhoff averages need not exist almost everywhere. Indeed, in the system pictured in Fig. 4 Birkhoff averages for the flow *do not exist almost everywhere*, they only exist for the sources s_3, s_4 and for the set of separatrixes and saddle equilibria $W = W_1 \cup W_2 \cup W_3 \cup W_4 \cup \{s_1, s_2\}$.

The orbit under this flow ϕ_t of every point $z \in S^1 \times [-1, 1] = M$ not in W accumulates on either side of the separatrixes, as suggested in the

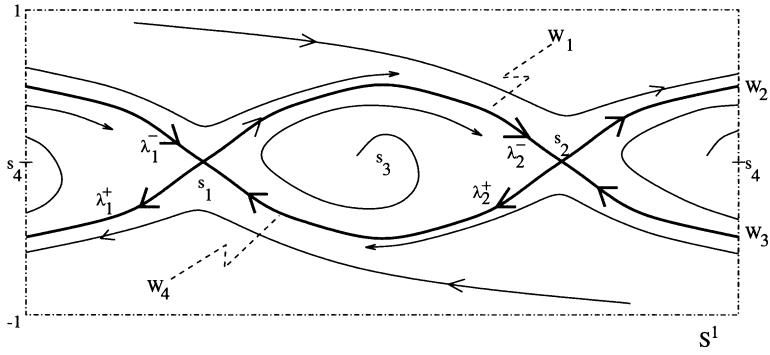

Fig. 4. A sketch of Bowen's example flow.

figure, if we impose the condition $\lambda_1^-\lambda_2^- > \lambda_1^+\lambda_2^+$ on the eigenvalues of the saddle fixed points s_1 and s_2 (for more specifics on this see [28] and references therein).

We apply Theorem 1 to this case. We remark that M is not a boundaryless manifold, but its border $S^1 \times \{\pm 1\}$ is sent by ϕ_1 into $S^1 \times [-1, 1]$. Moreover, Theorem 1 refers not to perturbations of flows, so we will consider the time one map ϕ_1 as our diffeomorphism $f : M \to M$ and, since M is parallelizable, we can make an absolutely continuous random perturbation, as in Example 1 of Section 2.4. In this circumstances the proof of Theorem 1 equally applies.

For everything to be properly defined, though, we must restrict the noise level $\varepsilon > 0$ to a small interval $]0, \varepsilon_0[$ such that the perturbed orbits stay in $S^1 \times]-1, 1[$. After this minor technicalities we proceed to prove

PROPOSITION 10.1. – *The system above, under random absolutely continuous noise of level* $\varepsilon \in]0, \varepsilon_0[$, *admits a single physical absolutely continuous probability measure* μ *whose support is a neighborhood of the separatrixes*: $\operatorname{int}(\operatorname{supp}\mu) \supset W$. *Moreover the ergodic basin of* μ *is the entire manifold*: $E(\mu) = M$, $\mu \bmod 0$.

Proof. – Let $\varepsilon \in]0, \varepsilon_0[$ be the fixed noise level from now on and let U be the ball of radius $\varepsilon/4$ around s_1. We will build fundamental domains for the action of $f = \phi_1$ over $M \setminus W$ in U, as explained below.

We choose two strait lines l_1, l_2 through s_1 crossing U and let l_1', l_2' be their images under ϕ_1 as sketched in Fig. 5. Now we choose two points in each line l_1, l_2 on either side of s_1: p_1, p_2, p_3 and p_4, and consider their orbits under the flow ϕ for positive time, until they return to U and cut l_1', l_2', as depicted in the abovementioned figure.

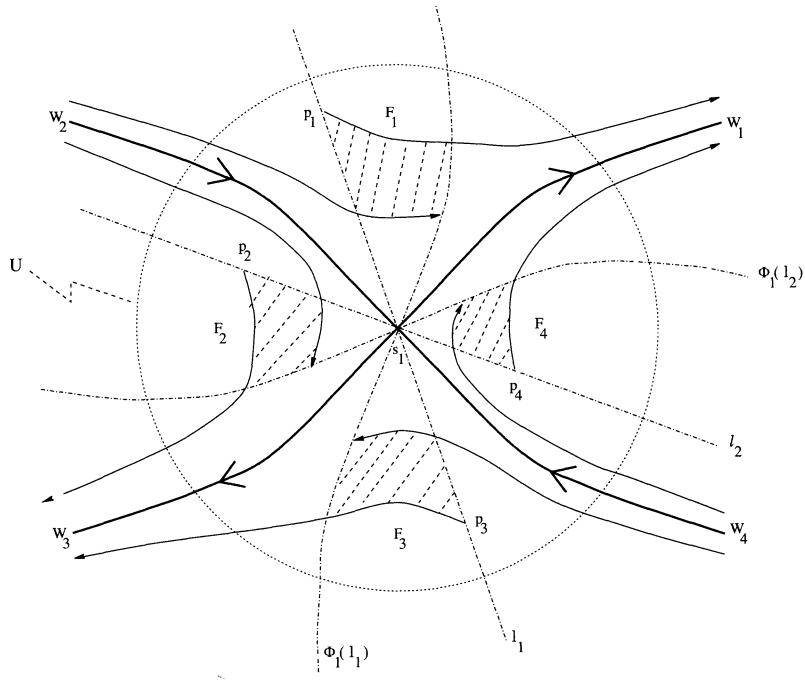

Fig. 5. How the fundamental domains are obtained.

The four intersections of the orbit of p_i with the proper l_j, l'_j, together with portions of the orbit and of l_j, l'_j define a "square" F_i (shadowed in Fig. 5) which is a fundamental domain for the dynamics of $f = \phi_1$ on the connected components of $M \setminus W$, $i = 1, 2, 3, 4$ and $j = 1$ or 2.

This means that every $z \in M \setminus W$ is such that there is a $k \geqslant 1$ with $z_k = f^k(z) \in F = F_1 \cup F_2 \cup F_3 \cup F_4$. Moreover, by the choice of U, z_k may be sent into any F_i, $i = 1, 2, 3, 4$, by adding to a vector of length smaller than ε. Thus we deduce that $f^k(z, \Delta) \supset F_1 \cup F_2 \cup F_3 \cup F_4$ and even more: $f^k(z, \Delta) \supset U$.

Keeping in mind that for $m \geqslant 1$ we have $f^{k+m}(z, \Delta) = f^m\big(f^k(z.\Delta), \Delta\big) = \{f^m(w, \Delta)\colon w \in f^k(z, \Delta)\}$, we see that $f^{k+m}(z, \Delta)$ will contain all the f-images of each F_1, F_2, F_3 and F_4, which will return to U infinitely many times. Furthermore, at each return the points may again be sent into any F_1, F_2, F_3 or F_4 by an ε-perturbation. Hence the sets of the sequence $\{f^n(z, \Delta)\}_{n=1}^{\infty}$ contain F_1, F_2, F_3 or F_4 for infinitely many $n's$ and also all their f-images.

We conclude that $\omega(z, \Delta)$ contains a neighborhood of W.

The same holds for $w \in W$, since $f^1(w, \Delta)$ is an open set and so contains some $z \in M \setminus W$. That is, *every* $z \in M$ *is such that* $\omega(z, \Delta)$ *contains a neighborhood of* W.

Therefore, there can be only one minimal $\mathcal{M}$ in the perturbed system, such that $\mathcal{M} \supset W$ and into which every point $z \in M$ finally falls by almost every perturbed orbit (this is a consequence of Sections 6.2, 7, 8 and 9). We have further that the characteristic probability $\mu_{\mathcal{M}}$ is *the physical probability* μ of the system, with $E(\mu) = M$, μ mod 0, and $\operatorname{supp}\mu \supset \mathcal{M} \supset W$, as stated. □

11. HOMOCLINIC BIFURCATIONS AND RANDOM PARAMETRIC PERTURBATIONS

We consider arcs (one-parameter families) of diffeomorphisms exhibiting a quadratic homoclinic tangency and derive similar properties for their random parametric perturbations to those stated in Theorem 1.

11.1. One-parameter families

The arcs we will be considering are given by a C^∞ function

$$f : M^2 \times]-1, 1[\to M^2$$

such that for every $-1 < t < 1$, $f_t : M^2 \to M^2$, $x \mapsto f(x, t)$ is a diffeomorphism of the boundaryless surface M^2. The family of diffeomorphisms $\mathcal{F} = (f_t)_{-1<t<1}$ satisfies the following conditions.

1. $\mathcal{F}$ has a *first tangency* at $t = 0$, that is (v. [22, Appendix 5])
   (a) for $t < 0$, f_t is persistently hyperbolic;
   (b) for $t = 0$ the nonwandering set $\Omega(f_0)$ consists of a closed hyperbolic set $\widetilde{\Omega}(f_0) = \lim_{t \nearrow 0} \Omega(f_t)$ together with a homoclinic orbit of tangency $\mathcal{O}$ associated with a hyperbolic fixed saddle point p_0, so that $\Omega(f_0) = \widetilde{\Omega}(f_0) \cup \mathcal{O}$;
   (c) the branches $W^s_+(p_0)$, $W^u_+(p_0)$ of the invariant manifolds $W^s(p_0)$, $W^u(p_0)$ have a quadratic tangency along $\mathcal{O}$ unfolding generically as pictured in Fig. 6 (v. [22, Chapter 3]): $\mathcal{O}$ is the only orbit of tangency between stable and unstable separatrixes of periodic orbits of f_0;
2. The saddle p_0 has eigenvalues $0 < \lambda_0 < 1 < \sigma_0$ satisfying the conditions for the existence of C^2 linearizing coordinates in a neighborhood of $(p_0, 0)$ in $M^2 \times]-1, 1[$ (v. [27]).

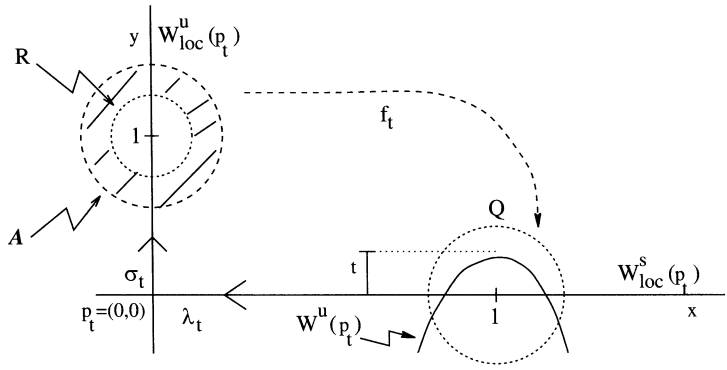

Fig. 6. A sketch of the situation to be considered.

Condition 1 imposes bounds on the region where new accumulation points can appear for $t > 0$ (small)—Section 11.3 will specify this (cf. [22, Appendix 5]).

We note that condition 2 above is generic in the space of all C^∞ one-parameter families satisfying 1. Moreover, those families that satisfy 1 are open (cf. [22, Chapter 3, Appendix 5] and references therein).

11.2. Statement of the results

For some small $t^\star > 0$, to be explained in the following sections, we fix $t_0 \in]0, t^\star[$, $\varepsilon_0 = \min\{|t_0|, |t^\star - t_0|\}$ and the *noise level* $\varepsilon \in]0, \varepsilon_0[$. We consider the system f_{t_0} under a random parametric perturbation of noise level ε, $\mathcal{F}_{t_0,\varepsilon}$, as defined in Section 2.1. We let $\Delta = \Delta_\varepsilon(t_0)$ be the perturbation space $[t_0 - \varepsilon, t_0 + \varepsilon]^{\mathbb{N}}$.

We will be interested in studying what happens in $\mathcal{Q}$, a closed neighborhood of q to be constructed. We need an effective definition of *interesting points*.

DEFINITION 11.1 (First Return Times). – *Given some* $z \in M^2$ *and* $\underline{t} \in \Delta$ *we let*

$$r(z, \underline{t}, 1) = \min\{k \geqslant 0\colon\ f^k(z, \underline{t}) \in \mathcal{Q}\}$$

and inductively define $r(z, \underline{t}, n+1) = \min\{k \geqslant 1\colon\ f^{R(z,\underline{t},n)+k}(z, \underline{t}) \in \mathcal{Q}\}$ *for every* $n \geqslant 1$, *where* $R(z, \underline{t}, n) = \sum_{i=1}^{n} r(z, \underline{t}, i)$, *with the convention* $\min \emptyset = +\infty$.

DEFINITION 11.2. – *A* V-recurrent point *is a* $z \in \mathcal{Q}$ *for which there exists a* $V \subset \Delta$ *satisfying*

1. $\nu^\infty(V) > 0$;
2. $r(z, \underline{t}, n) < \infty$ *for every* $n \geqslant 1$ *and* ν^∞*-a.e.* $\underline{t} \in V$.

In other words, $z \in \mathcal{Q}$ is *interesting* if its perturbed orbits pass through $\mathcal{Q}$ infinitely often under a positive measure set of perturbations.

We can now state

THEOREM 2. – *For every* C^∞ *arc of diffeomorphisms as described in Section* 11.1 *and any given homoclinic tangency point* q *associated to the saddle* p_0*, there are a closed neighborhood* $\mathcal{Q}$ *of* q *and* $t^\star > 0$ *such that, for each* $t_0, \varepsilon > 0$ *satisfying* $0 < t_0 < t^\star$ *and* $0 < \varepsilon < \varepsilon_0 = \min\{|t_0|, |t^\star - t_0|\}$*, the random parametric perturbation* $\mathcal{F}_{t_0,\varepsilon}$ *of* f_{t_0} *with noise level* ε *admits a finite number of probabilities* $\mu_1, \ldots, \mu_l$ *whose support intersects* $\mathcal{Q}$ *and that*

1. $\mu_1, \ldots, \mu_l$ *are physical absolutely continuous probability measures*;
2. $\operatorname{supp}\mu_i \cap \operatorname{supp}\mu_j = \emptyset$ *for all* $1 \leqslant i < j \leqslant l$;
3. *for all* $z \in \mathcal{Q}$ *and* $V \subset \Delta$ *such that* z *is* V*-recurrent there are open sets* $V_1 = V_1(z), \ldots, V_l = V_l(z) \subset V$ *such that*
   (a) $V_i \cap V_j = \emptyset$, $1 \leqslant i < j \leqslant l$;
   (b) $\nu^\infty(V \setminus (V_1 \cup \cdots \cup V_l)) = 0$;
   (c) *for all* $1 \leqslant i \leqslant l$ *and* ν^∞*-a.e.* $\underline{t} \in V_i$ *we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f^j(z, \underline{t})\big) = \int \varphi \, d\mu_i, \quad \text{for every } \varphi \in C(M, \mathbb{R}).$$

11.3. Adapting the linearization

As preparation for the proof of Theorem 2 by using Theorem 1 we study the adaptation of the linearizing coordinates to our setting.

Condition 2 enables us to consider a change of coordinates $\varphi_t : L \subset \mathbb{R}^2 \to M^2$ in a neighborhood L of every p_t, where $|t| < t^\star$ for some small $t^\star > 0$ and

$$f_t\big(\varphi_t(x, y)\big) = \varphi_t(\lambda_t \cdot x, \sigma_t \cdot y) \tag{20}$$

with $0 < \lambda_t < 1 < \sigma_t$ the eigenvalues of the hyperbolic saddle fixed point p_t. These coordinates will be adapted much like [22, p. 49 and Appendix 5]. Specifically, after choosing a homoclinic point q associated to p_0:

I) we suppose $q \in W^u(p_0) \cap W^s(p_0)$ to be in L—to achieve this we may extend L along $W^s(p_0)$ as explained in [22, Chapter 2];

II) we extend L along $W^u(p_0)$ in order that $r = f_0^{-1}(q)$ be in L;

III) we use the implicit function theorem and two independent rescalings of the x- and y-axis to get, because of condition 1(c):
   (a) $q = (1,0)$, $r = (0,1)$, $p_t = (0,0)$ and $W^s_{\mathrm{loc}}(p_t)$, $W^u_{\mathrm{loc}}(p_t)$ are the x- and y-axis, respectively;
   (b) $f_t(0,1)$ is a local maximum of the y-coordinate restricted to $W^u(p_t)$;
   (c) $\varphi_t^{-1} \circ f_t \circ \varphi_t(0,1) = (1,t)$;
   for every $|t| < t^\star$ in the coordinates defined by φ_t;

IV) writing Λ_0 the basic set to which p_0 belongs (possibly $\Lambda_0 = \{p_0\}$ trivially) by condition 1(b) we have $W^s(\Lambda_0) = W^s(\Lambda_0 \cup \mathcal{O})$ and $W^u(\Lambda_0) = W^u(\Lambda_0 \cup \mathcal{O})$ and there exists a filtration $\emptyset \neq M_1 \subset M_2 \subset M$ such that (v. [22, Appendix 5, pp. 212–214] and cf. [26, Chapter 1]):
   (a) M_i is closed and $f_0(M_i) \subset \mathrm{int}(M_i)$ for $i = 1, 2$;
   (b) $M_1 \subset \mathrm{int}(M_2)$, and
   (c) $\Lambda_0 \cup \mathcal{O} = (\bigcap_{j\geqslant 0} f_0^j(M_2)) \cap (\bigcap_{j\geqslant 0} f_0^{-j}(M_1^c))$;

V) since Λ_0 is a basic set (of saddle type) there is a small compact neighborhood U of Λ_0 where extensions $\mathcal{H}^s$, $\mathcal{H}^u$ of the stable and unstable foliations $W^s(\Lambda_0)$, $W^u(\Lambda_0)$ are defined (v. [22, Appendix 1] and references therein), and by IV(c) there is $N^\star \in \mathbb{N}$ such that:
   (a) $(\bigcap_{j=0}^{N^\star-1} f_0^j(M_2)) \cap (\bigcap_{j=0}^{N^\star-1} f_0^{-j}(M_1^c)) \subset U \cup \mathcal{Q}^\star$ where $\mathcal{Q}^\star \subset L$ is a neighborhood of the portion of $\mathcal{O}$ outside U with finitely many components $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_l$ and $\mathcal{Q}^\star \cap U = \emptyset$. Moreover we can assume they satisfy $f_0(\mathcal{Q}_1) \subset \mathcal{Q}_2, \ldots, f_0(\mathcal{Q}_{l-1}) \subset \mathcal{Q}_l$ with $q \in \widetilde{\mathcal{Q}} = \mathcal{Q}_i$, $i \in \{1, \ldots, l\}$;
   (b) making $t^\star > 0$ smaller if need be and $\mathcal{Q}^\star$ and U a little bigger, we get also for all $t_1, \ldots, t_N, t'_1, \ldots, t'_N \in]-t^\star, t^\star[$

$$\left(\bigcap_{j=0}^{N^\star-1} f_{t_j} \circ \cdots \circ f_{t_1}(M_2)\right) \cap \left(\bigcap_{j=0}^{N^\star-1} f_{t'_1}^{-1} \circ \cdots \circ f_{t'_N}^{-1}(M_1^c)\right) = U \cup \mathcal{Q}^\star$$

   and also $f_{t_1}(\mathcal{Q}_1) \subset \mathcal{Q}_2, \ldots, f_{t_{l-1}}(\mathcal{Q}_{l-1}) \subset \mathcal{Q}_l$ for all $t_1, \ldots, t_{l-1} \in]-t^\star, t^\star[$;
   (c) $f_t(M_i) \subset \mathrm{int}(M_i)$ for all $|t| < t^\star$ and $i = 1, 2$;
   (d) $\Lambda_t = \bigcap_{n\in\mathbb{Z}} f_t^n(U)$ is the analytic continuation of Λ_0 for all $|t| < t^\star$;

VI) for every closed neighborhood $\mathcal{Q} \subset \widetilde{\mathcal{Q}} \subset L$ of q and $t^\star > 0$ small we have that:
   (a) there is $N_Q \in \mathbb{N}$ such that $f_{t_i} \circ \cdots \circ f_{t_1}(\mathcal{Q}) \subset L$ for all $t_1, \ldots, t_i \in]-t^\star, t^\star[$ and $i = 1, \ldots, N_Q$;
   (b) in the neighborhood $\mathcal{R} = \overline{\bigcup_{|t|<t^\star} f_t^{-1}(\mathcal{Q})}$ of $r = (0,1)$—we may suppose $\mathcal{R} \subset L$ by making $\mathcal{Q}$ and $t^\star$ smaller, keeping (a) by increasing N_Q—the map $\tilde{f}_t = \varphi_t^{-1} \circ f_t \circ \varphi_t$ has the form

$$(x, 1+y) \mapsto \big(1 + \alpha y + \eta x + H_1(t,x,y);\ \beta y^2 + \gamma x + t + H_2(t,x,y)\big), \qquad (21)$$

   where $\alpha \cdot \beta \cdot \gamma \neq 0$, H_1 is of order 2 or higher and H_2 is of order 3 or higher in y and order 2 or higher in x, t and $y \cdot t$;
   (c) for all $|t| < t^\star$ we make $f_t(0,1) \in \operatorname{int}(\mathcal{Q})$ by taking $t^\star$ smaller if needed and keeping $\mathcal{Q}$ and N_Q unchanged satisfying (a) and (re)defining $\mathcal{R}$ as in (b);
   (d) for any given $\delta_0 > 0$ and all sufficiently small $\mathcal{Q}$ and $t^\star$, we may keep everything up until now increasing N_Q and imposing $|D_2 H_i|, |D_3 H_i| < \delta_0$, $i = 1, 2$;

VII) since all of the above holds for every small (compact) neighborhood $\mathcal{Q} \subset \widetilde{\mathcal{Q}}$ of q and $t^\star > 0$, except that N_Q increases, we may suppose $\mathcal{Q}$ is so small that $N_Q > N^\star$ and then make $t^\star$ so small that item V) holds with $\mathcal{Q}$ in the place of $\widetilde{\mathcal{Q}}$ for some integer $N > N^\star$. Furthermore writing $\mathcal{Q}'$ for this new neighborhood we may suppose that Λ_t still is the maximal invariant set inside $B(U, \rho) = \bigcup_{z \in U} B(z, \rho)$ for $|t| < t^\star$ and $\overline{B(U,\rho)} \cap \overline{B(\mathcal{Q}', \rho)} = \emptyset$ for some small $\rho > 0$;

VIII) we may suppose the extended foliations $\mathcal{H}^s, \mathcal{H}^u$, which are defined in a neighborhood of p_0 (since $p_0 \in \Lambda_0$), were extended by positive and negative iterations of f_0 to cover all of L. Moreover we may assume also that there are extended foliations $\mathcal{H}_t^s, \mathcal{H}_t^u$ defined all over L with respect to f_t for every $|t| < t^\star$;

IX) in a small neighborhood $\mathcal{A}$ of $\mathcal{R}$ given by $\mathcal{A} = (\bigcup_{z \in \mathcal{R}} B(z, \xi)) \setminus \mathcal{R}$, $\xi > 0$ small (we may think of it as a small *annulus* around $\mathcal{R}$), every point is sent by f_t outside of $U \cup \mathcal{Q}'$, for every $t \in T$, because U and $\mathcal{Q}'$ are separated according to item VII. $\mathcal{A}$ is open and will be called the *nonreturn* annulus.

We note that Fig. 6 was made having these items already in mind.

11.4. Another tour of another proof

To begin with, pick a V-recurrent point $z \in \mathcal{Q}$ and deal with its generic ω-limit points w, which are always *regular* by the following

PROPOSITION 11.1. – *There exists $J \in \mathbb{N}$ such that if $z \in \mathcal{Q}$ is V-recurrent for some $V \subset \Delta = \Delta_\varepsilon(t_0)$ with $\nu^\infty(V) > 0$, then the first return times of $w \in \omega(z, \underline{t})$, for all V-generic $\underline{t}$, do not depend on $\underline{s} \in \Delta$ and are bounded by J:*

$$r(w, \underline{s}, n) \equiv r(w, n) \leqslant J \quad \textit{for every } n \geqslant 1.$$

DEFINITION 11.3. – *The points $w \in M^2$ which satisfy the conclusion of the proposition above will be called* **regular points** *(with respect to $\mathcal{F}_{t_0,\varepsilon}$).*

Taking advantage of the regularity of w, the expression (21) for $f_t|_{\mathcal{R}}$ and condition 1, we will derive versions of hypothesis A) and B) of Theorem 1:

PROPOSITION 11.2. – *Let $w \in M^2$ be a regular point. Writing $r_n = r(w, n)$, $n \geqslant 1$, the following holds.*

1. *For every $s \in]t_0 - \varepsilon, t_0 + \varepsilon[$ there is a $\xi_0 = \xi_0(s) > 0$ such that for all $n \geqslant 2$*

$$f^{R_n}(w, \Delta) \supset B\big(f_s^{R_n}(w), \xi_0\big) \quad \textit{where } R_n = \sum_{i=1}^{n} r_i;$$

2. *For all $n \geqslant 2$ it holds that $f^{R_n}(w, \nu^\infty) \ll m$.*

In other words, we get conditions A) and B) of Theorem 1 for the return times of w, which do not depend on the perturbation chosen, since w is regular. Behind Proposition 11.2 is the geometrically intuitive idea of mixing expanding and contracting directions near q due to the homoclinic tangency, together with condition 1 that keeps the orbits of regular points confined in a neighborhood of $\Lambda_0 \cup \mathcal{O}$ (v. Section 12).

This is enough to prove Theorem 2.

Indeed, setting $K = 2(J + 1)$ then $R_2 = R_2(w) \leqslant K$ for every regular point w and for $k \geqslant K$ there are $n \geqslant 2$ and $0 \leqslant i \leqslant r_{n+1} - 1 \leqslant J$ (by Proposition 11.1) such that $k = R_n + i$. After item 1 of Proposition 11.2 we have $f^k(w, \Delta) = f^i(f^{R_n}(w, \Delta), \Delta) \supset f_{t_0}^i(B(f_{t_0}^{R_n}(w), \xi_0))$ and since $0 \leqslant i \leqslant J$ there is some $\xi_0' > 0$ such that $f_{t_0}^i(B(f_{t_0}^{R_n}(w), \xi_0)) \supset$

$B(f_{t_0}^{R_n+i}(w), \xi_0') = B(f_{t_0}^k(w), \xi_0')$ because f_t is a diffeomorphism. We have hypothesis A).

For hypothesis B) we let w and $k \geqslant K$ be as above. Then $k = R + i$ with $i \geqslant 0$ and $R = R_2 = R_2(w)$. We suppose $i \geqslant 1$ for otherwise item 2 of Proposition 11.2 does the job. We take a measurable set $E \subset M^2$ such that $m(E) = 0$ and observe that $f^{R+i}(w, \nu^\infty)E = \nu^k(F)$ where $F = \{(t_1, \ldots, t_k) \in T^k\colon\ f^{R+i}(w, t_1, \ldots, t_k) \in E\}$.

Defining for every $(t_{R+1}, \ldots, t_k) \in T^i$ the section $F(t_{R+1}, \ldots, t_k) = \{(s_1, \ldots, s_R) \in T^R\colon\ (s_1, \ldots, s_R, t_{R+1}, \ldots, t_k) \in F\}$ we have by Fubini's theorem

$$\nu^k(F) = \nu^{R+i}(F) = \int \nu^R\big(F(t_{R+1}, \ldots, t_k)\big)\, d\nu^i(t_{R+1}, \ldots, t_k). \tag{22}$$

However

$$\begin{aligned} F(t_{R+1}, \ldots, t_k) &= \big\{(s_1, \ldots, s_R) \in T^R\colon\ f^i_{t_{R+1}, \ldots, t_k} \circ f^R_{s_1, \ldots, s_R}(w) \in E\big\} \\ &= \big\{(s_1, \ldots, s_R)\colon\ f^R_{s_1, \ldots, s_R}(w) \in \big(f^i_{t_{R+1}, \ldots, t_k}\big)^{-1}(E)\big\} \end{aligned}$$

and each f_t is a diffeomorphism, so the inverse image of a set of measure zero is a set of measure zero. Hence $\nu^R(F(t_{R+1}, \ldots, t_k))$ is given by $f^R(w, \nu^\infty)[(f^i_{t_{R+1}, \ldots, t_k})^{-1}(E)] = 0$ since $f^{R_2}(w, \nu^\infty) \ll m$ by Proposition 11.2(2). We deduce from (22) that $f^{R+i}(w, \nu^\infty)E = f^k(w, \nu^\infty)(E) = \nu^k(F) = 0$ whenever $m(E) = 0$, i.e., $f^k(w, \nu^\infty) \ll m$ for every $k \geqslant K$.

It is clear that Theorem 2 holds by considering $(\mathcal{D}, \preceq)$ as the set of s-invariant domains $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ with respect to $\mathcal{F}_{t_0, \varepsilon}$ whose points $\mathcal{U}_0 \cup \cdots \cup \mathcal{U}_{r-1}$ are regular points, with the same relation $\preceq$ as before, and using Theorem 1.

We should explain how to get the decomposition of item 3 of Theorem 2 for V-recurrent point $z \in \mathcal{Q}$. We use two previous ideas:

(1) Going back to Section 9, taking a generic $w \in \omega(z, \underline{t})$ (i.e., $\underline{t}$ is V-generic) provides a stationary probability μ, as in Sections 7 and 8, which decomposes as in (17) and we get the sets $V_i = \{\underline{s} \in \Delta\colon\ \exists k \geqslant 1$ such that $f^k(w, \underline{s}) \in \mathcal{M}_i\}$ as in item 3 of Theorem 1.

(2) The previous item together with Proposition 11.1 just says that a V-recurrent point $z \in \mathcal{Q}$ satisfies Lemma 5.8, i.e., there are $W \subset V$ with $\nu^\infty(W) > 0$ and $m \in \mathbb{N}$ such that $f^m_{\underline{\theta}}(z) \in \mathcal{M}$ for every $\underline{\theta} \in W$, where $\mathcal{M}$ is some minimal of $(\mathcal{D}, \preceq)$. We know there is just a finite number $\mathcal{M}_1, \ldots, \mathcal{M}_l$ of minimals in $(\mathcal{D}, \preceq)$ and define $V_i = V_i(z) = \{\underline{s} \in V\colon\ \exists k \geqslant 1$ such that $f^k(z, \underline{s}) \in \mathcal{M}_i\}$, $1 \leqslant i \leqslant l$.

Repeating the arguments of Proposition 9.1 with Δ replaced by V throughout gives item 3 of Theorem 2 and completes the proof.

12. PHYSICAL PARAMETRIC NOISE WITH A SINGLE PARAMETER

We start the proof of Proposition 11.2 deducing the following consequence of condition 1 in Section 11.1 and items V and VI.

LEMMA 12.1. – *For every small $t^\star > 0$ and $\mathcal{Q}$ and every $z \in \mathcal{Q}$ recurrent under some vector $\underline{t} = (t_j)_{j=1}^{\infty}$ with $|t_j| < t^\star$, $j \geqslant 1$, i.e., such that $\omega(z, \underline{t}) \cap \mathcal{Q} \neq \emptyset$, the following holds*

$$f^j(z, \underline{t}) \in L \quad \text{for } 0 \leqslant j \leqslant N_Q \quad \text{and}$$
$$f^j(z, \underline{t}) \in U \cup \mathcal{Q}^\star \quad \text{for } j \geqslant N_Q. \tag{23}$$

Proof. – We let $z \in \mathcal{Q} \subset U \cup \mathcal{Q}^\star \subset M_2 \cap M_1^c$ be a recurrent point under $\underline{t}$ as stated, and suppose that $f^j(z, \underline{t}) \notin U \cup \mathcal{Q}^\star$ for some $j \geqslant N^\star$. Then by item Vb it must hold

$$f^j(z, \underline{t}) \in \bigcup_{i=0}^{N^\star - 1} f_{t_j} \circ \cdots \circ f_{t_{j-i}}(M_2^c) \quad \text{or}$$
$$f^j(z, \underline{t}) \in \bigcup_{i=0}^{N^\star - 1} f_{t_{j+1}}^{-1} \circ \cdots \circ f_{t_{j+i}}^{-1}(M_1).$$

Since $z \in U \cup \mathcal{Q}^\star \subset M_2$ we have by item Vc that $f^i(z, \underline{t}) \in M_2$ for every $i \geqslant 0$. Hence only the right hand side alternative above can hold, otherwise we would have for some $0 \leqslant i \leqslant N^\star - 1$ that $f_{t_j} \circ \cdots \circ f_{t_1}(z) \in f_{t_j} \circ \cdots \circ f_{t_{j-i}}(M_2^c)$ and so $f_{t_{j-i-1}} \circ \cdots \circ f_{t_1}(z) \in M_2^c$ with $j - i - 1 \geqslant 0$ because we took $j \geqslant N^\star$, a contradiction. But then we get $f_{t_j} \circ \cdots \circ f_{t_1}(z) \in f_{t_{j+1}}^{-1} \circ \cdots \circ f_{t_{j+i}}^{-1}(M_1)$, i.e., $f^{j+i}(z, \underline{t}) \in M_1$, and item Vc says $f^{j+i+k}(z, \underline{t}) \in M_1$ for all $k \geqslant 0$ with $\mathcal{Q} \subset U \cup \mathcal{Q}^\star \subset M_1^c$. That is, $\omega(z, \underline{t}) \cap \mathcal{Q} = \emptyset$, contradicting the choice of z and $\underline{t}$.

We have show (23) to hold for $j \geqslant N^\star$, since $\mathcal{Q}^\star \subset L$. However, by item VIa, we know $f^j(z, \underline{t}) \in L$ for $1 \leqslant j \leqslant N_Q$, where $N_Q > N^\star$ by item VII. □

Remark 12.1. – The arguments above show that if we replace $N^\star$ by N and assume $\widetilde{\mathcal{Q}} = \mathcal{Q}$ as in item VII, then writing $\mathcal{Q}'$ for this new neighborhood of the portion of $\mathcal{O}$ outside U, we may ensure under the same conditions of Lemma 12.1 that $f^j(z, \underline{t}) \in U \cup \mathcal{Q}'$ for all $j \geqslant N$.

This *confinement* property in turn implies

LEMMA 12.2. – *For every given $b_0 > 0$, $c_0 > 0$ and $\sigma > 1$ there are*
- *a sufficiently small compact neighborhood $\mathcal{Q} \subset \mathcal{Q}^\star \subset L$ of q, and*
- *a small enough $t^\star > 0$*

such that N_Q of item VIa *be big enough in order that whenever*
- *$v_0 \in T_{z_0}M^2$ with $z_0 \in \mathcal{Q}$;*
- *$\underline{t} = (t_j)_{j=1}^{\infty}$ is a sequence satisfying $|t_j| < t^\star$, $j \geqslant 1$, and*
- *there is $k \in \mathbb{N}$ such that $N_Q \leqslant k < \infty$ is the first integer satisfying $f_{\underline{t}}^k(z) \in \mathcal{R}$;*

then we have
1. $\text{slope}(v_0) \geqslant c_0 \Rightarrow \text{slope}(Df_{\underline{t}}^k(z_0)v_0) \geqslant b_0$, *and*
2. $\|Df_{\underline{t}}^k(z_0)v_0\| \geqslant \sigma\|v_0\|$,

where $\|\cdot\|$, the maximum norm on $L \subset \mathbb{R}^2$, and the slope are to be measured in the linearizing coordinates given by $\varphi_0 : L \to M^2$.

In other words, every vector sufficiently away from the tangent directions of $\mathcal{H}^s$ at $\mathcal{Q}$ will keep pointing away from $\mathcal{H}^s$ when it first arrives at $\mathcal{R}$, i.e., there are no folds in between by the action of f_t.

Proof. – By items I through VII of Section 11.3 there is an expanding cone field $\mathcal{C}^u$ defined over $U \cup \mathcal{Q}^\star \cup L$ respected by all f_t with $|t| < t^\star$ outside of $\mathcal{R}$. It may be seen as a cone field centered around the tangent vectors to $\mathcal{H}^u$, and we may assume that vectors in $\mathcal{C}^u$ at points of L have slope $\geqslant b_0$, since $\mathcal{H}^u$ is given by $x = cont.$ in the domain L of the coordinate chart φ_0.

We let $v_0 \in T_{z_0}M^2$, $z_0 \in \mathcal{Q}$, $\underline{t}$ and $N_Q \leqslant k < \infty$ be as in the statement of the lemma. If $\text{slope}(v_0) \geqslant c_0$, then by VIa it holds that $z_{N_Q} = f_{\underline{t}}^{N_Q}(z_0) \in L$ and $v_{N_Q} = Df_{\underline{t}}^{N_Q}(z_0)v_0 \in \mathcal{C}^u(z_{N_Q})$. Indeed by (20) we have $\text{slope}(v_{N_Q}) \geqslant C^{N_Q} \cdot \text{slope}(v_0)$, where $C \approx \sigma_0\lambda_0^{-1} > 1$, and N_Q may be taken sufficiently big according to item VI, by shrinking Q and $t^\star$. Likewise we may arrange for $\|v_{N_Q}\| \geqslant \sigma\|v_0\|$ to hold.

If $k = N_Q$, then the lemma is proved. Otherwise we can write $z_k = f_{\underline{t}}^k(z_0) = f_{\underline{s}}^{k-N_Q}(z_{N_Q}) \in \mathcal{R}$ where $\underline{s} = \sigma^{k-N_Q}\underline{t}$ and $v_k = Df_{\underline{t}}^k(z_0)v_0 = Df_{\underline{s}}^{k-N_Q}(z_{N_Q})v_{N_Q}$. Moreover, Lemma 12.1, the construction of $\mathcal{C}^u$ and the definition of $k \geqslant 1$ as the first iterate to arrive at $\mathcal{R}$ together imply that the iterates $v_{N_Q}, \ldots, v_{k-1}, v_k$ are all in the respective cones of $\mathcal{C}^u$, and therefore $\text{slope}(v_k) \geqslant b_0$ and $\|v_k\| \geqslant \|v_{N_Q}\| \geqslant \sigma\|v_0\|$. □

Now for the effect of the tangency in $\mathcal{Q}$, recalling that the slope and norm are measured in the φ_0 coordinates.

LEMMA 12.3. – *Given $\zeta > 0$ there is $b_0 > 0$ such that for all sufficiently small compact neighborhoods $\mathcal{Q}$ of q and small $t^\star > 0$ it holds for every $|t| < t^\star$ that*

$$\left.\begin{array}{l} z \in \mathcal{R},\ v \in T_z M^2 \\ \text{and } \operatorname{slope}(v) \geqslant b_0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \operatorname{slope}(Df_t(z)v) \leqslant \zeta \text{ and} \\ \|Df_t(z)v\| \geqslant \dfrac{\alpha}{100} \cdot \|v\|. \end{array}\right.$$

Proof. – We take $z \in \mathcal{R}$, $v \in T_z M^2$ and $\zeta > 0$. By the differentiability of φ_t with respect to t we know that $\overline{f}_t = \varphi_0^{-1} \circ f_t \circ \varphi_0$ has the same local expression (21) as $\tilde{f}_t$. We may suppose $\varphi_0(z) = (z, y+1)$ and $D\varphi_0(v) = (v_1, v_2)$ and derive from (21) that

$$\begin{aligned} &\operatorname{slope}\big(D\varphi_0^{-1}\big(f_t(x, y+1)\big) Df_t(x, y+1)(v_1, v_2)\big) \\ &\quad = \left| \frac{[2\beta y + D_3 H_2(t, x, y)] \cdot v_2 + [\gamma + D_2 H_2(t, x, y)] \cdot v_1}{[\alpha + D_3 H_1(t, x, y)] \cdot v_2 + [\rho + D_2 H_1(t, x, y)] \cdot v_1} \right| \\ &\quad \leqslant \frac{|2\beta y + D_3 H_2(t, x, y)| + |\gamma + D_2 H_2(t, x, y)| \cdot |v_1/v_2|}{||\alpha + D_3 H_1(t, x, y)| - |\rho + D_2 H_1(t, x, y)| \cdot |v_1/v_2||}. \end{aligned}$$

If $\operatorname{slope}(v_1, v_2) \geqslant b_0$ then we can write

$$\leqslant \frac{|2\beta y + D_3 H_2(t, x, y)| + |\gamma + D_2 H_2(t, x, y)| \cdot b_0^{-1}}{||\alpha + D_3 H_1(t, x, y)| - |\rho + D_2 H_1(t, x, y)| \cdot b_0^{-1}|}.$$

We easily see that if b_0 is big enough and $\delta_0 > 0$ in item VI is small enough, then since $\alpha \cdot \beta \cdot \gamma \neq 0$ the last quotient approximates $|2\beta y|/|\alpha| = |2\beta\alpha^{-1}| \cdot |y|$, which can be made smaller then any positive $\zeta > 0$ by shrinking $\mathcal{R}$ via taking $\mathcal{Q}$ and $t^\star > 0$ smaller. Moreover making the compact neighborhood $\mathcal{Q}$ of q and $t^\star > 0$ smaller just enables δ_0 to be smaller, so we are safe.

The denominator in the last quotient has a modulus bigger than

$$\begin{aligned} &\big||\alpha + D_3 H_1(t, x, y)| \cdot |v_2| - |\rho + D_2 H_1(t, x, y)| \cdot |v_1|\big| \\ &\quad \geqslant |v_2| \cdot \big||\alpha + D_3 H_1(t, x, y)| - |\rho + D_2 H_1(t, x, y)| \cdot |v_1/v_2|\big| \\ &\quad \geqslant \big\|(v_1, v_2)\big\| \cdot \big||\alpha + D_3 H_1(t, x, y)| - |\rho + D_2 H_1(t, x, y)| \cdot b_0^{-1}\big| \\ &\quad \geqslant \frac{\alpha}{100} \cdot \big\|(v_1, v_2)\big\| \end{aligned}$$

since $\alpha \neq 0$ and $|D_3 H_1|, |D_2 H_1|$ and b_0^{-1} may be made very small. Also $|v_2| = \max\{|v_1|, |v_2|\}$ because we may take $|v_2/v_1| \geqslant b_0 > 1$. This provides the result on the norm. □

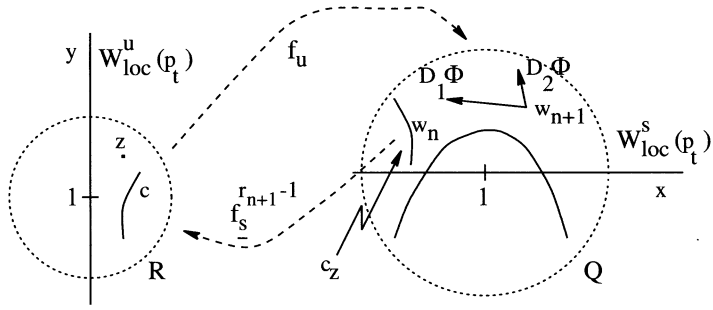

Fig. 7. The iterations in the proof of Proposition 11.2.

We let $t_0, \varepsilon > 0$ be such that $|t| < t^\star$ and $\varepsilon < \min\{|t|, |t^\star - t_0|\}$ as in the statement of Theorems 1 and 2 and observe the following.

Remark 12.2. – Expression (21) for $\tilde{f}_{t|\mathcal{R}}$ implies there are $l_0, \eta > 0$ such that the smooth curve $c_z : T = [t_0 - \varepsilon, t_0 + \varepsilon] \to M^2$, $t \mapsto f(z,t)$ has slope $\geqslant \eta$ and velocity $\geqslant l_0$ at every point $c_z(t)$ independently of $z \in \mathcal{R}$ and $t \in T$.

If we make $\zeta = \eta/3$ we get, by Lemma 12.3, a $b_0 > 0$ such that this lemma holds for all sufficiently small $\mathcal{Q}$ and $t^\star$. Setting $c_0 = \eta$ and using the b_0 just obtained, Lemma 12.2 holds for every sufficiently small $t^\star$ and $\mathcal{Q}$. We note that (23) of Lemma 12.1, on which both Lemmas 12.2 and 12.3 rest, still holds if we shrink $\mathcal{Q}$ and $t^\star$ and, moreover, Lemmas 12.2 and 12.3 are independent of each other.

Hence there are a compact neighborhood $\mathcal{Q}$ of q and $t^\star > 0$ such that both Lemmas 12.2 *and* 12.3 *hold with some $b_0 > 0$ and $c_0 = \eta$, $\zeta = \eta/3 > 0$.*

We are now ready for the

Proof of 11.2. – We let $w \in \mathcal{Q}$ be a regular point with respect to $\mathcal{F}_{t_0,\varepsilon}$ according to Definition 11.3 and pick some $\underline{t} \in \Delta = \Delta_\varepsilon(t_0)$ and $n \geqslant 1$. Then $w_n = f^{R_n}(w, \underline{t}) \in \mathcal{Q}$ and $z = f^{R_n - 1}(w, \underline{t}) \in \mathcal{R}$. Moreover since w is regular, its perturbed orbits $\mathcal{O}(w, \underline{s})$ have the same return times to $\mathcal{Q}$ independently of $\underline{s} \in \Delta$, and so c_z is a smooth curve in $\mathcal{Q}$ with slope $\geqslant c_0 = \eta$ and speed $\geqslant l_0$.

Setting $\underline{s} = \sigma^{r_n}\underline{t}$ then

$$c = f_{\underline{s}}^{r_{n+1}-1} \circ c_z : t \in T \mapsto f^{r_{n+1}-1}\big(c_z(t), t_{r_n+1}, \ldots, t_{r_{n+1}-1}\big)$$

is a curve in $\mathcal{R}$ with slope $\geqslant b_0$ and speed $\geqslant \sigma_0 l_0$ by Lemma 12.2, whereas, by Lemma 12.3, $f_u \circ c$ is a curve in $\mathcal{Q}$ with slope $\leqslant \zeta = \eta/3$ and speed $\geqslant \frac{\alpha}{100}\sigma_0 l_0$ for all $u \in T = [t_0 - \varepsilon, t_0 + \varepsilon]$.

The regularity of w implies $\Phi(t,u) = f(c(t),u)$ to be such that $\Phi(t,u) \in f^{R_{n+1}}(w,\Delta) \subset \mathcal{Q}$ for every $(t,u) \in T \times T$. In short we have

$$\begin{cases} \text{slope}(D_1\Phi) \leqslant \eta/3, \\ \|D_1\Phi\| \geqslant \dfrac{\alpha}{100} \cdot \sigma_0 l_0, \end{cases} \quad \text{and} \quad \begin{cases} \text{slope}(D_2\Phi) \geqslant \eta, \\ \|D_2\Phi\| \geqslant l_0. \end{cases} \tag{24}$$

Noting that $D\Phi$ is the derivative of $f^{R_{n+1}}(w,.)$ with respect to the R_nth and R_{n+1}th coordinates at $\underline{t}$, we have $D\Phi = D_{R_n,R_{n+1}} f^{R_{n+1}}(w,\underline{t}) : \mathbb{R}^2 \to T_{w_{n+1}} M^2$ is a surjection for every $\underline{t} \in \Delta$. We conclude that $f_w^{R_{n+1}} : \Delta \to M^2, \underline{t} \mapsto f^{R_{n+1}}(w,\underline{t})$ is a submersion. This immediately gives 11.2(2) by definition of $f^{R_{n+1}}(w,\nu^\infty)$, because the inverse image by a submersion preserves sets of measure zero.

Making $\underline{t} = (s,s,s,\ldots) \in \Delta$ for some $s \in T$, since the bounds in (24) do not depend on $\underline{t}$, we deduce from $f^{R_{n+1}}(w,\Delta) \supset \Phi(T \times T)$ that there is $\xi_0 = \xi_0(s) > 0$ such that $f^{R_{n+1}}(w,\Delta)$ contains a ball of radius ξ_0 around $\Phi(s,s) = f_s^{R_{n+1}}(w)$ as stated in 11.2(1). □

13. REGULARITY OF LIMIT POINTS

Let $z \in \mathcal{Q}$ be V-recurrent with $\nu^\infty(V) > 0$ and let $\underline{t}$ be a V-generic vector and $w \in \omega(z,\underline{t})$.

CLAIM 13.1. – *If some $\underline{\theta} \in \Delta$ takes w to $\mathcal{Q}$ after $k \geqslant 1$ iterates, then every other $\underline{\varphi} \in \Delta$ must do the same.*

Indeed, if $k \geqslant 1$ and $\underline{\theta} \in \Delta$ are such that $f^k(w,\underline{\theta}) \in \mathcal{Q}$ and there is $\underline{\varphi} \in \Delta$ such that $f^k(w,\underline{\varphi}) \notin \mathcal{Q}$, then we must have $f^{k-1}(w,\underline{\theta}) \in \mathcal{R}$ and $f^{k-1}(w,\underline{\varphi}) \notin \mathcal{R}$.

By connectedness of T^{k-1} and continuity of f^k (v. Property 2.1) there must be $\underline{\psi} \in \Delta$ such that $f^{k-1}(w,\underline{\psi}) \in \mathcal{A}$. Since $\mathcal{A}$ is open and $w \in \omega(z,\underline{t})$ with $\underline{t}$ V-generic, we may find for small $\delta > 0$ a $n \in \mathbb{N}$ (according to Lemma 5.2) such that for every $\underline{s} \in V$ satisfying $d(\underline{s},\underline{t}) < \delta$, $d_M(f^n(z,\underline{s}),w) < \delta$ and $d(\sigma^n\underline{s},\underline{\psi}) < \delta$ it holds that $f^{n+k-1}(z,\underline{s}) \in \mathcal{A}$, and so $f^{n+k}(z,\underline{s}) \in (U \cup \mathcal{Q}')^c$. Moreover, these points form a set of positive ν^∞-measure.

According to Remark 12.1 (the n above can be made arbitrarily big, bigger than N in particular), those $\underline{s}$ cannot define a perturbed

orbit $\mathcal{O}(z, \underline{s})$ with infinitely many returns to $\mathcal{Q}$, which contradicts the assumptions on z and V.

The previous arguments readily prove

CLAIM 13.2. – *The orbit of w under any $\underline{\theta} \in \Delta$ cannot fall outside of $U \cup \mathcal{Q}'$.*

CLAIM 13.3. – *If some $\underline{\theta} \in \Delta$ keeps the orbit $\mathcal{O}(w, \underline{\theta})$ inside U for all kth iterates with $k \geqslant k_0$, then every other $\underline{\varphi} \in \Delta$ must do likewise.*

In fact, if $\underline{\theta} \in \Delta$ is such that $f^k(w, \underline{\theta}) \in U$ for all $k \geqslant k_0$ for some $k_0 \in \mathbb{N}$ and there are $k_1 \geqslant k_0$ and $\underline{\varphi} \in \Delta$ such that $f^{k_1}(w, \underline{\varphi}) \notin U$, then by the connectedness of T^{k_1}, Property 2.1 and the separation between U and $\mathcal{Q}'$ given by item VII, there is $\underline{\psi} \in \Delta$ satisfying $f^{k_1}(w, \underline{\psi}) \in (U \cup \mathcal{Q}')^c$. We may now repeat the arguments proving the preceding claim.

For $w \in \omega(z, \underline{t})$ with $\underline{t}$ a V-generic vector we have the following alternatives:

1. w returns to $\mathcal{Q}$ a finite number of times only under every $\underline{\theta} \in \Delta$;
2. w never passes through $\mathcal{Q}$ under every $\underline{\theta} \in \Delta$;
3. w returns to $\mathcal{Q}$ infinitely often and $r(w, \underline{s}, n) = r(w, n)$, $\underline{s} \in \Delta$, $n \geqslant 1$.

Since w cannot get out from $U \cap \mathcal{Q}'$ by Claim 13.2, alternatives 1 and 2 imply that the orbits of w stay forever in U after some finite number of iterates or never leave U, respectively. For our purposes it is enough to suppose $w \in \omega(z, \underline{t}) \cap \mathcal{Q}$.

13.1. Finite number of returns

First we eliminate alternative 1. By Claims 13.1 and 13.3 the return times to $\mathcal{Q}$ and the iterate after which the orbits remain forever in U do not depend on the perturbation vector.

Let $r_0 \in \mathbb{N}$ be the last return iterate of w to $\mathcal{Q}$ under every $\underline{\theta} \in \Delta$. The point w is like a *regular point up to iterate* r_0 and so the arguments in Section 12 show that $f^{r_0}(w, \Delta)$ contains a curve c with slope $\geqslant \eta$ and speed $\geqslant l_0$ at every point. So its length is $\geqslant 2\varepsilon \cdot l_0 = a_0 > 0$ and since $w \in \omega(z, \underline{t})$, no orbit is allowed to leave $U \cup \mathcal{Q}'$. Hence $f^k(c, \Delta) \subset U$ for all $k \geqslant 1$. In particular, $c_k = f^k_{t_0}(c) = f^k(c, \underline{t_0}) \subset U$, $k \geqslant 1$.

According to the previous section, after N_Q iterates curve c will have all its tangent vectors in $\mathcal{C}^u$ and keep them this way for all iterates onward, because $c_k \subset U$ for all $k \geqslant 1$. Since $\mathcal{C}^u$ is a field of unstable cones, the length of c_k will grow without bound with c_k being an *unstable curve* always inside U.

This is a contradiction, since U is a small neighborhood of a hyperbolic set Λ_{t_0} of saddle type which is the maximal invariant set inside U.

13.2. No returns

Let w be as in alternative 2. Consequently $f_{t_0}^k(w) \in U$ for all $k \geqslant 1$. Since Λ_{t_0} is the maximal invariant set inside U, we deduce that if γ^u is a small segment of $\mathcal{H}_{t_0}^u(w)$ centered at w, then it is not possible that $f_{t_0}^k(\gamma^u) \subset U$ for all $k \geqslant 1$. Likewise if we replace U by $B(U, \rho)$, by item VII. Hence, writing γ_+^u, γ_-^u the two segments such that $\gamma_+^u \cup \gamma_-^u = \gamma^u$ and $\gamma_+^u \cap \gamma_-^u = \{w\}$, there are $k_\pm \geqslant 1$ and nonempty intervals $I_+ \subset \gamma_+^u$, $I_- \subset \gamma_-^u$ satisfying $f_{t_0}^i(I_\pm) \subset B(U, \rho)$ for $1 \leqslant i \leqslant k$ and $f_{t_0}^{k+1}(I_\pm) \subset (\overline{B(U, \rho)} \cup \overline{B(\mathcal{Q}', \rho)})^c$—because $\overline{B(U, \rho)} \cap \overline{B(\mathcal{Q}', \rho)} = \emptyset$ and by connectedness of $\gamma_\pm^u$ (v. Fig. 8).

Let $x \in I_\pm$ and $y \in \mathcal{H}_{t_0}^s(x)$. Then we have

$$d_M\big(f_{t_0}^k(x), f_{t_0}^k(y)\big) \leqslant C\lambda^k d_M(x, y)$$

where $1 > \lambda \geqslant |\lambda_t|$ for $|t| < t^\star$. So every $y \in \mathcal{H}_{t_0}^s(x)$ with $d_M(x, y) \leqslant C^{-1}\lambda^{-k} \cdot \rho/2$ satisfies $f_{t_0}^k(y) \in (U \cup \mathcal{Q}')^c$.

Geometrically this means that near w there are two strips $B_\pm$ made of $\mathcal{H}_{t_0}^s$-leaves with length $C^{-1}\lambda^{-k} \cdot \rho/2$ and whose intersection with γ^u is $I_\pm$ (cf. Fig. 8).

Making γ^u small and k big we can make the length of $B_\pm$ big and the distance to w small. The angle between leaves in $B_\pm$ and γ^u is near a straight angle in the φ_{t_0}-coordinates of $L \supset \mathcal{Q} \ni w$, since the slope of $\mathcal{H}_{t_0}^s$ is near 0.

Let $n_1 < n_2 < n_3 < \cdots$ be such that $z_j = f^{n_j}(z, \underline{t}) \to w$ when $j \to \infty$.

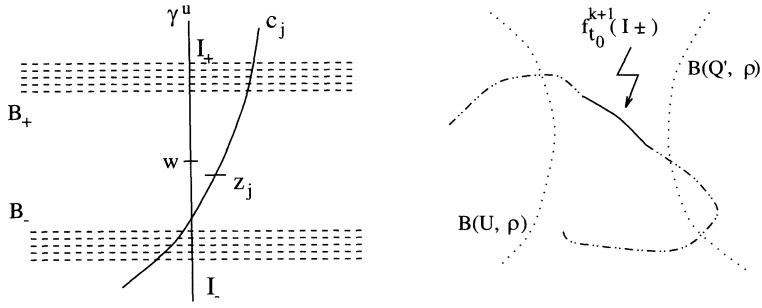

Fig. 8. The situation near w and the image of $I_\pm$.

We define $c_j : T \to L$, $u \mapsto f_u(f^{n_j-1}(z, \underline{t}))$, the *perturbation curve* through z_j and observe that either B_+ or B_- intersects $c_j(T)$ in a segment of positive length $\geqslant a_1 > 0$, since $\text{slope}(c_j'(u)) \geqslant \eta$ and the length of c_j is $\geqslant a_0 > 0$, for all $u \in T$ and $j \geqslant 1$.

This means there is a segment S_j of length $\geqslant a_2 > 0$ in T such that $c_j(S_j) \subset B_\pm$ and thus $f_{t_0}^{k+1}(c_j(S_j)) \subset (U \cup \mathcal{Q}')^c$.

According to Lemma 5.3, for every $0 < \gamma, \delta < 1$ we can find $k_0 \in \mathbb{N}$ such that for all $j \geqslant k_0$ we have $\nu(p_{n_j} V_{n_j-1}(\underline{t}, n_j - 1, \underline{s})) \geqslant 1 - \delta$ for a positive measure set $V_{n_j-1} \subset V$ and a set of $\underline{s} \in \Delta$ with ν^∞-measure $\geqslant 1 - \gamma$. Hence, since k is fixed, we may find for j big a $\underline{s} \in \Delta$ very close to $\underline{t_0} = (t_0, t_0, \ldots)$ (taking $\gamma > 0$ small) such that

$$\nu\big(S_j \cap p_{n_j} V_{n_j-1}(\underline{t}, n_j - 1, \underline{s})\big) > 0 \quad \text{and} \quad f_{\underline{s}}^{k+1}\big(c_j(S_j)\big) \subset (U \cup \mathcal{Q}')^c.$$

We have shown that inside V there is a positive measure set whose perturbation vectors send z into $(U \cup \mathcal{Q}')^c$ after $n_j + k + 1$ iterates, where j (and n_j) may be made arbitrarily big. This contradicts the assumption of V-recurrence on z, since those perturbed orbits will never again return to $\mathcal{Q}$. Alternative 2 is thus impossible.

13.3. Bounded first return times

The points $w \in \omega(z, \underline{t}) \cap \mathcal{Q}$ with $\underline{t}$ a V-generic vector satisfy alternative 3. Going back to the arguments in Section 13.1, we have an unstable curve c in $f^k(w, \Delta)$ whose length cannot grow unbounded. Therefore it must leave U and go to $\mathcal{Q}$ (since w no orbit may leave $U \cup \mathcal{Q}'$) after a finite number or iterates bounded by some $J \in \mathbb{N}$. We observe that since the length of c is $\geqslant 2\varepsilon \cdot l_0$ and the diameter of $\mathcal{R}$ is finite, we must have $(2\varepsilon l_0) \cdot \sigma^J \approx \text{diam}(\mathcal{R})$.

This proves Proposition 11.1 and Theorem 2.

Remark 13.1. – We may drop the first tangency condition of Section 11.1 if we strengthen Definition 11.2 of V-recurrent point by adding the following item

3. for ν^∞-a.e. $\underline{t} \in V$ there is $n = n(\underline{t}) \geqslant 1$ such that $f^k(z, \underline{t}) \in U \cup \mathcal{Q}$ for all $k \geqslant n$,

where U is a fixed neighborhood of the basic set p_0 belongs to and $\mathcal{Q}$ a neighborhood of the piece of the orbit of tangency outside U.

Lemma 12.1 is now needless and the rest of the proof is unchanged. The scope of the theorem is enlarged and next section shows how this extra condition on V-recurrence is not too restrictive.

14. INFINITELY MANY ATTRACTORS

We start with the particular case of perturbations of sinks.

DEFINITION 14.1. – *We say $f \in Diff^l(M)$, $l \geqslant 1$, has a* perturbation of a sink *in a finite collection $(\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ of pairwise disjoint open sets of M if there exists a neighborhood $\mathcal{V}$ of f in $Diff^l(M)$ such that, for every continuous arc $\mathcal{G} = \{g_t\}_{t\in B} \subset \mathcal{V}$ with $g_0 \equiv f$, the following holds:*

1. *$\overline{g_{\underline{t}}^n(\mathcal{U}_i)} \subset \mathcal{U}_{(i+n) \bmod r}$ for every $n \geqslant 1$, $\underline{t} \in B^{\mathbb{N}}$ and $0 \leqslant i \leqslant r-1$;*
2. *There is a constant $\beta > 0$ such that for every point $x \in \mathcal{U}_i$, $0 \leqslant i \leqslant r-1$, every $v \in T_xM \setminus \{0\}$ and every $\underline{t} \in B^{\mathbb{N}}$ it holds that*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \left\| Dg_{\underline{t}}^n(x) \cdot v \right\| \leqslant -\beta;$$

3. *With the notation introduced at Definition 5.2 we have*

$$\operatorname{diam}\left(\omega\left(\mathcal{U}_j, \Delta_\varepsilon(0)\right) \cap \mathcal{U}_i\right) \to 0 \quad \text{when } \varepsilon \to 0^+$$

for every $0 \leqslant i \leqslant r-1$.

[Where $B = \overline{B}^{\,j}(0,1)$ and $\Delta_\varepsilon(0) = (\overline{B}^{\,j}(0,\varepsilon))^{\mathbb{N}}$ as in Section 2.1.]

Next proposition characterizes this kind of invariant domains.

PROPOSITION 14.1. – *Let f be a C^l diffeomorphism of M, $l \geqslant 1$. Then f has a hyperbolic sink s_0 with period $k \geqslant 1$ if, and only if, f has a perturbation of a sink in a neighborhood $(\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ of the orbit $s_0, s_1 = f(s_0), \ldots, s_{r-1} = f^{r-1}(s_0)$ of s_0.*

Proof. – First some results that locate the limit points near a perturbed sink.

If $s_0 \in M$ is a hyperbolic sink for f with period r, then for some $0 < \lambda_1 < 1$ every eigenvalue $\lambda \in \mathbb{C}$ of $Df^r(s_0)$ satisfies $|\lambda| \leqslant \lambda_1$. Moreover, given some $\lambda_1 < \upsilon < 1$ there are $\delta > 0$ and a neighborhood $\mathcal{V}$ of f in $\mathrm{Diff}^l(M)$—both may be made arbitrarily small—such that each eigenvalue $\lambda \in \mathbb{C}$ of $Dg^r(x)$ satisfies $|\lambda| \leqslant \upsilon$ for every $g \in \mathcal{V}$ and $x \in B(s_0, \delta)$. Consequently

$$d_M\left(g^r(x), g^r(y)\right) \leqslant \upsilon \cdot d(x,y) \quad \text{for every } x, y \in B(s_0,\delta) \text{ and } g \in \mathcal{V}. \tag{25}$$

So, writing $s_i = f^i(s_0)$, we see that $(\mathcal{U}_0 = B(s_0,\delta), \ldots, \mathcal{U}_{r-1} = B(s_{r-1}, \delta))$ is a finite collection of pairwise disjoint (we may take $\delta < (1/2) \times$

$\min\{d_M(s_i, s_j)\colon\ 0 \leqslant i < j \leqslant r-1\}$) open sets of M that satisfies conditions 1 and 2 of Definition 14.1.

To get condition 3 we have the next

LEMMA 14.2. – *Let $\mathcal{G} = \{g_t\}_{t\in I} \subset \mathcal{V}$ be some continuous arc in $Diff^1(M)$ with $g_0 \equiv f$. Let $P_i = \{s_i(t)\colon\ t \in B\}$ be the set of analytic continuations of the orbit $\mathcal{O}(s_0)$ of the sink s_0 with respect to g_t, $t \in B$.*

If we fix $x \in \mathcal{U}_i$, $0 \leqslant i \leqslant r-1$, and $\underline{t} \in \Delta$, then we have

$$d_M(y, P_j) \leqslant \frac{\upsilon}{1-\upsilon} \cdot \max\{\operatorname{diam}(P_k)\colon\ 0 \leqslant k \leqslant r-1\},$$

for every $y \in \omega(x, \underline{t}) \cap \mathcal{U}_j$, $j = 0, \ldots, r-1$.

Proof. – This is an easy consequence of (25). □

We now know that $\omega(x, \underline{t}) \subset B(P, \gamma)$ where $\gamma = \frac{\upsilon}{1-\upsilon} \cdot \max\{\operatorname{diam}(P_h)\colon\ 0 \leqslant h \leqslant r-1\}$ and, since s_0 is an hyperbolic sink for $f \equiv g_0$, we have

$$\operatorname{diam}\big(\{s_h(t)\colon\ t \in \overline{B}^{\,j}(0, \varepsilon)\}\big) \to 0 \quad \text{when } \varepsilon \to 0^+$$

by the structural stability results for such attractor.

Therefore item 3 holds for $(\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ constructed above and we have shown that in a neighborhood of the orbit of every hyperbolic sink there is a perturbation of sink.

Conversely, let us suppose f has a perturbation of a sink in some collection $(\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ of pairwise disjoint open sets and take $\mathcal{G} = \{g_t\}_{t\in I}$ as in Definition 14.1. Then we will have by definition $\omega(\mathcal{U}_i, \Delta_{\varepsilon_1}(0)) \subseteq \omega(\mathcal{U}_i, \Delta_{\varepsilon_2}(0))$ for every small $0 < \varepsilon_1 < \varepsilon_2$ and every $0 \leqslant i \leqslant r-1$. Property 3 of Definition 14.1 now ensures there is a point s_0 such that $\{s_0\} = \bigcap_{\varepsilon>0}[\omega(\mathcal{U}_0, \Delta_\varepsilon(0)) \cap \mathcal{U}_0]$ since $\omega(\mathcal{U}_0, \Delta_\varepsilon(0))$ is a closed set.

Writing $\underline{0} = (0, 0, \ldots)$ then $\underline{0} \in \Delta_\varepsilon(0)$ and $\omega(s_0, \underline{0}) \subset \omega(\mathcal{U}_0, \Delta_\varepsilon(0))$ for every $\varepsilon > 0$. Thus $\{s_0\} = \omega(s_0, \underline{0}) \cap \mathcal{U}_0 = \omega_f(s_0) \cap \mathcal{U}_0$. Considering the dynamics induced in $(\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ by the arc $\mathcal{G}$ we see that $\omega_f(s_0) = \{s_0, \ldots, s_{r-1}\}$ where $s_i = f^i(s_0)$, $i = 0, \ldots, r-1$.

Since the limit is f-invariant, we have $f^r(s_0) = s_0$ and found a r-periodic orbit of f. In addition, Property 2 of Definition 14.1 guarantees that for each $v \in T^1_{s_0}M = \{u \in T_{s_0}M\colon\ \|u\| = 1\}$ such that v is an eigenvector of $Df^r(s_0)$ corresponding to the eigenvalue $\lambda \in \mathbb{C}$ (using the complexification of $Df^r(s_0)\colon T_{s_0}M \to T_{s_0}M$ if need be) the following holds

$$\begin{aligned} 0 > -\beta &\geqslant \limsup_{n\to+\infty} \frac{1}{rn} \log\big\|Df^{n\cdot r}(s_0) \cdot v\big\| \\ &= \frac{1}{r} \log|\lambda| \Rightarrow |\lambda| \leqslant \exp(-r\beta) < 1 \end{aligned}$$

and so $\mathrm{sp}(Df^r(s_0)) \subset \{z \in \mathbb{C}\colon |z| < 1\}$. Hence $s_0, \ldots, s_{r-1}$ is the orbit of an hyperbolic sink for f. □

14.1. Newhouse's and Colli's phenomena

Let us suppose the family f satisfying the conditions specified in Section 11.1 is also in the conditions of Newhouse's theorem (cf. [16, 17] and [22]) on the coexistence of infinitely many sinks, that is, p_0 is a dissipative ($|\det Df_0(p_0)| < 1$) saddle point.

We may now choose a parameter $a > 0$ such that f_a has infinitely many hyperbolic sinks in $\mathcal{Q}$. Moreover $a > 0$ may be taken arbitrarily close to zero (see [22, Chapter 6]) and thus all the results of previous sections apply to the present setting.

Let N be some positive integer and let us pick N distinct orbits of hyperbolic sinks for f_a in $\mathcal{Q}$: $\mathcal{O}(s^{(i)})$, $i = 1, \ldots, N$. Since they are hyperbolic attractors, they are isolated: there exist pairwise disjoint – even separated – open neighborhoods V_i of $\mathcal{O}(s^{(i)})$, $i = 1, \ldots, N$. Moreover, by the previous subsection, we may construct a perturbation of a sink inside each V_i associated to $\mathcal{O}(s^{(i)})$ with respect to an arc $\mathcal{F}_{a,\varepsilon_i}$, for some $\varepsilon_i > 0$, and every $1 \leqslant i \leqslant N$.

We now observe that a perturbation of a sink obviously is, in particular, a completely and symmetrically invariant domain. Specifically, each perturbation of a sink constructed in V_i is a completely and symmetrically invariant domain with respect to the arc $\mathcal{F}_{a,\varepsilon_i}$, $i = 1, \ldots, N$.

Hence, setting $\varepsilon_0 = \min\{\varepsilon_1, \ldots, \varepsilon_N\}$, we have $\varepsilon_0 > 0$ and the former invariant domains are also completely and symmetrically invariant with respect to the arc $\mathcal{F}_{a,\varepsilon}$ for every $0 < \varepsilon < \varepsilon_0$. Then, by Section 6.1, there is a minimal domain $\mathcal{M}_i^\varepsilon$ inside each perturbation of a sink V_i, for every $1 \leqslant i \leqslant N$ and noise level $0 < \varepsilon < \varepsilon_0$.

We have thus constructed N distinct minimal invariant domains in $\mathcal{Q}$ for the arc $\mathcal{F}_{a,\varepsilon}$ for every $0 < \varepsilon < \varepsilon_0$ and proved

PROPOSITION 14.3. – *Given an arc $\mathcal{F}$ as in Section 11.1 where p_0 is a dissipative saddle point, for every parameter $a > 0$ sufficiently close to zero such that f_a has infinitely many sinks in $\mathcal{Q}$, we have the following.*

For every $N \in \mathbb{N}$ there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, the number of minimal invariant domains in $\mathcal{Q}$ for the arc $\mathcal{F}_{a,\varepsilon}$ is no less than N.

We now remark that what enables us to build an invariant domain in a neighborhood of a sink is the fact that it is attractive: given any

neighborhood U of the orbit of a sink $s_0, \ldots, s_{r-1}$ there is another neighborhood $V \subset \overline{V} \subset U$ of the same orbit such that $f(\overline{V}) \subset V$ (a *trapping region*). By continuity, this persists for any diffeomorphism g close to f and hence we get an invariant domain.

In [7] E. Colli shows how to have infinitely many Hénon-like attractors when generically unfolding an homoclinic tangency under the same conditions of Newhouse's theorem. These attractors are separated like the infinity of sinks in the Newhouse phenomenon and each one admits a *trapping region* according to [2] and [30]. Specifically, the constructions described in [7] can be carried out verbatim within a restricted set of parameter values having this property, without altering the statements of any theorem in that paper.

Consequently we may state and prove a proposition analogous to 14.3 replacing *sink* by *Hénon-like attractor* in the paragraphs above.

15. SOME CONJECTURES

The methods used in this paper are prone to generalization. We propose some here.

(1) Is there some *similar* result to Theorem 1 for flows? The kind of perturbation to perform is part of the question.

(2) In Section 14 a characterization is given for invariant domains originating from a perturbation of a sink. Is there some *similar* characterization of an invariant domain obtained by a perturbation of an Hénon-like strange attractor?

(3) The same question regarding perturbations of elliptic islands. This is more subtle: we may ask whether there is some invariant domain near an elliptic island.

(4) We did not look at what happens to the physical probabilities when the noise level $\varepsilon > 0$ tends to zero. Does the limit exist? If it does then it must be an f-invariant probability measure. Is it an SRB-measure?

(5) Globally what can we say about the stochastic stability of the infinitely many Dirac (in Newhouse's phenomemon) or SRB (in Colli's phenomenon) measures in a neighborhood of a homoclinic tangency point? Here a global notion of stochastic stability is required, see, e.g., [31]: if μ_i are the SRB measures of f ($i = 1, 2, \ldots$), time averages of each continuous φ along almost all random orbits should be closed to the convex hull of the $\int \varphi \, d\mu_i$ for small $\varepsilon > 0$.

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