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Stationary solutions for the Cahn-Hilliard equation

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ABSTRACT. – We study the Cahn-Hilliard equation in a bounded domain without any symmetry assumptions. We assume that the mean curvature of the boundary has a nondegenerate critical point. Then we show that there exists a spike-like stationary solution whose global maximum lies on the boundary. Our method is based on Lyapunov-Schmidt reduction and the Brouwer fixed-point theorem. © Elsevier, Paris

RÉSUMÉ. – Nous étudions l'équation de Cahn et Hilliard dans un domaine ouvert en ne supposant de symétrie pour le domaine. Nous supposons que la courbure moyenne sur la frontière a un point critique non dégénéré. Nous montrons qu'il existe une solution stationnaire avec un pic qui atteint son maximum sur la frontière du domaine. Notre méthode utilise la réduction de Lyapunov et Schmidt et le théorème du point fixe de Brouwer. © Elsevier, Paris

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1. INTRODUCTION

The Cahn-Hilliard equation [5] is an accepted macroscopic field-theoretical model of processes such as phase separation in a binary alloy. In its original form it is derived from a Helmholtz free energy

$$E(u) = \int_\Omega [F(u(x)) + \frac{1}{2}\epsilon^2 |\nabla u(x)|^2] dx$$

where Ω is the region occupied by the body, $u(x)$ is a conserved order parameter representing for example the concentration of one of the components, and $F(u)$ is the free energy density which has a double well structure at low temperatures (see Figure 1). The most commonly used model is for $F(u) = (1 - u^2)^2$.

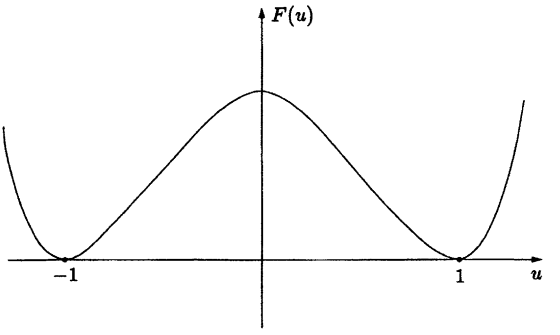

Fig. 1.

The constant ϵ is proportional to the range of intermolecular forces and the gradient term is a contribution to the free energy coming from spatial fluctuations of the order parameter. Moreover the mass $\overline{m} = \frac{1}{|\Omega|}\int_\Omega u dx$ is constant. Thus a stationary solution of $E(u)$ under $\overline{m} = \frac{1}{|\Omega|}\int_\Omega u dx$ takes the following form

$$\begin{cases} \epsilon^2 \Delta u - f(u) = \sigma_\epsilon & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \int_\Omega u = \overline{m}|\Omega| \end{cases} \tag{1.1}$$

where $f(u) = F'(u)$ (see Figure 2) and σ_ϵ is a constant.

There have been numerous studies of the Cahn-Hilliard equation. The global minimizer of $E(u)$ has a transition layer. More precisely there exists an open set $\Gamma \subset \Omega$ such that if u_ϵ is a global minimizer then $u_\epsilon \to 1$ on $\Omega \setminus \overline{\Gamma}$, $u_\epsilon \to -1$ on Γ and $\partial\Gamma \cap \overline{\Omega}$ is a minimal surface and has constant

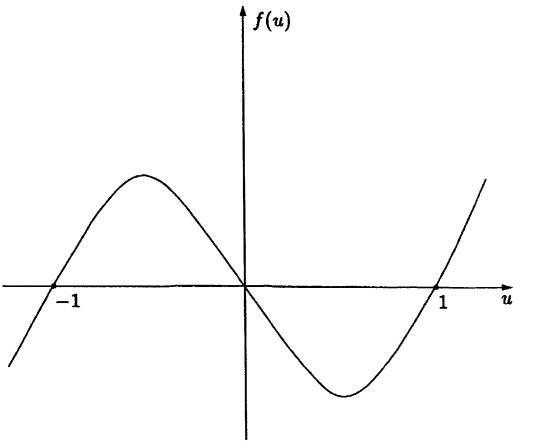

Fig. 2.

mean curvature, see [16]. The dynamics of the interface have been studied extensively, see for example [2], [3], [23]. Also local minimizers of $E(u)$ have been studied and their transition layer structure has been established in [6] and [13]. In particular, Chen and Kowalczyk in [6] used boundary mean curvature to construct local minimizers (therefore transition layer solutions) for equation (1.1).

In this paper we are concerned with solutions of (1.1) with spike layers. In the one dimensional case, Bates and Fife [4] studied nucleation phenomena for the Cahn-Hilliard equation and proved the existence of three monotone nondecreasing stationary solutions when $\overline{m}$ is in the metastable region ($\sqrt{1/3} < \overline{m} < 1$), (a) the constant solution $u \equiv \overline{m}$, (b) a boundary spike layer solution where the layer is located at the left-hand endpoint, (c) a transition layer solution with a layer in the interior of the material.

Motivated by the results of [4], we shall construct a boundary spike layer solution to (1.1) for $\epsilon << 1$ in the higher dimensional case when $\overline{m}$ is in the metastable region.

The existence of spike layer solutions as well as the location and the profile of the peaks for other problems arising in various models such as chemotaxis, pattern formation, chemical reactor theory, etc. have been studied by Lin, Ni, Pan, and Takagi [14, 17, 18, 19] for the Neumann problem and by Ni and Wei [20] for the Dirichlet problem. However, they do not have the volume constraint and the nonlinearity is simpler than here. To our knowledge the present paper is the first to establish this kind of results for the Cahn-Hilliard equation in higher dimensions without any symmetry assumptions on Ω.

Naturally these stationary solutions are essential for the understanding of the dynamics of the corresponding evolution process. While Bates and Fife [4] prove some results in this direction for the one dimensional case these questions are open for higher dimensions.

In [11] in the one dimensional case the number of all stationary solutions is counted by arguments using transversality.

First we make the following transformation.

$$v = \overline{m} - u,$$

$$g(v) = -f(\overline{m}) + f(\overline{m} - v).$$

Rewrite

$$g'(0) = -m,\ g(v) = -mv + h(v).$$

Then equation (1.1) becomes

$$\begin{cases} \epsilon^2 \Delta v - mv + h(v) - \frac{1}{|\Omega|} \int_\Omega h(v) = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

(Figure 3 shows qualitatively how the graph of g looks like.)

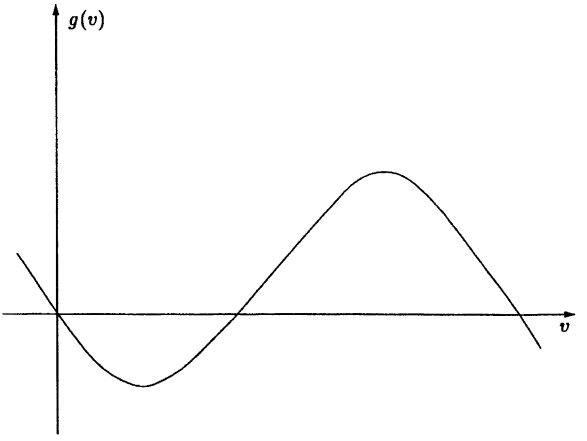

Fig. 3.

To accommodate more general g we assume that

(1) $g'(0) < 0,\ g(0) = 0,\ g \in C^3(R, R)$.

(2) $g(v)$ has only two zeroes for $v > 0,\ 0 < a_1 < a_2$ and

$$\int_0^{a_2} g(s)\, ds > 0,\ g'(a_2) < 0.$$

(3) The function $v \to \frac{g(v)}{v-v_0}$ is nonincreasing in the interval (v_0, a_2) where v_0 is defined as the unique number in (a_1, a_2) such that $\int_0^{v_0} g(s)\,ds = 0$.

(4) $|h'(v)|, |h''(v)| \leq C$ for any v.

Remarks. – (1) Condition (3) can be weakened further. For example, the conditions in [7] will be enough since we just need the uniqueness and weak nondegeneracy of the ground state solutions of (1.3).

(2) Condition (4) is not a restriction physically since in the physical world v is always bounded. Hence we can modify h near infinity so that h satisfies (4).

It is easy to see that for $f(u) = -2u(1-u^2)$ conditions (1), (2), (3), and (4) are satisfied. Our main result can be stated as follows.

THEOREM 1.1. – *Let Ω be a bounded smooth domain in $R^N (N \geq 2)$ and $P_0 \in \partial\Omega$ be such that $\nabla_{\tau_{P_0}} H(P_0) = 0$ and $(\nabla^2_{\tau_{P_0}} H(P_0)) \neq 0$ where $H(P_0)$ is the mean curvature of $P_0 \in \partial\Omega$ and ∇_{P_0} is the tangential derivative at P_0. Then for $\epsilon << 1$ there exists a solution v_ϵ of (1.2) such that $v_\epsilon \to 0$ in $C^1_{loc}(\overline{\Omega} \setminus P_0)$, v_ϵ has only one local (hence global) maximum point P_ϵ and $P_\epsilon \in \partial\Omega, P_\epsilon \to P_0, v_\epsilon(P_\epsilon) \to V(0) > 0$. Moreover*

$$\epsilon^{-N}\left\{\int_\Omega \epsilon^2 \left|\nabla v_\epsilon - \nabla V\left(\frac{x-P_\epsilon}{\epsilon}\right)\right|^2 + \int_\Omega \left|v_\epsilon - V\left(\frac{x-P_\epsilon}{\epsilon}\right)\right|^2\right\} \to 0$$

as $\epsilon \to 0$ where $V(y)$ is the unique solution of

$$\begin{cases} \Delta V - mV + h(V) = 0, \\ V(0) = \max_{y \in R^N} V(y),\ V > 0, \\ V(y) \to 0 \text{ at } \infty. \end{cases} \tag{1.3}$$

(By the results of [9] and [24], (1.3) has a unique radial solution).

The method of our construction evolves from that of [8], [21] and [22] on the semi-classical (i.e. for small parameter h) solution of the nonlinear Schrödinger equation

$$\frac{h^2}{2}\Delta U - (V-E)U + U^p = 0 \tag{1.4}$$

in R^N where V is a potential function and E is a real constant. The method of Lyapunov-Schmidt reduction was used in [8], [21] and [22] to construct solutions of (1.4) close to nondegenerate critical points of V for h sufficiently small.

Following the strategy of [9], [21] and [22] we shall construct a solution v_ϵ of (1.2) with maximum near a given nondegenerate critical point of the mean curvature P_0 on $\partial\Omega$. Heuristically we rescale (1.2) to obtain

$$\begin{cases} \Delta u_\epsilon - m u_\epsilon + h(u_\epsilon) - \frac{1}{|\Omega_{\epsilon,P}|}\int_{\Omega_{\epsilon,P}} h(u_\epsilon) = 0 & \text{in } \Omega_{\epsilon,P}, \\ \frac{\partial u_\epsilon}{\partial \nu_\epsilon} = 0 & \text{on } \partial\Omega_{\epsilon,P} \end{cases} \tag{1.5}$$

where $u_\epsilon(z) = v_\epsilon(x)$ for $z = (x - P)/\epsilon$, $z \in \Omega_{\epsilon,P}$ and $\Omega_{\epsilon,P} = \{z \in R^N \mid \epsilon z + P \in \Omega\}$ and ν_ϵ is the unit outer normal to $\partial\Omega_{\epsilon,P}$.

Taking the limit $\epsilon \to 0$, $u_\epsilon \to V$ where V is the unique solution of

$$\begin{cases} \Delta w - m w + h(w) = 0 & \text{in } R^N_+, \\ w > 0 & \text{in } R^N_+, \\ \frac{\partial w}{\partial y_N} = 0 & \text{on } R^{N-1} \times \{0\} \end{cases} \tag{1.6}$$

with $V(0) = \max_{R^N_+} V$. Therefore the ground state solution V restricted to R^N_+ can be an approximate solution for u_ϵ. Since the linearized problem arising from (1.6) has the $(N-1)$-dimensional kernel span$\{\frac{\partial V}{\partial y_1}, \ldots, \frac{\partial V}{\partial y_{N-1}}\}$ we first "solve" (1.6) up to this kernel and then use the nondegeneracy of $H(P_0)$ to take care of the kernel separately.

The paper is organized as follows. Notation, preliminaries and some useful estimates are explained in Section 2. Section 3 contains the setup of our problem and we solve (1.2) up to approximate kernel and cokernel, respectively. Finally in Section 4 we solve the reduced problem.

2. TECHNICAL ANALYSIS

In this section we introduce a projection and derive some useful estimates.

Throughout the paper we shall use the letter C to denote a generic positive constant which may vary from term to term. We denote $R^N_+ = \{(x', x_N) | x_N > 0\}$. Let V be the unique solution of (1.3).

Let $P \in \partial\Omega$. We can define a diffeomorphism straightening the boundary in a neighborhood of P. After rotation of the coordinate system we may assume that the inward normal to $\partial\Omega$ at P is pointing in the direction of the positive x_N-axis. Denote $x' = (x_1, \ldots, x_{N-1})$, $B'(R_0) = \{x' \in R^{N-1} \mid |x'| < R_0\}$ and $\Omega_1 = \Omega \cap B(P, R_0) = \{(x', x_N) \in B(P, R_0) | x_N - P_N > \rho(x' - P')\}$ where $B(P, R_0) = \{x \in R^N \mid |x - P| < R_0\}$. Then, since $\partial\Omega$ is smooth, we can find a constant $R_0 > 0$ such that $\partial\Omega \cap \overline{\Omega_1}$ can be represented by the graph of a smooth function $\rho_P : B'(R_0) \to R$ where

$\rho_P(0) = 0, \nabla\rho_P(0) = 0$. From now on we omit the use of P in ρ_P and write ρ instead if this can be done without causing confusion. The sum of the principal curvatures of $\partial\Omega$ at P is $H(P) = \sum_{i=1}^{N-1} \rho_{ii}(0)$ where

$$\rho_i = \frac{\partial\rho}{\partial x_i}, \quad i = 1, \ldots, N-1$$

and higher derivatives will be defined in the same way. By Taylor expansion we have

$$\rho(x' - P') = \frac{1}{2}\sum_{i,j=1}^{N-1} \rho_{ij}(0)(x_i - P_i)(x_j - P_j)$$

$$+\frac{1}{6}\sum_{i,j,k=1}^{N-1} \rho_{ijk}(0)(x_i - P_i)(x_j - P_j)(x_k - P_k) + \mathcal{O}(|x' - P'|^4).$$

In the following we use ρ_α to denote the multiple differentiation $\frac{\partial^{|\alpha|}\rho}{\partial x^\alpha}$, where α is a multiple index.

For $x \in \partial\Omega$, let $\nu(x)$ denote the unit outward normal at x and $\partial/\partial\nu$ the normal derivative. Let $(\tau_1(x), ..., \tau_{N-1}(x))$ denote $(N-1)$ linearly independent tangential vectors and $(\frac{\partial}{\partial\tau_1}, .., \frac{\partial}{\partial\tau_{n-1}})$ the tangential derivatives.

In our coordinate system, for $x \in \omega_1 := \partial\Omega \cap B(P, R_0)$, we have

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla_{x'}\rho|^2}}(\nabla_{x'}\rho, -1),$$

$$\frac{\partial}{\partial\nu} = \frac{1}{\sqrt{1 + |\nabla_{x'}\rho|^2}}\left\{\sum_{j=1}^{N-1}\rho_j\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_N}\right\}\Bigg|_{x_N - P_N = \rho(x'-P')},$$

$$\tau_i(x) = (0, \ldots, 1, \ldots, 0, \rho_i(x')),$$

$$\frac{\partial}{\partial\tau_i} = \frac{1}{\sqrt{1 + |\nabla_{x'}\rho|^2}}\left\{\frac{\partial}{\partial x_i} + \rho_i\frac{\partial}{\partial x_N}\right\}\Bigg|_{x_N - P_N = \rho(x'-P')}.$$

For a smooth bounded domain U we now introduce a projection P_U of $H^2(U)$ onto $\{v \in H^2(U) | \partial v/\partial\nu = 0 \text{ at } \partial U\}$ as follows: For $v \in H^2(U)$ let $w = P_U v$ be the unique solution of the boundary value problem

$$\begin{cases} \Delta w - mw + h(v) = 0 & \text{in U,} \\ \frac{\partial w}{\partial\nu} = 0 & \text{on } \partial U. \end{cases}$$

Let $h_{\epsilon,P}(x) = V\left(\frac{x-P}{\epsilon}\right) - P_{\Omega_{\epsilon,P}} V\left(\frac{x-P}{\epsilon}\right)$ where

$$\Omega_{\epsilon,P} = \{z \in R^n | P + \epsilon z \in \Omega\}.$$

Then $h_{\epsilon,P}$ satisfies

$$\begin{cases} \epsilon^2 \Delta v - mv = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial V}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We denote

$$\|v\|_\epsilon^2 = \epsilon^{-N} \int_\Omega [\epsilon^2 |\nabla v|^2 + mv^2].$$

For $x \in \Omega_1$ set now

$$\begin{cases} \epsilon y' = x' - P', \\ \epsilon y_N = x_N - P_N - \rho(x' - P'). \end{cases} \tag{2.2}$$

Furthermore, for $x \in \Omega_1$ we introduce the transformation

$$\begin{cases} T_i(x') = x_i, & i = 1, \ldots, N-1 \\ T_N(x') = x_N - P_N - \rho(x' - P'). \end{cases} \tag{2.3}$$

Note that then

$$y = \frac{1}{\epsilon} T(x).$$

The Laplace operator and the boundary derivative operator become

$$\epsilon^2 \Delta_x = \Delta_y + |\nabla_{x'} \rho|^2 \frac{\partial^2}{\partial y_N^2} - 2 \sum_{i=1}^{N-1} \rho_i \frac{\partial^2}{\partial y_i \partial y_N} - \epsilon \Delta_{x'} \rho \frac{\partial}{\partial y_N} \quad \text{for } x \in \Omega_1, \tag{2.4}$$

$$\sqrt{1 + |\nabla_{x'} \rho|^2} \frac{\partial}{\partial \nu_x} = \frac{1}{\epsilon} \left\{ \sum_{j=1}^{N-1} \rho_j \frac{\partial}{\partial y_j} - (1 + |\nabla_{x'} \rho|^2) \frac{\partial}{\partial y_N} \right\} \quad \text{for } x \in \omega_1. \tag{2.5}$$

Let v_1 be the unique solution of

$$\begin{cases} \Delta v - mv = 0 & \text{in } R_+^N, \\ \frac{\partial v}{\partial y_N} = -\frac{V'}{|y|} \frac{1}{2} \sum_{i,j=1}^{N-1} \rho_{ij}(0) y_i y_j & \text{on } \partial R_+^N \end{cases} \tag{2.6}$$

where V' is the radial derivative of V, i.e. $V' = V_r(r)$, and $r = \left|\frac{x-P}{\epsilon}\right|$. Let v_2 be the unique solution of

$$\begin{cases} \Delta v - mv - 2 \sum_{i,j=1}^{N-1} \rho_{ij}(0) y_i \frac{\partial^2 v_1}{\partial y_j \partial y_N} = 0 & \text{in } R_+^N, \\ \frac{\partial v}{\partial y_N} = \sum_{i.j=1}^{N-1} \rho_{ij}(0) y_i \frac{\partial v_1}{\partial y_j} & \text{on } \partial R_+^N. \end{cases} \tag{2.7}$$

Let v_3 be the unique solution of

$$\begin{cases} \Delta v - mv = 0 & \text{in } R^N_+, \\ \frac{\partial v}{\partial y_N} = -\frac{V'}{|y|}\frac{1}{3}\sum_{i,j,k=1}^{N-1} \rho_{ijk}(0) y_i y_j y_k & \text{on } \partial R^N_+. \end{cases} \tag{2.8}$$

Note that v_1, v_2 are even functions in $y' = (y_1, ..., y_{N-1})$ and v_3 is an odd function in $y' = (y_1, ..., y_{N-1})$ (i.e. $v_1(y', y_N) = v_1(-y', y_N), v_3(y', y_N) = -v_3(-y', y_N)$). Moreover, it is easy to see that $|v_1|, |v_2|, |v_3| \leq Ce^{-\mu|y|}$ for some $0 < \mu < \sqrt{m}$. Let $\chi(x)$ be a smooth cutoff function such that $\chi(x) = 1$, $x \in B(0, R_0 - \delta)$ and $\chi(x) = 0$ for $x \in B(0, R_0)^C$ (for a positive number δ). Set

$$h_{\epsilon,P}(x) = \epsilon v_1(y)\chi(x-P) + \epsilon^2(v_2(y)\chi(x-P) + v_3(y)\chi(x-P)) + \epsilon^3 \Psi_{\epsilon,P}(x).$$

Then we have

PROPOSITION 2.1.

$$\|\Psi_{\epsilon,P}\|_\epsilon \leq C.$$

To prove Proposition 2.1, we begin with

LEMMA 2.2. – *Let u be a solution of*

$$\begin{cases} \Delta u - mu + f = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases}$$

Assume that $\int_\Omega |f|^2 \leq C\epsilon^N$, $\int_{\partial\Omega} |g|^2 \leq C\epsilon^{N-1}$. Then

$$\|u\|_\epsilon \leq C.$$

Proof. – Multiplying the equation by u, we have

$$\epsilon^2 \int_\Omega |\nabla u|^2 + m\int_\Omega u^2 = \int_\Omega fu + \epsilon^2 \int_{\partial\Omega} gu.$$

Lemma 2.2 follows easily by the following interpolation inequality (the proof of it is delayed to Appendix A),

$$\|u\|_{L^2(\partial\Omega_{\epsilon,P})} \leq C\|u\|_\epsilon$$

where $\Omega_{\epsilon,P} = \{z |\ x = P + \epsilon z \in \Omega\}$ for a fixed $P \in \partial\Omega$. □

Proof of Proposition 2.1. – We first compute the equation for $\Psi_{\epsilon,P}(x)$:

$$-\epsilon^2\Delta\Psi_{\epsilon,P}(x) + m\Psi_{\epsilon,P}(x)$$

$$= \frac{1}{\epsilon^3}\left[\epsilon^2\left\{\Delta_x(\epsilon v_1\chi + \epsilon^2(v_2\chi + v_3\chi))\right\} - m\epsilon v_1\chi - m\epsilon^2 v_2\chi - m\epsilon^2 v_3\chi\right]$$

$$= \frac{1}{\epsilon^2}\left[\left\{\Delta_y v_1 + |\nabla_{x'}\rho|^2\frac{\partial^2 v_1}{\partial y_N^2} - 2\sum_{i=1}^{N-1}\rho_i\frac{\partial^2 v_1}{\partial y_i\partial y_N} - \epsilon\Delta_{x'}\rho\frac{\partial v_1}{\partial y_N} - mv_1\right\}\chi\right.$$

$$+\epsilon\left\{\Delta_y v_2 + |\nabla_{x'}\rho|^2\frac{\partial^2 v_2}{\partial y_N^2} - 2\sum_{i=1}^{N-1}\rho_i\frac{\partial^2 v_2}{\partial y_i\partial y_N} - \epsilon\Delta_{x'}\rho\frac{\partial v_2}{\partial y_N} - mv_2\right\}\chi$$

$$+\epsilon\left\{\Delta_y v_3 + |\nabla_{x'}\rho|^2\frac{\partial^2 v_3}{\partial y_N^2} - 2\sum_{i=1}^{N-1}\rho_i\frac{\partial^2 v_3}{\partial y_i\partial y_N} - \epsilon\Delta_{x'}\rho\frac{\partial v_3}{\partial y_N} - mv_3\right\}\chi$$

$$\left.+E_\epsilon(\chi)\right]$$

$$= \frac{1}{\epsilon^2}\left[\chi\left\{|\nabla\rho|^2\frac{\partial^2 v_1}{\partial y_N^2} - \epsilon\Delta\rho\frac{\partial v_1}{\partial y_N} - 2\sum_{i,j=1}^{N-1}(\rho_i - \epsilon\rho_{ij}(0)y_j)\frac{\partial^2 v_1}{\partial y_i\partial y_N}\right\}\right.$$

$$+\chi\left\{\epsilon|\nabla\rho|^2\frac{\partial^2 v_2}{\partial y_N^2} - 2\epsilon\sum_{i=1}^{N-1}\rho_i\frac{\partial^2 v_2}{\partial y_i\partial y_N} - \epsilon^2\Delta\rho\frac{\partial v_2}{\partial y_N}\right\}$$

$$\left.+\chi\left\{\epsilon|\nabla\rho|^2\frac{\partial^2 v_3}{\partial y_N^2} - 2\epsilon\sum_{i=1}^{N-1}\rho_i\frac{\partial^2 v_3}{\partial y_i\partial y_N} - \epsilon^2\Delta\rho\frac{\partial v_3}{\partial y_N}\right\}\right]$$

$$+\frac{1}{\epsilon^2}E_\epsilon(\chi)$$

$$= f_\epsilon$$

where $E_\epsilon(\chi)$ denotes all the terms involving derivatives of χ. Since $|v_1|, |v_2|, |v_3| \le \exp(-\mu|y|)$ for some $\mu < \sqrt{m}$ we have $f_\epsilon \in L^2(\Omega_{\epsilon,P})$ and $\int_{\Omega_{\epsilon,P}} f_\epsilon^2 \le C$. On the other hand, for $x \in \partial\Omega$ it holds that

$$\frac{\partial\Psi_{\epsilon,P}}{\partial\nu}(x) = \frac{1}{\epsilon^3}\left\{\frac{\partial V}{\partial\nu} - \epsilon\frac{\partial(v_1\chi)}{\partial\nu} - \epsilon^2\left(\frac{\partial(v_2\chi)}{\partial\nu} + \frac{\partial(v_3\chi)}{\partial\nu}\right)\right\}.$$

Note that

$$\frac{\partial V}{\partial \nu}\sqrt{1+|\nabla\rho|^2} = V'\frac{< x-P, \nu >}{\epsilon|x-P|}\sqrt{1+|\nabla\rho|^2}$$

$$= V'\left(\frac{x-P}{\epsilon}\right)\frac{1}{\epsilon|x-P|}\left\{\frac{1}{2}\sum_{i,j=1}^{N-1}\rho_{ij}(0)(x_i-P_i)(x_j-P_j)\right.$$

$$\left.+\frac{1}{3}\sum_{i,j,k=1}^{N-1}\rho_{ijk}(0)(x_i-P_i)(x_j-P_j)(x_k-P_k)+\mathcal{O}(|x'-P'|^4)\right\}$$

$$= \frac{V'(y)}{|y|}\left\{\frac{1}{2}\sum_{i,j=1}^{N-1}\rho_{ij}(0)y_iy_j+\frac{\epsilon}{3}\sum_{i,j,k=1}^{N-1}\rho_{ijk}(0)y_iy_jy_k\right\}+\mathcal{O}(\epsilon^2\exp(-\mu|z|)).$$

Furthermore,

$$\sqrt{1+|\nabla\rho|^2}\frac{\partial v_1}{\partial\nu} = \frac{1}{\epsilon}\left\{\sum_{k=1}^{N-1}\rho_k\frac{\partial v_1}{\partial y_k}-(1+|\nabla\rho|^2)\frac{\partial v_1}{\partial y_N}\right\},$$

$$\epsilon\frac{\partial\Psi_{\epsilon,P}}{\partial\nu}(x) = \frac{1}{\sqrt{1+|\nabla\rho|^2}\epsilon^2}\left[\frac{V'}{|y|}\left\{\frac{1}{2}\sum_{i,j=1}^{N-1}\rho_{ij}y_iy_j+\frac{\epsilon}{3}\sum_{i,j,k=1}^{N-1}\rho_{ijk}(0)y_iy_jy_k\right\}\right.$$

$$+\mathcal{O}(\epsilon^2(\exp(-\mu|y|)))$$

$$+\chi\left\{-\sum_{k=1}^{N-1}\rho_k\frac{\partial v_1}{\partial y_k}+\frac{\partial v_1}{\partial y_N}+|\nabla\rho|^2\frac{\partial v_1}{\partial y_N}\right.$$

$$-\epsilon\sum_{k=1}^{N-1}\rho_k\frac{\partial v_2}{\partial y_k}+\epsilon\frac{\partial v_2}{\partial y_N}+\epsilon|\nabla\rho|^2\frac{\partial v_2}{\partial y_N}$$

$$\left.\left.-\epsilon\sum_{k=1}^{N-1}\rho_k\frac{\partial v_3}{\partial y_k}+\epsilon\frac{\partial v_3}{\partial y_N}+|\nabla\rho|^2\frac{\partial v_3}{\partial y_N}\right\}+E_\epsilon(\chi)\right]$$

$$= g_\epsilon(y)$$

where again $E_\epsilon(\chi)$ denotes all the terms involving derivatives of χ. This implies

$$g_\epsilon \le C\exp(-\mu|z|) \quad \text{for } |z'| \le \frac{R_0-\delta}{\epsilon}.$$

Therefore

$$\left|\epsilon\frac{\partial\Psi_{\epsilon,P}}{\partial\nu}(x)\right| \le C\exp(-\mu|z|) \quad \text{for } z=\frac{x-P}{\epsilon}.$$

Let $\tilde{\Psi}_{\epsilon,P}(z) = \Psi_{\epsilon,P}(x), x = P + \epsilon z$. Then $\tilde{\Psi}_{\epsilon,P}$ satisfies

$$\Delta\tilde{\Psi}_{\epsilon,P} - \tilde{\Psi}_{\epsilon,P} + f_\epsilon = 0 \quad \text{in } \Omega_{\epsilon,P},$$

$$\frac{\partial\tilde{\Psi}_{\epsilon,P}}{\partial\nu_\epsilon} = g_\epsilon \quad \text{on } \partial\Omega_{\epsilon,P}$$

where $f_\epsilon \in L^2(\Omega_{\epsilon,P})$, $g_\epsilon \in L^2(\partial\Omega_{\epsilon,P})$ and both the corresponding norms are bounded independent of ϵ. Hence by Lemma 2.2

$$\|\Psi_{\epsilon,P}\|_\epsilon \leq C.$$

Therefore Proposition 2.1 is proved. □

We next analyze $\partial/\partial\tau_{P_j} P_{\Omega_{\epsilon,P}} V\left(\frac{x-P}{\epsilon}\right)$. After choosing a suitable coordinate system we can assume that $\partial/\partial\tau_{P_j} = \partial/\partial P_j$. Then $\partial/\partial P_j h_{\epsilon,P}(x)$ satisfies

$$\epsilon^2\Delta v - mv = 0 \quad \text{in } \Omega,$$

$$\frac{\partial v}{\partial\nu} = \frac{\partial}{\partial\nu}\frac{\partial}{\partial P_j} V\left(\frac{x-P}{\epsilon}\right) \quad \text{on } \partial\Omega.$$

We compute

$$(1 + |\nabla_{x'}\rho|^2)\frac{\partial}{\partial\nu}\frac{\partial}{\partial P_j} V\left(\frac{x-P}{\epsilon}\right) = \sum_{i=1}^{N-1} \frac{\partial}{\partial x_i}\frac{\partial}{\partial P_j} V\left(\frac{x-P}{\epsilon}\right)\rho_i$$

$$-\frac{\partial}{\partial x_N}\frac{\partial}{\partial P_j} V\left(\frac{x-P}{\epsilon}\right)$$

$$= -\left[\sum_{i=1}^{N-1} \frac{\partial^2 V}{\partial x_i \partial x_j}\left(\frac{x-P}{\epsilon}\right)\rho_i - \frac{\partial^2 V}{\partial x_N \partial x_j}\left(\frac{x-P}{\epsilon}\right)\right].$$

Now we have (let $x = P + \epsilon z$)

$$\frac{\partial V}{\partial z_j}(z) = V'\frac{z_j}{|z|},$$

$$\frac{\partial^2 V}{\partial z_i \partial z_j} = V''\frac{z_i z_j}{|z|^2} + V'\left\{\frac{\delta_{ij}}{|z|} - \frac{z_i z_j}{|z|^3}\right\},$$

$$\frac{\partial^2 V((x-P)/\epsilon)}{\partial x_N \partial x_j} = \frac{1}{\epsilon^2}\left\{V''\frac{z_j\rho/\epsilon}{|z|^2} - V'\frac{z_j\rho/\epsilon}{|z|^3}\right\}$$

$$= \left\{ V'' \frac{y_j}{|y|^2} - V' \frac{y_j}{|y|^3} \right\} \frac{1}{\epsilon^3} \rho + \text{h.o.t.},$$

$$\frac{\partial^2 V((x-P)/\epsilon)}{\partial x_i \partial x_j} = \frac{1}{\epsilon^2} \left\{ V'' \frac{z_i z_j}{|z|^2} + V' \left\{ \frac{\delta_{ij}}{|z|} - \frac{z_i z_j}{|z|^3} \right\} \right\} \rho_i,$$

$$(1 + |\nabla \rho|^2) \frac{\partial}{\partial \nu} \frac{\partial}{\partial P_j} V\left(\frac{x-P}{\epsilon} \right)$$

$$= -\left[\frac{1}{\epsilon^2} \left\{ V'' \frac{y_i y_j}{|y|^2} + V' \left\{ \frac{\delta_{ij}}{|y|} - \frac{y_i y_j}{|y|^3} \right\} \right\} \epsilon \sum_{k=1}^{N-1} \rho_{ik} y_k \right.$$

$$\left. - \frac{1}{\epsilon^3} \left\{ V'' \frac{y_j}{|y|^2} - V' \frac{y_j}{|y|^3} \right\} \frac{\epsilon^2}{2} \sum_{k,l=1}^{N-1} \rho_{kl} y_k y_l \right] + \text{h.o.t.}.$$

$$= \frac{1}{\epsilon} \left[\frac{1}{2} \sum_{k,l=1}^{N-1} \rho_{kl} \left(\frac{V''}{|y|^2} - \frac{V'}{|y|^3} \right) y_k y_j y_l + \frac{V'}{|y|} \sum_{k=1}^{N-1} \rho_{jk} y_k \right] + \text{h.o.t.}.$$

Let

$$\left[\frac{\partial V}{\partial \tau_{P_j}} - \frac{\partial P_{\Omega_{\epsilon,P}} V}{\partial \tau_{P_j}} \right] \left(\frac{x-P}{\epsilon} \right) = w_1(y) \chi(x-P) + \epsilon w_2^\epsilon(x).$$

Here w_1 is the unique solution of

$$\begin{cases} \Delta v - mv = 0 & \text{in } R_+^N, \\ \frac{\partial v}{\partial y_N} = -\frac{1}{2} \left(\frac{V''}{|y|^2} - \frac{V'}{|y|^3} \right) \sum_{k,l=1}^{N-1} \rho_{kl}(0) y_k y_l y_j \\ \quad - \frac{V'}{|y|} \sum_{k=1}^{N-1} \rho_{jk}(0) y_k & \text{on } \partial R_+^N. \end{cases} \tag{2.9}$$

Note that $|w_1| \leq C \exp(-\mu |y|)$ for some $\mu < \sqrt{m}$ and w_1 is an odd function in y'. Then w_2 satisfies

$$\begin{cases} \epsilon^2 \Delta w_2 - m w_2 + \frac{1}{\epsilon} [\epsilon^2 \Delta w_1 \chi_1 - w_1 \chi_1] = 0, \\ \frac{\partial w_2}{\partial \nu} = \frac{1}{\epsilon} \left(\frac{\partial}{\partial \nu} \frac{\partial V}{\partial \tau_{P_j}} - \frac{\partial}{\partial \nu} [w_1(y) \chi(x-P)] \right) \end{cases} \tag{2.10}$$

Note that $|w_2| \leq C \exp(-\mu |y|)$ for some $\mu < \sqrt{m}$. Similar to the proof of Proposition 2.1, we have

PROPOSITION 2.3. –

$$\left[\frac{\partial V}{\partial \tau_{P_j}} - \frac{\partial P_{\Omega_{\epsilon,P}} V}{\partial \tau_{P_j}} \right] \left(\frac{x-P}{\epsilon} \right) = w_1(y) \chi(x-P) + \epsilon w_2^\epsilon(x)$$

where w_1 is defined above and

$$\|w_2^\epsilon\|_\epsilon \leq C.$$

Finally, let

$$L_0 = \Delta - m + h'(V).$$

We have

LEMMA 2.4. –

$$Ker(L_0) \cap H_N^2(R_+^N) = span\left\{\frac{\partial V}{\partial y_1}, \ldots, \frac{\partial V}{\partial y_{N-1}}\right\}.$$

where $H_N^2(R_+^N) = \{u \in H^2(R_+^N), \frac{\partial u}{\partial y_N} = 0 \text{ on } \partial R_+^N\}$.

Proof. – See Lemma 4.2 in [19]. □

3. REDUCTION TO FINITE DIMENSIONS

Let $P \in \Omega$ and

$$\Omega_{\epsilon,P} = \{z \in R^N | \epsilon z + P \in \Omega\}.$$

Let $H_N^2(\Omega_{\epsilon,P})$ be a Hilbert space defined by

$$H_N^2(\Omega_{\epsilon,P}) = \left\{u \in H^2(\Omega_{\epsilon,P}) \middle| \frac{\partial u}{\partial \nu_\epsilon} = 0 \text{ on } \partial\Omega_{\epsilon,P}\right\}.$$

For $u \in H_N^2(\Omega_{\epsilon,P})$, set

$$S_\epsilon(u) = \Delta u - mu + h(u) - \frac{1}{|\Omega|}\int_\Omega h(u).$$

Then solving equation (1.2) is equivalent to

$$S_\epsilon(u) = 0, u \in H_N^2(\Omega_{\epsilon,P}).$$

To this end, we first study the linearized operator

$$\tilde{L_\epsilon} : u(z) \mapsto \Delta u(z) - mu(z) + h'(P_{\Omega_{\epsilon,P}} V(z))u(z),$$

$$H_N^2(\Omega_{\epsilon,P}) \to L^2(\Omega_{\epsilon,P}).$$

$\tilde{L}_\epsilon$ is not invertible due to the approximate kernel

$$\mathcal{K}_{\epsilon,P} = \text{span}\left\{ \frac{\partial P_{\Omega_{\epsilon,P}} V(z)}{\partial \tau_{P_j}} \middle| j = 1, \ldots, N-1 \right\}$$

in $H^2_N(\Omega_{\epsilon,P})$. It is easy to see (integration by parts) that the cokernel of $\tilde{L}_\epsilon$ coincides with its kernel. We choose approximate cokernel and kernel as follows:

$$\mathcal{C}_{\epsilon,P} = \mathcal{K}_{\epsilon,P} = \text{span}\left\{ \frac{\partial P_{\Omega_{\epsilon,P}} V(z)}{\partial \tau_{P_j}} \middle| j = 1, \ldots, N-1 \right\}.$$

Let $\pi_{\epsilon,P}$ denote the projection in $L^2(\Omega_{\epsilon,P})$ onto $\mathcal{C}^\perp_{\epsilon,P}$. Our goal in this section is to show that the equation

$$\pi_{\epsilon,P} \circ S_\epsilon(P_{\Omega_{\epsilon,P}} V + \Phi_{\epsilon,P}) = 0$$

has a unique solution $\Phi_{\epsilon,P} \in \mathcal{K}^\perp_{\epsilon,P}$ if ϵ is small enough.

As a preparation in the following two propositions we show invertibility of the corresponding linearized operator.

PROPOSITION 3.1. – *Let $L_{\epsilon,P} = \pi_{\epsilon,P} \circ \tilde{L}_\epsilon$. There exist positive constants $\bar{\epsilon}, \lambda$ such that for all $\epsilon \in (0, \bar{\epsilon})$*

$$\|L_{\epsilon,P}\Phi\|_{L^2(\Omega_{\epsilon,P})} \geq \lambda \|\Phi\|_{H^2(\Omega_{\epsilon,P})} \tag{3.1}$$

for all $\Phi \in \mathcal{K}^\perp_{\epsilon,P}$.

PROPOSITION 3.2. – *There exists a positive constant $\bar{\bar{\epsilon}}$ such that for all $\epsilon \in (0, \bar{\bar{\epsilon}})$ and $P \in \partial\Omega$ the map*

$$L_{\epsilon,P} = \pi_{\epsilon,P} \circ \tilde{L}_\epsilon : \mathcal{K}^\perp_{\epsilon,P} \to \mathcal{C}^\perp_{\epsilon,P}$$

is surjective.

Proof of Proposition. 3.1. – We will follow the method used in [9], [21] and [22]. Suppose that (3.1) is false. Then there exist sequences $\{\epsilon_k\}$, $\{P_k\}$, and $\{\Phi_k\}$ with $P_k \in \partial\Omega$, $\Phi_k \in \mathcal{K}^\perp_{\epsilon_k,P_k}$ such that

$$\|L_{\epsilon_k,P_k}\Phi_k\|_{L^2} \to 0, \tag{3.2}$$

$$\|\Phi_k\|_{H^2} = 1, \quad k = 1, 2, \ldots \tag{3.3}$$

We omit the argument Ω_{ϵ_k,P_k} where this can be done without confusion. Denote

$$e_{k,j} = \frac{\partial}{\partial \tau_{P_j}} P_{\Omega_{\epsilon,P}} V \Big/ \left\| \frac{\partial}{\partial \tau_{P_j}} P_{\Omega_{\epsilon,P}} V \right\|_{L^2}.$$

Note that

$$< e_{k,i}, e_{k,j} >= \delta_{ij} + \mathcal{O}(\epsilon_k) \quad \text{as } k \to \infty$$

by Proposition 2.3 and because of the symmetry of the function w_1, which was defined in (2.9), where δ_{ij} is the Kronecker symbol. Furthermore because of (3.2),

$$\|\tilde{L}_{\epsilon_k}\Phi_k\|_{L^2}^2 - \sum_{j=1}^{N-1}\left(\int_{\Omega_{\epsilon_k,P_k}} \tilde{L}_{\epsilon_k}\Phi_k e_{k,j}\right)^2 \to 0 \tag{3.4}$$

as $k \to \infty$. Let Ω_1, χ, ρ and T be as defined in Section 2. Then T has an inverse T^{-1} such that

$$T^{-1} : T(B(P,R_0)\cap\overline{\Omega}) \to B(P,R_0)\cap\overline{\Omega}.$$

Recall that $\epsilon y = T(x)$. We introduce a new sequence $\{\varphi_k\}$ by

$$\varphi_k(y) = \chi(T^{-1}(\epsilon_k y))\Phi_k\big(T^{-1}(\epsilon_k y)\big) \tag{3.5}$$

for $y \in R_+^N$. Since T and T^{-1} have bounded derivatives it follows from (3.3) and the smoothness of χ that

$$\|\varphi_k\|_{H^2(R_+^N)} \le C$$

for all k sufficiently large. Therefore there exists a subsequence, again denoted by $\{\varphi_k\}$ which converges weakly in $H^2(R_+^N)$ to a limit φ_∞ as $k \to \infty$. We are now going to show that $\varphi_\infty \equiv 0$. As a first step we deduce

$$\int_{R_+^N} \varphi_\infty \frac{\partial V}{\partial P_j} = 0, \quad j = 1,\dots,N-1. \tag{3.6}$$

This statement is shown as follows (note that $\det DT = \det DT^{-1} = 1$)

$$\int_{R_+^N} \varphi_k(y)\left[\frac{\partial P_{\Omega_{\epsilon,P}} V}{\partial \tau_{P_{k,j}}}\left(\frac{T^{-1}(\epsilon_k y) - P_k}{\epsilon_k}\right)\right] dy$$

$$= \epsilon_k^{-N}\int_{\Omega_1} \chi(x)\Phi_k(x)\frac{\partial P_{\Omega_{\epsilon,P}} V\left(\frac{x-P_k}{\epsilon_k}\right)}{\partial \tau_{P_{k,j}}} dx$$

$$= \epsilon_k^{-N}\int_{\Omega} \Phi_k(x)\frac{\partial P_{\Omega_{\epsilon,P}} V\left(\frac{x-P_k}{\epsilon_k}\right)}{\partial \tau_{P_{k,j}}}$$

$$-\epsilon_k^{-N}\int_{\Omega\setminus\Omega_1}\Phi_k(x)\frac{\partial P_{\Omega_{\epsilon,P}}V\left(\frac{x-P_k}{\epsilon_k}\right)}{\partial\tau_{P_{k,j}}}$$

$$-\epsilon_k^{-N}\int_{\Omega_1}[1-\chi(x)]\Phi_k(x)\frac{\partial P_{\Omega_{\epsilon,P}}V\left(\frac{x-P_k}{\epsilon_k}\right)}{\partial\tau_{P_{k,j}}}$$

$$=0-\epsilon_k^{-N}\int_{\Omega\setminus\Omega_1}\Phi_k(x)\left[\frac{\partial V}{\partial P_{k,j}}-\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_{k,j}}}\right]\left(\frac{x-P_k}{\epsilon_k}\right)$$

$$-\epsilon_k^{-N}\int_{\Omega_1}[1-\chi(x)]\Phi_k(x)\left[\frac{\partial V\left(\frac{x-P_k}{\epsilon_k}\right)}{\partial P_{k,j}}-\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_{k,j}}}\right]\left(\frac{x-P_k}{\epsilon_k}\right)$$

$$-\epsilon_k^{-N}\int_{\Omega\setminus\Omega_1}\Phi_k(x)\frac{\partial V\left(\frac{x-P_k}{\epsilon_k}\right)}{\partial P_{k,j}}$$

$$-\epsilon_k^{-N}\int_{\Omega_1}[1-\chi(x)]\Phi_k(x)\frac{\partial V\left(\frac{x-P_k}{\epsilon_k}\right)}{\partial P_{k,j}}$$

where Ω_1 is as defined in section 2. In the last expression the first two terms tend to zero as $k\to\infty$ since ${\epsilon_k}^{-N}\Phi_k$ is bounded in $L^2(\Omega)$ and $[\ldots]\to 0$ strongly in $L^2(\Omega)$. The last two terms tend to zero as $k\to\infty$ because of the exponential decay of $\partial V/\partial P_{k,j}$ at infinity.

We conclude

$$\limsup_{k\to\infty}\left|\int_{R_+^N}\varphi_k(y)\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_{k,j}}}\left(\frac{T^{-1}(\epsilon_k y)-P_k}{\epsilon_k}\right)\right|=0,\quad j=1,\ldots,N-1. \tag{3.7}$$

This implies (3.6).

Let $\mathcal{K}_0$ and $\mathcal{C}_0$ be the kernel and cokernel, respectively, of the linear operator $S_0'(V)$ which is the Fréchet derivative at V of

$$S_0(v)=\Delta v-mv+h(v),$$

$$S_0: H_N^2(R_+^N)\to L^2(R_+^N),$$

$$H_N^2(R_+^N)=\left\{u\in H_N^2(R_+^N)\,\middle|\,\frac{\partial u}{\partial y_N}=0\right\}.$$

Note that

$$\mathcal{K}_0=\mathcal{C}_0=\operatorname{span}\left\{\frac{\partial V}{\partial y_j}\,|\,j=1,\ldots,N-1\right\}.$$

Equation (3.6) implies that $\varphi_\infty \in \mathcal{K}_0^\perp$. By the exponential decay of V and by (3.2) we have after possibly taking a further subsequence that

$$\Delta\varphi_\infty - m\varphi_\infty + h'(V)\varphi_\infty = 0,$$

i.e. $\varphi_\infty \in \mathcal{K}_0$. Therefore $\varphi_\infty = 0$.

Hence

$$\varphi_k \rightharpoonup 0 \quad \text{weakly in } H^2(R_+^N) \tag{3.8}$$

as $k \to \infty$. By the definition of φ_k we get $\Phi_k \rightharpoonup 0$ in H^2 and

$$\|h'(P_{\Omega_{\epsilon,P}}V)\Phi_k\|_{L^2} \to 0 \quad \text{as } k \to \infty.$$

Furthermore,

$$\|(\Delta - m)\Phi_k\|_{L^2} \to 0 \quad \text{as } k \to \infty.$$

Since

$$\int_{\Omega_{\epsilon_k,P_k}} |\nabla\Phi_k|^2 + m\Phi^2 = \int_{\Omega_{\epsilon_k,P_k}} [(m - \Delta)\Phi_k]\Phi_k$$
$$\leq C\|(\Delta - m)\Phi_k\|_{L^2}$$

we have that

$$\|\Phi_k\|_{H^1} \to 0 \quad \text{as } k \to \infty.$$

In summary:

$$\|\Delta\Phi_k\|_{L^2} \to 0 \text{ and } \|\Phi_k\|_{H^1} \to 0. \tag{3.9}$$

From (3.9) and the following elliptic regularity estimate (for a proof see Appendix B)

$$\|\Phi_k\|_{H^2} \leq C(\|\Delta\Phi_k\|_{L^2} + \|\Phi_k\|_{H^1}) \tag{3.10}$$

for $\Phi_k \in H_N^2$ we imply that

$$\|\Phi_k\|_{H^2} \to 0 \quad \text{as } k \to \infty.$$

This contradicts the assumption

$$\|\Phi_k\|_{H^2} = 1$$

and the proof of Proposition 3.1 is completed. □

Proof of Proposition. 3.2. – Assume that the statement is not true. Then there exist sequences $\{\epsilon_k\}$, $\{P_k\}$ such that $\epsilon_k \to 0$ as $k \to \infty$ and $P_k \in \partial\Omega$ and such that for all k, $L_{\epsilon_k,P_k} : \mathcal{K}^{\perp}_{\epsilon_k,P_k} \to \mathcal{C}^{\perp}_{\epsilon_k,P_k}$ is not surjective. Let $\mathcal{K}_{\epsilon,P}$ and $\mathcal{C}_{\epsilon,P}$ be the kernel and cokernel of $\tilde{L}_\epsilon$, respectively. Then $\pi_{\epsilon_k,P_k} : C^{\perp}_{\epsilon_k,P_k} \to \mathcal{C}^{\perp}_{\epsilon_k,P_k}$ is not surjective, i.e. for all k there exists a $\Phi_k \in \mathcal{C}^{\perp}_{\epsilon_k,P_k}$ with $\Phi_k \neq 0$ such that $\Psi + \Phi_k \notin C^{\perp}_{\epsilon_k,P_k}$ for all $\Psi \in C^{\perp}_{\epsilon_k,P_k}$. This is equivalent to $\Phi_k \in C_{\epsilon_k,P_k}$ and $\Phi_k \neq 0$. Because we can assume that w.l.o.g. $\Phi_k = 1$ this can be rewritten as follows. For all k there exists a $\Phi_k \in \mathcal{C}_{\epsilon_k,P_k}$ such that

$$\|\Phi_k\|_{L^2} = 1, \tag{3.11}$$

$$\int_{\Omega_{\epsilon_k,P_k}} \Phi_k \frac{\partial P_{\Omega_{\epsilon,P}} V}{\partial \tau_{P_k,j}} = 0 \quad j = 1, \ldots, N-1.$$

Now since

$$\Delta\Phi_k - m\Phi_k + h'(P_{\Omega_{\epsilon,P}} V)\Phi_k = 0$$

and because of the elliptic estimate (3.10) it follows that

$$\|\Phi_k\|_{H^2} \leq C$$

for some constant C independent of k. Extract a subsequence (again denoted by $\{\Phi_k\}$) such that φ_k as defined in (3.5) converges weakly in $H^2(R^N_+)$ to φ_∞ as $k \to \infty$ and φ_∞ satisfies

$$\Delta\varphi_\infty - m\varphi_\infty + h'(V)\varphi_\infty = 0 \quad \text{in } R^N_+,$$

$$\frac{\partial \varphi_\infty}{\partial y_n} = 0 \quad \text{in } R^{N-1} \times \{0\} \tag{3.12}$$

with

$$\int_{R^N_+} \varphi_\infty \frac{\partial V}{\partial y_j} = 0 \quad j = 1, \ldots, N-1. \tag{3.13}$$

From (3.12) we deduce that φ_∞ belongs to the kernel of $S_0'(V)$ and (3.13) implies that φ_∞ lies in the orthogonal complement of the kernel of $S_0'(V)$.

Therefore $\varphi_\infty = 0$. As in the proof of Proposition 3.1 we show by the elliptic regularity estimate (3.10) that $\|\Phi_k\|_{H^2} \to 0$ as $k \to \infty$. This contradicts (3.11) and the proof of Proposition 3.2 is finished. □

We are now in a position to solve the equation

$$\pi_{\epsilon,P} \circ S_\epsilon(P_{\Omega_{\epsilon,P}} V + \Phi_{\epsilon,P}) = 0. \tag{3.14}$$

Since $L_{\epsilon,P}|_{\mathcal{K}^{\perp}_{\epsilon,P}}$ is invertible (call the inverse $L^{-1}_{\epsilon,P}$) we can rewrite

$$\Phi = -(L^{-1}_{\epsilon,P} \circ \pi_{\epsilon,P})(S_\epsilon(P_{\Omega_{\epsilon,P}} V)) - (L^{-1}_{\epsilon,P} \circ \pi_{\epsilon,P}) N_{\epsilon,P}(\Phi) \equiv M_{\epsilon,P}(\Phi) \tag{3.15}$$

where

$$N_{\epsilon,P}(\Phi) = S_\epsilon(P_{\Omega_{\epsilon,P}} V + \Phi) - [S_\epsilon(P_{\Omega_{\epsilon,P}} V) + S'_\epsilon(P_{\Omega_{\epsilon,P}} V)\Phi]$$

and the operator $M_{\epsilon,P}$ is defined by the last equation for $\Phi \in H^2_N(\Omega_{\epsilon,P})$. We are going to show that the operator $M_{\epsilon,P}$ is a contraction on

$$B_{\epsilon,\delta} \equiv \{\Phi \in H^2(\Omega_{\epsilon,P}) | \|\Phi\|_{H^2(\Omega_{\epsilon,P})} < \delta\}$$

if δ is small enough. We have

$$\begin{aligned} &\|M_{\epsilon,P}(\Phi)\|_{H^2(\Omega_{\epsilon,P})} \\ &\quad \leq \lambda^{-1}(\|\pi_{\epsilon,P} N_{\epsilon,P}(\Phi)\|_{L^2(\Omega_{\epsilon,P})} + \|\pi_{\epsilon,P}(P_{\Omega_{\epsilon,P}} V - V)\|_{L^2(\Omega_{\epsilon,P})}) \\ &\quad \leq \lambda^{-1} C(c(\delta)\delta + \epsilon) \end{aligned}$$

where $\lambda > 0$ is independent of $\delta > 0$ and $c(\delta) \to 0$ as $\delta \to 0$. Similarly we show

$$\|M_{\epsilon,P}(\Phi) - M_{\epsilon,P}(\Phi')\|_{H^2(\Omega_{\epsilon,P})} \leq \lambda^{-1} C(\epsilon + c(\delta)\delta) \|\Phi - \Phi'\|_{H^2(\Omega_{\epsilon,P})}$$

where $c(\delta) \to 0$ as $\delta \to 0$. Therefore $M_{\epsilon,P}$ is a contraction on B_δ. The existence of a fixed point $\Phi_{\epsilon,P}$ now follows from the Contraction Mapping Principle and $\Phi_{\epsilon,P}$ is a solution of (3.15).

Because of

$$\begin{aligned} \|\Phi_{\epsilon,P}\|_{H^2(\Omega_{\epsilon,P})} &\leq \lambda^{-1}(\|N_{\epsilon,P}(\Phi_{\epsilon,P})\|_{L^2(\Omega_{\epsilon,P})} + \|P_{\Omega_{\epsilon,P}} V - V\|_{L^2}) \\ &\leq \lambda^{-1}(c\epsilon + c(\delta)\|\Phi_{\epsilon,P}\|_{H^2(\Omega_{\epsilon,P})}) \end{aligned}$$

we have

$$(1 - \lambda^{-1} c(\delta))\|\Phi_{\epsilon,P}\|_{H^2} \leq C\epsilon.$$

We have proved

LEMMA 3.3. – *There exists $\bar{\epsilon} > 0$ such that for every pair of ϵ, P with $0 < \epsilon < \bar{\epsilon}$ and $P \in \partial\Omega$ there exists a unique $\Phi_{\epsilon,P} \in \mathcal{K}^{\perp}_{\epsilon,P}$ satisfying $S_\epsilon(P_{\Omega_{\epsilon,P}} V + \Phi_{\epsilon,P}) \in \mathcal{C}_{\epsilon,P}$ and*

$$\|\Phi_{\epsilon,P}\|_{H^2(\Omega_{\epsilon,P})} \leq C\epsilon. \tag{3.16}$$

We need another statement about the asymptotic behavior of the function $\Phi_{\epsilon,P}$ as $\epsilon \to 0$, which gives an expansion in ϵ and is stated as follows.

PROPOSITION 3.4.

$$\Phi_{\epsilon,P}(x) = \epsilon(\Phi_0(y)\chi(x-P)) + \epsilon^2\Psi_{\epsilon,P}(x) \tag{3.17}$$

where

$$\|\Psi_{\epsilon,P}\|_\epsilon \leq C$$

and Φ_0 *is the unique solution of*

$$\Delta\Phi_0 - m\Phi_0 + h'(V)\Phi_0 - h'(V)v_1 = 0, \quad \text{in } R^N_+,$$

$$\frac{\partial\Phi_0}{\partial y_N} = 0 \quad \text{on } \partial R^N_+,$$

$$\Phi_0 \text{ is orthogonal to the kernel of } L_0 \tag{3.18}$$

where $L_0 = \Delta - m + h'(V)$, $L_0 : H^2_N(R^N_+) \to L^2(R^N_+)$.

Proof. – Note that the kernel of L_0 is

$$\left\{ \frac{\partial V}{\partial y_j} \middle| j = 1, \ldots, N-1 \right\}.$$

Furthermore we have

$$|\Phi_0| \leq C\exp(-\mu|y|) \quad \text{for } \mu < \sqrt{m}.$$

The notations for Ω_1, χ, ρ and T are as in section 2. Our strategy is to decompose $\Psi_{\epsilon,P}$ into three parts and show that each of them is bounded in $\|\cdot\|_{H^1(\Omega_{\epsilon,P})}$ as $\epsilon \to 0$. That means we make the ansatz

$$\Psi_{\epsilon,P}(x) = \Psi^1_\epsilon(x) + \Psi^{2,1}_\epsilon(x) + \Psi^{2,2}_\epsilon(x)$$

where the functions Ψ^1_ϵ, $\Psi^{2,1}_\epsilon$, $\Psi^{2,2}_\epsilon$ will be defined as follows. Let Ψ^1_ϵ be the unique solution of

$$\epsilon^2\Delta\Psi^1_\epsilon - m\Psi^1_\epsilon = 0 \quad \text{in } \Omega,$$

$$\frac{\partial\Psi^1_\epsilon}{\partial\nu} = g^\epsilon \quad \text{on } \partial\Omega \tag{3.19}$$

where

$$g_\epsilon(x) = -\frac{\partial}{\partial\nu_x}[\Phi_0(y)\chi(x)].$$

Since $\|g_\epsilon\|_{L^2} \leq C$ there exists a constant $C > 0$ such that

$$\|\Psi_\epsilon^1\|_{H^1} \leq C. \tag{3.20}$$

Define $\Psi_\epsilon^{2,1}$ by

$$\Psi_\epsilon^{2,1} = -\frac{1}{\epsilon}\tilde{\pi}\Phi_0(x)\chi - \tilde{\pi}\Psi_\epsilon^1 \tag{3.21}$$

where $\tilde{\pi}$ is the projection in $L^2(\Omega_{\epsilon,P})$ onto $\mathcal{K}_{\epsilon,P}$. Because of the exponential decay of Φ_0, the smoothness of χ and and by (3.20) it follows that

$$\|\Psi_\epsilon^{2,1}\|_\epsilon \leq C. \tag{3.22}$$

Finally, define $\Psi_\epsilon^{2,2}(x)$ to be the unique solution in $H_N^2(\Omega)$ of the following equation

$$\epsilon^2\Delta\Psi_\epsilon^{2,2} - m\Psi_\epsilon^{2,2} + h'(P_{\Omega_{\epsilon,P}}V)\Psi_\epsilon^{2,2} = -\frac{1}{\epsilon^2}f_\epsilon \quad \text{in } \Omega, \tag{3.23}$$

$$\frac{\partial\Psi_\epsilon^{2,2}}{\partial\nu} = 0 \quad \text{on } \partial\Omega \tag{3.24}$$

where

$$f_\epsilon = \tilde{L}_\epsilon(\Phi_{\epsilon,P} - \epsilon\Phi_0\chi - \epsilon^2(\Psi_\epsilon^1 + \Psi_\epsilon^{2,1})).$$

Note that the right-hand side of the last equation lies in $\mathcal{C}_{\epsilon,P}^\perp$ since

$$\Phi_{\epsilon,P} - \epsilon\Phi_0\chi - \epsilon^2(\Psi_\epsilon^1 + \Psi_\epsilon^{2,1}) \in H_N^2.$$

This is clear for $\Phi_{\epsilon,P}$ by definition. By construction we have that $-\epsilon\Phi_0\chi - \epsilon^2(\Psi_\epsilon^1 + \Psi_\epsilon^{2,1})$ satisfies the Neumann boundary condition. By (3.18) and the smoothness of χ we conclude that $\Phi_0\chi \in H^2$. By (3.19), $\Psi_\epsilon^1 \in H^2$. Finally, since $e_j \in H^2$ where

$$e_j = \frac{\partial V}{\partial\tau_{P_j}} \Big/ \left\|\frac{\partial V}{\partial\tau_{P_j}}\right\|_{L^2(\Omega_{\epsilon,P})} \qquad j = 1, \ldots, N-1$$

we have $\Psi_\epsilon^{2,1} \in H^2$. Therefore $f_\epsilon \in \mathcal{C}_{\epsilon,P}^\perp$. Furthermore, the following lemma is true.

LEMMA 3.5.

$$\|f_\epsilon\|_{L^2(\Omega_{\epsilon,P})} \leq C\epsilon^2.$$

Proof. – We have

$$f_\epsilon = S'_\epsilon(P_{\Omega_{\epsilon,P}}V)(\Phi_\epsilon - \epsilon\Phi_0\chi - \epsilon^2(\Psi^1_\epsilon + \Psi^{2,1}_\epsilon))$$

$$= -h(P_{\Omega_{\epsilon,P}}V) + h(V) + \epsilon h'(V)v_1\chi + N'_{\epsilon,P}(\Phi_\epsilon)$$

where

$$N'_{\epsilon,P}(\Phi) = \frac{1}{|\Omega|}\int_\Omega h'(P_{\Omega_{\epsilon,P}}V)\Phi_\epsilon + \frac{1}{|\Omega|}\int_\Omega [h(P_{\Omega_{\epsilon,P}}V) - h(V)]$$

$$-[h(P_{\Omega_{\epsilon,P}}V + \Phi_\epsilon) - h(P_{\Omega_{\epsilon,P}}V) - h'(P_{\Omega_{\epsilon,P}}V)\Phi_\epsilon]$$

$$+\frac{1}{|\Omega|}\int_\Omega [h(P_{\Omega_{\epsilon,P}}V + \Phi_\epsilon) - h(P_{\Omega_{\epsilon,P}}V) - h'(P_{\Omega_{\epsilon,P}}V)\Phi_\epsilon]$$

$$+\epsilon\Phi_0(y)[\Delta - m + h'(P_{\Omega_{\epsilon,P}}V)]\chi(x) + \epsilon < \nabla_x\Phi_0(y), \nabla_x\Phi(x) >$$

$$+\epsilon^2 h'(P_{\Omega_{\epsilon,P}}V)\Psi^1_\epsilon + \epsilon^2[\Delta - m + h'(P_{\Omega_{\epsilon,P}}V)]\Psi^{2,1}_\epsilon.$$

Note that

$$\| - h(P_{\Omega_{\epsilon,P}}V) + h(V) + \epsilon h'(V)v_1\chi(x)\|_{L^2}$$

$$\leq \| - h(P_{\Omega_{\epsilon,P}}V) + h(V) + \epsilon h'(V)v_1\|_{L^2}$$

$$+\|\epsilon(-h'(V)v_1 + h'(V)v_1\chi)\|_{L^2}$$

$$\leq C(\epsilon^2 + \exp(-\mu R_0))$$

by the definition of χ and the exponential decay of V. Furthermore

$$\|N'_{\epsilon,P}(\Phi)\|_{L^2} \leq C\epsilon^2.$$

This proves Lemma 3.5. □

By Lemma 3.5 and the invertibility of

$$\tilde{L_\epsilon} : H^2_N \cap \mathcal{K}^\perp_{\epsilon,P} \to \mathcal{C}^\perp_{\epsilon,P}$$

Proposition 3.4 follows. □

4. THE REDUCED PROBLEM

In this section we solve the reduced problem and prove our main theorem. By Lemma 3.3 there exists a unique solution $\Phi_{\epsilon,P} \in \mathcal{K}^{\perp}_{\epsilon,P}$ such that

$$S_\epsilon(u_\epsilon) = S_\epsilon\left(P_{\Omega_{\epsilon,P}} V\left(\frac{x-P}{\epsilon}\right) + \Phi_{\epsilon,P}\right)$$

$$= \epsilon^2 \Delta u_\epsilon - m u_\epsilon + h(u_\epsilon) - \frac{1}{|\Omega|}\int_\Omega h(u_\epsilon) \in \mathcal{C}_{\epsilon,P}.$$

Our idea is to find P such that

$$S_\epsilon(u_\epsilon) \perp \mathcal{C}_{\epsilon,P}.$$

Let

$$W_{\epsilon,j}(P) = \frac{1}{\epsilon^{N+1}}\int_\Omega \left(S_\epsilon(u_\epsilon)\frac{\partial P_{\Omega_{\epsilon,P}} V}{\partial \tau_{P_j}}\right),$$

$$W_\epsilon(P) = (W_{\epsilon,1}(P), ..., W_{\epsilon,N-1}(P)).$$

Then $W_\epsilon(P)$ is a continuous map of P.

Let us now calculate $W_\epsilon(P)$. First of all, from condition (4) on h, we have

$$|h(t)| \le Ct^2.$$

Therefore

$$\int_\Omega h(u_\epsilon) \le C\epsilon^N.$$

Hence by Proposition 2.3

$$\frac{1}{\epsilon^{N+1}}\int_\Omega \left(\int_\Omega h(u_\epsilon)\right)\frac{\partial P_{\Omega_{\epsilon,P}} V}{\partial \tau_{P_j}} = \int_\Omega h(u_\epsilon)\frac{1}{\epsilon^{N+1}}\int_\Omega \frac{\partial P_{\Omega_{\epsilon,P}} V}{\partial \tau_{P_j}}$$

$$= \mathcal{O}(\epsilon^N)\left(\frac{1}{\epsilon^{N+1}}\int_\Omega \left(\frac{\partial V}{\partial P_j} + w_1(y)\chi(x-P) + \epsilon w_2^\epsilon(x)\right)\right)$$

$$= \mathcal{O}(\epsilon^N)\frac{1}{\epsilon^{N+1}}\left[\mathcal{O}(\exp(-\sigma/\epsilon)) + \epsilon\int_\Omega w_2^\epsilon\right]$$

$$= \mathcal{O}(\epsilon^{N/2})$$

because

$$\frac{1}{\epsilon^N}\int_\Omega w_2^\epsilon \le \frac{1}{\epsilon^{N/2}}\|w_2^\epsilon\|_{L^2(\Omega)}$$

and Proposition 2.3. On the other hand, since

$$\epsilon^2\Delta\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_j}} - m\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_j}} + h'(V)\frac{\partial V}{\partial P_j} = 0,$$

we conclude

$$\int_\Omega [\epsilon^2\Delta u_\epsilon - mu_\epsilon + h(u_\epsilon)]\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_j}}$$

$$= \int_\Omega \left\{ h(u_\epsilon)\frac{\partial P_{\Omega_{\epsilon,P}}V\left(\frac{x-P}{\epsilon}\right)}{\partial\tau_{P_j}} + \left[\epsilon^2\Delta\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_j}} - m\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_j}}\right]u_\epsilon \right\}$$

$$= \int_\Omega \left[h(u_\epsilon)\frac{\partial P_{\Omega_{\epsilon,P}}V\left(\frac{x-P}{\epsilon}\right)}{\partial\tau_{P_j}} - h'(V)\frac{\partial V\left(\frac{x-P}{\epsilon}\right)}{\partial P_j}u_\epsilon \right]$$

$$= \int_\Omega \left[h(P_{\Omega_{\epsilon,P}}V + \Phi_{\epsilon,P}) - h(P_{\Omega_{\epsilon,P}}V) - h'(P_{\Omega_{\epsilon,P}}V)\Phi_{\epsilon,P}\right]\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_j}}$$

$$+ \int_\Omega \left[h'(P_{\Omega_{\epsilon,P}}V)\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_j}} - h'(V)\frac{\partial V}{\partial P_j}\right]\Phi_{\epsilon,P}$$

$$+ \int_\Omega \left[h(P_{\Omega_{\epsilon,P}}V) - h(V)\right]\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_j}}$$

$$= I_\epsilon^1 + I_\epsilon^2 + J_\epsilon$$

where I_ϵ^1, I_ϵ^2, and J_ϵ are defined by the last equality. We first calculate I_ϵ^2.

$$I_\epsilon^2 = \int_\Omega \left[h'(P_{\Omega_{\epsilon,P}}V)\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_j}} - h'(V)\frac{\partial V}{\partial\tau_{P_j}}\right](\epsilon\Phi_0(y)\chi(x-P)+\epsilon^2\Psi_\epsilon(x))dx$$

$$= \int_\Omega \left[h'(P_{\Omega_{\epsilon,P}}V)\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_j}} - h'(V)\frac{\partial V}{\partial\tau_{P_j}}\right]\epsilon\Phi_0\chi$$

$$+\epsilon^2\int_\Omega \left[h'(P_{\Omega_{\epsilon,P}}V)\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial\tau_{P_j}} - h'(V)\frac{\partial V}{\partial\tau_{p_j}}\right]\Psi_\epsilon$$

$$= \epsilon I_\epsilon^{2,1} + \epsilon^2 I_\epsilon^{2,2}.$$

Note that

$$h'(P_{\Omega_{\epsilon,P}}V)\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}} - h'(V)\frac{\partial V}{\partial \tau_{P_j}}$$

$$= \left[h'(P_{\Omega_{\epsilon,P}}V) - h'(V)\right]\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}} + h'(V)\left[\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}} - \frac{\partial V}{\partial \tau_{P_j}}\right]$$

and

$$\int_\Omega [h'(P_{\Omega_{\epsilon,P}}V) - h'(V)]\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}}\Phi_0\chi$$

$$= \int_\Omega h''(V)(P_{\Omega_{\epsilon,P}}V - V)\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}}\Phi_0 + \int_\Omega h'''(v_1)(P_{\Omega_{\epsilon,P}}V - V)^2\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}}\Phi_0$$

$$+\mathcal{O}(\exp(-\delta/\epsilon))$$

$$= \mathcal{O}(\epsilon^{N+1})$$

since Φ_0 is even and $V - P_{\Omega_{\epsilon,P}}V = \epsilon V_1$ where V_1 is even. By Proposition 2.1

$$\int |\Psi_\epsilon|^2 \leq C\epsilon^N.$$

Hence

$$|I_\epsilon^{2,2}| \leq \mathcal{O}(\epsilon^N).$$

So

$$|I_\epsilon^2| \leq \mathcal{O}(\epsilon^{N+2}).$$

We next compute I_ϵ^1.

$$I_\epsilon^1 = \int_\Omega h''(P_{\Omega_{\epsilon,P}}V)\Phi_{\epsilon,P}^2\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}} + \int_\Omega h'''(v_1 + P_{\Omega_{\epsilon,P}}V)\Phi_{\epsilon,P}^3\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}}$$

$$= \int_\Omega h''(P_{\Omega_{\epsilon,P}}V)\epsilon^2[\Phi_0^2\chi^2 + 2\epsilon\Phi_0\chi\Psi_{\epsilon,P} + \epsilon^2\Psi_{\epsilon,P}^2]\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}} + \mathcal{O}(\epsilon^{N+2})$$

$$= \mathcal{O}(\epsilon^{N+2})$$

since Φ_0 is even. Finally, we compute the term J_ϵ.

$$J_\epsilon = \int_\Omega [h(P_{\Omega_{\epsilon,P}}V) - h(V)]\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}}$$

$$= \int_\Omega h'(V)(P_{\Omega_{\epsilon,P}}V - V)\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}} + h''(V)(P_\Omega V - V)^2\frac{\partial P_{\Omega_{\epsilon,P}}V}{\partial \tau_{P_j}} + \mathcal{O}(\epsilon^{N+2})$$

$$= \epsilon \int_\Omega h'(V)(v_1\chi + \epsilon(v_2\chi + v_3\chi) + \epsilon^2\Psi_\epsilon)\left(\frac{\partial V}{\partial P_j} + w_1 + \epsilon w_2^\epsilon(x)\right)$$

$$+\epsilon^2 \int_\Omega h''(V)(v_1^2\chi^2 + \epsilon(\Psi_\epsilon^1)^2)\frac{\partial P_{\Omega_{\epsilon,P}} V}{\partial \tau_{P_j}} + \mathcal{O}(\epsilon^{N+2})$$

$$= \epsilon^2 \int_\Omega h'(V)v_3\frac{\partial V}{\partial P_j} + \mathcal{O}(\epsilon^{N+2})$$

$$= -\epsilon^{N+1}\left(\int_{\Omega_{\epsilon,P}} h'(V)v_3\frac{\partial V}{\partial y_j}\right) + \mathcal{O}(\epsilon^{N+2})$$

$$= -\epsilon^{N+1}\int_{R_+^N} h'(V)v_3\frac{\partial V}{\partial y_j} + \mathcal{O}(\epsilon^{N+2}).$$

But

$$\int_{R_+^N} h'(V)v_3\frac{\partial V}{\partial y_j} = -\int_{R_+^N}\left(\Delta\frac{\partial V}{\partial y_j} - m\frac{\partial V}{\partial y_j}\right)v_3$$

$$= \int_{\partial R_+^N}\frac{\partial v_3}{\partial y_N}\frac{\partial V}{\partial y_j} - v_3\frac{\partial}{\partial y_N}\frac{\partial V}{\partial y_j}$$

$$= -\frac{1}{3}\int_{R^{N-1}}\left(\frac{V'}{|y|}\right)^2\sum_{k,l,m=1}^{N-1}\rho_{klm}(0)y_k y_l y_m y_j dy$$

$$= -\frac{1}{3}\int_{R^{N-1}}\left(\frac{V'}{|y|}\right)^2\sum_{k,l,m=1}^{N-1} y_k y_l y_m y_j \rho_{klm}(0)dy$$

$$= -\frac{1}{3}\int_{R^{N-1}}\left(\frac{V'}{|y|}\right)^2 y_j^2\sum_{l,m=1}^{N-1} y_l y_m \rho_{jlm}(0)dy$$

$$= \nu\rho_{jkk}(0)$$

$$= \nu\nabla_j H(P)$$

where

$$\nu = -\sum_{k=1}^{N-1}\frac{1}{3}\int_{R^{N-1}}\left(\frac{V'}{|y|}\right)^2 y_j^2 y_k^2\, dy \neq 0.$$

Combining I_ϵ^1, I_ϵ^2, J_ϵ, we obtain

$$W_\epsilon(P) = \nu\nabla_{P_0} H(P) + W_\epsilon'(P)$$

where $W_\epsilon'(P)$ is continuous in P and $W_\epsilon'(P) = \mathcal{O}(\epsilon)$ uniformly in P. Suppose at P_0, we have $\det(\nabla_j\nabla_k H(P_0)) \neq 0$ then standard Brouwer's

fixed point theorem shows that for $\epsilon << 1$ there exists a P_ϵ such that $W_\epsilon(P_\epsilon) = 0, P_\epsilon \to P_0$.

Thus we have proved the following proposition.

PROPOSITION 4.1. – *For ϵ sufficiently small there exist points P_ϵ with $P_\epsilon \to P_0$ such that $W_\epsilon(P_\epsilon) = 0$.*

By Lemma 3.3 and Proposition 4.1 we have

$$S_\epsilon(v_\epsilon) = 0,$$

i.e.

$$\epsilon^2 \Delta v_\epsilon - m v_\epsilon + h(u_\epsilon) - \frac{1}{|\Omega|} \int_\Omega h(v_\epsilon) = 0 \quad \text{in } \Omega,$$

$$\frac{\partial v_\epsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Hence $\int_\Omega v_\epsilon = 0$. Let $u_\epsilon = \overline{m} - v_\epsilon$. We have

$$\epsilon^2 \Delta u_\epsilon - f(u_\epsilon) = \sigma_\epsilon,$$

$$\partial u_\epsilon / \partial \nu = 0 \quad \text{on } \partial\Omega$$

$$\int_\Omega u_\epsilon = \overline{m}|\Omega|,$$

i.e. u_ϵ is a solution of the Cahn-Hilliard equation. Moreover

$$\left\| v_\epsilon - V\left(\frac{x - P_\epsilon}{\epsilon}\right) \right\|_\epsilon \to 0$$

and $P_\epsilon \to P_0 \in \partial\Omega$.

Finally, we study the shape of the solutions v_ϵ. Let P_ϵ be any local maximum point of v_ϵ. Then by (1.1),

$$m v_\epsilon - h(v_\epsilon) + \frac{1}{|\Omega|} \int_\Omega h(v_\epsilon) \leq 0.$$

But $\epsilon^{-N} \int_\Omega h(v_\epsilon) \to \int_{R^N_+} h(V) > 0$, hence

$$m v_\epsilon - h(v_\epsilon) < 0.$$

So $v_\epsilon(P_\epsilon) \geq a_1 > 0$. On the other hand, from our construction,

$$\|v_\epsilon\|^2_\epsilon \to \frac{1}{2}\left(\int_{R^N} |\nabla V|^2 + m V^2\right).$$

Similar proof as in Theorem 1.2 of [18], we conclude $P_\epsilon \in \partial\Omega$ and there is only one such P_ϵ.

Appendix A: Trace Inequality

LEMMA A.1. – *Let* $0 < \epsilon \leq 1$. *Then*

$$\|\Phi\|_{L^2(\partial\Omega_{\epsilon,P})} \leq C\|\Phi\|_{H^1(\Omega_{\epsilon,P})} \tag{A.1}$$

for all $\Phi \in H^1(\Omega)$ *where the constant* C *is independent of* ϵ.

Note that the constant C in (A.1) is required to be independent of ϵ. Therefore Lemma A.1 is special although trace inequalities are quite standard.

Proof of Lemma A.1. – For $\Phi \in H^1(\Omega_{\epsilon,P})$ define $\Psi \in H^1(\Omega)$ by a linear transformation:

$$\Psi(x) = \Phi(z) \quad \text{where } z = \frac{x - P}{\epsilon}.$$

Observe that $\|\Phi\|^2_{L^2(\partial\Omega_{\epsilon,P})} = \epsilon^{1-N}\|\Psi\|^2_{L^2(\partial\Omega)}$, $\|\Phi\|^2_{L^2(\Omega_{\epsilon,P})} = \epsilon^{-N}\|\Psi\|^2_{L^2(\Omega)}$, and $\|\nabla\Phi\|^2_{L^2(\Omega_{\epsilon,P})} = \epsilon^{2-N}\|\nabla\Psi\|^2_{L^2(\Omega)}$. Therefore (and after translation) (A.1) is equivalent to

$$\|\Psi\|^2_{L^2(\partial\Omega)} \leq C(\epsilon\|\nabla\Psi\|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\|\Psi\|^2_{L^2(\Omega)}) \tag{A.2}$$

for all $\Psi \in H^1(\Omega)$ and $0 < \epsilon \leq 1$ where C is independent of ϵ. The proof of (A.2) is standard and is omitted here (see for example the proof of Theorem 3.1 in [1]). □

Appendix B: An elliptic regularity estimate

In this section we prove the following inequality

$$\|\Phi\|_{H^2(\Omega_{\epsilon,P})} \leq C(\|\Delta\Phi\|_{L^2(\Omega_{\epsilon,P})} + \|\Phi\|_{H^1(\Omega_{\epsilon,P})}) \tag{B.1}$$

for all $\Phi \in H^2_N(\Omega_{\epsilon,P})$, $0 < \epsilon \leq \epsilon_0$ where $\Omega_{\epsilon,P}$ is as defined in Section 2 and C is a constant independent of ϵ. For a point P on $\partial\Omega$ we can find a constant $R_0 > 0$ and a smooth function $\rho : B'(R_0) \to R$ such that in $B(P, R_0)$ the boundary $\partial\Omega$ is described by the graph of ρ where ρ satisfies $\rho(0) = 0$, $\nabla\rho(0) = 0$ (compare Section 2). Furthermore there exists a map $\eta = T(\xi)$ with $DT(0) = I$ (the identity map) from a neighborhood U_P of P onto a ball $B(0, R_1)$ (compare Section 3). By a linear transformation we naturally get a map T^ϵ from $U_P^\epsilon = \{(x - P)/\epsilon | x \in U_P\}$ onto a ball

$B(R_1/\epsilon)$ with center at 0. We set $y = \eta/\epsilon$. Then the Laplace operator becomes $\epsilon^2\Delta_x = \Delta_y + A^\epsilon$ where

$$A^\epsilon = |\nabla_{x'}\rho|^2\frac{\partial^2}{\partial y_N^2} - 2\sum_{i=1}^{N-1}\rho_i\frac{\partial^2}{\partial y_i \partial y_N} - \epsilon\Delta_{x'}\rho\frac{\partial}{\partial y_N}.$$

Observe that for given $\delta > 0$ we can find $R_1 > 0$ and ϵ_0 such that for $0 < \epsilon \leq \epsilon_0$

(B.2)

$$\||\nabla_{x'}\rho|^2\|_{L^\infty(B(R_0/\epsilon))} \leq \delta,\ \|\rho\|_{L^\infty(B(R_0/\epsilon))} \leq \delta,\ \|\epsilon\Delta_{x'}\rho\|_{L^\infty(B(R_0/\epsilon))} \leq \delta.$$

In the same way we transform

$$\epsilon\frac{\partial}{\partial \nu_x} = \{1 + |\nabla_{x'}\rho|^2\}^{-1/2}\left\{\sum_{k=1}^{N-1}\rho_k\frac{\partial}{\partial y_k} - (1 + |\nabla_{x'}\rho|^2)\frac{\partial}{\partial y_N}\right\}$$

$$= -\frac{\partial}{\partial y_N} + B^\epsilon$$

where B^ϵ is a differential operator on $B(R_1/\epsilon)\cup\{y_N = 0\}$ with coefficients which are bounded in L^∞ for $0 < \epsilon \leq \epsilon_0$ (compare section 2). From $\{U_P | P \in \partial\Omega\}$ we select a finite subcovering of $\partial\Omega$ and denote it by $\{U_1, \ldots, U_n\}$. Choosing $U_0 = \Omega$ the set $\{U_0, \ldots, U_n\}$ is a finite covering of $\overline{\Omega}$ consisting of open sets. We keep this covering fixed from now on. Let $\{\theta_0, \ldots, \theta_n\}$ be a partition of unity subordinate to this open covering. Denote $\theta_i^\epsilon(y) = \theta_i \circ T^{-1}(\epsilon y)$. Since

$$u = \sum_{i=0}^{n}\theta_i^\epsilon u$$

we have

$$\|u\|^2_{H^2(\Omega_{\epsilon,P})} \leq \|\theta_0^\epsilon u\|^2_{H^2(\Omega_{\epsilon,P})} + \sum_{i=1}^{n}\|\theta_i^\epsilon u\|^2_{H^2(\Omega_{\epsilon,P})}. \tag{B.3}$$

Since θ_0^ϵ has compact support in R^N

$$\|\theta_0^\epsilon u\|^2_{H^2(R^N)} = \|\Delta(\theta_0^\epsilon u)\|^2_{L^2(R^N)} + \|\theta_0^\epsilon u\|^2_{H^1(R^N)}$$

(see for example [10], Corollary 9.10). Because of

$$\Delta(\theta_0^\epsilon) = \theta_0^\epsilon \Delta u + 2\nabla u \cdot \theta_0^\epsilon + u\Delta\theta_0^\epsilon$$

and

$$\|\nabla\theta_0^\epsilon\|_{L^\infty(R^N)} \leq C\epsilon, \quad \|\Delta\theta_0^\epsilon\|_{L^\infty(R^N)} \leq C\epsilon^2,$$

we obtain

$$\|\theta_0^\epsilon u\|^2_{H^2(\Omega_{\epsilon,P})} \leq C(\|\theta_0^\epsilon \Delta u\|^2_{L^2(\Omega_{\epsilon,P})} + \|u\|^2_{H^1(\Omega_{\epsilon,P})}). \tag{B.4}$$

We are now going to estimate $\theta_i^\epsilon u$, $i = 1, \ldots, n$. Note that

$$\frac{1}{C}\|(\theta_i^\epsilon u)^*\|_{H^k(R^N_+)} \leq \|\theta_i^\epsilon u\|_{H^k(\Omega_{\epsilon,P})} \leq C\|(\theta_i^\epsilon u)^*\|_{H^k(R^N_+)} \tag{B.5}$$

where $k = 0, 1,$ or 2 and

$$v^*(y) \equiv v(\frac{1}{\epsilon}T^{-1}(\epsilon y))$$

for $v \in H^2(U_i^\epsilon)$. Then

$$\|(\theta_i^\epsilon u)^*\|^2_{H^2(R^N_+)} \leq \overline{C}\Bigg(\|\Delta(\theta_i^\epsilon u)^*\|^2_{L^2(R^N_+)} + \left\|\frac{\partial}{\partial y_N}(\theta_i^\epsilon u)^*\right\|^2_{H^{1/2}(R^{N-1}\times\{0\})}$$

$$+\|(\theta_i^\epsilon u)^*\|^2_{H^1(R^N_+)}\Bigg) \tag{B.6}$$

(see for example [15], Theorem 4.1). Now (B.2) implies that

$$\|A^\epsilon(\theta_i^\epsilon u)^*\|^2_{L^2(R^N_+)} \leq \delta^2\|(\theta_i^\epsilon u)^*\|^2_{H^1(R^N_+)}.$$

Therefore from (B.6)

$$(1 - \overline{C}\delta^2)\|(\theta_i^\epsilon u)^*\|^2_{H^2(R^N_+)}$$

$$\leq \overline{C}\Bigg(\|(\Delta + A^\epsilon)(\theta_i^\epsilon u)^*\|^2_{L^2(R^N_+)} + \left\|\frac{\partial}{\partial y_N}(\theta_i^\epsilon u)^*\right\|^2_{H^{1/2}(R^{N-1}\times\{0\})}$$

$$+\|(\theta_i^\epsilon u)^*\|^2_{H^1(R^N_+)}\Bigg).$$

For the operator B^ϵ we can calculate in an analogous way. The trace theorem implies

$$(1-\tilde{C}\delta^2)\|(\theta_i^\epsilon u)^*\|^2_{H^2(R^N_+)}$$

$$\leq C\Bigg(\|(\Delta+A^\epsilon)(\theta_i^\epsilon u)^*\|^2_{L^2(R^N_+)}+\left\|\left(\frac{\partial}{\partial y_N}+B^\epsilon\right)(\theta_i^\epsilon u)^*\right\|^2_{H^{1/2}(R^{N-1}\times\{0\})}$$

$$+\|(\theta_i^\epsilon u)^*\|^2_{H^1(R^N_+)}\Bigg).$$

Since $\tilde{C}$ is by construction independent of ϵ we can choose δ so small that $1-\tilde{C}\delta^2 \geq 1/2$. This implies

$$\text{(B.7)}\quad \|\theta_i^\epsilon u\|^2_{H^2(\Omega_{\epsilon,P})} \leq C\Bigg(\|\Delta(\theta_i^\epsilon u)\|^2_{L^2(\Omega_{\epsilon,P})}+\left\|\frac{\partial}{\partial\nu_\epsilon}(\theta_i^\epsilon u)\right\|^2_{H^{1/2}(\partial\Omega_{\epsilon,P})}$$

$$\|\theta_i^\epsilon u\|^2_{H^1(\Omega_{\epsilon,P})}\Bigg).$$

Similarly as before

$$\text{(B.8)}\qquad \Delta(\theta_i^\epsilon u)\|^2_{L^2(\Omega_{\epsilon,P})} \leq C(\|\theta_i^\epsilon\Delta u\|^2_{L^2(\Omega_{\epsilon,P})}+\|u\|^2_{H^1(\Omega_{\epsilon,P})})$$

and

$$\text{(B.9)}\qquad \left\|\frac{\partial}{\partial\nu_\epsilon}(\theta_i^\epsilon u)\right\|^2_{H^{1/2}(\partial\Omega_{\epsilon,P})} \leq C\|u\|^2_{H^1(\Omega_{\epsilon,P})}$$

because of $\partial u/\partial\nu_\epsilon = 0$. Combining (B.7) - (B.9) we get

$$\text{(B.10)}\qquad \|\theta_i^\epsilon u\|^2_{H^2(\Omega_{\epsilon,P})} \leq C(\|\theta_i^\epsilon\Delta u\|^2_{L^2(\Omega_{\epsilon,P})}+\|u\|^2_{H^1(\Omega_{\epsilon,P})}).$$

We conclude, using (B.3), (B.4) and (B.10), that

$$\|u\|^2_{H^2(\Omega_{\epsilon,P})} \leq C\left(\sum_{i=0}^{n}\|\theta_\epsilon^i\Delta u\|^2_{L^2(\Omega_{\epsilon,P})}+(n+1)\|u\|^2_{H^1(\Omega_{\epsilon,P})}\right)$$

$$\leq C_n(\|\Delta u\|^2_{L^2(\Omega_{\epsilon,P})}+\|u\|^2_{H^1(\Omega_{\epsilon,P})})$$

where C_n depends on n. Since n is independent of ϵ the proof of (B.1) is finished. □

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