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Ergodic problem for the Hamilton-Jacobi-Bellman equation. II

by

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ABSTRACT. – We study the ergodic problem for the first-order Hamilton-Jacobi-Equations (HJBs), from the view point of controllabilities of underlying controlled deterministic systems. We shall give sufficient conditions for the ergodicity by the estimates of controllabilities.

Next, we shall give some results on the Abelian-Tauberian problem for the solutions of HJBs. Our solutions of HJBs satisfy the equations in the sense of viscosity solutions. © Elsevier, Paris

RÉSUMÉ. – Nous étudions le problème ergodique pour les équations de Hamilton-Jacobi-Bellmans (HJBs). Nous utiliserons les notions des contrôlabilités dans les systèmes déterministes contrôlés pour donner des conditions suffisantes pour la convergence ergodique.

Ensuite, nous donnons des résultats du problème de Abel et de Tauber pour les solutions des HJBs. Nos solutions des HJBs satisfont les équations au sens de la solution de la viscosité. © Elsevier, Paris

1. INTRODUCTION

The so-called ergodic problem for the Hamilton-Jacobi-Bellman equations concerns in studying the convergence of the terms $\lambda u_\lambda(x)$, $\frac{1}{T}u(x,T)$ (as λ goes to $+0$, T goes to $+\infty$ respectively) in the following equations.

(Stationary problem - infinite horizon control problem)

$$(1)\quad \sup_{\alpha\in A}\{-<b(x,\alpha),\nabla u_\lambda(x)>+\lambda u_\lambda(x)-f(x,\alpha)\} = 0 \quad , \quad x\in\Omega \, .$$

(Time dependent problem - finite horizon control problem)

$$(2)\quad \frac{\partial u}{\partial t}(x,t)+\sup_{\alpha\in A}\{-<b(x,\alpha),\nabla u(x,t)>-f(x,\alpha)\} = 0\,, \qquad t>0\,,\ x\in\Omega\,,$$

$$u(x,0) = 0\,, \qquad x\in\Omega\,,$$

with either one of the following boundary conditions.

(Periodic B.C.)

$\overline{\Omega}$ is assumed to be a n-dimensional torus

$$T^n = \mathbf{R}^n / \prod_{i=1}^{n}(T_i\mathbf{Z}) \approx \prod_{i=1}^{n}[0,T_i]\,,$$

where T_i $(1\le i\le n)$ are real numbers, in which case $b(x,\alpha)$, $f(x,\alpha)$ are periodic in x_i $(1\le i\le n)$ with the period T_i $(1\le i\le n)$.

(Neumann and oblique type B.C.)

$$(3)\qquad <\nabla u_\lambda(x),\gamma(x)> = 0\quad,\quad x\in\partial\Omega\,,$$

$$(3')\qquad <\nabla u(x,t),\gamma(x)> = 0\quad,\quad t>0\,,\ x\in\partial\Omega\,,$$

where $\gamma(x)$ is a smooth vector field on $\partial\Omega$ pointing outwards, i.e. denoting $n(x)$ the unit outward normal at $x\in\partial\Omega$, $\gamma(x)$ satisfies

$$(4)\qquad \exists\,\nu>0\ \text{ such that }\ <n(x),\gamma(x)>\ge\nu\quad,\quad \forall\,x\in\partial\Omega\,.$$

(State constraints B.C.)

(5) $u_\lambda(x)$, $u(x,t)$ are viscosity supersolutions of (1), (2) in $\overline{\Omega}$, $\overline{\Omega}\times(0,\infty)$ respectively.

Here, Ω is a bounded, connected, open smooth subset in $\mathbf{R}^n$; $u_\lambda(x)$ $(\lambda>0)$ and $u(x,t)$ are real-valued unknown functions defined in $\overline{\Omega}$, $\overline{\Omega}\times[0,\infty)$ respectively ; A is a metric set corresponding to the values of the controls for the underlying controlled dynamical system ; $b(x,\alpha)$ is a continuous function on $\overline{\Omega}\times A$ with values in $\mathbf{R}^n$ which is bounded, Lipschitz continuous in x uniformly in α ; $f(x,\alpha)$ is bounded continuous on $\overline{\Omega}\times A$ with real values.

The relationship between the convergence of $\lim_{\lambda\to 0}\lambda u_\lambda(x)$, $\lim_{T\to\infty}\frac{1}{T}u(x,T)$ and the notion of "ergodic problem" was mentioned in our previous paper [1]. Here, we only note that the unique viscosity solutions (for the definition of the viscosity solution, we refer M.G. Crandall, P.L. Lions [5]) $u_\lambda(x)$, $u(x,t)$ of the equations (1), (2) are the value functions of the following control problems. (See P.L. Lions [6], [7], I. Capuzzo-Dolcetta - P.L. Lions [3], I. Capuzzo-Dolcetta - J.L. Menaldi [4], H.M. Soner [9]).

$$(6) \quad u_\lambda(x) = \inf_{\alpha(\cdot)} \int_0^\infty e^{-\lambda t} f(x_\alpha(t), \alpha(t))\, dt \quad , \quad \lambda > 0 \, , \, x \in \overline{\Omega} \, ,$$

$$(7) \quad u(x,t) = \inf_{\alpha(\cdot)} \int_0^t f(x_\alpha(s), \alpha(s))\, ds \quad , \quad t > 0 \, , \, x \in \overline{\Omega} \, ,$$

where $\alpha(\cdot)$ is any measurable function "called control" from $[0,\infty)$ to A, $x_\alpha(t)$ is the solution of the following ordinary differential equation corresponding to $\alpha(\cdot)$:

(Controlled deterministic system - Periodic B.C.)

$$(8) \quad \begin{cases} \dfrac{d}{dt} x_\alpha(t) = b(x_\alpha(t), \alpha(t)) & t \geq 0 \, , \\ x_\alpha(0) = x & x \in \overline{\Omega} \, , \; \forall \, \alpha(\cdot) \text{ control,} \\ x_\alpha(t) \in \overline{\Omega} & \forall \, t \geq 0 \, , \; \forall \, \alpha(\cdot) \text{ control.} \end{cases}$$

(Controlled deterministic system - Neumann and oblique type B.C.)

$$(9) \quad \begin{cases} x_\alpha(t) = x + \displaystyle\int_0^t b\big(x_\alpha(s), \alpha(s)\big)\, ds - \int_0^t \gamma\big(x_\alpha(s)\big)\, d\beta_s & \forall \, t \geq 0 \, , \\ x_\alpha(0) = x & x \in \overline{\Omega} \, , \; \forall \, \alpha \text{ control} \, , \\ x_\alpha(t) \in \overline{\Omega} & \forall \, t \geq 0 \, , \; \forall \, \alpha \text{ control} \, , \\ B_t = \displaystyle\int_0^t 1_{\partial\Omega}(x_\alpha(s))\, d\beta_s & \forall \, t \geq 0 \, , \\ \qquad \text{is continuous and nondecreasing.} \end{cases}$$

(Controlled deterministic system - State constraints B.C.)

$$(10) \quad \begin{cases} \dfrac{d}{dt} x_\alpha(t) = b(x_\alpha(t), \alpha(t)) & t \geq 0 \, , \\ x_\alpha(0) = x & x \in \overline{\Omega} \, , \; \forall \, \alpha(\cdot) \text{ control,} \\ x_\alpha(t) \in \overline{\Omega} & \forall \, t \geq 0 \, , \; \forall \, \alpha(\cdot) \text{ control.} \\ \text{At any } y \in \partial\overline{\Omega}, \text{there exists } \beta \in A \text{ such that} \\ \qquad < n(y), b(y, \beta) > \, < 0. \end{cases}$$

Since our study in this paper is closely related to the results of our former paper [1], we recall them here.

THEOREM A. – *Let* $f(x,\alpha)$ *in (1), (2) be of the following form* $f(x,\alpha) = g(x) + h(x,\alpha)$, *where* $g(x)$ *is an arbitrary real-valued Lipschitz continuous function on* $\overline{\Omega}$ *and* $h(x,\alpha)$ *is a bounded continuous function in* $\overline{\Omega} \times A$. *If for any Lipschitz continuous function* $g(x)$ *there is a constant* d_g *such that*

$$\lim_{\lambda\to 0} \lambda u_\lambda(x) = d_g \quad , \quad \textit{for all} \quad x \in \overline{\Omega}\,, \tag{11}$$

$$\textit{(resp.} \quad \lim_{T\to\infty} \frac{1}{T} u(x,T) = d_g \quad , \quad \textit{for all} \quad x \in \overline{\Omega}\,,) \tag{12}$$

then there exists a subset Z *of* $\overline{\Omega}$ *which satisfies the following properties* (Z), (P), (A).

(Z) Z *is non-empty and* $z \in Z$ *if and only if for any* $y \in \overline{\Omega}$ *and for any* $\varepsilon > 0$ *there exist* $T_\varepsilon > 0$ *and a control* α_ε *such that*

$$\lim_{\varepsilon\downarrow 0} T_\varepsilon = +\infty \quad , \quad |z - y_{\alpha_\epsilon}(T_\varepsilon)| < \varepsilon\,.$$

(P) Z *is closed, connected and positively invariant, i.e.*

$$z_\alpha(t) \in Z \quad , \quad \forall\, z \in Z\,,\ \forall\, \alpha \textit{ control}\,,\ \forall\, t \geq 0\,. \tag{I}$$

(A) Z has the following time averaged attracting property, i.e. for any open neighborhood U of Z,

$$\lambda \int_0^\infty e^{-\lambda t} \chi_U(x_\alpha(t))\, dt \to 1\,, \quad \text{as } \lambda \downarrow 0, \text{ uniformly in } \alpha(\cdot)\,,\ \forall\, x \in \overline{\Omega}\,, \tag{13}$$

$$\text{(resp.} \quad \frac{1}{T}\int_0^T \chi_U(x_\alpha(t))\, dt \to 1\,, \quad \text{as } T \to \infty, \text{ uniformly in } \alpha(\cdot)\,,\ \forall\, x \in \overline{\Omega}\ ) \tag{14}$$

where $\chi_U (U \subset \overline{\Omega})$ denotes the characteristic function of the set U.

THEOREM B. – *Let* $f(x,\alpha)$ *in (1), (2) be in the form of* $f(x,\alpha) = g(x) + h(x,\alpha)$ *where* $g(x)$ *is an arbitrary real-valued Lipschitz continuous function on* $\overline{\Omega}$ *and* $h(x,\alpha)$ *is a bounded continuous function in* $\overline{\Omega} \times A$. *Assume that there exists a maximal subset* Ω_0 *of* Ω *such that for any Lipschitz continuous function* $g(x)$ *there exists a constant number* d_g *such that*

$$\lim_{\lambda\downarrow 0} \lambda u_\lambda(x) = d_g \quad , \quad \textit{for all} \quad x \in \Omega_0\,, \tag{15}$$

$$\textit{(resp.} \quad \lim_{T\to\infty} \frac{1}{T} u(x,T) = d_g \quad , \quad \textit{for all} \quad x \in \Omega_0\ ). \tag{16}$$

Then, there exists a subset Z_0 of $\overline{\Omega}$ which satisfies the following properties (Z_0), (P_0), (A_0).

(Z_0) *Z_0 is non-empty and $z \in Z_0$ if and only if for any $y \in \overline{\Omega}_0$ and for any $\varepsilon > 0$ there exist $T_\varepsilon > 0$ and a control α_ε such that*

$$\lim_{\varepsilon \downarrow 0} T_\varepsilon = +\infty \quad , \quad |z - y_{\alpha_\varepsilon}(T_\varepsilon)| < \varepsilon \, .$$

(P_0) *Z_0 is closed and positively invariant, i.e.*

$$\text{(I)} \qquad z_\alpha(t) \in Z_0 \quad , \quad \forall\, z \in Z_0 \; , \; \forall\, \alpha \textit{ control} \; , \; \forall\, t \geq 0 \, .$$

(A_0) *For any open neighborhood U of Z_0,*

$$\text{(17)} \qquad \varliminf_{\lambda \downarrow 0} \inf_{\alpha(\cdot)} \lambda \int_0^\infty e^{-\lambda t} \chi_{U^c}(x_\alpha(t))\, dt = 0 \quad , \quad \forall\, x \in \Omega_0 \, ,$$

$$\text{(18)} \quad \textit{(resp.} \quad \varliminf_{T \to \infty} \inf_{\alpha(\cdot)} \frac{1}{T} \int_0^T \chi_{U^c}(x_\alpha(t))\, dt = 0 \quad , \quad \forall\, x \in \Omega_0 \;),$$

where $\chi_U (U \subset \overline{\Omega})$ denotes the characteristic function of the set U.

Roughly speaking, Theorem A (resp. B) asserts that the convergence property (11), (12) (resp. (15), (16)) in $\overline{\Omega}$ (resp. in Ω_0) leads to the existence of the ergodic attractor Z (resp. Z_0). In other words, the existence of such a subset Z (resp. Z_0) is a necessary condition for the convergence property (11), (12) (resp. (15), (16)).

Our first goal in this paper is to study the converse, i.e. does the existence of Z leads the convergence (11), (12) ? For this, we introduce a notion of controllability (the exact controlability and the approximate controllability).

Next, we shall study the equivalence between the averages :

$$\liminf_{\lambda \downarrow 0} \inf_{\alpha} \lambda \int_0^\infty e^{-\lambda t} f(x_\alpha(t), \alpha(t))\, dt$$

and

$$\lim_{T \to \infty} \inf_{\alpha} \frac{1}{T} \int_0^T f(x_\alpha(t), \alpha(t))\, dt,$$

which can be called an Abelian-Tauberian problem. More precisely, in the linear case, i.e. $b(x, \alpha) = b(x)$, $\forall\, x \in \overline{\Omega}$, $f(x, \alpha) = f(x)$, $\forall\, x \in \overline{\Omega}$, it is known that if for a continuous function $f(x)$ there exists $\lim_{\lambda \downarrow 0} \lambda \int_0^\infty e^{-\lambda t} f(x_0(t))\, dt$ at some point $x_0 \in \overline{\Omega}$ ($x_0(t)$ denotes the solution of (8), (9), (10)), then $\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x_0(t))\, dt$ exists and

$$\text{(19)} \qquad \lim_{\lambda \downarrow 0} \lambda \int_0^\infty e^{-\lambda t} f(x_0(t))\, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x_0(t))\, dt$$

holds, which is called the Abelian Theorem ; on the other hand, if for a continuous function $f(x)$ there exists $\lim_{T\to 0} \frac{1}{T}\int_0^T f(x_0(t))\,dt$ at some point $x_0 \in \overline{\Omega}$, then $\lim_{\lambda\downarrow 0} \lambda \int_0^\infty e^{-\lambda t} f(x_0(t))\,dt$ exists and (19) holds, which is called the Tauberian Theorem. We refer for instance to B. Simon [8] for these results. Here, we generalize these relationships in a nonlinear case when $b(x,\alpha)$, $f(x,\alpha)$ possibly depend on $\alpha \in A$. These results will be given in section 3.

Finally, we shall see the relationship between the convergence property (11) of Theorem A and the following first order partial differential equation

$$\sup_{\alpha\in A}\{< -b(x,\alpha)\,,\, \nabla\omega(x)>\} = 0 \quad , \quad x\in\overline{\Omega}\,, \tag{20}$$

with the same boundary condition as for (1). We shall call the equation (20) the equation of first integrals by analogy with the dynamical systems case, i.e. $b(x,\alpha) = b(x)$, $\forall\ x \in \overline{\Omega}$. Roughly speaking, we shall assert that the convergence (11) implies that the unique viscosity solutions of the equations of first integrals are the constants, and the converse is also true. These results will be given in Proposition 9, Theorem 10 in section 4.

In the following, we use the notations $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ for the sets of real, integer, natural numbers respectively. The usual distance between two points $x, y \in \overline{\Omega}$ is given by $|x-y|$; the usual scalar product of $n\times n$ is denoted by $<\cdot,\cdot>$. We use the letters $C(C_1, C_2, \ldots)$ for positive constants. We shall write the solution of the ordinary differential equations (8) or (9) or (10) as $x_\alpha(t), y_\beta(t)$, $t \geq 0$, etc... which correspond to the initial value $x_\alpha(0) = x$, $y_\beta(0) = y$. We denote by $\mathcal{A}$ the set of all measurable functions from $[0,\infty)$ to A; by $\mathcal{A}_x$ the subsets of $\mathcal{A}$ such that

$$\mathcal{A}_x = \{\alpha(\cdot) \in \mathcal{A} \mid x_\alpha(t) \in \overline{\Omega} \qquad \forall t \geq 0.\}$$

We shall sometimes write

$$H(x,p) = \sup_{\alpha\in A}\{-<b(x,\alpha),p> - f(x,\alpha)\}\,,$$

where the right-hand side appears in (1), (2).

From the Lipschitz continuity of $b(x,\alpha)$ in $x \in \overline{\Omega}$ and $\alpha \in \mathcal{A}$, in Periodic B.C., and Neumann B.C. cases, we have

$$\begin{gathered} |x_\alpha(t) - y_\alpha(t)| \leq e^{\lambda_0 t}|x-y|\,, \\ \forall\ x,y \in \overline{\Omega}\,,\ \forall\ \alpha \text{ control}\,,\ \forall\ t\geq 0\ \lambda_0 \geq 0 \text{ is a constant,} \end{gathered} \tag{21}$$

and in the State constraints B.C. case, we have for any $x, y \in \overline{\Omega}$, and for any $\alpha(\cdot) \in \mathcal{A}_x$, there exists $\beta(\cdot) \in \mathcal{A}_y$ such that

$$(21') \qquad |x_\alpha(t) - y_\beta(t)| \leq e^{\lambda_0 t}|x-y| , \quad \forall\, t \geq 0 \ \lambda_0 \geq 0 \text{ is a constant.}$$

To be more specific, in the Periodic B.C. case λ_0 is given as follows (see P.L. Lions [6])

$$(22) \qquad \lambda_0 = \sup_{\substack{x,x' \in \overline{\Omega} \\ \alpha \in A}} \left\{ -<b(x,\alpha) - b(x',\alpha) \, , \, x - x'>|x-x'|^{-2} \right\} .$$

For Neumann type boundary conditions, (21) is proved in P.L. Lions [5]. In the case of State constraints boundary condition, we refer to M. Arisawa-P.L. Lions [2]. In particular, if $\lambda_0 = 0$ in (22), we have respectively in place of (21) (in Periodic and Neumann type B.C. cases)

$$(23) \qquad |x_\alpha(t) - y_\alpha(t)| \leq C_0|x - y| , \quad \forall\, x, y \in \overline{\Omega} \, , \ \forall\, \alpha \text{ control} \, , \ \forall\, t \geq 0 \, ,$$

and in place of (21') (in State constraint B.C. case) for any $x, y \in \overline{\Omega}$, $\alpha(\cdot) \in \mathcal{A}_x$, there exists $\beta(\cdot) \in \mathcal{A}_y$, such that

$$(23') \qquad |x_\alpha(t) - y_\beta(t)| \leq C_0|x-y| \qquad \forall\, t \geq 0$$

where C_0 is a constant in each cases, we shall call that the system is Lipschitz continuous.

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2. CONTROLLABILITY

We recall the definitions of the exact controllability and the approximated controllability.

DEFINITION 1 (Exact controllability). – *A point $x \in \overline{\Omega}$ is said to be exactly controllable to a point $y \in \overline{\Omega}$ if there exist a control $\alpha(\cdot) \in \mathcal{A}_x$ and $T(x,y) > 0$ such that $x_\alpha(T(x,y)) = y$.*

DEFINITION 2 (Approximated controllability). – *A point $x \in \overline{\Omega}$ is said to be approximately controllable to a point $y \in \overline{\Omega}$ with the estimate $\delta(\varepsilon; x, y)$*

if for any $\varepsilon > 0$ *there exist a control* $\alpha \in \mathcal{A}_x$ *and* $T(\varepsilon; x, y) \geq 0$ *such that* $|x_\alpha(T(\varepsilon; x, y)) - y| < \varepsilon$, $T(\varepsilon; x, y) \leq \delta(\varepsilon; x, y)$.

From the above definitions, one sees that the properties (Z) and (Z_0) of the ergodic attractors Z and Z_0 in Theorems A,B mean that all points in $\overline{\Omega}$ (resp. $\overline{\Omega}_0$) are exactly or approximately controllable to any points in Z (resp. Z_0). That is, the convergence (11), (12) or (15), (16) implies the controllability of the system towards Z or Z_0. In the rest of this section, we shall study the converse, i.e. does the exact or approximated controllability imply the convergence (11), (12) or (15), (16) ? We present two results in Theorems 1 and 2 in this direction.

THEOREM 1. – *Assume that any point* $x \in \overline{\Omega}$ *is either exactly controllable or approximately controllable with some estimate* $\delta(\varepsilon; x, y)$ *to any point* $y \in \overline{\Omega}$, *and that the controllability is satisfied in the following uniform sense, i.e. either one of the following conditions (i), (ii), (iii) holds.*

(i) (Uniform exact controllability). *There exists a number* $T > 0$ *such that any point* $x \in \overline{\Omega}$ *is exactly controllable to any point* $y \in \overline{\Omega}$ *with* $T(x, y) \leq T$.

(ii) (Uniform approximated controllability). *There exist some* $\gamma \in [0, 1)$ *and some* $C > 0$ *such that any point* $x \in \overline{\Omega}$ *is approximately controllable to any point* $y \in \overline{\Omega}$ *with the estimate* $\delta(\varepsilon; x, y)$ *such that*

$$\delta(\varepsilon; x, y) \leq C(-\log \varepsilon)^\gamma \quad , \quad \forall\, \varepsilon > 0\, ,\, \forall\, x, y \in \overline{\Omega}\, . \tag{24}$$

(iii) (Uniform approximated controllability for the Lipschitz continuous system). *Let the system be Lipschitz continuous ((23)). There exists a continuous function* $\delta(\varepsilon)$ *defined in* $\varepsilon \geq 0$ *such that* $\lim_{\varepsilon \downarrow 0} \delta(\varepsilon) = +\infty$ *and any point* $x \in \overline{\Omega}$ *is approximately controllable to any point* $y \in \overline{\Omega}$ *with the estimate* $\delta(\varepsilon; x, y)$ *such that*

$$\delta(\varepsilon; x, y) = \delta(\varepsilon) \quad , \quad \forall\, \varepsilon > 0\, ,\, \forall\, x, y \in \overline{\Omega}\, .$$

Let $f(x, \alpha)$ *be Lipschitz continuous in* $\overline{\Omega} \times A$ *uniformly in* $\alpha \in A$. *Then, for any such function* $f(x, \alpha)$, *there exists a constant* d_f *such that*

$$\lim_{\lambda \downarrow 0} \lambda u_\lambda(x) = d_f \quad , \quad \textit{uniformly in} \quad x \in \overline{\Omega}\, . \tag{25}$$

$$\lim_{T \to \infty} \frac{1}{T}\, u(x, T) = d_f \quad , \quad \textit{uniformly in} \quad x \in \overline{\Omega}\, . \tag{26}$$

Next, we shall consider the situation such that any point $x \in \overline{\Omega}$ is exactly or approximately controlable to any point $y \in Z_1$, where Z_1 is strictly contained in $\overline{\Omega}$.

THEOREM 2. – *Assume that there exists a nonempty closed invariant subset $Z_1 \subset \overline{\Omega}$ which satisfies the following properties.*

(A_0) *For any point $x \in \overline{\Omega}$, for any $\alpha(\cdot) \in \mathcal{A}_x$, $\{x_\alpha(t)\}(t \geq 0)$ is attracted to Z_1 in the following sense.*

(i) (General case of (21) or (21')). *For any point $x \in \overline{\Omega}\backslash Z_1$, for any $\alpha(\cdot)$ control in $\mathcal{A}_x$, and for any $\varepsilon > 0$, there exist $z_\varepsilon \in Z_1$ and a number $T_\varepsilon > 0$ such that*

$$|x_\alpha(T_\varepsilon) - z_\varepsilon| < \varepsilon,$$

$$T_\varepsilon \leq C(-log\varepsilon)^\gamma,$$

where $C > 0$, $\gamma \in [0, 1)$ are constants depending on x.

(ii) (Lipschitz continuous case of (23) or (23')). *Let the system be Lipschitz continuous. For any point $x \in \overline{\Omega}\backslash Z_1$, for any $\alpha(\cdot)$ control in $\mathcal{A}_x$, and for any $\varepsilon > 0$, there exist $z_\varepsilon \in Z_1$ and a number $T_\varepsilon > 0$ such that*

$$|x_\alpha(T_\varepsilon) - z_\varepsilon| < \varepsilon.$$

(C) Any point $x \in \overline{\Omega}\backslash Z_1$ is approximately controllable to any point $z \in Z_1$ with the estimates $\delta(\varepsilon; x, z)$, in each of the following cases.

(i) (General case of (21) or (21')).

$$\delta(\varepsilon; x, z) \leq C(-log\varepsilon)^\gamma,$$

where $C > 0$, $\gamma \in [0, 1)$ are constants depending on x, z.

(ii) (Lipschitz continuous case of (23) or (23')).

$$\delta(\varepsilon; x, z) \leq \delta(\varepsilon),$$

where $\delta(\varepsilon)$ is a non-increasing continuous function defined in $\varepsilon \geq 0$.

(UCZ) *Any point $z \in Z_1$ is either exactly controllable or approximately controllable with some estimate $\delta(\varepsilon; z, w)$ to any point $w \in Z_1$, and the controllability is satisfied in either one of the following uniform sense: (i) uniform exact controllability, (ii) uniform approximated controllability, or (iii) uniform approximated controllability in the Lipschitz continuous system in Theorem 1, where in the statement $\overline{\Omega}$ is replaced by Z_1.*

Let $f(x,\alpha)$ be Lipschitz continuous in $\overline{\Omega} \times A$ uniformly in $\alpha \in A$. Then, for any such function $f(x,\alpha)$ there exists a constant d_f such that

$$\lim_{\lambda \downarrow 0} \lambda u_\lambda(x) = d_f \quad , \quad \forall\, x \in \overline{\Omega}\,, \tag{29}$$

$$\lim_{T\to\infty} \frac{1}{T} u(x,T) = d_f \quad , \quad \forall\, x \in \overline{\Omega}\,. \tag{30}$$

The followings are simple examples of the systems satisfying the assumptions in Theorems 1 and 2.

Example 1 (for Theorem 1). – Let $H(x,p) = \sup_{\alpha\in A}\{-<b(x,\alpha),p> - f(x,\alpha)\} = |p| - f(x)$, where $f(x)$ is an arbitrary Lipschitz continuous function in $x \in \overline{\Omega}$. Consider the equation (1) with this Hamiltonian and either one of the boundary conditions of Periodic B.C., Neumann and oblique type B.C., or State constraints B.C. Then, this is the case of the uniformly exact controllability in Theorem 1, (i), and we have the convergence property (25), (26).

Example 2 (for Theorem 2). – Let $\Omega = (-1,1) \times (-1,1) \subset \mathbb{R}^2$, and put $H(x,p) = \sup_{|\alpha|\le 1}\{x_1 p_1 + \alpha p_2\} - f(x)$, where $x = (x_1,x_2) \in \Omega$, $p = (p_1,p_2) \in \mathbb{R}^2$, and $f(x)$ is an arbitrary Lipschitz continuous function in $\overline{\Omega}$. Consider the equation (1) with this Hamiltonian and either one of the boundary conditions of Periodic B.C., Neumann and oblique type B.C., or State constraints B.C. Then, we can apply Theorem 2, because the system satisfies (23) and $Z_1 = \{(0,x_2) \mid -1 \le x_2 \le 1\}$ satisfies (A), (C) and (UCZ).

The proof of Theorems 1, 2 are based on the following two Lemmas.

LEMMA 3. – *Let d_1, d_2 be the real numbers, and assume that $u_1(x), u_2(x)$ are respectively a viscosity subsolution and a viscosity supersolution of the problem*

$$H(x,\nabla u_i) + d_i = 0 \quad , \quad x \in \Omega\,,$$

where $H(x,p) = \sup_{\alpha\in A}\{-<b(x,\alpha),p> - f(x,\alpha)\}$, with either one of the boundary conditions of Periodic B.C., Neumann and oblique type B.C., or State constraints B.C. Then, $d_1 \le d_2$.

LEMMA 4. – *Let $f(x,\alpha)$ in (1), (2) be Lipschitz continuous in $x \in \overline{\Omega}$ uniformly in $\alpha \in A$. Let $\lambda_0 \ge 0$ be the number given in (21), (21'). Then, we have the following.*

(i) *(In the case when $\lambda_0 > 0$)*

$$|\lambda u_\lambda(x) - \lambda u_\lambda(y)| \le C|x-y|^{\frac{\lambda}{\lambda_0}} \quad , \quad \forall\, \lambda > 0\,,\ \forall\, x,y \in \overline{\Omega}\,, \tag{31}$$

where $C > 0$ is a constant.

(ii) *(In the case when $\lambda_0 = 0$)*

$$(32)\qquad |\lambda u_\lambda(x) - \lambda u_\lambda(y)| \leq C|x-y| \quad , \quad \forall\, \lambda > 0\,,\ \forall\, x, y \in \overline{\Omega}\,,$$

$$(33)\qquad \left|\frac{1}{T}\, u(x,T) - \frac{1}{T}\, u(y,T)\right| \leq C|x-y| \quad , \quad \forall\, T > 0\,,\ \forall\, x, y \in \overline{\Omega}\,,$$

where $C > 0$ is a constant.

Proof of Lemma 3. – Let $d_1 > d_2$, and we shall look for a contradiction. If necessary, by adding a positive constant to u_1, we can assume that $u_1 > u_2$. Let $\varepsilon > 0$ be small enough such that $d_1 - \varepsilon u_1 > d_2 - \varepsilon u_2$ holds in $\overline{\Omega}$. Since u_2 is a supersolution of the above problem, u_2 is also a supersolution of

$$H(x, \nabla u_2) + \varepsilon u_2 + d_1 - \varepsilon u_1 \geq 0 \quad , \quad x \in \Omega\,,$$

with the corresponding boundary condition. Since u_1 is a subsolution of

$$H(x, \nabla u_1) + \epsilon u_1 + d_1 - \epsilon u_1 \leq 0, \quad x \in \Omega,$$

with the same boundary condition, the standard comparison theorems of the viscosity solution theory (for the periodic B.C. case, see P.L. Lions [6]; for the Neumann B.C. case, see P.L. Lions [7] ; for the state constraints B.C. case, see H.M. Soner [9], I. Capuzzo-Dolcetta, P.L. Lions [3]), we have $u_1 \leq u_2$, in $x \in \overline{\Omega}$, which is a contradiction. Therefore, $d_1 = d_2$.

Proof of Lemma 4. – (i) In the state constraints case, the estimate (31) on λu_λ follows from I. Capuzzo-Dolcetta and P.-L. Lions [3]. In the other cases, the proof is straightforward and we reproduce it here for the sake of completeness. So, in order to prove (31) for $\lambda_0 > 0$ (for the cases of Periodic B.C. and Neumann B.C.), first we shall prove

$$(34)\qquad \begin{aligned} &|\lambda u_\lambda(x) - \lambda u_\lambda(y)| \leq C|x-y|^{\frac{\lambda}{\lambda_0}}\,, \\ &\qquad \forall\, \lambda > 0\,,\ \forall\, x, y \in \overline{\Omega} \quad \text{such that} \quad |x-y| \leq \frac{2}{\lambda_0 - \lambda}\,. \end{aligned}$$

Let $x, y \in \overline{\Omega}$ be arbitrary, $\varepsilon > 0$ be an arbitrary small number, and take a control β_0 of y such that for any $T \geq 0$, any $\lambda > 0$, the following holds

$$\lambda u_\lambda(y) + \varepsilon \geq \lambda \int_0^T e^{-\lambda t} f\big(y_{\beta_0}(t), \beta_0(t)\big)\, dt + \lambda e^{-\lambda T} u_\lambda\big(y_{\beta_0}(T)\big)\,.$$

Then, by the dynamic programming principle we have

$$(35)\quad \lambda u_\lambda(x) - \lambda u_\lambda(y) - \varepsilon \\ \leq \lambda u_\lambda(x) - \lambda \int_0^T e^{-\lambda t} f\big(y_{\beta_0}(t), \beta_0(t)\big)\, dt - \lambda e^{-\lambda T} u_\lambda\big(y_{\beta_0}(T)\big) .$$

By the continuity (21) of the system,

$$\big|x_{\beta_0}(t) - y_{\beta_0}(t)\big| \leq e^{\lambda_0 t}|x-y| \quad , \quad \forall\, t \geq 0 ,$$

and putting this into (35) we have

$$\lambda u_\lambda(x) - \lambda u_\lambda(y) - \varepsilon \leq \lambda C\Big(e^{(\lambda_0-\lambda)T}|x-y| + \frac{e^{-\lambda T}}{\lambda}\Big) , \\ 0 < \forall\, \lambda < \lambda_0 \, , \; \forall\, T \geq 0 ,$$

and since $\varepsilon > 0$, $x, y \in \overline{\Omega}$ are arbitrary, we get

(36)

$$|\lambda u_\lambda(x) - \lambda u_\lambda(y)| \leq \lambda C\, G(T) \quad , \quad \forall\, \lambda > 0 \, , \; \forall\, x, y \in \overline{\Omega} \, , \; \forall\, T > 0 ,$$

where $C > 0$ is a constant, $G(T) = e^{(\lambda_0-\lambda)T}|x-y| + \frac{e^{-\lambda T}}{\lambda}$. It is easy to see that $G(T)$ $(T \geq 0)$ takes its minimum at $T_0 = \frac{1}{\lambda_0} \log\big(\frac{1}{(\lambda_0-\lambda)|x-y|}\big)$ provided that $|x-y| < \frac{1}{\lambda_0-\lambda}$. Inserting T_0 in (36), we have (34).

For $x, y \in \overline{\Omega}$ such that $|x-y| \geq \frac{1}{\lambda_0-\lambda}$, since

$$|x-y|^{\frac{\lambda}{\lambda_0}} \geq \Big(\frac{1}{\lambda_0-\lambda}\Big)^{\frac{\lambda}{\lambda_0}} ,$$

where the right-hand side converges to 1 as $\lambda \to 0$,

$$|\lambda u_\lambda(x) - \lambda u_\lambda(y)| \leq C' \leq C|x-y|^{\frac{\lambda}{\lambda_0}} ,$$

and (31) is proved.

(ii) The inequality (32) is proved by repeating the preceding proof by taking $T = +\infty$ in (35). The relationship (33) is also quite similarly shown.

Now, we shall prove Theorems 1 and 2.

Proof of Theorem 1. – First, we shall prove

$$(37)\quad \lim_{\lambda\to 0} |\lambda u_\lambda(x) - \lambda u_\lambda(y)| = 0 \quad , \quad \text{uniformly in } x, y \in \overline{\Omega} ,$$

for each cases of (i), (ii) and (iii).

In the case of (i), for an arbitrary pair of $x, y \in \overline{\Omega}$ there exists $T > 0$ such that for a control $\overline{\alpha}$ of x, $x_{\overline{\alpha}}(T(x,y)) = y$, $T(x,y) < T$. Thus, by the dynamic programming principle

$$\begin{aligned}
&\lambda u_\lambda(x) - \lambda u_\lambda(y) \\
&\quad \le \lambda \int_0^{T(x,y)} e^{-\lambda t} f\big(x_{\overline{\alpha}}(t), \overline{\alpha}(t)\big)\, dt + \lambda e^{-\lambda T(x,y)} u_\lambda(y) - \lambda u_\lambda(y) \ ,
\end{aligned}$$

and by the boundedness of f and λu_λ,

$$\lambda u_\lambda(x) - \lambda u_\lambda(y) \ \le \ C\big(\lambda T + e^{-\lambda T} - 1\big) \quad , \quad \forall\, \lambda > 0 \, , \, \forall\, x, y \in \overline{\Omega} \, ,$$

where $C > 0$ is a constant. Since $x, y \in \overline{\Omega}$ are arbitrary, by letting $\lambda \to 0$ we get (37).

In the case of (ii), there exists a number γ, $0 < \gamma < 1$ which satisfies the statement. Let $x, y \in \overline{\Omega}$ be an arbitrary pair of points, $\varepsilon > 0$ be arbitrary, and take a control $\overline{\alpha}$ of x such that $|x_{\overline{\alpha}}(T(\varepsilon; x, y)) - y| < \varepsilon$. Then, by the dynamic programming principle

$$\begin{aligned}
\lambda u_\lambda(x) - \lambda u_\lambda(y) \ &\le \ \lambda \int_0^{T(\varepsilon;x,y)} e^{-\lambda t} f\big(x_{\overline{\alpha}}(t), \overline{\alpha}(t)\big)\, dt + \\
&\qquad + \lambda e^{-\lambda T(\varepsilon;x,y)} u_\lambda\big(x_{\overline{\alpha}}(T(\varepsilon;x,y)\big) - \lambda u_\lambda(y) \\
&\le \ C\big(1 - e^{-\lambda T(\varepsilon;x,y)}\big) + \\
&\qquad + e^{-\lambda T(\varepsilon;cx,y)}\big(\lambda u_\lambda(x_{\overline{\alpha}}(T(\varepsilon;x,y))) - \lambda u_\lambda(y)\big) \ ,
\end{aligned}$$

where $C > 0$ is a constant independent of the choice of $x, y \in \overline{\Omega}$, $\varepsilon > 0$, $\lambda > 0$. From this and Lemma 4, we have

$$\begin{aligned}
\lambda u_\lambda(x) - \lambda u_\lambda(y) \ &\le \ C\big(1 - e^{-\lambda T(\varepsilon;x,y)} + \varepsilon^{\frac{\lambda}{\lambda_0}}\big) \ , \\
&\forall\, \lambda > 0 \, , \, \forall\, x, y \in \overline{\Omega} \, , \, \forall\, \varepsilon > 0 \, .
\end{aligned} \tag{38}$$

Now, we recall the estimate (24). Then, putting

$$\varepsilon \ = \ \exp\big(-\lambda^{-(1+\omega)}\big) \quad , \quad \text{for some} \quad 0 < \omega < \frac{1}{\gamma} - 1$$

in (38), we have (37).

In the case of (iii), from the same argument as in the case of (ii), by using Lemma 4, we have

$$\begin{aligned}
\lambda u_\lambda(x) - \lambda u_\lambda(y) \ &\le \ C\big(1 - e^{-\lambda T(\varepsilon;x,y)} + \varepsilon\big) \ , \\
&\forall\, \lambda > 0 \, , \, \forall\, x, y \in \overline{\Omega} \, , \, \forall\, \varepsilon > 0 \, ,
\end{aligned}$$

where $C > 0$ is a constant. And since $T(\varepsilon; x, y) = O(\delta(\varepsilon))$, $\lim_{\varepsilon\to 0}\delta(\varepsilon) = +\infty$, we have (37).

Therefore, in each cases of (i), (ii) and (iii), we have proved (37).

Now, from (37) there are a subsequence $\lambda' \to 0$ of $\lambda \to 0$ and a number d_1 such that

(39) $$\lim_{\lambda'\to 0} \lambda' u_{\lambda'}(x) = d_1 \quad , \quad \text{uniformly in} \quad x \in \overline{\Omega}\,.$$

We shall prove the uniqueness of the number d_1. For this purpose, suppose that there are another subsequence $\mu' \to 0$ of $\lambda \to 0$ and a number d_2 such that $d_1 \neq d_2$,

(40) $$\lim_{\mu'\to 0} \mu' u_{\mu'}(x) = d_2 \quad , \quad \text{uniformly in} \quad x \in \overline{\Omega}\,.$$

Assume that $d_1 > d_2$, and we shall look for a contradiction. If $d_1 > d_2$, by the uniform convergence of $\lambda' u_{\lambda'}$, $\mu' u_{\mu'}$, for $\varepsilon > 0$ small enough we can take λ', μ' small enough such that

$u_{\lambda'}$ is a viscosity subsolution of
$$H(x, \nabla u_{\lambda'}) + d_1 \leq \varepsilon \quad , \quad x \in \Omega\,,$$
and a viscosity supersolution of
$$H(x, \nabla u_{\lambda'}) + d_1 \geq -\varepsilon \quad , \quad x \in \Omega\,,$$
$u_{\mu'}$ is a viscosity subsolution of
$$H(x, \nabla u_{\mu'}) + d_2 \leq \varepsilon \quad , \quad x \in \Omega\,,$$
and a viscosity supersolution of
$$H(x, \nabla u_{\mu'}) + d_2 \geq -\varepsilon \quad , \quad x \in \Omega\,,$$

where $H(x,p) = \sup_{\alpha\in A}\{-<b(x,\alpha), p> - f(x,\alpha)\}$, with the appropriate boundary conditions. Then, by Lemma 3, since $\varepsilon > 0$ is arbitrary we have $d_1 = d_2$. Therefore, we have proved the convergence property (25). We do not give the proof for the convergence property (26), here. This can be obtained from (25) by using the Theorem 5 below in §3.

Proof of Theorem 2. – First of all, we remark that from Theorem 1, the assumption (UCZ) leads the following : for any bounded continuous function $f(x, \alpha)$ in $\overline{\Omega} \times A$, Lipschitz in $x \in \overline{\Omega}$ uniformly in $\alpha \in A$, there exists a constant d_f such that

(41) $$\lim_{\lambda\downarrow 0} \lambda u_\lambda(z) = d_f \quad , \quad \text{uniformly in} \quad z \in Z_1\,,$$

(42) $$\lim_{T\to\infty} \frac{1}{T}\, u(z, T) = d_f \quad , \quad \text{uniformly in} \quad z \in Z_1\,.$$

In the following, we only prove the statement for the general case of (21) or (21'), for the Lipschitz case can be proved similarly. Let $x \in \Omega$, and choose $\alpha(\cdot)$ so that for arbitrary $\varepsilon > 0, \quad T > 0$,

$$(43) \qquad \lambda u_\lambda(x) + \varepsilon \geq \lambda \int_0^T e^{-\lambda t} f(x_{\alpha(t)}), \alpha(t))dt + \lambda e^{-\lambda T} u_\lambda(x_{\alpha(T)}).$$

Then from the condition (A_0), (i), for any $\delta > 0$ there exist $z_\delta \in Z_1$, $T_\delta > 0$ such that

$$|x_\alpha(T_\delta) - z_\delta| < \delta,$$

$$T_\delta \leq C(-log\delta)^\gamma,$$

where $C > 0, \quad \gamma \in [0, 1)$ are constants depending on x. From Lemma 4, by putting $T = T_\delta$ in (43) we have

$$\lambda u_\lambda(x) - \lambda u_\lambda(z_\delta) + \varepsilon \geq -C\lambda T_\delta + (e^{-\lambda T_\delta} - 1)\lambda u_\lambda(z_\delta) - C\delta^{\lambda\lambda_0^{-1}},$$

and from the estimate of T_δ, we know that the right-hand side of the above inequality converges to 0, provided that we take

$$\delta = exp(-\lambda^{-(1+w)}),$$

for some $0 < w < \gamma^{-1} - 1$. Since $\varepsilon > 0$ is arbitrary, from (41) we have

$$(44) \qquad \varliminf_{\lambda \to 0} \lambda u_\lambda(x) \geq d_f \quad , \quad \forall\, x \in \overline{\Omega}\,.$$

Next, we shall prove the converse relationship

$$(45) \qquad \varlimsup_{\lambda \to 0} \lambda u_\lambda(x) \leq d_f \quad , \quad \forall\, x \in \overline{\Omega}\,.$$

Let $x \in \overline{\Omega}$, $z \in Z_1$ be arbitrary, $\varepsilon > 0$ be arbitrary. By the assumption (C), there exist a control α of x, $T(\varepsilon; x, z) > 0$ such that $|x_\alpha(T(\varepsilon; x, z)) - z| < \varepsilon$, $T(\varepsilon; x, z) \leq \delta(\varepsilon; x, z)$. Therefore, as in the proof of Theorem 1, by using the dynamic programming principle, we have

(46)

$$\lambda u_\lambda(x) - \lambda u_\lambda(z) \; \leq \; C\big(1 - e^{-\lambda T(\varepsilon; x, z)} + \varepsilon\big)^{\lambda\lambda_0^{-1}} \; , \quad \forall\, \lambda > 0\,, \forall\, \varepsilon > 0\,,$$

here we used Lemma 4.

Then, by putting $\varepsilon = exp(-\lambda^{-(1+w)})$, for some $0 < w < \gamma^{-1} - 1$ in (46), from (41) we have (45).

Therefore from (43), (45), we have proved the convergence property (29). The relationship (30) can be proved similarly, and we omit the proof.

3. ABELIAN-TAUBERIAN PROBLEM

In this section, we shall study the relationship between the two averages $\lim_{\lambda\to 0}\lambda u_\lambda(x)$ and $\lim_{T\to\infty}\frac{1}{T}\,u(x,T)$. The following result (Theorem 5) concerns the case when one of the limits exists uniformly in $x\in\overline{\Omega}$, in which case the two convergences are equivalent. Next, in Theorem 6 we treat the case where the convergences hold, but not necessarily uniformly in $x\in\overline{\Omega}$, in which case we can show that the existence of $\lim_{T\to\infty}\frac{1}{T}\,u(x,T)$ implies $\lim_{\lambda\to 0}\lambda u_\lambda(x)=\lim_{T\to\infty}\frac{1}{T}\,u(x,T)$. We shall state the Theorems.

THEOREM 5. – *Let $u_\lambda(x), u(x,t)$ be the solutions of (1), (2) respectively which satisfy the same boundary condition, either one of Periodic B.C., Neumann and oblique type B.C., or State constraints B.C. respectively. Then, the following holds.*

(i) *If $\lambda u_\lambda(x)$ converges uniformly in $x\in\overline{\Omega}$ to a real number d as λ goes to 0_+, then $\frac{1}{T}\,u(x,T)$ converges uniformly in $x\in\overline{\Omega}$ to d as T goes to $+\infty$.*

(ii) *If $\frac{1}{T}\,u(x,T)$ converges uniformly in $x\in\overline{\Omega}$ to a real number d as T goes to $+\infty$, then $\lambda u_\lambda(x)$ converges uniformly in $x\in\overline{\Omega}$ to d as λ goes to 0_+.*

For the case of pointwise convergence, first we shall give the following Lemmas.

LEMMA 6. – *Let $u_\lambda(x); u(x,t)$ be the solutions of (1), (2) respectively which satisfy the same boundary condition, either one of Periodic B.C., Neumann and oblique type B.C., or State constraints B.C. respectively. Then, the following holds.*

$$\underline{\lim}_{\lambda\to 0_+}\ \lambda u_\lambda(x)\ \geq\ \underline{\lim}_{T\to\infty}\ \frac{1}{T}\,u(x,T)\quad,\quad \forall\, x\in\overline{\Omega}\,, \tag{47}$$

$$\underline{\lim}_{\lambda\to 0_+}\ \lambda u_\lambda(x)\ \leq\ \underline{\lim}_{T\to\infty}\ \frac{1}{T}\,u(x,T)\quad,\quad \forall\, x\in\overline{\Omega}\,. \tag{48}$$

LEMMA 7. – *Let $f(x,\alpha)$ in (1), (2) be in the form of $f(x,\alpha)=f(x)+g(x,\alpha)$, where $f(x)$, $g(x,\alpha)$ are continuous functions defined in $\overline{\Omega}$, $\overline{\Omega}\times A$ respectively. Let $u_\lambda(t)$, $u(x,t)$ be the solutions of (1), (2) respectively which satisfy the same boundary condition, either one of Periodic B.C., Neumann and oblique type B.C., or State constraints B.C. respectively. We denote*

$$\overline{H}(x,p)\ =\ \sup_{\alpha\in A}\big\{-\langle b(x,\alpha),p\rangle-g(x,\alpha)\big\}\,. \tag{49}$$

And we assume

$$(50)\qquad \overline{H}(x,p) \geq 0, \quad \forall\, x \in \overline{\Omega}\,,\ \forall\, p \in \mathbf{R}^n,\ \overline{H}(x,0) = 0, \quad \forall\, x \in \overline{\Omega},$$

$$(51)\qquad \overline{H}(x,p) \quad \textit{is convex in } p \in \mathbf{R}^n.$$

Then, there exists $0 < \tau \leq 1$ *such that the following holds*

$$(52)\qquad \begin{aligned} &\lambda u_\lambda(x) \leq \frac{1}{T}\, u(x,T)\,, \\ &\quad \forall\, x \in \overline{\Omega}\,,\ \forall\, \lambda > 0 \quad \textit{and} \quad \forall\, T > 0 \quad \textit{such that} \quad \lambda T \leq \tau\,. \end{aligned}$$

We then obtain the following Theorem from Lemmas 6 and 7.

THEOREM 8. – *Let* $u_\lambda(x), u(x,t)$ *be the solutions of (1), (2) respectively which satisfy the same boundary condition, either one of Periodic B.C., Neumann and oblique type B.C., or State constraints B.C. respectively. Then, the following holds.*

(i) *If at a point* $\overline{x} \in \overline{\Omega}$ $\lim_{T\to\infty} \frac{1}{T}\, u(\overline{x},T)$ *exists, then* $\lim_{\lambda\to 0} \lambda u_\lambda(\overline{x})$ *exists and*

$$(53)\qquad \lim_{\lambda\to 0} \lambda u_\lambda(\overline{x}) = \lim_{T\to\infty} \frac{1}{T}\, u(\overline{x},T)\,.$$

(ii) Let $f(x,\alpha)$ in (1), (2) satisfy the assumptions in Lemma 7. Then, we have

$$(54)\qquad \underline{\lim}_{\lambda\to 0} \lambda u_\lambda(x) = \underline{\lim}_{T\to\infty} \frac{1}{T}\, u(x,T) \quad , \quad \forall\, x \in \overline{\Omega}\,.$$

Now, se shall prove the Theorems and Lemmas.

Proof of Theorem 5. – (i) Let $M = \sup_{\overline{\Omega}\times A} |f(x,\alpha)|$, and let $\varepsilon > 0$ be an arbitrary small number (which is fixed). Let $x \in \overline{\Omega}$ be an arbitrary point and let $T > 0$. We choose $\lambda = \varepsilon \cdot T^{-1}$. Then, by the dynamic programming principle

$$(55)\qquad \begin{aligned} \lambda u_\lambda(x) &= \inf_\alpha \left\{ \lambda \int_0^T e^{-\lambda t} f(x_\alpha(t),\alpha(t))\, dt + \lambda e^{-\lambda T}\, u_\lambda(x_\alpha(T)) \right\} \\ &= \inf_\alpha \left\{ \frac{\varepsilon}{T} \int_0^T f(x_\alpha(t),\alpha(t))\, dt \right. \\ &\quad \left. + \frac{\varepsilon}{T} \int_0^T (e^{-\lambda t} - 1) f(x_\alpha(t),\alpha(t))\, dt + \lambda e^{-\varepsilon}\, u_\lambda(x_\alpha(T)) \right\}. \end{aligned}$$

In (55),

$$\left|\frac{\varepsilon}{T}\int_0^T (e^{-\lambda t}-1)\, f(x_\alpha(t),\alpha(t))\,dt\right| \leq M(1-\varepsilon-e^{-\varepsilon}) ,$$

and dividing the both hand-side of (55) by ε, letting $T \to \infty$, by the uniform convergence of λu_λ to d we get the following

$$(56)\quad \begin{aligned} &\left|\varliminf_{T\to\infty}\frac{1}{T}\inf_\alpha\int_0^T f(x_\alpha(t),\alpha(t))\,dt + \frac{d(e^{-\varepsilon}-1)}{\varepsilon}\right| \leq \frac{M(1-\varepsilon-e^{-\varepsilon})}{\varepsilon} ,\\ &\left|\varlimsup_{T\to\infty}\frac{1}{T}\inf_\alpha\int_0^T f(x_\alpha(t),\alpha(t))\,dt + \frac{d(e^{-\varepsilon}-1)}{\varepsilon}\right| \leq \frac{M(1-\varepsilon-e^{-\varepsilon})}{\varepsilon} .\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have proved

$$\lim_{T\to\infty}\inf_\alpha\frac{1}{T}\int_0^T f(x_\alpha(t),\alpha(t))\,dt = d \quad , \quad \forall\, x\in\overline{\Omega} ,$$

and from the above argument it is clear that the convergence is uniform in $x \in \overline{\Omega}$. And the proof of (i) is complete.

(ii) Let $v(x,t) = e^{-\lambda t}u(x,t)$, $\forall\, x \in \overline{\Omega}$, $\forall\, t > 0$. Since $u(x,t)$ is the viscosity solution of

$$\begin{aligned} &\frac{\partial u}{\partial t} + \sup_\alpha\{-\langle b(x,\alpha),\nabla u\rangle - f(x,\alpha)\} = 0 \quad , \quad x\in\overline{\Omega} ,\ t>0 ,\\ &u(x,0) = 0 \quad , \quad x\in\overline{\Omega} ,\end{aligned}$$

with the appropriate boundary condition, $v(x,t)$ is the viscosity solution of

$$(57)\quad \begin{aligned} &\frac{\partial v}{\partial t} + \sup_\alpha\{-\langle b(x,\alpha),\nabla v\rangle - e^{-\lambda t} f(x,\alpha)\} + \lambda e^{-\lambda t}u(x,t) = 0 ,\\ &\qquad\qquad x\in\overline{\Omega} ,\ t>0 ,\\ &v(x,0) = 0 \quad , \quad x\in\overline{\Omega} ,\end{aligned}$$

with the same boundary condition as $u(x,t)$. Therefore, by the dynamic programming principle we can write

$$(58)\quad \begin{aligned} v(x,t) &= e^{-\lambda t}\,u(x,t)\\ &= \inf_\alpha\left\{\int_0^t e^{-\lambda s}\big(f(x_\alpha(s),\alpha(s)) - \lambda u(x_\alpha(s),s)\big)\,ds\right\} .\end{aligned}$$

Let $M = \sup_{\overline{\Omega}\times A} |f(x,\alpha)|$. Since $u(x,t) \leq Mt$, we deduce

$$\left|\lambda \int_t^\infty e^{-\lambda s}\big(f(x_\alpha(s),\alpha(s)) - \lambda u(x_\alpha(s),s)\big)\, ds\right| \leq Ce^{-\lambda t}(1+\lambda t)\,,$$

where $C > 0$ is a constant.

Hence, multiplying the right-hand side of (58) by λ, letting $t \to \infty$, we have

$$\text{(59)}\qquad \lambda \inf_\alpha \int_0^\infty e^{-\lambda t}\big(f(x_\alpha(t),\alpha(t)) - \lambda u(x_\alpha(t),t)\big)\, dt = 0\,, \quad \forall\,\lambda > 0\,,\ \forall\, x \in \overline{\Omega}\,.$$

Let $\varepsilon > 0$ be an arbitrary fixed number, and let $\lambda > 0$, $T > 0$ satisfy $\varepsilon = \lambda T$. By the uniform convergence of $\frac{1}{T}\, u(x,T)$ to d as $T \to \infty$, there exists $T_0 > 0$ such that

$$\text{(60)}\qquad \left|\frac{1}{T}\, u(x,T) - d\right| < \varepsilon \quad , \quad \forall\, x \in \overline{\Omega}\,,\ \forall\, T > T_0\,.$$

For an arbitrary $T > T_0$, we rewrite (59) as follows

$$\text{(61)}\qquad \begin{aligned} &\inf_\alpha \Big\{ \lambda \int_0^\infty e^{-\lambda t} f(x_\alpha(t),\alpha(t))\, dt - \lambda^2 \int_0^T e^{-\lambda t} u(x_\alpha(t),\alpha(t))\, dt \\ &\qquad - \lambda^2 \int_T^\infty e^{-\lambda t} u(x_\alpha(t),\alpha(t))\, dt \Big\} = 0\,, \\ &\forall\,\lambda > 0\,,\ \forall\, x \in \overline{\Omega}\,,\ \forall\, T > T_0 \quad \text{such that} \quad \lambda T = \varepsilon\,. \end{aligned}$$

We estimate the second and the third terms of the left hand-side of the above equality.

$$\text{(62)}\qquad \begin{aligned} &\left|\lambda^2 \int_0^T e^{-\lambda t} u(x_\alpha(t),\alpha(t))\, dt\right| \leq M\,\varepsilon\,\lambda \int_0^T e^{-\lambda t}\, dt = M\varepsilon(1-e^{-\varepsilon})\,, \\ &\forall\, x \in \overline{\Omega}\,,\ \forall\, \alpha \text{ control}\,. \end{aligned}$$

$$\text{(63)}\qquad \begin{aligned} \varepsilon e^{-\varepsilon}(d-1-\varepsilon) &\leq \lambda^2 \int_T^\infty e^{-\lambda t} u(x_\alpha(t),\alpha(t))\, dt - de^{-\varepsilon} \\ &\leq \varepsilon e^{-\varepsilon}(d+1+\varepsilon) \quad , \quad \forall\, x \in \overline{\Omega}\,,\ \forall\, \alpha \text{ control}\,. \end{aligned}$$

From (61), (62), (63), we obtain

$$\left|\overline{\lim_{\lambda\to 0}}\ \inf_\alpha \lambda \int_0^\infty e^{-\lambda t} f(x_\alpha(t),\alpha(t))\,dt - de^{-\varepsilon}\right| \le K(\varepsilon)\,,$$
$$\forall\, x \in \overline{\Omega}\,,\ \forall\, \varepsilon > 0\,,$$
$$\left|\underline{\lim}_{\lambda\to 0}\ \inf_\alpha \lambda \int_0^\infty e^{-\lambda t} f(x_\alpha(t),\alpha(t))\,dt - de^{-\varepsilon}\right| \le K(\varepsilon)\,,$$
$$\forall\, x \in \overline{\Omega}\,,\ \forall\, \varepsilon > 0\,,$$

where $K(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore, we have

$$\lim_{\lambda\to 0}\ \inf_\alpha \lambda \int_0^\infty e^{-\lambda t} f(x_\alpha(t),\alpha(t))\,dt = d \quad , \quad \forall\, x \in \overline{\Omega}\,,$$

and from the above discussion, the convergence is uniform in $x \in \overline{\Omega}$.

Proof of Lemma 6. – Let $M = \sup_{\overline{\Omega}\times A} f(x,\alpha)$, and x be an arbitrary point in $\overline{\Omega}$. We write

$$\begin{aligned}\lambda u_\lambda(x) &= \inf_\alpha \lambda \int_0^\infty e^{-\lambda t} f(x_\alpha(t),\alpha(t))\,dt\\ &= \inf_\alpha \lambda^2 \int_0^\infty e^{-\lambda t} \int_0^t f(x_\alpha(s),\alpha(s))\,ds\,dt\,,\end{aligned}$$

$$\text{(64)}\quad \begin{aligned}\lambda u_\lambda(x) = \inf_\alpha \Big\{ &\lambda^2 \int_0^T e^{-\lambda t} \int_0^t f(x_\alpha(s),\alpha(s))\,ds\,dt\\ &+ \lambda^2 \int_T^\infty e^{-\lambda t} \int_0^t f(x_\alpha(s),\alpha(s))\,ds\,dt \Big\}, \forall\, T \ge 0\,.\end{aligned}$$

By using (64), we shall prove (47), (48) in order.

First, for (47), we get the following inequality from (64)

$$\text{(65)}\quad \begin{aligned}\lambda u_\lambda(x) \ge \inf_\alpha\ &\lambda^2 \int_0^T e^{-\lambda t} \int_0^t f(x_\alpha(s),\alpha(s))\,ds\,dt\\ &+ \lambda^2 \int_T^\infty e^{-\lambda t} \inf_\alpha \int_0^t f(x_\alpha(s),\alpha(s))\,ds\,dt, \quad \forall\, T \ge 0\,.\end{aligned}$$

Denote $\underline{d}(x) = \underline{\lim}_{T\to\infty}\ \frac{1}{T} u(x,T)$, and for an arbitrary $\varepsilon_0 > 0$ choose $T_0 > 0$ such that

$$\text{(66)}\qquad \inf_\alpha \frac{1}{t} \int_0^t f(x_\alpha(s),\alpha(s))\,ds \ge \underline{d}(x) - \varepsilon_0 \quad , \quad \forall\, t \ge T_0\,.$$

In the inequality (65), we set $T = T_0$ and we have

$$\lambda u_\lambda(x) \geq \inf_\alpha \lambda^2 \int_0^{T_0} e^{-\lambda t} \int_0^t f(x_\alpha(s), \alpha(s))\, ds\, dt + \big(\underline{d}(x) - \varepsilon_0\big)\big(\lambda T_0 e^{-\lambda T_0} + e^{-\lambda T_0}\big) . \tag{67}$$

Now, let $\varepsilon = \lambda T_0$ and we have the following

$$\Big|\lambda^2 \int_0^{T_0} e^{-\lambda t} \int_0^t f(x_\alpha(s), \alpha(s))\, ds\, dt\Big| \leq M(1-e^{-\varepsilon}-\varepsilon e^{-\varepsilon}) , \quad \forall\, \alpha \text{ control} . \tag{68}$$

From (67), (68),

$$\lambda u_\lambda(x) - \underline{d}(x)(\varepsilon e^{-\varepsilon}+e^{-\varepsilon}) \geq -M(1-e^{-\varepsilon}-\varepsilon e^{-\varepsilon}) - \varepsilon_0(\varepsilon e^{-\varepsilon}+e^{-\varepsilon}) .$$

Since this inequality holds for any $\lambda > 0$ independently of the choice of $\varepsilon_0 > 0$, $T_0 > 0$ taken in (66), as $\varepsilon = \lambda T_0 \to 0$,

$$\varliminf_{\lambda\to 0} \lambda u_\lambda(x) \geq \underline{d}(x) - \varepsilon_0 ,$$

and we have (47), for $\varepsilon_0 > 0$ is arbitrary.

Next, for (48), we get the following inequality from (64),

$$\lambda u_\lambda(x) \leq \inf_\alpha \lambda^2 \int_T^{\infty} e^{-\lambda t} \int_0^t f(x_\alpha(s), \alpha(s))\, ds\, dt + M\big(1-e^{-\lambda T}-\lambda T e^{-\lambda T}\big) \quad , \quad \forall\, T > 0 , \tag{69}$$

here we used the same estimate as in (68). Denote $\overline{d}(x) = \limsup_{T\to\infty} \frac{1}{T}\, u(x,T)$, and for an arbitrary number $\varepsilon_1 > 0$ choose $T_1 > 0$ such that

$$\inf_\alpha \frac{1}{t} \int_0^t f(x_\alpha(s), \alpha(s))\, ds < \overline{d}(x) + \varepsilon_1 \quad , \quad \forall\, x \in \overline{\Omega} , \forall\, t > T_1 . \tag{70}$$

In the inequality (69), we set $T = T_1$ and by (70) we have

$$\lambda u_\lambda(x) \leq (\overline{d}(x)+\varepsilon_1)\big(\lambda T_1 e^{-\lambda T_1}+e^{-\lambda T_1}\big) + M\big(1-e^{-\lambda T_1}-\lambda T_1 e^{-\lambda T_1}\big) .$$

Putting $\varepsilon = \lambda T_1$, we get

$$\lambda u_\lambda(x) \leq (\overline{d}(x)+\varepsilon_1)\big(\varepsilon e^{-\varepsilon}+e^{\varepsilon}\big) + M\big(1-e^{-\varepsilon}-\varepsilon e^{-\varepsilon}\big) .$$

By the same argument as the one used to prove (47), we get (48) from the above inequality.

Proof of Lemma 7. – Let $v_\lambda(x,t)=\lambda t u_\lambda(x)$, $\forall\ \lambda>0$, $\forall\ t\geq 0$, $\forall\ x\in\overline{\Omega}$. Since $u_\lambda(x)$ is the viscosity solution of

$$\overline{H}(x,\nabla u_\lambda(x))+\lambda u_\lambda(x)-f(x)\ =\ 0\quad,\quad x\in\Omega\,, \tag{71}$$

with the appropriate boundary condition, $v_\lambda(x,t)$ is the viscosity solution of
(72)

$$\frac{\partial v_\lambda}{\partial t}+\overline{H}(x,\nabla v_\lambda)-f(x)\ =\ \overline{H}(x,\nabla v_\lambda)-\overline{H}(x,\nabla u_\lambda),\quad x\in\Omega\,,\ t>0\,,$$
$$v_\lambda(x,0)\ =\ 0\quad,\quad x\in\overline{\Omega}\,,$$

with the same boundary condition to (71). By the assumption (50), (51), there exists $0<\tau\leq 1$ with which the following holds : for any $\lambda>0$, $t>0$ such that $\lambda t<\tau\leq 1$,

$$\begin{aligned}\overline{H}(x,\nabla v_\lambda)-\overline{H}(x,\nabla u_\lambda)\ &=\ \overline{H}(x,\lambda t\nabla u_\lambda)-\overline{H}(x,\nabla u_\lambda)\\ &\leq\ \{\lambda t-1\}\,\overline{H}(x,\nabla u_\lambda)\,.\end{aligned} \tag{73}$$

Hence, from (50), (72), (73) $v_\lambda(x,t)$ satisfies

$$\frac{\partial v_\lambda}{\partial t}+\overline{H}(x,\nabla v_\lambda)-f(x)\ \leq\ 0\quad,\quad x\in\Omega\,,\ 0<t<\frac{\tau}{\lambda}\,,$$
$$v_\lambda(x,0)\ =\ 0\quad,\quad x\in\overline{\Omega}\,,$$

with the same boundary condition to (72). Therefore, by comparison theorem, we have

$$\lambda t u_\lambda(x)\ =\ v_\lambda(x,t)\ \leq\ u(x,t)\quad,\quad x\in\Omega\,,\ 0<t<\frac{\tau}{\lambda}\,,$$
$$\lambda\, u_\lambda(x)\ \leq\ \tfrac{1}{t}\,u(x,t)\quad,\quad x\in\Omega\,,\ 0<t<\tfrac{\tau}{\lambda}\,.$$

Proof of Theorem 8. – We have (i), directly from (47), (48) in Lemma 6. For (ii), if $f(x,\alpha)$ satisfies the assumption in Lemma 7, we can combine (47) in Lemma 6 with (52) in Lemma 7, and we have (54).

4. FIRST-INTEGRAL EQUATION

In this section, we study the relationship between the convergence $\lim_{\lambda\to 0}\lambda u_\lambda(x)$, for all $x\in\overline{\Omega}$, $\lim_{T\to\infty}\frac{1}{T}u(x,T)$, for all $x\in\overline{\Omega}$ and the unique solvability (in the viscosity solutions' sense) of the first-integral equation

$$\sup_{\alpha\in A}\{-\langle b(x,\alpha),\nabla\omega(x)\rangle\}\ =\ 0\quad,\quad x\in\Omega\,, \tag{20}$$

with the same boundary condition as (1),(2). Our results are the following.

PROPOSITION 9. – *We assume that any continuous solution of (20) is constant. Assume that for a bounded continuous function* $f(x, \alpha)$, $x \in \overline{\Omega}$, $\alpha \in A$, $\lambda u_\lambda(x)$ *converges to a continuous function uniformly in* $x \in \overline{\Omega}$ *as* λ *goes to* 0. *Then*

$$\lim_{\lambda \to 0} \lambda u_\lambda(x) = d_f \quad , \quad \forall\, x \in \overline{\Omega}\,,$$

where d_f *is a constant.*

THEOREM 10. – *If for any continuous function* $f(x)$ *in* $\overline{\Omega}$, *for the solution* $u_\lambda(x)$ *of (1) with* $f(x, \alpha) = f(x)$, *there exist a real-number* d_f *and a sequence* $\{\lambda_n\}$ *such that* $\lim_{n \to \infty} \lambda_n = 0$,

$$\lim_{n \to \infty} \lambda_n u_{\lambda_n}(x) = d_f \quad , \quad \forall\, x \in \overline{\Omega}\,, \tag{74}$$

then any continuous solution of (20) is constant.

Remark. – The claim in Theorem 10 holds for more general controlled system than stated above. In fact, let the Hamiltonian $\overline{H}(x, p)$, $x \in \overline{\Omega}$, $p \in \mathbb{R}^n$ satisfy

$$\overline{H}(x, kp) = a(k)\,\overline{H}(x, p) \quad , \quad \forall\, k \geq 0\,, \forall\, x \in \overline{\Omega}\,, \forall\, p \in \mathbb{R}^n\,, \tag{75}$$

where $a(k) \neq 0$, $\forall\, k > 0$. Let us denote by u_λ the solution of the equation

$$\overline{H}(x, \nabla u_\lambda) + \lambda u_\lambda - f = 0 \quad , \quad \forall\, x \in \overline{\Omega}\,, \tag{76}$$

with an appropriate boundary condition of Periodic B.C., Neumann and oblique type B.C. or State constraints B.C.

In this situation, if for any continuous function $f(x)$ in $\overline{\Omega}$, there exist a real number d_f and a sequence $\{\lambda_n\}$ such that $\lim_{n \to \infty} \lambda_n = 0$,

$$\lim_{n \to \infty} \lambda_n u_{\lambda_n}(x) = d_f \quad , \quad \forall\, x \in \overline{\Omega}\,, \tag{74}$$

then any continuous solution of

$$\overline{H}(x, \nabla \omega) = 0 \quad , \quad \forall\, x \in \overline{\Omega}\,, \tag{77}$$

with the same boundary condition to (76), is constant. An example of such Hamiltonian $\overline{H}(x, p)$ is $|p|^m$ $(m > 1)$.

Proof of Proposition 9. – Let $\lambda u_\lambda(x)$ converge to a continuous function $d(x)$ uniformly in $\overline{\Omega}$. By the dynamic programming principle

$$\lambda u_\lambda(x) = \inf_\alpha \left\{ \lambda \int_0^T e^{-\lambda t} f(x_\alpha(t), \alpha(t))\, dt + \lambda e^{-\lambda T} u_\lambda(x_\alpha(T)) \right\}, \quad \forall\, \lambda > 0\,,\ \forall\, x \in \overline{\Omega}\,,\ \forall\, T > 0\,.$$

Since $f(x,\alpha)$ is bounded, by letting $\lambda \downarrow 0$ in the above, we have

$$d(x) = \inf_\alpha \{d(x_\alpha(T))\} \quad , \quad \forall\, x \in \overline{\Omega}\,,\ \forall\, T > 0\,.$$

This implies that $d(x)$ is a solution of (20), and thus $d(x)$ is a constant.

Proof of Theorem 10 and Remark. – Theorem 10 is a special case of Remark, where $a(k) = k$ in (75). So we shall prove the claim in Remark. Let $\omega(x)$ be a continuous viscosity solution of (77). Then, for $u_\lambda(x) = \lambda^{-1}\omega(x)$, $u_\lambda(x)$ is the unique viscosity solution of

$$\overline{H}(x, \nabla u_\lambda) + \lambda u_\lambda - \omega = a(\lambda^{-1})\,\overline{H}(x, \nabla \omega) = 0 \quad , \quad x \in \Omega\,,$$

with the same boundary condition as (77). From the statement, there exist a real number d_ω and a sequence $\{\lambda_n\}_{n\in}$ such that $\lim_{n\to\infty} \lambda_n = 0$, $\lim_{n\to\infty} \lambda_n u_{\lambda_n}(x) = d_\omega = \omega(x)$, $\forall\, x \in \overline{\Omega}$. Therefore, $\omega(x)$ is constant.

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