

# ANNALES DE L'I. H. P., SECTION C

A. CAPOZZI

D. FORTUNATO

G. PALMIERI

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*Annales de l'I. H. P., section C*, tome 2, n° 6 (1985), p. 463-470

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## An existence result for nonlinear elliptic problems involving critical Sobolev exponent

by

A. CAPOZZI, D. FORTUNATO, G. PALMIERI

Dipartimento di Matematica dell'Università Bari

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ABSTRACT. — In this paper we consider the following problem:

$$(1) \quad \begin{cases} -\Delta u - \lambda u = |u|^{2^*-2} \cdot u \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \quad 2^* = 2n/(n-2)$$

where  $\Omega \subset \mathbb{R}^n$  is a bounded domain and  $\lambda \in \mathbb{R}$ .

We prove the existence of a nontrivial solution of (1) for any  $\lambda > 0$ , if  $n \geq 4$ .

RÉSUMÉ. — Soient  $\Omega$  un sous-ensemble ouvert borné de  $\mathbb{R}^n$  et  $\lambda$  un nombre positif, le but de cette note c'est de montrer que le problème suivant :

$$\begin{cases} -\Delta u - \lambda u = |u|^{2^*-2} \cdot u \\ u|_{\partial\Omega} = 0 \end{cases} \quad 2^* = 2n/(n-2)$$

admet, au moins, une solution non triviale, si  $n \geq 4$ .

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Work supported by G. N. A. F. A. or C. N. R. and by Ministero Pubblica Istruzione (fondi 40 % e 60 %).

## 0. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be an open bounded set with smooth boundary. Consider the problem

$$(0.1) \quad \begin{cases} -\Delta u - \lambda u - u \cdot |u|^{2^*-2} = 0 \\ u \in H_0^1(\Omega) \end{cases}$$

where  $\lambda$  is a real parameter and  $2^* = 2n/(n-2)$  is the critical Sobolev exponent for the Sobolev embedding  $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ .

The solutions of (0.1) are the critical points of the energy functional

$$(0.2) \quad f_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx.$$

Since the embedding  $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$  is not compact the functional  $f_\lambda$  does not satisfy the Palais-Smale condition in the energy range  $]-\infty, +\infty[$  (cf. remark 2.3 of [4]).

Moreover if  $\lambda \leq 0$  and  $\Omega$  is starshaped (0.1) has only the trivial solution (cf. [6]).

Recently Brezis and Nirenberg in [2] have proved that if  $n \geq 4$  and  $0 < \lambda < \lambda_1$  ( $\lambda_1$  is the first eigenvalue of  $-\Delta$ ) then (0.1) has a positive solution. In [4] Cerami, Fortunato and Struwe have obtained multiplicity results for (0.1) in the case in which  $\lambda$  belongs to a suitable left neighbourhood of an arbitrary eigenvalue of  $-\Delta$  (cf. also [3]).

In this paper we prove the following theorem:

**THEOREM 0.1.** — *If  $n \geq 4$  the problem (0.1) possesses at least one non trivial solution for any  $\lambda > 0$ .*

A weaker result related to theorem 0.1 has been announced in [5].

We observe that if  $n = 3$  and  $\Omega$  is a ball, Brezis and Nirenberg [2] have proved that the problem (0.1) does not have nontrivial radial solutions if  $0 < \lambda < \frac{\lambda_1}{4}$ .

## 1. SOME PRELIMINARIES

Let  $\|\cdot\|, |\cdot|_p$  denote respectively the norms in  $H_0^1(\Omega)$  and  $L^p(\Omega)$  ( $1 \leq p \leq \infty$ ), and let

$$S = \inf \{ \|u\|^2 / |u|_{2^*}^2 : u \in H_0^1(\Omega) \setminus \{0\} \}$$

denote the best constant for the embedding  $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ .

The following lemma shows that  $f_\lambda$  satisfies a local P. S. condition.

LEMMA 1.1. — For any  $\lambda \in \mathbb{R}$  the functional  $f_\lambda$  (see (0.2)) satisfies the Palais-Smale condition in  $\left] -\infty, \frac{1}{n} S^{n/2} \right[$  in the following sense:

If  $c < \frac{1}{n} S^{n/2}$  and  $\{u_m\}$  is a sequence in  $H_0^1(\Omega)$  such that

(P. S.)  $\left\{ \begin{array}{l} \text{as } m \rightarrow \infty f_\lambda(u_m) \rightarrow c, f'_\lambda(u_m) \rightarrow 0 \text{ strongly in } H^{-1}(\Omega), \text{ then } \{u_m\} \\ \text{contains a subsequence converging strongly in } H_0^1(\Omega). \end{array} \right.$

The proof of this lemma is in [2] and in [4]. We recall that a deeper compactness result has been proved in [7].

We recall a critical point Theorem (cf. [1, Theorem 2.4]) which is a variant of some results contained in [0].

THEOREM 1.2. — Let  $H$  be a real Hilbert space and  $f \in C^1(H, \mathbb{R})$  be a functional satisfying the following assumptions:

- (f<sub>1</sub>)  $f(u) = f(-u), f(0) = 0$  for any  $u \in H$
- (f<sub>2</sub>) there exists  $\beta > 0$  such that  $f$  satisfies (P. S.) in  $]0, \beta[$
- (f<sub>3</sub>) there exist two closed subspaces  $V, W \subset H$  and positive constants  $\rho, \delta$  such that

- (i)  $f(u) < \beta$  for any  $u \in W$
- (ii)  $f(u) \geq \delta$  for any  $u \in V, \|u\| = \rho$
- (iii)  $\text{codim } V < +\infty$ .

Then there exist at least  $m$  pairs of critical points, with

$$m = \dim(V \cap W) - \text{codim}(V + W).$$

## 2. PROOF OF THEOREM 0.1

Our aim is to define two suitable closed subspaces  $V$  and  $W$ , with  $V \cap W \neq \{0\}$  and  $V + W = H$ , such that  $f_\lambda$  satisfies the assumptions  $f_2$  and  $f_3$  of Theorem 1.2 with  $\beta = \frac{1}{n} S^{n/2}$ .

In the sequel we denote by  $\lambda_j$  the eigenvalues of  $-\Delta$  and by  $M(\lambda_j)$  the corresponding eigenspaces.

Given  $\lambda > 0$ , we set

$$(2.1) \quad \lambda^+ = \min \{ \lambda_j \mid \lambda < \lambda_j \}$$

$$H_1 = \overline{\bigoplus_{\lambda_j \geq \lambda^+} M(\lambda_j)} \quad H_2 = \bigoplus_{\lambda_j < \lambda^+} M(\lambda_j)$$

where the closure is taken in  $H_0^1(\Omega)$ .

If  $r > 0$  we set

$$N_r(0) = \{ x \in \mathbb{R}^n \mid \|x\| < r \}.$$

Without loss of generality we can suppose that  $0 \in \Omega$  and that  $N_1(0) \subset \Omega$ . Given  $\mu > 0$  we set (cf. [2] [7])

$$\psi_\mu(x) = \phi(x) \cdot u_\mu(x)$$

where  $\phi \in C_0^\infty(N_1(0))$ ,  $\phi(x) = 1$  on  $N_{\frac{1}{2}}(0)$ , and

$$u_\mu(x) = \frac{|n(n-2)\mu|^{(n-2)/4}}{|\mu + |x|^2|^{(n-2)/2}}.$$

The following lemma holds:

LEMMA 2.1. — *If  $\psi_\mu(x)$  is defined as in (2.1), then for any  $\mu$*

$$(2.2) \quad \|\psi_\mu\|^2 = S^{n/2} + O(\mu^{(n-2)/2}) \quad (1)$$

$$(2.3) \quad |\psi_\mu|_{2^*}^{2^*} = S^{n/2} + O(\mu^{n/2})$$

$$(2.4) \quad |\psi_\mu|_2^2 = \begin{cases} K_1\mu + O(\mu^{(n-2)/2}) & \text{if } n \geq 5 \\ K_1\mu |\log \mu| + O(\mu) & \text{if } n = 4 \end{cases}$$

$$(2.5) \quad |\psi_\mu|_1 \leq K_2\mu^{(n-2)/4}$$

$$(2.6) \quad |\psi_\mu|_{2^*-1}^{2^*-1} \leq K_3\mu^{(n-2)/4}$$

where  $K_1, K_2, K_3$  are suitable positive constants.

*Proof.* — The proof of (2.2), (2.3), (2.4) is contained in [2], moreover (2.5) and (2.6) can be straightforwardly verified.

Now we shall prove some technical lemmas. We set

$$\overline{W}(\mu) = \{u \in H_0^1 \mid u = u^- + t\psi_\mu, u^- \in H_2, t \in \mathbb{R}\}.$$

The following lemma holds:

LEMMA 2.2. — *If  $u \in \overline{W}(\mu)$ , then for any  $\mu > 0$*

$$(2.7) \quad |u|_{2^*}^{2^*} \geq |t\psi_\mu|_{2^*}^{2^*} + \frac{1}{2} |u^-|_{2^*}^{2^*} - K_4 t^{2^*} \mu^{n(n-2)/(2n+4)} \quad \text{for any } t \in \mathbb{R}.$$

*Proof.* — By the identity

$$(2.8) \quad |u|_{2^*}^{2^*} = 2^* \int_\Omega dx \int_0^u |s|^{2^*-2} s ds$$

(1) In the sequel we denote by  $O(\mu^\alpha)$ ,  $\alpha > 0$  a function  $|f(\mu)| \leq \text{const } \mu^\alpha$  near  $\mu = 0$ , and by  $o(\mu)$ , a function such that  $f(\mu)/\mu \rightarrow 0$  as  $\mu \rightarrow 0$ .

it follows that

$$\begin{aligned}
 (2.9) \quad & |u^- + t\psi_\mu|_{2^*}^{2^*} - |t\psi_\mu|_{2^*}^{2^*} - |u^-|_{2^*}^{2^*} = \\
 & = 2^* \int_0^1 d\tau \int_\Omega [ |t\psi_\mu + \tau u^-|^{2^*-2} \cdot (t\psi_\mu + \tau u^-) - |\tau u^-|^{2^*-2} \cdot \tau u^- ] u^- dx = \\
 & = 2^*(2^* - 1) \int_0^1 d\tau \int_\Omega |\tau u^- + t\psi_\mu \theta|^{2^*-2} \cdot t\psi_\mu \cdot u^- dx
 \end{aligned}$$

where  $\theta = \theta(x)$  is a measurable function such that  $0 < \theta(x) < 1$ .

By (2.9) and by (2.5), (2.6) we have that

$$\begin{aligned}
 (2.10) \quad & | |u|_{2^*}^{2^*} - |t\psi_\mu|_{2^*}^{2^*} - |u^-|_{2^*}^{2^*} | \\
 & \leq c_1 \int_0^1 d\tau \int_\Omega \{ |u^-| \cdot |t\psi_\mu|^{2^*-1} + \tau^{2^*-2} \cdot |t\psi_\mu| \cdot |u^-|^{2^*-1} \} dx \leq \\
 & \leq c_2 \{ |t\psi_\mu|_{2^*-1}^{2^*-1} \cdot |u^-|_\infty + |t\psi_\mu|_1 \cdot |u^-|_\infty^{2^*-1} \} \leq \\
 & \leq c_3 \{ |t\psi_\mu|_{2^*-1}^{2^*-1} \cdot |u^-|_2 + |t\psi_\mu|_1 \cdot |u^-|_{2^*}^{2^*-1} \} \leq \\
 (2.10)a \quad & \leq e_3 \cdot t^{2^*-1} \cdot \mu^{(n-2)/4} |u^-|_2 + \frac{1}{4} |u^-|_{2^*}^{2^*} + c_4 \cdot t^{2^*} \cdot \mu^{n/2} \leq \\
 & \leq \frac{1}{2} |u^-|_{2^*}^{2^*} + k_4 t^{2^*} \cdot \mu^{n(n-2)/2n+4}
 \end{aligned}$$

and the lemma is proved.

LEMMA 2.3. — *If  $\mu$  is sufficiently small, then*

$$(2.11) \quad \frac{\|\psi_\mu\|^2 - \lambda |\psi_\mu|_2^2}{|\psi_\mu|_{2^*}^{2^*}} = \begin{cases} S - K_5 \mu + O(\mu^{\frac{n-2}{2}}) & \text{if } n \geq 5 \\ S + K_5 \mu \log \mu + O(\mu) & \text{if } n = 4 \end{cases} \quad (2.11)a$$

$$(2.11)b$$

*Proof.* — The evaluation (2.11) follows immediately by (2.2), (2.3) and (2.4).

REMARK 2.4. — Suppose that  $\lambda = \lambda_j$ , with  $\lambda_j \in \sigma(-\Delta)$  and denote by  $P_j$  the projector on the eigenspace  $M_j$  corresponding to  $\lambda_j$ .

We set

$$(2.12) \quad \tilde{\psi}_\mu = \psi_\mu - P_j \psi_\mu.$$

Let  $\{v_k\}$  an orthonormal family spanning  $M_j$ , then by (2.5) we have

$$(2.13) \quad |P_j \psi_\mu|_2^2 = \sum_k \left( \int_\Omega \psi_\mu v_k dx \right)^2 \leq \text{const} | \psi_\mu |_1^2 \leq K_6 \mu^{\frac{n-2}{2}}$$

then

$$(2.14) \quad |P_j \psi_\mu|_\infty \leq K_7 \mu^{\frac{n-2}{4}}.$$

Moreover we have

$$\begin{aligned} \left| \int_{\Omega} \{ |\tilde{\psi}_{\mu}|^{2^*} - |\psi_{\mu}|^{2^*} \} dx \right| &= 2^* \left| \int_0^1 d\tau \int_{\Omega} |\psi_{\mu} - \tau P_j \psi_{\mu}|^{2^*-2} (\psi_{\mu} - \tau P_j \psi_{\mu}) P_j \psi_{\mu} dx \right| \leq \\ &\leq 2^* \cdot 2^{2^*-1} \int_0^1 d\tau \int_{\Omega} \{ |\psi_{\mu}|^{2^*-1} + \tau^{2^*-1} |P_j \psi_{\mu}|^{2^*-1} \} |P_j \psi_{\mu}| dx \\ &\leq \text{const} \{ |\psi_{\mu}|_{2^*-1}^{2^*-1} \cdot |P_j \psi_{\mu}|_{\infty} + |P_j \psi_{\mu}|_{2^*}^{2^*} \}. \end{aligned}$$

Then by (2.14) and (2.6) it follows that

$$(2.15) \quad \left| |\tilde{\psi}_{\mu}|_{2^*}^{2^*} - |\psi_{\mu}|_{2^*}^{2^*} \right| \leq c_1 \mu^{\frac{n-2}{2}}.$$

Moreover by (2.14) and (2.6) we have

$$(2.16) \quad \begin{aligned} |\tilde{\psi}_{\mu}|_{2^*-1}^{2^*-1} = |\psi_{\mu} - P_j \psi_{\mu}|_{2^*-1}^{2^*-1} &\leq \text{const} \{ |\psi_{\mu}|_{2^*-1}^{2^*-1} + |P_j \psi_{\mu}|_{2^*-1}^{2^*-1} \} \\ &\leq \text{const} \mu^{\frac{n-2}{4}}. \end{aligned}$$

Analogously by (2.14) and (2.5) we have

$$(2.17) \quad |\tilde{\psi}_{\mu}|_1 \leq \text{const} \mu^{\frac{n-2}{4}}.$$

By (2.15), (2.16), (2.17) it easily follows that (2.11) holds if we replace  $\psi_{\mu}$  with  $\tilde{\psi}_{\mu}$ .

Moreover, by (2.15), (2.16), (2.17), also (2.7) holds (for  $\mu$  small) if we replace  $\psi_{\mu}$  with  $\tilde{\psi}_{\mu}$  and  $\overline{W}(\mu)$  with

$$\overline{\overline{W}}(\mu) = \{ u \in H_0^1 \mid u = u^- + t\tilde{\psi}_{\mu}, u^- \in H_2, t \in \mathbb{R} \}.$$

Now we can prove a crucial lemma:

LEMMA 2.5. — For  $\mu$  sufficiently small

$$(2.18) \quad \sup_{\mathbf{W}} f(u) < \frac{1}{n} S^{n/2}$$

where  $\mathbf{W} = \overline{W}(\mu)$  (resp.  $\overline{\overline{W}}(\mu)$ ) if  $\lambda \notin \sigma(-\Delta)$  (resp.  $\lambda \in \sigma(-\Delta)$ ).

*Proof.* — Observe that if we fix  $u \in H_0^1(\Omega)$ ,  $u \neq 0$ , then

$$(2.19) \quad \max_t f_{\lambda}(tu) = \frac{1}{n} \left( \frac{\|u\|^2 - \lambda |u|_2^2}{|u|_{2^*}^2} \right)^{n/2}.$$

Then in order to prove (2.18) we need to evaluate

$$(2.20) \quad \sup_{\substack{u \in \mathbf{W}(\mu) \\ |u|_{2^*} = 1}} \{ \|u\|^2 - \lambda |u|_2^2 \}.$$

We distinguish two cases:

Case i)  $\lambda \notin \sigma(-\Delta)$ .

Let  $u = u^- + t\psi_{\mu} \in \overline{W}(\mu)$  with  $|u|_{2^*} = 1$ .

Observe that  $t$  is bounded if  $\mu$  is small, in fact by (2.7) and (2.3) we get

$$1 = |u|_{2^*}^{2^*} \geq |t\psi_\mu|_{2^*}^{2^*} - K_4 t^{2^*} \mu^{n/2} + \frac{1}{2} |u^-|_{2^*}^{2^*} = t^{2^*} [S^{n/2} + O(\mu^{n/2})] + \frac{1}{2} |u^-|_{2^*}^{2^*}.$$

Then by (2.5) we have that

$$(2.21) \quad \|u\|^2 - \lambda |u|_2^2 = |\nabla u^-|_2^2 - \lambda |u^-|_2^2 + |\nabla t\psi_\mu|_2^2 - \lambda |t\psi_\mu|_2^2 - 2 \int_{\Omega} \{ |t\psi_\mu| |\Delta u^-| + \lambda |u^-| |t\psi_\mu| \} dx \leq \\ \leq |\nabla u^-|_2^2 - \lambda |u^-|_2^2 + |\nabla t\psi_\mu|_2^2 - \lambda |t\psi_\mu|_2^2 + c_1 \{ |\Delta u^-|_\infty |t\psi_\mu|_1 + |u^-|_\infty |t\psi_\mu|_1 \} \leq \\ \leq |\nabla u^-|_2^2 - \lambda |u^-|_2^2 + |\nabla t\psi_\mu|_2^2 - \lambda |t\psi_\mu|_2^2 + c_2 |u^-|_2 \cdot \mu^{\frac{n-2}{4}} \leq \\ \leq (\bar{\lambda} - \lambda) |u^-|_2^2 + \frac{|\nabla t\psi_\mu|_2^2 - \lambda |t\psi_\mu|_2^2}{|t\psi_\mu|_{2^*}^{2^*}} \cdot |t\psi_\mu|_{2^*}^{2^*} + c_2 |u^-|_2 \cdot \mu^{\frac{n-2}{4}}$$

where  $\bar{\lambda} = \max \{ \lambda_j \mid \lambda_j < \lambda \}$ .

We set  $A(u^-, \mu, c) = (\bar{\lambda} - \lambda) |u^-|_2^2 + C |u^-|_2 \mu^{\frac{n-2}{4}}$  and observe that

$$(2.22) \quad A(u^-, \mu, c) \leq 0 \quad \text{or} \quad A(u^-, \mu, c) \leq c^2 / (\lambda - \bar{\lambda}) \mu^{(n-2)/2}$$

If  $|u^-|_{2^*}^{2^*} \leq 2K_4 t^{2^*} \mu^{\frac{n(n-2)}{2n+4}}$ , by (2.10)a and the boundness of  $t$ ,

$$|t\psi_\mu|_{2^*}^{2^*} \leq \left( 1 - \frac{3}{4} |u^-|_{2^*}^{2^*} + c_3 \mu^{\frac{n-2}{4}} |u^-|_2 + c_4 \mu^{\frac{n}{2}} \right)^{\frac{2}{2^*}} \\ \leq 1 + \frac{2}{2^*} (c_3 \mu^{\frac{n-2}{4}} |u^-|_2 + c_4 \mu^{\frac{n}{2}})^{\frac{2}{2^*}},$$

then, if  $n \geq 5$ , by (2.11)a, (2.21)

$$(2.23) \quad \|u\|^2 - \lambda |u|_2^2 \leq (S - K_5 \mu + O(\mu^{\frac{n-2}{2}}))(1 + c_5 \mu^{n/2}) + A(u^-, \mu, c_6).$$

If  $|u^-|_{2^*}^{2^*} > 2K_4 t^{2^*} \mu^{\frac{n(n-2)}{2n+4}}$ , by (2.7),  $|t\psi_\mu|_{2^*} < 1$ , then, by (2.21)

$$(2.24) \quad \|u\|^2 - \lambda |u|_2^2 \leq (S - K_5 \mu + O(\mu^{\frac{n-2}{2}})) + A(u^-, \mu, c_2),$$

then, by (2.22), the conclusion follows in the case  $n \geq 5$ .

If  $n = 4$  the proof is the same. In this case (2.11)b replaces (2.11)a in (2.22).

Case ii)  $\lambda = \lambda_{\bar{j}} \in \sigma(-\Delta)$ .

Let  $u = u^- + t\tilde{\psi}_\mu \in \overline{W}(\mu)$  with  $|u|_{2^*} = 1$ . We set  $u = u^- + t\tilde{\psi}_\mu = \tilde{u} + P_j u^- + t\tilde{\psi}_\mu$ , then

$$\|u\|^2 - \lambda_{\bar{j}} |u|_2^2 = |t\nabla \tilde{\psi}_\mu|_2^2 - \lambda_{\bar{j}} |t\tilde{\psi}_\mu|_2^2 + |\nabla \tilde{u}|_2^2 - \lambda_{\bar{j}} |\tilde{u}|_2^2 - 2 \int_{\Omega} (t\tilde{\psi}_\mu \Delta u^- + \lambda_{\bar{j}} \tilde{\psi}_\mu u^-) dx.$$



Observe that

$$\begin{aligned} \int_{\Omega} (t\tilde{\psi}_{\mu}\Delta u^{-} + \lambda_{\bar{j}}t\tilde{\psi}_{\mu}u^{-})dx &= \int_{\Omega} (t\tilde{\psi}_{\mu}\Delta\tilde{u}_{-} + \lambda_{\bar{j}}t\tilde{\psi}_{\mu}\tilde{u}_{-})dx \leq \\ &\leq |\Delta\tilde{u}_{-}|_{\infty} |t\tilde{\psi}_{\mu}|_1 + \lambda_{\bar{j}}|\tilde{u}_{-}|_{\infty} |t\tilde{\psi}_{\mu}|_1 \leq c_3 |\tilde{u}_{-}|_2 \mu^{\frac{n-2}{4}}. \end{aligned}$$

Now the proof follows by using the previous arguments.

*Proof of theorem 0.1.* — If  $\lambda \notin \sigma(-\Delta)$  ( $\lambda > 0$ ) we set  $V = H_1$  and  $W = \bar{W}(\mu)$  with  $\mu$  suitably small in order that (2.18) is verified. We see that the assumptions of Theorem 1.2 are satisfied. Obviously  $(f_1)$  and  $(f_3.iii)$  are verified. Moreover  $(f_2)$  is verified with  $\beta = \frac{1}{n}S^{n/2}$  by lemma 1.1 and  $(f_3.i)$  (with  $\beta = \frac{1}{n}S^{n/2}$ ) is verified by lemma 2.5.

Finally observe that if  $u \in H_1$ , then

$$\begin{aligned} (2.25) \quad f_{\lambda}(u) &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} dx \geq \\ &\geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda^+} \right) \|u\|^2 - \frac{1}{2^*} \|u\|^{2^*} \geq \frac{1}{2} \left( \lambda - \frac{\lambda}{\lambda^+} \right) \|u\|^2 - \text{const} \|u\|^{2^*} \geq \delta > 0 \end{aligned}$$

if  $\|u\| = \rho$  with  $\rho$  suitably small.

Hence by (2.27) also  $(f_3.ii)$  is verified. Since  $\dim V \cap W = 1$  and  $V + W = H_0^1(\Omega)$ , then by Theorem 1.2, we deduce that problem (0.1) has at least one non trivial solution.

If  $\lambda \in \sigma(-\Delta)$  we set  $W = \bar{W}(\mu)$  with  $\mu$  suitably small in order that (2.18) is verified and, by repeating the above arguments, the conclusion follows.

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(Manuscrit reçu le 15 novembre 1984)

Annales de l'Institut Henri Poincaré - Analyse non linéaire