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Periodic solutions with prescribed energy for some Keplerian N-body problems [1]

by

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ABSTRACT. – We prove the existence of periodic solutions with prescribed energy for a class of N-body type problems with Keplerian like interaction.

Key words: Periodic solutions, N-body problems.

RÉSUMÉ. – Nous prouvons l'existence de solutions périodiques d'énergie prescrite pour une classe de problèmes à N corps avec interaction de type képlérien.

1. INTRODUCTION

In this paper we seek for periodic solutions of

$$\left.\begin{aligned} \ddot{x} + V'(x) = 0, \\ \frac{1}{2}|\dot{x}|^2 + V(x) = h, \end{aligned}\right\} \tag{1.1}$$

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where $x = (x_1, \cdots, x_N)$, $x_j \in \mathbf{R}^k$ and

$$V(x) = \frac{1}{2} \sum_{i \neq j} V_{ij}(x_i - x_j). \tag{1.2}$$

Roughly, we deal with potentials like

$$V_{ij}(\xi) \simeq -\frac{1}{|\xi|^\alpha}, \qquad 0 < \alpha < 2. \tag{1.3}$$

When $V_{ij}(\xi)$ is as in (1.3) and $\alpha = 1$, (1.2) is the interaction potential of N-bodies of masses $m_1 = \cdots = m_N = 1$, and (1.1) is nothing but the Kepler N-body problem.

Periodic solutions of (1.1) with $N = 2$ have been widely investigated. *See* [3] and references therein.

When $N > 2$, the problem is more difficult because the lack of compactness arises in a stronger form. The breakdown of the Palais-Smale condition has been bypassed either assuming $V_{ij}(\xi) = V_{ji}(\xi)$, (*cf.* [5]) or using critical point at infinity and Morse theory in [4, 9], or employing critical point theory with boundary condition in [7]. Using this latter tool, solutions with fixed energy have been found in [8] for a class of $V_{ij}(\xi) \simeq -|\xi|^{-\alpha}$, $\alpha > 2$ and $h > 0$. However this does not cover the Kepler N-body problem, where, among other things, the natural value of energy is negative. When $V_{ij}(\xi) = V_{ji}(\xi)$, $V_{ij}(\xi) \simeq -|\xi|^{-\alpha}$, $0 < \alpha < 2$ and $h < 0$, the existence of periodic solutions of (1.1) has been proved in [2], but no results dealing with the general case, are known. In the present paper we address this situation and prove the existence of (generalized) solutions of (1.1) for a class of Keplerian-like K-body problem.

The usual functional framework to study (1.1) is to look for critical points of the Maupertuis-like functional:

$$I(u) = \frac{1}{2} \|\dot{u}\|_{L^2}^2 \int_0^1 (h - V(u))\, dt$$

defined on

$$\Lambda = \{ u \in H^{1,2}(S^1, \mathbf{R}^{kN});$$
$$u_i(t) \neq u_j(t) \quad \text{for all } t \in \mathbf{R} \text{ and } i \neq j \}.$$

Our approach consists of 4 steps:

1° In order to control the behavior of I on $\partial\Lambda$, we consider the perturbed potentials $V_\varepsilon = V - \frac{1}{2} \cdot \varepsilon \sum_{i \neq j} |u_i - u_j|^{-2}$ and the corresponding functional I_ε.

2° In constrast with [8], I_ε is not bounded from below on Λ, because $h < 0$. To bypass this difficulty we use a device like in [1]. Namely we consider the manifold

$$M = \{ u \in \Lambda;\ \|\dot{u}\|_{L^2}^2 = 1 \}$$

and a suitable, related functional $J_\varepsilon(u)$ (*see* section 3) which is bounded below on M and such that the critical points of $J_\varepsilon(u)$ constrained on M correspond to critical points of $I_\varepsilon(u)$.

3° We show that the arguments used in [7, 8] to overcome the lack of Palais-Smale condition can be adapted here to obtain approximate solutions $x^\varepsilon(t)$ of (1.1) with V_ε instead of V.

4° We show that $x^\varepsilon(t) \rightharpoonup x(t)$, a weak solution of (1.1) in the sense of [3]. *See* also Definition 3.1 below.

We point out that the regularity of weak solutions will not be studied here. For this kind of results, when $N = 2$ and $k \geqq 3$, *see* [10].

2. MAIN RESULT

We assume that the potential $V(x) = \dfrac{1}{2} \sum_{i \neq j} V_{ij}(x_i - x_j)$ satisfies the following conditions:

(V1) $V_{ij} \in C^2(\mathbf{R}^k \backslash \{0\}, \mathbf{R})$, $V_{ij}(\xi) < 0$ for all $\xi \neq 0$;

(V2) $3 V_{ij}'(\xi)\xi + V_{ij}''(\xi)\xi \cdot \xi \neq 0$ for all $\xi \neq 0$;

(V3) There exists an $\alpha \in (0, 2)$ such that

$$V_{ij}'(\xi)\xi \geqq -\alpha V_{ij}(\xi) \quad \text{for all } \xi \neq 0;$$

(V4) There exist $\beta \in (0, 2)$ and $r_1 > 0$ such that

$$V_{ij}'(\xi)\xi \leqq -\beta V_{ij}(\xi) \quad \text{for all } 0 < |\xi| \leqq r_1;$$

(V5) $V_{ij}(\xi) + \dfrac{1}{2} V_{ij}'(\xi)\xi \to 0$ as $|\xi| \to \infty$;

(V6) There exist $\theta \in [0, \pi/2)$ and $r_2 > 0$ such that

$$\operatorname{ang}(V_{ij}'(\xi), \xi) \leqq \theta \quad \text{for all } |\xi| \geqq r_2.$$

Here

$$\operatorname{ang}(\xi, \eta) = \begin{cases} \arccos \dfrac{\xi \cdot \eta}{|\xi|\,|\eta|}, & \text{if } |\xi||\eta| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.1. – Without loss of generality, we can assume that

$$V_{ij}(\xi) = V_{ji}(-\xi).$$

Otherwise, consider $\bar{V}_{ij}(\xi) = \frac{1}{2}(V_{ij}(\xi) + V_{ji}(-\xi))$ instead of $V_{ij}(\xi)$.

Remark 2.2. – From (V1)-(V4), it follows that

(i) $V'_{ij}(\xi)\,\xi > 0$ for all $\xi \neq 0$, (2.1)

(ii) For some $a > 0$,

$$V_{ij}(\xi) \leqq -\frac{a}{|\xi|^\alpha} \quad \text{for all } 0 < |\xi| \leqq r_1, \tag{2.2}$$

(iii) $V_{ij}(\xi) + \frac{1}{2} V'_{ij}(\xi)\,\xi \to -\infty$ as $|\xi| \to 0$. (2.3)

To state our result, we need to introduce the concept of weak periodic solutions of (1.1) as in Definition 10.1 of [3]. Roughly, it is a special class of generalized solutions which are found as limits of non-collision solutions of approximate problems. Since we need some notations to give a precise definition of weak solutions, we will give it in the next section.

Remark 2.3. – It is shown in [3] that every weak T-periodic solution $x(t)$ satisfies

(i) meas $\mathcal{C}(x) = 0$, where

$$\mathcal{C}(x) = \{ t \in \mathbf{R};\ x_i(t) \neq x_j(t) \text{ for some } i \neq j \},$$

(ii) $x(t) \in C^2(\mathbf{R}\backslash\mathcal{C}(x), \mathbf{R}^{kN})$,

(iii) $x(t)$ satisfies (1.1) for all $t \in \mathbf{R}\backslash\mathcal{C}(x)$,

(iv) $\int_0^T \left[\frac{1}{2}|\dot{x}|^2 - V(x(t))\right] dt < \infty$.

In the sequel, a solution $x(t)$ will be called a *noncollision* solutions of (1.1) if $\mathcal{C}(x) = \emptyset$.

Now we can state our main result.

THEOREM 2.1. – *Suppose that* (V1)-(V6) *hold. Then for all* $h < 0$, *the problem* (1.1) *has a weak periodic solution.*

In the following sections, we will give a proof to Theorem 2.1.

3. VARIATIONAL FORMULATION

Throughout this paper, we use the following notation:

NOTATION:

$$H = H^{1,2}(S^1, \mathbf{R}^k),$$

$$[u] = \int_0^1 u(t)\,dt \qquad \text{for} \quad u \in H,$$

$$\|u\|_2^2 = \int_0^1 |u|^2\,dt \qquad \text{for} \quad u \in L^2(S^1, \mathbf{R}^{kN}),$$

$$\|u\|^2 = \int_0^1 |\dot{u}|^2\,dt + |[u]|^2 = \sum_{i=1}^N \left(\int_0^1 |\dot{u}_i|^2\,dt + |[u_i]|^2\right)$$

for all $u = (u_1, \cdots, u_N) \in H^N$,

$$E = \left\{ u = (u_1, \cdots, u_N) \in H^N;\ \sum_{i=1}^N [u_i] = 0 \right\},$$

$$\Lambda = \{ u \in E;\ u_i(t) \neq u_j(t) \text{ for all } t \text{ and all } i \neq j \},$$

$$(u|v) = \int_0^1 uv\,dt \quad \text{for } u,\ v \in L^2(S^1, \mathbf{R}^{kN}),$$

$\langle f|u \rangle$ = the duality product of $f \in E^*$ and $u \in E$.

For a sequence $(u^n)_{n=1}^\infty \subset E$, we write

$$u^n \rightharpoonup u^0$$

to indicate that u^n converges to u^0 weakly in E and uniformly on [0, 1].

We consider the following functional, $I : \Lambda \to \mathbf{R}$,

$$I(u) = \frac{1}{2}\|\dot{u}\|_2^2 \int_0^1 (h - V(u))\,dt.$$

It is well-known that critical point of $I(u)$ on Λ, such that $I(u) > 0$, would give a rise – after a suitable time scaling – to a non-collision periodic solution of (1.1). However unfortunately, it is difficult to deal with $I(u)$ directly and we need to introduce a modified functional $I_\varepsilon(u)$ for $\varepsilon \in (0, 1]$, by setting

$$V_\varepsilon(x) = V(x) - \frac{\varepsilon}{2}\sum_{i\neq j} \frac{1}{|x_i - x_j|^2} \qquad \text{for} \quad x \in \mathbf{R}^{kN},$$

$$I_\varepsilon(u) = \frac{1}{2}\|\dot{u}\|_2^2 \int_0^1 (h - V_\varepsilon(u))\,dt$$

$$= I(u) + \frac{\varepsilon}{4}\|\dot{u}\|_2^2 \int_0^1 \sum_{i\neq j} \frac{1}{|u_i - u_j|^2}\,dt.$$

The main different feature of V_ε is that: for every sequence $(u^n) \subset \Lambda$ such that $u^n \rightharpoonup u \in \partial\Lambda$, we have

$$\int_0^1 \sum_{i \neq j} \frac{1}{|u_i^n - u_j^n|^2} \, dy \to \infty, \tag{3.1}$$

that is, for $\varepsilon \in (0, 1]$

$$\int_0^1 -V_\varepsilon (u^n) \, dt \to \infty. \tag{3.2}$$

We remark here that if $v^\varepsilon \in \Lambda$ satisfies $I'_\varepsilon (v^\varepsilon) = 0$ and $I'_\varepsilon (v^\varepsilon) > 0$, then

$$x^\varepsilon (t) = v^\varepsilon(\omega_\varepsilon t), \qquad \omega_\varepsilon = \frac{2\sqrt{I_\varepsilon (v^\varepsilon)}}{\|\dot{v}^\varepsilon\|_2^2} \tag{3.3}$$

is a periodic solution of the perturbed problem:

$$\left.\begin{aligned} \ddot{x} + V'_\varepsilon (x) = 0, \\ \frac{1}{2}|\dot{x}|^2 + V_\varepsilon (x) = h, \end{aligned}\right\} \tag{3.4}$$

Now we can give a precise definition of a weak periodic solution $x(t)$ of (1.1).

DEFINITION 3.1. – (*cf.* Definition 10.1 of [3]). $x(t)$ is said to be a *weak periodic solution* of (1.1) if there exist sequences $(v^n) \subset \Lambda$ and $\varepsilon_n \to 0$ such that

1° $v^n \in \Lambda$ is a critical point of I_{ε_n} such that $I_{\varepsilon_n} (v^n) > 0$, that is, if we set $x^n (t)$ as in (3.3), $x^n (t)$ is a periodic solution of (3.4).

2° There exists a constant $a > 0$ such that

$$0 < I_{\varepsilon_n} (v^n) \leqq a < \infty.$$

3° $\omega_n \to \omega \neq 0$, $v^n \rightharpoonup v \in E$ and $x(t) = v(\omega t)$.

4° There exists a $t_0 \in (0, 1/\omega]$ such that

$$x_i (t_0) \neq x_j (t_0) \quad \text{for all } i \neq j.$$

As anticipated before, it has been proved in Theorem 10.7 of [3] that any weak periodic solution satisfies the properties (i)-(iv) of Remark 2.3.

Next we define for $u \in \Lambda$ with $\dot{u} \not\equiv 0$ a positive number $\rho = \rho(u) > 0$ by

$$\frac{d}{d\rho} I_\varepsilon (\rho u) = 0. \tag{3.5}$$

LEMMA 3.1. – *For any* $u \in \Lambda$ *with* $\dot{u} \not\equiv 0$, *the equation* (3.5) *has a unique solution* $\rho = \rho(u) > 0$, *which is independent of* $\varepsilon \in (0, 1]$ *and satisfies*

$$h = \int_0^1 \left[V(\rho u) + \frac{1}{2} V'(\rho u) \, \rho u \right] dt. \tag{3.6}$$

Proof. – For $u \in \Lambda$, a direct calculation gives us

$$\frac{d}{d\rho} I(\rho u) = \rho \|\dot{u}\|_2^2 \int_0^1 \left[h - V(\rho u) - \frac{1}{2} V'(\rho u) \rho u \right] dt.$$

Thus, for $u \in \Lambda$, with $\dot{u} \not\equiv 0$, (3.5) is equivalent to (3.6). We set for $u \in \Lambda$, and $\rho > 0$

$$\phi_u(\rho) = \int_0^1 \left[V(\rho u) + \frac{1}{2} V'(\rho u) \rho u \right] dt.$$

From (V2), (V5) and (2.3), it follows

$$\phi_u'(\rho) > 0 \quad \text{for all } \rho \in (0, \infty), \tag{3.7}$$

$$\phi_u(\rho) \to -\infty \quad \text{as } \rho \to 0,$$

$$\phi_u(\rho) \to 0 \quad \text{as } \rho \to \infty.$$

Thus there is a unique $\rho = \rho(u) > 0$ such that $\phi_u(\rho) = h$ for all $u \in \Lambda$ and $h < 0$. ■

Remark 3.1. – In what follows, we define $\rho(u) > 0$ for all $u \in \Lambda$ by (3.6). We state some properties of $\rho(u)$.

LEMMA 3.2. – (i) $\rho(u) \in C^1(\Lambda, \mathbf{R})$;

(ii) *If* $u^n \rightharpoonup u \in \Lambda$, *then* $\rho(u^n) \to \rho(u)$;

(iii) *For all* $u \in \Lambda$,

$$\begin{aligned} \int_0^1 [h - V(\rho(u) u)] \, dt &= \frac{1}{2} \int_0^1 V'(\rho(u) u) \rho(u) u \, dt \\ &\geqq \frac{1}{2} \left(\frac{1}{\alpha} - \frac{1}{2} \right)^{-1} (-h) \equiv C_0 > 0 \end{aligned} \tag{3.8}$$

Proof. – Properties (i) and (ii) easily follow from the implicit function theorem, using (3.6) and (3.7).

(iii) From (3.6) and (V3), we have

$$\begin{aligned} h &= \int_0^1 \left[V(\rho(u) u) + \frac{1}{2} V'(\rho(u) u) \rho(u) u \right] dt \\ &\leqq -\left(\frac{1}{\alpha} - \frac{1}{2} \right) \int_0^1 V'(\rho(u) u) \rho(u) u \, dt, \end{aligned}$$

and (3.8) follows. ■

We set

$$M = \{ u \in \Lambda;\ \|\dot u\|_2^2 = 1 \}$$

and

$$\begin{aligned} J_\varepsilon (u) &= I_\varepsilon (\rho (u) u) \\ &= \frac{1}{2} \rho (u)^2 \int_0^1 [h - V_\varepsilon (\rho (u)\, u)]\, dt \\ &= \frac{1}{2} \rho (u)^2 \int_0^1 [h - V (\rho (u) u)]\, dt + \frac{\varepsilon}{4} \int_0^1 \sum_{i \neq j} \frac{1}{|u_i - u_j|^2}\, dt \\ &= \frac{1}{4} \rho (u)^2 \int_0^1 V' (\rho (u)\, u) \rho (u)\, u\, dt \\ &\quad + \frac{\varepsilon}{4} \int_0^1 \sum_{i \neq j} \frac{1}{|u_i - u_j|^2}\, dt. \end{aligned} \tag{3.9}$$

We remark that $\rho (u) > 0$ and (3.8) imply

$$J_\varepsilon (u) \geqq \frac{C_0}{2} \rho (u)^2 > 0 \quad \text{on } M. \tag{3.10}$$

We also remark here that M is a submanifold of E of codimension 1 and

$$T_u M = \{ \varphi \in E;\ (\dot u | \dot\varphi) = 0 \},$$

$$(T_u M)^\perp = \operatorname{span} \{ u - [u] \}.$$

In particular, all constant functions belong to $T_u M$ for all $u \in M$.

In what follows, for a functional $F (u) \in C^1 (\Lambda, \mathbf{R})$, we denote by $F' (u)$ its gradient in Λ, and for $u \in M$ we denote by $\nabla_M F (u)$ its gradient constrained on M, *i.e.*, $\nabla_M F (u) \in E^*$ is a vector satisfying

$$\langle \nabla_M F (u) | \varphi \rangle = \langle F' (u) | \varphi \rangle \quad \text{for all } \varphi \in T_u M,$$

$$\langle \nabla_M F (u) | \varphi \rangle = 0 \quad \text{for all } \varphi \in (T_u M)^\perp.$$

More precisely, it is given by

$$\langle \nabla_M F (u) | \varphi \rangle = \langle F' (u) | \varphi \rangle - \langle F' (u) | u - [u] \rangle (\dot u | \dot\varphi).$$

Here we state some properties of $I_\varepsilon (u)$ and $J_\varepsilon (u)$.

1° It follow from (3.5) that

$$\langle I_\varepsilon' (\rho (u) u) | u \rangle = 0 \quad \text{for all } u \in \Lambda. \tag{3.11}$$

2° For $u \in \Lambda$ and $\varphi \in E$,

$$\begin{aligned}\langle J_\varepsilon'(u)|\varphi\rangle &= \rho(u)\langle I_\varepsilon'(\rho(u)u)|\varphi\rangle + \langle I_\varepsilon'(\rho(u)u)|u\rangle\langle\rho'(u)|\varphi\rangle\\ &= \rho(u)\langle I_\varepsilon'(\rho(u)u)|\varphi\rangle,\end{aligned}$$

that is,

$$J_\varepsilon'(u) = \rho(u)\, I_\varepsilon'(\rho(u)\, u). \tag{3.12}$$

3° For $u \in E$, define $f_u \in E^*$ by

$$\langle f_u|\varphi\rangle = (\dot u|\dot\varphi).$$

Then $\nabla_M J_\varepsilon(u)$ can be written as

$$\begin{aligned}\nabla_M J_\varepsilon(u) &= J'(u) - \nu f_u\\ &= \rho(u)\, I_\varepsilon'(\rho(u)\, u) - \nu f_u\end{aligned} \tag{3.13}$$

with $\nu = \langle J_\varepsilon'(u)|u - [u]\rangle$.

4° Assume $\nabla_M J_\varepsilon(u) = 0$ for $u \in M$. By 1° and 3°, we have

$$\begin{aligned}0 &= \langle \nabla_M J_\varepsilon(u)|u\rangle\\ &= \rho(u)\langle I_\varepsilon'(\rho(u)\,u)|u\rangle - \nu = -\nu.\end{aligned}$$

Hence $I_\varepsilon'(\rho(u)\, u) = 0$.

Thus we have

LEMMA 3.3. – *Let $u^\varepsilon \in M$ be a critical point of J_ε on M, that is, $\nabla_M J_\varepsilon(u^\varepsilon) = 0$. Then*

(i) $I_\varepsilon'(\rho_\varepsilon u^\varepsilon) = 0$, *where* $\rho_\varepsilon = \rho(u^\varepsilon)$.

(ii) *Set* $\omega_\varepsilon^2 = 4 I_\varepsilon(\rho_\varepsilon u^\varepsilon)/\rho_\varepsilon^4$ *and* $x^\varepsilon(t) = \rho_\varepsilon u^\varepsilon(\omega_\varepsilon t)$. *Then* $x^\varepsilon(t)$ *is a noncollision solution of* (3.4). ■

Remark 3.2. – Since $\rho_\varepsilon > 0$ and (3.10) holds, ω_ε is well-defined and $\omega_\varepsilon > 0$.

In the following sections, we shall find a critical point u^ε of J_ε on M.

4. A CRITICAL POINT LEMMA

It is known that $J_\varepsilon(u)$ does not satisfy the Palais-Smale condition on M (*cf.* [4, 7, 8]). To overcome this difficulty, we follow the procedure of [7, 8] (*see* also [6]). We set

$$g(u) = \sum_{i=1}^{N} |[u_i - u_j]|^2 \qquad \text{for} \quad u \in E,$$
$$M^b = \{ u \in M;\ g(u) \leqq b \}.$$

The following is nothing but Lemma 2.1 of [8] in our setting.

LEMMA 4.1. – *Assume that there are constants c and $\tilde{c}$ with $c < \tilde{c}$ and $b \in \mathbf{R}$ such that*

(H1) *If $(u^n) \subset M$ satisfies $u^n \rightharpoonup u^0 \in \partial\Lambda$ and $g(u^n)$ is bounded, then*

$$J_\varepsilon(u^n) \to \infty.$$

(H2) $\nabla_M g(u) \neq 0$ *for all* $g(u) = b$, $J_\varepsilon(u) = c$.

(H3) *If $(u^n) \subset M$ satisfies $J_\varepsilon(u^n) \to c$, $\limsup g(u^n) \leqq b$ and $\nabla_M J_\varepsilon(u^n) \to 0$, then (u^n) possesses a convergent subsequence.*

(H4) *If $(u^n) \subset M$ satisfies $J_\varepsilon(u^n) \to c$, $g(u^n) \to b$ and*

$$\nabla_M J_\varepsilon(u^n) - \lambda_n \nabla_M (u^n) \to 0$$

for some $\lambda_n \geqq 0$, then (u^n) possesses a convergent subsequence.

(H5) $\nabla_M J_\varepsilon(u) \neq \lambda \nabla_M g(u)$ *for all $u \in M$ with $J_\varepsilon(u) = c$, $g(u) = b$ and for all $\lambda > 0$.*

(H6) *For any $\delta > 0$ with $c + \delta < \tilde{c}$, the set*

$$\{ u \in M;\ J_\varepsilon(u) \leqq c+\delta \} \cup (\{ u \in M;\ J_\varepsilon(u) \leqq \tilde{c} \} \cap \{ u \in M;\ g(u) \geqq b \})$$

is not deformable in M into

$$\{ u \in M;\ J_\varepsilon(u) \leqq c-\delta \} \cup (\{ u \in M;\ J_\varepsilon(u) \leqq \tilde{c} \} \cap \{ u \in M;\ g(u) \geqq b \}).$$

Then $J_\varepsilon(u)$ has a least one critical point $u \in M$ such that $J_\varepsilon(u) = c$ and $g(u) \leqq b$. ■

We are going to verify the conditions (H1)-(H6) for suitable c, $\tilde{c}$, $b > 0$. First of all, we remark by (3.1) and (3.9) that if $u^n \rightharpoonup u^0 \in \partial\Lambda$ then

$$J_\varepsilon(u^n) \geqq \frac{\varepsilon}{4} \int_0^1 \sum_{i \neq j} \frac{1}{|u_i - u_j|^2}\, dt \to \infty. \tag{4.1}$$

Therefore (H1) holds. Moreover, since

$$\langle \nabla_M g(u) | [u] \rangle = 2g(u) = 2b \neq 0 \qquad \text{and} \qquad [u] \in T_u M$$

for all $g(u) = b > 0$,

$$\nabla_M g(u) \neq 0 \quad \text{for all } g(u) = b > 0.$$

That is, (H2) holds for all $b > 0$ and $\tilde{c} \in \mathbf{R}$.

In Section 5, we verify (H3) and (H4) which are local versions of Palais-Smale condition, and in Section 6 we will get (H5) and (H6).

5. PALAIS-SMALE CONDITION

To verify (H3), (H4), some lemmas are in order. First we need some properties of $\rho(u)$.

LEMMA 5.1. – *For $c > 0$, there are $k_1 = k_1(c) > 0$ and $k_2 = k_2(c) > 0$ independent of $\varepsilon \in (0, 1]$ such that for $u \in M$*

(i) $J_\varepsilon(u) \leqq c$ *implies* $\rho(u) \leqq k_1(c)$,

(ii) $J_\varepsilon(u) \geqq c$ *implies* $\rho(u) \geqq k_2(c)$.

Proof. – (i) follows from (3.10) easily. We prove (ii) here. We argue indirectly and assume there are sequences $(u^n) \subset M$ and $(\varepsilon_n) \subset (0, 1]$ such that

$$J_{\varepsilon_n}(u^n) \geqq c \qquad \text{and} \qquad \rho_n \equiv \rho(u^n) \to 0.$$

We set $w^n = \rho_n u^n$. Since $\rho^n \to 0$ and

$$c \leqq J_{\varepsilon_n}(u^n) = \frac{1}{2}\rho_n^2 \int_0^1 \left[h - V(w^n) + \frac{\varepsilon_n}{4} \sum_{i \neq j} \frac{1}{|w_i^n - w_j^n|^2}\right] dt,$$

it follows

$$\int_0^1 \left[h - V(w^n) + \frac{\varepsilon_n}{4} \sum_{i \neq j} \frac{1}{|w_i^n - w_j^n|^2}\right] dt \to \infty.$$

In particular, for some $i \neq j$, one has

$$\min_{t \in [0, 1]} |w_i^n(t) - w_j^n(t)| \to 0.$$

On the other hand, $\|\dot{w}^n\|_2 = \rho_n \to 0$ and hence

$$\max_{t \in [0, 1]} |w_i^n(t) - w_j^n(t)| \to 0.$$

Then $w_i^n - w_j^n \to 0$ uniformly in [0, 1] and therefore, by (3.6) and (2.3)

$$h = \int_0^1 \left[V(w^n) + \frac{1}{2} V'(w^n)\, w^n \right] dt \to -\infty.$$

This is a contradiction. ■

Recall that $\sum_{i=1}^{N} [u_i] = 0$ for $u \in E$. Therefore M^b is a bounded set of E for all b. Thus using also (4.1), we infer

LEMMA 5.2. – *For* $\varepsilon \in (0, 1]$, *suppose* $(u^n) \subset M$ *satisfies for* $b > 0$, $c > 0$
(i) $u^n \in M^b$, i.e., $g(u^n) \leqq b$,
(ii) $J_{\varepsilon_n}(u^n) \leqq c$.
Then (u^n) *has a subsequence – still denoted by* u^n *– such that*

$$u^n \rightharpoonup u^0 \in \Lambda. \quad ■$$

Next we prove (H3) and (H4).

LEMMA 5.3. – *Suppose* $\varepsilon \in (0, 1]$. *Then*
(i) (H3) *holds for all* $b > 0$ *and* $c > 0$,
(ii) (H4) *holds for all* $b > 0$ *and* $c > 0$.

Proof. – (i) We assume

$$u^n \in M^b,$$

$$J_\varepsilon(u^n) \to c > 0,$$

$$\nabla_M J_\varepsilon(u^n) \to 0 \quad \text{strongly in } E^*.$$

Our goal is to prove there is a strongly convergent subsequence of (u^n) such that $u^n \to u^0 \in M$. By Lemma 5.2, (u^n) possesses a weakly convergent subsequence $u^n \rightharpoonup u^0 \in \Lambda$. Thus it suffices to show the convergence is strong, that is, $\|\dot{u}^n\|_2 \to \|\dot{u}^0\|_2$, *i.e.*, $\|\dot{u}^0\|_2 = 1$.

First, by Lemma 5.1, we remark that

$$\rho_n \equiv \rho(u^n) \in [k_1, k_2],$$

where $k_1, k_2 > 0$ are independent of n and hence $\rho_n \to \rho_0 = \rho(u^0) \neq 0$.

Since $\nabla_M J_\varepsilon(u^n) \to 0$, by (3.13) there exists a sequence $(\nu_n) \subset \mathbf{R}$ such that

$$\rho_n I'_\varepsilon(\rho_n u^n) - \nu_n f_{u^n} \to 0 \quad \text{strongly in } E^*.$$

Taking a scalar product with u^n, we infer $\nu_n \to 0$ by (3.11). Thus,

$$I'_\varepsilon(\rho_n u^n) \to 0 \quad \text{strongly in } E^*.$$

In particular, we have $\langle I'_\varepsilon(\rho_n u^n)|u^0\rangle \to 0$, *i.e.*,

$$\rho_n(\dot u^n|\dot u^0)\int_0^1[h - V_\varepsilon(\rho_n u^n)]\,dt - \frac{1}{2}\rho_n^2\int_0^1 V'_\varepsilon(\rho_n u^n)\,u^0\,dt \to 0.$$

Then

$$\rho_0\|\dot u^0\|_2^2\int_0^1[h - V_\varepsilon(\rho_0 u^0)]\,dt - \frac{1}{2}\rho_0^2\int_0^1 V'_\varepsilon(\rho_0 u^0)\,u^0\,dt = 0. \tag{5.1}$$

On the other hand, by (3.11), $\langle I'_\varepsilon(\rho_n u^n)|u^n\rangle = 0$, *i.e.*,

$$\rho_n\int_0^1[h - V_\varepsilon(\rho_n u^n)]\,dt - \frac{1}{2}\rho_n^2\int_0^1 V'_\varepsilon(\rho_n u^n)\,u^0\,dt = 0.$$

Taking a limit as $n \to \infty$, we have

$$\rho_0\int_0^1[h - V_\varepsilon(\rho_0 u^0)]\,dt - \frac{1}{2}\rho_0^2\int_0^1 V'_\varepsilon(\rho_0 u^0)\,u^0\,dt = 0. \tag{5.2}$$

Comparing (5.1) and (5.2) and recalling (3.8), we have $\|\dot u^0\|_2 = 1$, *i.e.*, $u^n \to u^0 \in \Lambda$ strongly in E.

(ii) Next we assume $(u^n) \subset M$ satisfies

$$g(u^n) \to b > 0, \tag{5.3}$$

$$J_\varepsilon(u^n) \to c > 0, \tag{5.4}$$

$$\nabla_M J_\varepsilon(u^n) - \mu_n\nabla_M g(u^n) \to 0, \tag{5.5}$$

with $\mu_n \geqq 0$.

By Lemma 5.2, we may assume $u^n \to u^0 \in \Lambda$ and again it suffices to show $\|\dot u^0\|_2 = 1$. Again we note that

$$\rho_n \to \rho_0 \neq 0.$$

By the definition of ∇_M and (5.5), there exists $(\nu_n) \subset \mathbf{R}$ such that

$$\rho_n I'_\varepsilon(\rho_n u^n) - \mu_n g'(u^n) + \nu_n f_{u^n} \to 0 \quad \text{strongly in } E^*. \tag{5.6}$$

Taking a product of (5.6) and u^n, we get from (3.11)

$$-\mu_n\langle g'(u^n)|u^n\rangle + \nu_n\langle f_{u^n}|u^n\rangle \to 0,$$

i.e.,

$$-2\mu_n g(u^n) + \nu_n \to 0. \tag{5.7}$$

Taking a product of (5.6) and $[u^n]$, we also get

$$\rho_n \langle I'_\varepsilon(\rho_n u^n) | [u^n] \rangle - 2\mu_n g(u^n) \to 0. \tag{5.8}$$

Since $\rho_n u^n \rightharpoonup \rho_0 u^0 \in \Lambda$ uniformly in [0, 1] and weakly in E, we can see $\langle I'_\varepsilon(\rho_n u^n) | [u^n] \rangle$ stays bounded as $n \to \infty$. Thus by (5.3), (5.7), (5.8), μ_n and ν_n stay bounded as $n \to \infty$. Therefore we may assume $\mu = \lim_{n\to\infty} \mu_n$ and $\nu = \lim_{n\to\infty} \nu_n$ exist. Here we remark

$$\nu = \lim_{n\to\infty} \nu_n = 2b \lim_{n\to\infty} \mu_n \geqq 0. \tag{5.9}$$

As in the proof of (i), we take scalar products of (5.6) and u^n (resp. u^0) and take limits as $n \to \infty$. Then we have

$$\rho_0^2 \int_0^1 [h - V_\varepsilon(\rho_0 u^0)]\, dt - \rho_0^3 \int_0^1 V'_\varepsilon(\rho_0 u^0)\, u^0\, dt - 2b\mu + \nu = 0$$

and

$$\rho_0^2 \|\dot{u}^0\|_2^2 \int_0^1 [h - V_\varepsilon(\rho_0 u^0)]\, dt - \rho_0^3 \times \int_0^1 V'_\varepsilon(\rho_0 u^0)\, u^0\, dt - 2b\mu + \nu\, \|\dot{u}^0\|_2^2 = 0.$$

Recalling (3.8) and (5.9), we get $\|\dot{u}^0\|_2 = 1$. ■

6. SOLUTIONS OF (3.4)

Next we deal with (H5) and (H6). The arguments of the proofs are similar to those of [8].

LEMMA 6.1. – *For any* $0 < c_1 < c_2$ *there exists* $B_0 = B_0(c_1, c_2) > 0$ *independent of* $\varepsilon \in (0, 1]$ *such that*

$$\nabla_M J_\varepsilon(u^n) \neq \lambda \nabla_M g(u)$$

for all $\lambda > 0$ *and* $u \in M$ *with* $J_\varepsilon(u) \in [c_1, c_2]$, $g(u) \geqq B_0$.

Proof. – Arguing indirectly, we assume that there exist $(\varepsilon_n) \subset (0, 1]$, $(u^n) \subset M$ and $(\lambda_n) \subset (0, \infty)$ such that

$$\nabla_M J_{\varepsilon_n}(u^n) = \lambda_n \nabla_M g(u^n), \tag{6.1}$$

$$J_{\varepsilon_n}(u^n) \in [c_1,\, c_2], \tag{6.2}$$

$$g(u^n) \to \infty. \tag{6.3}$$

We set $z^n = g(u^n)^{-1/2}[u^n]$. Clearly (z^n) is a bounded sequence and we may assume $z^0 = \lim_{n\to\infty} z^n$ exists. We remark $g(z^0) = 1$ and $z^0 \in T_{u^n} M$ for all n. We will show

$$\langle \nabla_M J_{\varepsilon_n}(u^n) | z^0 \rangle \leqq 0, \tag{6.4}$$

$$\langle \nabla_M g(u^n) | z^0 \rangle > 0, \tag{6.5}$$

for large n. Clearly they are incompatible with (6.1) and $\lambda_n > 0$. By (6.2) and Lemma 5.1, we have

$$\rho_n \equiv \rho(u^n) \in [k_1(c_1),\, k_2(c_2)].$$

Note that if $z_i^0 \neq z_j^0$, then $|[u_i^n] - [u_j^n]| \to \infty$. For such $i \neq j$, we have from (6.3)

$$\begin{aligned} |\rho_n(u_i^n(t) - u_j^n(t))| &\geqq \rho_n |[u_i^n] - [u_j^n]| - 2\rho_n \|\dot{u}^n\|_2 \\ &\geqq k_1(c_1)|[u_i^n] - [u_j^n]| - k_2(c_2) \\ &\to \infty \quad \text{as } n \to \infty, \end{aligned} \tag{6.6}$$

$$\begin{aligned} &|g(u^n)^{-1/2}\rho_n(u_i^n(t) - u_j^n(t)) - \rho_n(z_i^0 - z_j^0)| \\ &\quad \leqq k_2(c_2)\, g(u^n)^{-1/2}\|\dot{u}^n\|_2 + 2k_2(c_2)|z^n - z^0| \to 0 \end{aligned} \tag{6.7}$$

as $n \to \infty$ uniformly in t.

Thus from (6.6)-(6.7), we can see

$$\operatorname{ang}(\rho_n(u_i^n(t) - u_j^n(t)),\, z_i^0 - z_j^0) \to 0,$$

$$\min_{t\in[0,1]} |\rho_n(u_i^n(t) - u_j^n(t))| \to \infty$$

as $n \to \infty$ uniformly in t. Thus by (V6)

$$\begin{aligned} \langle \nabla_M J_{\varepsilon_n}(u^n) | z \rangle &= \rho_n \langle I'_\varepsilon(\rho_n u^n) | z \rangle \\ &= -\frac{1}{2}\rho_n^2 \int_0^1 \Bigg[\sum_{i\neq j} V_{ij}(\rho_n(u_i^n - u_j^n))(z_i^0 - z_j^0) \\ &\quad + \frac{\varepsilon_n}{4} \sum_{i\neq j} \frac{(u_i^n - u_j^n)(z_i^0 - z_j^0)}{|u_i^n - u_j^n|^4} \Bigg] dt \\ &< 0 \end{aligned}$$

for large n. Thus we get (6.4). On the other hand,

$$\langle \nabla_M g(u^n) | z^0 \rangle = g(u^n)^{1/2} \langle g'(z^n) | z^0 \rangle \to \infty.$$

Therefore we get (6.5). Thus (6.1) cannot take place. ■

As in [7, 8], we define admissible sets. Let $\mathcal{H}$ be the set of deformations of Λ in E into the space of constant functions $\mathbf{R}^{kN}$;

$$\mathcal{H} = \{ \eta \in C([0, 1] \times \Lambda, E);\ \eta(0, \cdot) = \mathrm{id},\ \eta(1, \Lambda) \subset \mathbf{R}^{kN} \}.$$

We also use the notation, for $i \neq j$,

$$\Gamma_{ij} = \{ u \in E;\ u_i(t) = u_j(t) \text{ for some } t \}.$$

DEFINITION 6.1. – Let A be a closed subset of Λ. We say A is *admissible* if for any $\eta \in \mathcal{H}$ there exists $u \in A$ such that for any $i \neq j$ there exists a sequence $i_1, \cdots, i_m \in \{1, \cdots, N\}$ satisfying

1° $i_1 = i,\ i_m = j$;

2° $i_k \neq i_{k+1}$ for all $k = 1, \cdots, m-1$;

3° $\eta([0, 1] \times \{u\}) \cap \Gamma_{i_k i_{k+1}} \neq \emptyset$ for all $k = 1, \cdots, m-1$.

We denote by $\mathcal{A}$ the class of admissible sets.

It is shown in [7, 8] that there is a compact admissible set, which does not contain constant functions, and

(A1) If $A \subset B$ and $A \in \mathcal{A}$, then $B \in \mathcal{A}$,

(A2) If B is a deformation of $A \in \mathcal{A}$ in Λ, then $B \in \mathcal{A}$.

We set

$$\mathcal{A}_M = \{ A \subset M;\ A \in \mathcal{A} \}.$$

Plainly $\mathcal{A}_M \neq \emptyset$, indeed it contains any radial projection on M of $A \in \mathcal{A}$ with $A \cap \{ u \in \Lambda;\ \dot{u} \equiv 0 \} = \emptyset$.

The following property is important for our argument.

LEMMA 6.2. – *There exists $B_1 > 0$ such that*

$$\{ u \in M;\ g(u) \geqq B_1 \} \notin \mathcal{A}_M.$$

Proof. – It suffices to show for any $\mathcal{A} \in \mathcal{A}_M$ there exists $u \in A$ such that $g(u) \leqq 2N(N-1)^3$.

Let $\eta_0 \in \mathcal{H}$ be a deformation such that

$$\eta_0(s, u) = [u] + (1 - s)(u - [u]).$$

By the definition of admissible sets, there is a $u \in A$ such that for any $i \neq j$ there exists a sequence $i_1, \cdots, i_m$ satisfying the properties 1°-3° of Definition 6.1. We remark that we may assume $m \leqq N$.

By 3° of Definition 6.1, $\eta_0([0, 1] \times \{u\}) \cap \Gamma_{i_k i_{k+1}} \neq \varnothing$ for all $k = 1, \cdots, m-1$. Thus for some $s_k \in [0, 1]$ and t_k, we have

$$[u_{i_k} - u_{i_{k+1}}] + (1 - s_k)(u_{i_k}(t_k) - u_{i_{k+1}}(t_k) - [u_{i_k} - u_{i_{k+1}}]) = 0.$$

Thus

$$|[u_{i_k} - u_{i_{k+1}}]| \leqq (1 - s_k)\|\dot{u}_{i_k} - \dot{u}_{i_{k+1}}\|_2 \leqq \sqrt{2}.$$

Therefore

$$|[u_i - u_j]| \leqq \sum_{k=1}^{m-1} |[u_{i_k} - u_{i_{k+1}}]| \leqq \sqrt{2}(N-1).$$

Since the pair (i, j) with $i \neq j$ is arbitrary, we have

$$g(u) = \sum_{i \neq j} |[u_i - u_j]|^2 \leqq 2N(N-1)^3.$$

We also have

LEMMA 6.3. – *For any given $b' > 0$ there is a $\gamma = \gamma(b') > 0$ independent of $\varepsilon \in (0, 1]$ such that*

$$\{u \in M;\ J_\varepsilon(u) \leqq \gamma\} \subset \{u \in M;\ g(u) \geqq b'\}.$$

Proof. – We argue indirectly and assume there are sequences $(u^n) \subset M$ and $(\varepsilon_n) \subset (0, 1]$ such that

$$J_{\varepsilon_n}(u^n) \to 0,$$

$$g(u^n) \leqq b'.$$

By (3.10), we have

$$\rho_n \equiv \rho(u^n) \to 0.$$

Since $M^{b'}$ is bounded in E, we can see

$$\rho_n u^n \to 0 \quad \text{strongly in } E.$$

But this is incompatible with (3.6) and (2.3). ■

COROLLARY 6.4. – *Let $B_1 > 0$ be a number given in Lemma 6.2. Then*

$$\{u \in M; J_\varepsilon(u) \leqq \gamma(B_1)\} \cup \{u \in M; g(u) \geqq B_1\} = \{u \in M;\ g(u) \geqq B_1\}.$$

Now we define minimax values c_ε^* by

$$c_\varepsilon^* = \inf_{A \in \mathcal{A}_M} \sup_{u \in A} J_\varepsilon(u).$$

By the definition of c_ε^*, it is clear from (A1)-(A2) that for any $\delta > 0$

$$\{ u \in M;\ J_\varepsilon(u) \leqq C_\varepsilon^* + \delta \} \in \mathcal{A}_M. \tag{6.8}$$

Thus we see from Corollary 6.4 that

$$\gamma(B_1) \leqq c_\varepsilon^* + \delta.$$

Fix $C^* > c_1^*$ and let $B_2 = B_0(\gamma(B_1), C^*)$ [let $B_0(\cdot, \cdot)$ be given in Lemma 6.1] and set

$$d = \max\{ B_1, B_2\}.$$

LEMMA 6.5. – *$J_\varepsilon|_M$ has a critical point u^ε such that*
(i) $J_\varepsilon(u^\varepsilon) \in [\gamma(B_1), C^*]$,
(ii) $u^\varepsilon \in M^d$.

Proof. – Set

$$c_\varepsilon = \inf\{ c \in \mathbf{R};\ \{ u \in M;\ J_\varepsilon(u) \leqq c \} \cup \{ u \in M;\ g(u) \geqq d \} \in \mathcal{A}_M \}.$$

Then clearly

$$c_\varepsilon \leqq c_\varepsilon^* < C^* \quad \text{for all } \varepsilon \in (0, 1].$$

From Corollary 6.4 it follows

$$\begin{aligned} \{ u \in M;\ J_\varepsilon(u) \leqq \gamma(B_1) \} &\cup \{u \in M;\ g(u) \geqq d\} \\ &\subset \{u \in M;\ g(u) \leqq B_1 \} \cup \{u \in M;\ g(u) \geqq d\} \\ &= \{u \in M;\ g(u) \geqq d\}. \end{aligned}$$

Since $\{u \in M;\ g(u) \geqq d\} \notin \mathcal{A}_M$, (A1) yields

$$\{ u \in M;\ J_\varepsilon(u) \leqq \gamma(B_1) \} \cup \{u \in M;\ g(u) \geqq d\} \notin \mathcal{A}_M.$$

Thus we have

$$\gamma(B_1) < c_\varepsilon \quad \text{for all } \varepsilon \in (0, 1].$$

Now it is easy to see all assumptions (H1)-(H6) of Lemma 4.1 are satisfied with $c = c_\varepsilon$, $\tilde{c} = C^*$ and $b = d$. ■

7. LIMITING PROCESS

In previous sections, we have shown that for any $\varepsilon \in (0, 1]$ there exists a critical point u^ε such that

$$\nabla_M J_\varepsilon(u^\varepsilon) = 0,$$

$$J_\varepsilon(u^\varepsilon) = c_\varepsilon,$$

$$u^\varepsilon \in M^d.$$

We set

$$\rho_\varepsilon \equiv \rho(u^\varepsilon)$$
$$v^\varepsilon(t) = \rho_\varepsilon u^\varepsilon(t),$$
$$\omega_\varepsilon^2 = \frac{\int_0^1 [h - V_\varepsilon(v^\varepsilon)]\, dt}{\frac{1}{2}\rho_\varepsilon^2} = \frac{J_\varepsilon(u^\varepsilon)}{\frac{1}{4}\rho_\varepsilon^4}$$

From the arguments of Sections 5 and 6 one deduces:

$$c_\varepsilon \in [\gamma(B_1),\ C^*] \tag{7.1}$$

$$\rho(u^\varepsilon) \in [k_1(\gamma(B_1)),\ k_2(C^*)], \tag{7.2}$$

$$\omega_\varepsilon^2 \in \left[\frac{4\gamma(B_1)}{k_2(C^*)^4},\ \frac{C^*}{k_1(\gamma(B_1))^4}\right]. \tag{7.3}$$

Since $u^\varepsilon \in M^d$ and M^d is a bounded subset of E, then, up to a subsequence,

$$u^\varepsilon \rightharpoonup u. \tag{7.4}$$

Moreover, by (7.2), it follows that

$$\rho_\varepsilon \to \rho \neq 0, \tag{7.5}$$

while, by (7.3), $\omega_\varepsilon \to \omega$. We set

$$x(t) = \rho\, u(\omega t).$$

LEMMA 7.1. – *There exists a $t_0 \in (0, 1]$ such that*

$$x_i(t_0) \neq x_j(t_0) \quad \textit{for all } i \neq j. \tag{7.6}$$

Proof. – By (7.4) and (7.5), $v^\varepsilon \rightharpoonup v := \rho u$. Since

$$J_\varepsilon(u^\varepsilon) = \frac{1}{2}\rho_\varepsilon^2 \int_0^1 [h - V_\varepsilon(z^\varepsilon)]\, dt,$$

we have by (7.1) and (7.2)

$$\int_0^1 [h - V_\varepsilon(v^\varepsilon)]\,dt = \frac{2\,J_\varepsilon(u^\varepsilon)}{\rho_\varepsilon^2} \in \left[\frac{2\,\gamma(B_1)}{k_2\,(C^*)^2}, \frac{2\,C^*}{k_1\,(\gamma(B_1))^2}\right]$$

for all $\varepsilon \in (0, 1]$. It is easy to see that, *via* the Fatou's Lemma, this implies the existence of $t_0 \in (0, 1]$ satisfying (7.6). ■

Proof of Theorem 2.1 completed. – It suffices to note that $v_\varepsilon \rightharpoonup v$, $v = \rho\, u$ and properties 1°-3° of Definition 3.1 are satisfied, while, property 4° is nothing but Lemma 7.1. ■

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