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Quasi-minima

by

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ABSTRACT. — The aim of this paper is to introduce the new notion of quasi-minima (Q-minima) of regular functionals in the calculus of variations.

The interest of this notion lies mainly in its unifying features; it includes among other things minima of variational integrals, solutions of elliptic partial differential equations and systems, quasi-regular mappings.

We prove some regularity results for Q-minima in L^p and $C^{0,\alpha}$-spaces as well as qualitative features: Liouville property, weak maximum principle, removal of singularities.

RÉSUMÉ. — Le but de cet article est d'introduire la notion de quasi-minima (Q-minima) de fonctionnelles régulières du calcul des variations.

L'intérêt principal de cette notion consiste en son caractère unificateur; elle contient, entre autres choses, les minima d'intégrales variationnelles, les solutions d'équations et de systèmes d'équations aux dérivées partielles de type elliptique, les applications quasi-régulières.

Nous démontrons des résultats de régularité pour les Q-minima dans les espaces L^p et $C^{0,\alpha}$; et aussi des propriétés qualitatives comme la propriété de Liouville, le principe du maximum faible, la suppression des singularités.

1. INTRODUCTION

Direct methods in the Calculus of Variations have been one of the principal tools in the theory of existence of minima of multiple integrals, and of solutions to elliptic differential equations and inequalities.

It would be impossible to mention the multifarious applications of the method; the essential idea being that we have a minimum whenever the class of competing functions can be endowed with a topology such that the functional in question is lower semicontinuous and there exists a convergent minimizing sequence.

To be definite, let us consider the functional

$$\mathcal{F}(u\,;\Omega) = \int_\Omega f(x, u(x), \mathrm{D}u(x))dx \tag{1.1}$$

in which Ω is a bounded domain in $\mathbb{R}^n$, $u = (u^1, u^2, \ldots, u^{\mathrm{N}})$ is a function mapping Ω into $\mathbb{R}^{\mathrm{N}}$, $\mathrm{N} \geq 1$, and $f(x, u, p)$ is a Carathéodory function, namely measurable in x for every (u, p) and continuous in (u, p) for almost all $x \in \Omega$.

It is well known that if in addition f is convex in p, then the functional (1.1) is lower semicontinuous in the weak topology $\mathrm{H}^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^{\mathrm{N}})$ for every $q > 1$. In this case, the solvability of a minimum problem for the functional (1.1) depends on the existence of a weakly convergent minimizing sequence (or, what is the same, of a bounded minimizing sequence) in $\mathrm{H}^{1,q}_{\mathrm{loc}}$. The existence of such a sequence depends on growth conditions on f and on the class of functions u competing for the minimum, and has been established in a variety of situations.

When we pass from the problem of existence to that of the regularity of minima, the main path goes through the Euler equation of $\mathcal{F}$, and the regularity theorem of De Giorgi [*4*] for solutions of elliptic equations with discontinuous coefficients.

De Giorgi's result has been improved by various authors. It will be outside our scope to discuss here the various contributions, and we refer to the book by Ladyzenskaya and Ural'tseva [*16*]. The main technical tool is here the introduction of the De Giorgi classes $\mathcal{B}_m$, introduced in [*4*] and then brought to the optimal generality in [*16*], and the proof of the Hölder-continuity of the functions in $\mathcal{B}_m$.

In a recent paper [*12*] we have proved that the same tools can be used to prove the Hölder-continuity of the minima of the functional $\mathcal{F}$ in a direct way, and without passing through its Euler equation. This permits to give regularity results when f is not differentiable or else—perhaps more interesting—without assuming growth conditions on the derivatives of f, i. e. when $\mathcal{F}$ is not Gateaux-differentiable. Additional regularity results are proved in [*13*].

At the end of the same paper [*12*] we introduced the notion of quasi-minima and we remarked that several results proved there for minima could be extended to quasi-minima.

Before recalling the definition of Q-minimum, we specify further our functional $\mathcal{F}$ by requiring that the function f satisfy the inequalities

$$|p|^m - b|u|^\gamma - g(x) \leq f(x, u, p) \leq \mu|p|^m + b|u|^\gamma + g(x) \tag{1.2}$$

where g is a given non-negative function, and m, γ, b, μ are non-negative constants satisfying

$$(1.3) \qquad m > 1 ; 1 \leq \gamma < \begin{cases} m^* = \dfrac{nm}{n-m} & \text{if } m < n \\ +\infty & \text{if } m \geq n \end{cases}$$

We shall limit ourselves to the case $m < n$; however all the significant results hold in the case $m \geq n$, with minor changes in the proofs.

DEFINITION 1.1. — *A function* $u \in H^{1,m}_{loc}(\Omega, \mathbb{R}^N)$ *is a* Q-*minimum*, $Q \geq 1$, *for the functional* $\mathcal{F}$ *if for every open set* $A \subset\subset \Omega$ *and for every* $v \in H^{1,m}_{loc}(\Omega, \mathbb{R}^N)$ *with* $v = u$ *outside* A *we have*

$$\mathcal{F}(u ; A) \leq Q\mathcal{F}(v ; A)$$

Since $\mathcal{F}$ *is an integral functional, u is a* Q-*minimum for* $\mathcal{F}$ *if and only if for every* $\phi \in H^{1,m}(\Omega, \mathbb{R}^m)$ *with* supp $\phi = K \subset \Omega$ *we have*

$$(1.4) \qquad \mathcal{F}(u ; K) \leq Q\mathcal{F}(u + \phi ; K)$$

The aim of this paper is twofold. On one side we shall discuss the notion of quasi-minimum, showing that it includes among other things solutions of elliptic equations and systems in divergence form, thus providing a unified treatment of minima of functionals and solutions of elliptic partial differential equations. On the other hand, we will show that a number of results proved by several authors for solutions of elliptic equations extend to quasi-minima.

This extension is not quite complete: some properties of elliptic equations—e. g. the comparison principle—are false for Q-minima; of others, first of all the Harnack's inequality, we have not been able to find a proof in dimension $n > 1$. Nevertheless, we believe that the new notion of Q-minimum may provide a better understanding of the behaviour of the solutions of partial differential equations, and of the minima of functionals.

2. QUASI-MINIMA AND ELLIPTIC SYSTEMS

We begin our discussion with the remark that in many cases it is sufficient to consider the simple functional

$$(2.1) \qquad \mathcal{I}(u ; \Omega) = \int_\Omega (|Du|^m + b|u|^\gamma + h)dx$$

Actually, every quasi-minimum $u \in H^{1,m}(\Omega, \mathbb{R}^N)$ of the functional (1.1) with conditions (1.2) and (1.3) is a quasi-minimum (with a different constant) of (2.1) where $h = g + 1$.

To see that, let $v \in H^{1,m}(\Omega, \mathbb{R}^N)$ with $S = \operatorname{supp}(u - v) \subset \Omega$. We have

$$(2.2)\quad \int_S |Du|^m dx \leq Q \int_S (|Dv|^m + b|v|^\gamma + g)dx + \int_S (b|u|^\gamma + g)dx$$

On the other hand

$$(2.3)\quad |u|^\gamma \leq c_1(\gamma)(|v|^\gamma + |v-u|^\gamma) \leq \varepsilon |v-u|^{m^*} + c_2(\varepsilon, \gamma) + c_1(\gamma)|v|^\gamma$$

From the Sobolev imbedding theorem:

$$(2.4)\quad \int_S |u-v|^{m^*} dx \leq c_3 \left\{ \int_S (|Du|^m + |Dv|^m)dx \right\}^{\frac{m^*}{m}-1} \int_S (|Du|^m + |Dv|^m)dx$$

We remark now that we can suppose that

$$(2.5)\quad \int_S (|Du|^m + b|u|^\gamma)dx > \int_S (|Du|^m + |Dv|^m)dx$$

since otherwise we have trivially

$$\mathscr{I}(u; S) \leq \mathscr{I}(v; S)$$

From (2.5) and (2.4) we get

$$\int |u - v|^{m^*} dx \leq c_3 \left\{ \int_S (2|Du|^m + b|u|^\gamma)dx \right\}^{\frac{m^*}{m}-1} \int (|Du|^m + |Dv|^m)dx$$

$$\leq c_4(|Du|_{L^m(\Omega)}, |u|_{L^\gamma(\Omega)}) \int (|Du|^m + |Dv|^\gamma)dx$$

Taking $\varepsilon > 0$ small enough in (2.3), the result now follows from (2.2).

In particular, if $b = g = 0$ and $m = 2$, every quasi-minimum of the functional (1.1) with

$$|p|^2 \leq f(x, u, p) \leq \mu |p|^2$$

is a quasi-minimum of the Dirichlet integral.

To the same integral we reduce in the case of solutions of linear elliptic equations (and systems) in divergence form:

$$(2.7)\quad D_\alpha(a^{\alpha\beta}(x)D_\beta u) = 0$$

with measurable coefficients $a^{\alpha\beta}$ satisfying:

$$(2.8)\quad \begin{aligned} &a^{\alpha\beta}(x)\xi_\alpha\xi_\beta \leq \nu |\xi|^2 \qquad \forall \xi \in \mathbb{R}^n; \quad \nu > 0 \\ &|a^{\alpha\beta}| \leq L \end{aligned}$$

(as usual, summation over repeated indices is understood). Let $u \in H^{1,2}_{loc}(\Omega)$ be a solution of (2.7), and let $v \in H^{1,2}_{loc}(\Omega)$ with $S = \operatorname{supp}(u - v) \subset \Omega$.

Multiplying (2.7) by $u - v$ and integrating by parts we get

$$\nu \int_S |Du|^2 dx \leq \int_S a^{\alpha\beta} D_\alpha u D_\beta u dx = \int_S a^{\alpha\beta} D_\beta u D_\alpha v dx$$
$$\leq c_5 \left(\int_S |Du|^2 dx \right)^{\frac{1}{2}} \left(\int_S |Dv|^2 dx \right)^{\frac{1}{2}}$$

and therefore

$$\int_S |Du|^2 dx \leq Q \int_S |Dv|^2 dx$$

so that u is a Q-minimum of the Dirichlet integral.

The same argument works for solutions of quasilinear elliptic systems in divergence form:

$$\text{(2.9)} \qquad D_\alpha A_i^\alpha(x, u, Du) = B_i(x, u, Du) \qquad i = 1, 2, \ldots, N.$$

We shall suppose that the system (2.9) is elliptic in the sense that

$$\text{(2.10)} \quad \begin{cases} A_i^\alpha(x, u, p)p_\alpha^i \geq |p|^m - b|u|^\gamma - f(x) & m \leq \gamma < m^* \\ |A(x, u, p)| \leq L|p|^{m-1} + b|u|^\sigma + g(x) & \sigma = \gamma \dfrac{m-1}{m} \end{cases}$$

We will distinguish two cases, depending on the behaviour of B. The simplest situation is that of *non-natural* or *controlled growth conditions:*

$$\text{(2.11)} \qquad |B(x, u, p)| \leq L|p|^\tau + b|u|^\delta + l(x)$$

with
$$\tau = \frac{\gamma - 1}{\gamma} m, \qquad \delta = \gamma - 1.$$

We have:

THEOREM 2.1. — *Let* $u \in H^{1,m}_{\text{loc}}(\Omega, \mathbb{R}^N)$ *be a weak solution of the system* (2.9); *with condition* (2.10) *and* (2.11). *Then u is a quasi-minimum of the functional*

$$\mathscr{I}(u; \Omega) = \int_\Omega (|Du|^m + b|u|^\gamma + h(x))dx$$

with

$$h(x) = f(x) + g(x)^{\frac{m}{m-1}} + l(x)^{\frac{m^*}{m^*-1}}$$

Proof. — Let v be a function in $H^{1,m}(\Omega, \mathbb{R}^N)$ with $S = \operatorname{supp}(u - v) \subset \Omega$. Multiplying (2.9) by $(u - v)$ and integrating by parts we get

$$\int_S A_i^\alpha(x, u, Du) D_\alpha u^i dx = \int_S A_i^\alpha(x, u, Du) D_\alpha v^i dx + \int_S B_i(x, u, Du)(v^i - u^i) dx$$

and therefore, using (2.10) and (2.11):

$$\int |Du|^m dx \leq b\int |u|^\gamma dx + \int f dx + L\int |Du|^{m-1}|Dv|\,dx$$
$$+ b\int |u|^\sigma |Dv|\,dx + \int g|Dv|\,dx + L\int |Du|^\tau |u-v|\,dx$$
$$+ b\int |u|^\delta |u-v|\,dx + \int l|u-v|\,dx$$

all the integrals being taken on S.
Now:

$$\int |Du|^{m-1}|Dv|\,dx \leq \varepsilon\int |Du|^m dx + c(\varepsilon)\int |Dv|^m dx$$
$$\int |u|^\sigma |Dv|\,dx \leq c\left\{\int |Dv|^m dx + \int |u|^\gamma dx\right\}$$
$$\int g|Dv|\,dx \leq c\left\{\int |Dv|^m dx + \int g^{\frac{m}{m-1}} dx\right\}$$
$$\int |Du|^\tau |u-v|\,dx \leq \varepsilon\int |Du|^m dx + c(\varepsilon)\int |u-v|^\gamma dx$$
$$\int |u|^\delta |u-v|\,dx \leq c\left\{\int |u|^\gamma dx + \int |u-v|^\gamma dx\right\}$$
$$\int l|u-v|\,dx \leq \varepsilon\int |u-v|^{m^*} dx + c(\varepsilon)\int l^{\frac{m^*}{m^*-1}} dx$$

and the result follows arguing as above. q. e. d.

More complex is the case of *natural growth conditions.* In this case we consider bounded weak solutions of the system (2.9):

(2.12) $$|u(x)| \leq M$$

with right-hand side B satisfying

(2.13) $$|B(x, u, p)| \leq a|p|^m + l(x)$$

where the constant a and the function l may depend on M. We can also take $b = 0$ in (2.10) by allowing the functions f, g, and the constant L, to depend on M.

We shall consider separately the case of a single equation (N = 1) and of a system of equations (N > 1).

THEOREM 2.2. — *Let u be a bounded weak solution of equation* (2.9)

(N = 1), *and suppose that* (2.10), (2.12) *and* (2.13) *hold. Then u is a quasi-minimum of the functional*

$$\mathscr{I}(u; \Omega) = \int_\Omega (|Du|^m + h(x))dx \tag{2.14}$$

with

$$h = f + g^{\frac{m}{m-1}} + l$$

Proof. — Let $v \in H^{1,m}(\Omega)$ with $|v| \leq M$ and $S = \operatorname{supp}(u - v) \subset \Omega$. If we multiply both sides of (2.9) by $\phi = (u - v)^+ \; e^{a(u-v)}$, where $\alpha^+ = \max(\alpha, 0)$, and we integrate by parts, we obtain

$$\int A^\alpha D_\alpha u\,[a(u - v) + 1]e^{a(u-v)}dx = \int A^\alpha D_\alpha v\,[a(u - v) + 1]e^{a(u-v)}dx - \int B(u - v)e^{a(u-v)}dx$$

where all integrals are taken on $S^+ = \operatorname{supp}\phi \subset S$ and the coefficients A^α and B are computed at (x, u, Du). Using (2.10) and (2.13) we get

$$\int |Du|^m e^{a(u-v)}dx \leq \int f\,[a(u - v) + 1]e^{a(u-v)}dx + \int (L\,|Du|^{m-1} + g)\,[a(u - v) + 1]e^{a(u-v)}\,|Dv|\,dx + \int l(x)(u - v)e^{a(u-v)}dx\,.$$

Recalling that $|u - v| \leq 2M$ we easily conclude that

$$\mathscr{I}(u; S^+) \leq Q\mathscr{I}(v; S^+)$$

for some constant Q depending on M.

In a similar way, taking $\phi = (v - u)^+ e^{a(v-u)}$ we get $\mathscr{I}(u; S^-) \leq Q\mathscr{I}(v; S^-)$ and therefore

$$\mathscr{I}(u; S) \leq Q\mathscr{I}(v; S) \tag{2.15}$$

provided $|v| \leq M$.

If now w is a generic function of $H^{1,m}(\Omega)$, setting $v = \min\{M, \max(w, -M)\}$ we have $|v| \leq M$ and therefore (2.15) holds. On the other hand $|Dv| \leq |Dw|$ and hence

$$\mathscr{I}(u; S) \leq Q\mathscr{I}(v; S) \leq Q\mathscr{I}(w; S)$$

so that u is a Q-minimum for $\mathscr{I}$. q. e. d.

Let us consider now elliptic systems. In this case, theorem 2.2 cannot hold without further assumptions. In fact, bounded solutions can be singular even in dimension $n = 2$ and with $m = 2$ [7], while Q-minima have first derivatives higher integrable, i. e. $Du \in L^r_{loc}$ for some $r > 2$ (see theorem 3.1 later) and therefore are Hölder-continuous in dimension

$n = 2$. Moreover, a simple modification of Frehse's example [15] shows that (2.16) below is necessary except perhaps for the factor 2.
We have

THEOREM 2.3. — *Let u be a bounded solution of the system* (2.9). *Suppose that* (2.10), (2.12) *and* (2.13) *hold, and moreover*

$$2\mathrm{M}a(\mathrm{M}) < 1\,. \tag{2.16}$$

Then u is a quasi-minimum for the functional (2.14).

Proof. — Let as usual $v \in \mathrm{H}^{1,m}(\Omega, \mathbb{R}^{\mathrm{N}})$ with $\mathrm{S} = \operatorname{supp}(u - v) \subset \Omega$. If $|v| \leq \mathrm{M}$ it is sufficient to integrate by parts the left-hand side of (2.9) after multiplication by $(u - v)$. The term involving $|\mathrm{D}u|^m$ on the right-hand side can be estimated using (2.16).
In general we set

$$w = \begin{cases} v & \text{if } |v| \leq \mathrm{M} \\ \mathrm{M}\dfrac{v}{|v|} & \text{if } |v| > \mathrm{M} \end{cases}$$

We have $|w| \leq \mathrm{M}$, and $|\mathrm{D}w| \leq 2|\mathrm{D}v|$ therefore

$$\mathscr{I}(u\,;\mathrm{S}) \leq \mathrm{Q}\mathscr{I}(w\,;\mathrm{S}) \leq \mathrm{Q}2^m\mathscr{I}(v\,;\mathrm{S})$$

q. e. d.

The above results cover most of the cases of elliptic equations and systems studied in the literature, with the obvious exception of those systems in which the special structure of the coefficients enters in an essential way, as for instance diagonal systems, coefficients depending only on $|\mathrm{D}u|$ and so on. It is clear from the above that all the peculiarities depending on the structure or else on the continuity of the coefficients cannot be preserved when passing to quasi-minima.

By consequence, all the results that we shall obtain for Q-minima hold for solutions of elliptic systems (or equations) with discontinuous coefficients. This explains why most of the results of the next section are valid only in the scalar case ($\mathrm{N} = 1$), but also why the results for the general vector case ($\mathrm{N} > 1$) are perhaps more subtle and interesting.

Before passing to the properties of quasi-minima, let us discuss some additional examples.

a) **Variational inequalities with obstacles.**

Let $\psi \in \mathrm{H}^{1,m}(\Omega)$, and let $u \geq \psi$ be such that

$$\int \{ \mathrm{A}^\alpha(x, u, \mathrm{D}u)\mathrm{D}_\alpha(u - v) + \mathrm{B}(x, u, \mathrm{D}u)(u - v) \}\, dx \leq 0$$

for every $v \in \mathrm{H}^{1,m}(\Omega)$, $v \geq \psi$.

We use again the method of theorem 2.2 in order to show that if supp $(v - u) = S \subset \Omega$ and $v \geq \psi$ we have

$$\mathcal{I}(u; S) \leq Q \mathcal{I}(v; S). \tag{2.17}$$

We want to drop now the condition $v \geq \psi$. For that, let $w = \max(v, \psi)$. We have $w \geq \psi$ and supp $w \subset S$, so that (2.17) holds for w. On the other hand $|Dw| \leq |Dv| + |D\psi|$ and therefore u is a quasi-minimum of the functional

$$\mathcal{I}(u; \Omega) + \int_\Omega |D\psi|^m dx.$$

b) Quasi-regular mappings.

We recall that a mapping $u: \Omega \to \mathbb{R}^n$ is called *quasi-regular* if there exists a constant k such that for almost every $x \in \Omega$

$$|Du|^n \leq k \det(Du) \tag{2.18}$$

if in addition u is a homeomorphism, it is called *quasi-conformal* (see e. g. [*21*] [*9*]).
We have

THEOREM 2.4. — *A quasi-regular mapping is a quasi-minimum of the functional*

$$\int_\Omega |Du|^n dx$$

Proof. — We remark that

$$\int_\Omega \det(D\phi) dx = 0 \tag{2.19}$$

for every $\phi \in H_0^{1,m}(\Omega, \mathbb{R}^n)$. Moreover, for every $n \times n$ matrices A and B we have

$$\det(A + B) \leq \det B + c \sum_{i=0}^{n-1} \|A\|^{n-i} \|B\|^i$$

where, if $A = (a_{ij})$, $\|A\|^2 = \sum_{ij} a_{ij}^2$.

Taking $A = Dv$ and $B = D(u - v)$ we get

$$\det(Du) \leq \det(D(u - v)) + c \sum_{i=0}^{n-1} |Dv|^{n-i} |D(u - v)|^i$$

$$\leq \det(D(u - v)) + c \sum_{i=0}^{n-1} |Dv|^{n-i} (|Du|^i + |Dv|^i)$$

Set now $S = \operatorname{supp}(u - v) \subset \Omega$; integrating over S and taking into account (2.18), we get

$$\int_S |Du|^n dx \leq k \int_S \det(Du) dx \leq k_1 \left\{ \sum_{i=0}^{n-1} \int_S |Dv|^{n-i} |Du|^i dx + \int_S |Dv|^n dx \right\}$$

and the conclusion follows from the standard inequality

$$|Du|^i |Dv|^{n-i} \leq \varepsilon |Du|^n + c(\varepsilon) |Dv|^n .$$

q. e. d.

c) Quasi-minima in one independent variable.

We shall consider here the case where Ω is an interval in $\mathbb{R}$, and u is a quasi-minimum of the integral

$$\int |u'|^2 dx \tag{2.20}$$

It is true, in general, that it is sufficient to satisfy inequality (1.4) when the support of ϕ is connected. In our case that means that we can consider only variations whose support is a subinterval $[a, b] \subset \Omega$. Moreover, since the Dirichlet integral

$$\int_a^b |v'|^2 dx$$

is minimum when v' is constant, we can conclude that u is a Q-minimum of (2.20) if and only if

$$\int_a^b |u'|^2 dx \leq Q \frac{[u(b) - u(a)]^2}{b - a} \tag{2.21}$$

for every interval $[a, b] \subset \Omega$, or, what is the same

$$\fint_a^b |u'|^2 dx \leq Q \left(\fint_a^b u' dx \right)^2 \tag{2.22}$$

where

$$\fint_E f dx = \frac{1}{|E|} \int_E f dx$$

It follows at once from (2.21) that a quasi-minimum u must be a monotone function in Ω; to be definite we shall suppose that u is non-decreasing.

The inequality (2.22) is a reverse Hölder inequality, with the integral computed on the same set. This is a very strong inequality, and implies in particular that u' cannot have a zero of infinite order at a point x_0, without being identically zero near x_0 ([10], chap. 5, Proposition 1.3). If $u \in C^\infty$ this means that u must be either strictly increasing or constant. This is actually true in general for $u \in H^{1,2}(\Omega)$. Let us suppose on the contrary that $u = 0$ for $x \leq 0$ and $u > 0$ for $x > 0$, and let $a < 0 < b$. We have from (2.22):

$$\frac{1}{b-a}\int_0^b u'^2 dx \leq Q\left(\frac{1}{b-a}\int_0^b u' dx\right)^2 \leq \frac{Qb}{(b-a)^2}\int_0^b u'^2 dx$$

Letting $b \to 0$ we get a contradiction.

We have thus proved that *if $u(x)$ is a quasi-minimum for the integral* (2.20) *then* (*i*) *u is strictly monotone and* (*ii*) *u' has no zeros of infinite order.*

We remember that a non-negative function f has a zero of order $> s$ at x_0 if

$$\lim_{R \to 0} R^{-s} \fint_{B_R(x_0)} f dx = 0$$

The order of the zero is defined as the supremum of such s. The proof of the proposition 1.3 of [10] gives also a bound for the order of the zeros of u' in terms of the constant Q in (2.22).

In general, conditions (*i*) and (*ii*) are not sufficient to ensure that u is a Q-minimum of (2.20). To see that, define

$$u(x) = \begin{cases} x^{2k+1} & \text{if } x \geq 0 \\ x^{2h+1} & \text{if } x < 0 \end{cases} \qquad k > h > 0$$

and let $a < 0 < b$. It is a matter of calculation to show that if we let $a, b \to 0$ in such a way that $a/b \to 0$ and $b^{2k+1}/a^{2h+1} \to 0$ (for instance $a = -b^{1+\tau}$ with $0 < \tau < \dfrac{2(k-h)}{2h+1}$) the ratio

$$\frac{(b-a)\displaystyle\int_a^b u'^2 dx}{[u(b)-u(a)]^2} \tag{2.23}$$

tends to $+\infty$ and therefore u cannot be a quasi-minimum of (2.20).

In the example the different behaviour of the derivative on the two sides of the origin enters in an essential way. In fact we have

PROPOSITION 2.5. — *Let $u(x)$ be a strictly increasing function in a bounded interval* $[-L, L]$, *such that u' is bounded and essentially different from zero for $x \neq 0$. Suppose furthermore that there exist positive constants* A, B, *k and δ such that*

$$Ax^{2k} \leq u'(x) \leq Bx^{2k} \tag{2.24}$$

for $|x| < 2\delta$. Then u is a quasi-minimum for $\int u'^2 dx$.

Proof. — Let $I = [a, b]$ be any subinterval of $[-L, L]$. We split the proof in three cases as follows

(1) $|I| = b - a > \delta$. Since u is increasing there exists $\varepsilon_0 > 0$ such that $u(b) - u(a) > \varepsilon_0$. We have therefore

$$\int_I u'^2 dx \leq \frac{2LM^2}{\varepsilon_0^2} \frac{[u(b) - u(a)]^2}{b - a}$$

M being a bound for u'.

(2) $I \cap \{ |x| < \delta \} = \phi$. Here we use the hypothesis that u' is bounded away from zero far from the origin. We have

$$\varepsilon_1 < u'(x) \leq M \qquad \forall x \in I$$

for some $\varepsilon_1 > 0$, and therefore

$$\fint_I u'^2 dx \leq \frac{M}{\varepsilon_1}\left(\fint_I u' dx\right)^2$$

(3) $I \subset \{ |x| < 2\delta \}$. We can use here (2.24) getting

$$\int_a^b u'^2 dx \leq \frac{B}{4k+1}(b^{4k+1} - a^{4k+1})$$

$$u(b) - u(a) = \int_a^b u' dx \geq \frac{A}{2k+1}(b^{2k+1} - a^{2k+1})$$

and hence the ratio (2.23) is bounded by some constant Q depending only on k, A and B. q. e. d.

It is clear that the same method will work if u' has a finite number of zeros, and satisfies inequalities of the type of (2.24) near each of them.

In particular this is the case if u is C^∞ and u' has no zero of infinite order, so that conditions (*i*) and (*ii*) are sufficient in this case.

Finally, in the one-dimensional case we can prove the Harnack's inequality:

PROPOSITION 2.6. — *Let u be a non-negative* Q*-minimum for the functional* (2.20) *in* $\Omega = [0, 1]$, *and let* $x_0 \in \Omega$ *and* $R < \frac{1}{2Q}$ dist $(x_0, \partial\Omega)$. *Then*

$$\sup_{B(x_0,R)} u \leqslant \frac{1}{1 - \left[\dfrac{2QR}{\operatorname{dist}(x_0, \partial\Omega)}\right]^{1/2}} \inf_{B(x_0,R)} u .$$

Proof. — Changing possibly $u(t)$ into $u(1 - t)$ we can suppose that u is increasing; moreover we can assume that $u(0) = 0$. If $0 \leqslant a < b \leqslant 1$ we have

$$\int_a^b u'dt \leqslant (b-a)^{1/2}\left(\int_a^b u'^2 dt\right)^{1/2}$$
$$\leqslant (b-a)^{1/2}\left(\int_0^b u'^2 dt\right)^{1/2} \leqslant \mathrm{Q}^{1/2}(1-a/b)^{1/2}\int_0^b u'dt$$

where we have used (2.22) in the last step.

If a and b are such that $\beta^2 = \mathrm{Q}(1-a/b) < 1$ we get easily:

$$\int_0^b u'dt \leqslant \int_0^a u'dt + \beta\int_0^b u'dt$$

and therefore

$$u(b) = \int_0^b u'dt \leqslant \frac{1}{1-\beta}u(a).$$

In particular, taking $(a, b) = (x_0 - \mathrm{R}, x_0 + \mathrm{R})$ we get easily the conclusion since dist $(x_0, \partial\Omega) \leqslant x_0$.

d) Spherical quasi-minima.

We can define *spherical quasi-minima* $u(x)$ by the requirement that for every ball $\mathrm{B} \subset \Omega$ we have

$$(2.25) \qquad \int_{\mathrm{B}} |\,\mathrm{D}u\,|^2 dx \leq \mathrm{Q}\int_{\mathrm{B}} |\,\mathrm{D}v\,|^2 dx$$

where v is the harmonic function coinciding with u on $\partial\mathrm{B}$. Since such a v minimizes the Dirichlet integral, the inequality (2.25) holds for every function v agreeing with u on $\partial\mathrm{B}$.

We have observed that in the case of one independent variable this is equivalent to our original definition 1.1. On the other hand, when the independent variables are two or more, this is not true anymore, as we show by the following example.

We take a function $u(x)$, homogeneous of degree β and smooth in $\mathbb{R}^n - \{0\}$. We suppose further that u has no stationary points, and that u is not constant on the boundary of any ball. A function with these properties is for instance

$$u(x', x_n) = (\,|\,x'\,|^2 + ax_n^2)^{\beta/2}$$

where $x' = (x_1, \ldots, x_{n-1})$ and $0 < a < 1$. We shall prove that $u(x)$ satisfies (2.25) for some constant Q.

We shall observe first that if v is harmonic in B_R and $v = u$ on ∂B_R we have (see, e. g. [*20*])

$$(2.26)\quad R\int_{\partial B_R}\int_{\partial B_R}\frac{|u(x)-u(z)|^2}{|x-z|^{n+1}}\,dH_{n-1}(x)dH_{n-1}(z)\le c_1\int_{B_R}|Dv|^2dx.$$

It will be therefore sufficient to show that

$$\int_{B_R}|Du|^2dx\le c_2R\int_{\partial B_R}\int_{\partial B_R}\frac{|u(x)-u(z)|^2}{|x-z|^{n+1}}\,dH_{n-1}(x)dH_{n-1}(z)$$

for any ball $B_R = B_R(x_0) \subset \Omega$.

We can reduce to $R = 1$ by setting $x_0 = Ry_0$ and $x = R(y_0 + y)$; taking into account the homogeneity of u and of its gradient the above inequality becomes

$$\int_B|Du(y_0+y)|^2dy\le c_2\int_{\partial B}\int_{\partial B}\frac{|u(y_0+y)-u(y_0+w)|^2}{|y-w|^{n+1}}\,dH_{n-1}(y)dH_{n-1}(w).$$

Moreover, since $|y - w| \le 2$ it will be sufficient to show that

$$\int_B|Du(y_0+y)|^2dy\le c_3\int_{\partial B}\int_{\partial B}|u(y_0+y)-u(y_0+w)|^2dH_{n-1}(y)dH_{n-1}(w)$$

or, what is the same:

$$(2.27)\quad \int_B|Du(y_0+y)|^2dy$$
$$\le c_4\left\{ \fint_{\partial B}u^2(y_0+y)dH_{n-1}(y)-\left(\fint_{\partial B}u(y_0+y)dH_{n-1}(y)\right)^2\right\}.$$

Let us call $F(y_0)$ the left hand-side of (2.27), and $G(y_0)$ the quantity within brackets on the right-hand side. F and G are continuous functions in $\mathbb{R}^n$, and G is *strictly* positive since u is not constant on $\partial B_1(y_0)$.
The ratio F/G is therefore bounded on compact sets, and we have only to estimate it for large $|y_0|$.

We have for $|y| \le 1$:

$$u(y_0+y)=u(y_0)+\langle Du(y_0),y\rangle+\frac{1}{2}\langle D^2u(y_0)y,y\rangle+0(|y_0|^{\beta-3})$$

$$Du(y_0+y)=Du(y_0)+0(|y_0|^{\beta-2}).$$

From the last equation we get

$$F(y_0)=\int_B|Du(y_0+y)|^2dy=|B|\,|Du(y_0)|^2+0(|y_0|^{2\beta-3})$$

whereas from the first

$$\fint_{\partial B} u(y_0+y)dH_{n-1}(y)=u(y_0)+\frac{1}{2}\fint_{\partial B}\langle D^2u(y_0)y,y\rangle dH_{n-1}(y)+0(|y_0|^{\beta-3}).$$

On the other hand

$$u^2(y_0+y)=u^2(y_0)+\langle Du(y_0),y\rangle^2+2u(y_0)\langle Du(y_0),y\rangle$$
$$+u(y_0)\langle D^2u(y_0)y,y\rangle+0(|y_0|^{2\beta-3})$$

and therefore

$$G(y_0)=\fint_{\partial B}\langle Du(y_0),y\rangle^2 dH_{n-1}(y)+0(|y_0|^{2\beta-3})=c_5|Du(y_0)|^2+0(|y_0|^{2\beta-3})$$

We remark now that since Du is never zero we have

$$|Du(y_0)|\geq c_6|y_0|^{\beta-1}$$

and therefore $F/G\leq c_7$ if $|y_0|$ is sufficiently large, thus proving (2.27).

The function u is therefore a spherical quasi-minimum for the Dirichlet integral. On the other hand it is not a quasi-minimum, since it does not have the properties stated in theorem 4.3 (maximum principle) or in theorem 4.1 (local boundedness).

We remark that the requirement that $u\in H^{1,2}(\Omega)$ implies that $\beta>1-\frac{n}{2}$; and therefore our u is bounded (and even Hölder-continuous) if $n=2$ and may be unbounded if $n\geq 3$. This is not a coincidence, as will be clear from theorem 3.1.

e) General growth conditions.

More generally, we can consider integrands $f(x,u,p)$ satisfying instead of (2.1) the condition

(2.28) $\varphi(|u|)^m|p|^m-\theta(x,u)\leq f(x,u,p)\leq\mu\varphi(|u|)^m|p|^m+\theta(x,u)$

where φ and θ are positive functions such that

(2.29) $$\int_0^\infty\varphi(t)dt=+\infty$$

(2.30) $$|\theta(x,u)|\leq b\Phi^\alpha(|u|)+g(x)$$

with $\alpha<m^*$ and

$$\Phi(t)=\int_0^t\varphi(s)ds.$$

In this case, we reduce immediately to (2.1) setting

(2.31) $$w=\eta(|u|)u$$

with $\eta(t) = t^{-1}\Phi(t)$. In this way we have

$$\mathrm{D}w = \varphi(|u|)\mathrm{D}u \tag{2.32}$$

and therefore if u is a Q-minimum of the functional

$$\mathscr{F}(u;\Omega) = \int_\Omega f(x, u, \mathrm{D}u)dx$$

the transformed function w is a Q-minimum of

$$\mathscr{F}^*(w;\Omega) = \int_\Omega f^*(x, w, \mathrm{D}w)dx$$

where

$$f^*(x, w, \mathrm{D}w) = f(x, u, \mathrm{D}u)$$

satisfies

$$|q|^m - b|w|^\alpha - g \leq f^*(x, w, q) \leq \mu|q|^m + b|w|^\alpha + g\,. \tag{2.33}$$

We note that no assumption has been made on the function $\varphi(t)$, except (2.29) and $\varphi > 0$ in $(0, +\infty)$. In particular φ can have arbitrary growth, and can be zero for $t = 0$.

In the vector case some complications arise from the fact that instead of (2.32) we have

$$\mathrm{D}w^h = \eta(|u|)\left(\delta^{hi} + \frac{|u|\eta'(|u|)}{\eta(|u|)}\frac{u^hu^i}{|u|^2}\right)\mathrm{D}u^i$$

Since

$$\mathrm{D}u^i = \frac{1}{\eta(|u|)}\left(\delta^{ih} - \frac{|u|\eta'(|u|)}{\varphi(|u|)}\frac{u^iu^h}{|u|^2}\right)\mathrm{D}w^h$$

we can set as above

$$f^*(x, w, \mathrm{D}w) = f(x, u, \mathrm{D}u)\,.$$

Let

$$\mathrm{B}^{ih} = \delta^{ih} - \frac{|u|\eta' u^iu^h}{\varphi\quad|u|^2}$$

Remarking that $t\eta' = \varphi - \eta$ we get

$$|\mathrm{B}q|^2 = |q|^2 - \left(1 - \frac{\eta^2}{\varphi^2}\right)\frac{(u, q)^2}{|u|^2}$$

and therefore

$$|q|^2 \min\left\{1, \frac{\eta^2}{\varphi^2}\right\} \leq |\mathrm{B}q|^2 \leq |q|^2 \max\left\{1, \frac{\eta^2}{\varphi^2}\right\}.$$

In conclusion, we have

$$|\mathrm{D}w| \min\left\{1, \frac{\varphi}{\eta}\right\} \leq \phi(|u|)|\mathrm{D}u| \leq |\mathrm{D}w| \max\left\{1, \frac{\varphi}{\eta}\right\}$$

and therefore f^* satisfies (2.33) if

$$0 < \sigma \leq \frac{\varphi(t)}{\eta(t)} = \frac{t\varphi(t)}{\Phi(t)} \leq \beta .$$

We remark that the first inequality is not really restrictive. Actually it is satisfied if φ has only a finite number of relative maxima and minima near zero and if

$$\lim_{t \to \infty} \varphi(t) t^{1-\alpha} = +\infty$$

for some $\alpha > 0$, a condition similar to (2.29). On the contrary, the second inequality is equivalent to the requirement that $t^{-\beta}\Phi(t)$ decreases. In particular, it is satisfied if for some positive γ we have

$$t^{-\gamma}\varphi(t) \to \begin{cases} 0 & t \to \infty \\ +\infty & t \to 0 \end{cases} .$$

3. REGULARITY OF QUASI-MINIMA, N ≥ 1

In this and in the next section we shall prove a number of regularity results for quasi-minima of functionals. Most of these results are already known for solutions of elliptic equations and systems; others have recently been proved for minima of variational integrals [*12*].

Since solutions of elliptic equations and systems with measurable coefficients are quasi-minima of the Dirichlet integral, we can expect at most an extension of the results valid for such solutions. This is particularly restrictive for the case of vector-valued functions (N > 1), since J. Souček has shown with an example (see [*11*]) that solutions to linear elliptic systems with measurable coefficients may have singularities on a dense set.

The following theorem, essentially the only general result for vector valued quasi-minima, has a long story. It was proved first by B. V. Boyarskii [*2*] [*3*] for solutions to first order elliptic systems of Beltrami's type in dimension two. Later, N. G. Meyers [*18*] proved the same conclusion for solutions of second order linear elliptic equations with L^∞ coefficients in arbitrary dimension. More recently, F. W. Gehring [*9*] proved higher integrability of the derivatives of quasi-conformal mappings, using different methods. Gehring's techniques have been improved by M. Giaquinta and G. Modica [*14*], who proved a similar result for solutions of non-linear elliptic systems (see also N. G. Meyers and E. Elcrat [*19*]).

THEOREM 3.1. — *Let $u \in H^{1,m}(\Omega, \mathbb{R}^N)$ be a spherical quasi-minimum of the functional*

$$\mathcal{F}(u;\Omega) = \int_\Omega f(x, u, Du)dx \tag{3.1}$$

with f satisfying

$$|p|^m - b|u|^\gamma - g \le f(x, u, p) \le \mu |p|^m + b|u|^\gamma + g, \tag{3.2}$$

$$1 < m \le \gamma < m^* = \frac{nm}{n-m};\ g \in L^s(\Omega),\ s > 1. \tag{3.3}$$

Then there exists an exponent $r > m$ such that $u \in H^{1,r}_{loc}(\Omega, \mathbb{R}^N)$. Moreover, there exists $R_0 > 0$, depending on $\int_\Omega |Du|^m dx$ and on the various constants appearing in (3.2) *and* (3.3) ($R_0 = +\infty$ *if* $b = 0$) *such that for every ball* $B_R \subset \Omega$ *with* $R < R_0$:

$$\left\{ \fint_{B_{R/2}} (|Du|^m + b|u|^\gamma)^{\frac{r}{m}} dx \right\}^{\frac{1}{r}} \le c \left\{ \left[\fint_{B_R} (|Du|^m + |u|^\gamma)dx \right]^{1/m} + \left[\fint_{B_R} g^{r/m} dx \right]^{m/r} \right\}.$$

Proof. — We remark that the assumption $\gamma \ge m$ is not restrictive since if $\gamma < m$ we can use the inequality

$$|u|^\gamma \le |u|^m + 1$$

reducing to (3.2) with $\gamma = m$ and g replaced by $g + b$. Let $B_R \subset \Omega$ and let $\frac{R}{2} < t < s < R$. We compare u with the function

$$v = u - \eta(u - u_s)$$

where $u_s = \fint_{B_s} u dx$ and η is a C^∞ function with support in B_s and satisfying $0 \le \eta \le 1$, $\eta = 1$ on B_t, $|D\eta| \le 2(s-t)^{-1}$.

From $\mathcal{F}(u; B_s) \le Q\mathcal{F}(v; B_s)$ we get at once

$$\int_{B_s} (|Du|^m + b|u|^\gamma)dx \le Q_1 \left\{ \int_{B_s} (1-\eta)^m |Du|^m dx + \int_{B_s} |D\eta|^m |u - u_s|^m dx + b\int_{B_s} |u - u_s|^\gamma dx + b\int_{B_s} |u|^\gamma dx + \int_{B_s} g dx \right\}. \tag{3.5}$$

We have

$$\int_{B_s} |u|^\gamma dx \le Q_2 \left[\int_{B_s} |u - u_s|^\gamma dx + |u_s|^\gamma |B_s| \right] \le Q_2 \left[\int_{B_s} |u - u_s|^\gamma dx + \left(\fint_{B_s} |u| dx \right)^\gamma |B_s| \right].$$

Moreover

$$\int_{B_s} |u - u_s|^\gamma dx \le |B_R|^{1+\frac{\gamma(m-n)}{nm}} \left(\int_{B_s} |Du|^m dx \right)^{\gamma/m}$$

and hence, if

$$Q_3 b |B_R|^{1+\frac{\gamma(m-n)}{nm}} \left(\int_\Omega |Du|^m dx \right)^{\frac{\gamma}{m}-1} < 1 \qquad (Q_3 = Q_1 Q_2)$$

we can subtract the term $\int_{B_s} |u - u_s|^\gamma dx$ from the left-hand side, leaving

$$\int_{B_t} (|Du|^m + b|u|^\gamma) dx \le Q_4 \left\{ \int_{B_s - B_t} |Du|^m dx + \frac{1}{(s-t)^m} \int_{B_R} |u - u_R|^m dx \right.$$
$$\left. + \int_{B_R} g dx + b \left(\fint_{B_R} |u| dx \right)^\gamma |B_R| \right.$$

where we have used the properties of η.

Adding to both sides Q_4 times the quantity on the left, and dividing by $Q_4 + 1$, we get

$$\int_{B_t} (|Du|^m + b|u|^\gamma) dx \le \frac{Q_4}{1 + Q_4} \left\{ \int_{B_s} (|Du|^m + |u|^\gamma) dx \right.$$
$$\left. + \frac{1}{(s-t)^m} \int_{B_R} |u - u_R|^m dx + \int_{B_R} g dx + b \left(\fint_{B_R} |u| dx \right)^\gamma |B_R| \right\}.$$

We can now use the lemma 3.2 below to obtain

$$(3.6) \quad \int_{B_{R/2}} (|Du|^m + b|u|^\gamma) dx \le Q_5 \left\{ R^{-m} \int_{B_R} |u - u_R|^m dx \right.$$
$$\left. + \int_{B_R} g dx + b \left(\fint_{B_R} |u| dx \right)^\gamma |B_R| \right\}.$$

For $1 < q < m$ we estimate

$$\left(\fint_{B_R} |u| dx \right)^\gamma \le \left(\fint_{B_R} |u|^{\gamma \frac{q}{m}} dx \right)^{\frac{m}{q}}$$

moreover, if $q \ge \dfrac{nm}{n+m}$, we have

$$R^{-m} \int_{B_R} |u - u_R|^m dx \le Q_6 \left(\int_{B_R} |Du|^q dx \right)^{\frac{m}{q}} |B_R|^{1 - \frac{m}{q}}.$$

In conclusion setting

$$w = (|Du|^m + b|u|^\gamma)^{q/m}$$

we have

$$\fint_{B_{R/2}} w^{\frac{m}{q}} dx \leq Q_7 \left\{ \left(\fint_{B_R} w dx \right)^{\frac{m}{q}} + \fint_{B_R} g dx \right\}.$$

The result now follows from proposition 5.1 of [*14*]. q. e. d.

We have used the following result:

LEMMA 3.2. — *Let $f(t)$ be a bounded non-negative function in* $[T_0, T_1]$, *such that for every s, t, $T_0 \leq t < s \leq T_1$:*

$$f(t) \leq A(s-t)^{-\alpha} + B + \theta f(s) \tag{3.7}$$

with A, α, B, θ *non-negative, and $\theta < 1$.*
Then there exists a constant c, depending only on α and θ, such that for every ρ, R, $T_0 \leq \rho \leq R \leq T_1$

$$f(\rho) \leq c[A(R-\rho)^{-\alpha} + B]. \tag{3.8}$$

We shall not repeat the simple proof, which can be found in [*12*]. We only remark here that we can drop both the assumptions that f is non-negative and bounded. Actually, if (3.7) holds for f it holds as well for $g = \max(f, 0)$. Moreover, any f satisfying (3.7) is automatically bounded on compact subsets of (T_0, T_1). To see that, suppose that there exists a sequence $t_k \to t_0 \in (T_0, T_1)$ such that $f(t_k) \to +\infty$. Writing (3.7) for $t = t_k$ and $s = T_1$ and passing to the limit we get immediately a contradiction. We exclude similarly the existence of a sequence $s_k \to s_0 > T_0$ such that $f(s_k) \to -\infty$, although this is not necessary for our purposes. The conclusion (3.8) holds therefore for $T_0 < \rho < R < T_1$ and hence also for $R = T_1$ by continuity, and for $\rho = T_0$, as one can easily see combining (3.7) with $t = T_0$ and $s = \dfrac{T_0 + R}{2}$ with (3.8) with $\rho = \dfrac{T_0 + R}{2}$.

The next result is a weak version of a well-known theorem concerning the removability of singularities.

THEOREM 3.3. — *Let $u \in H^{1,m}(\Omega, \mathbb{R}^N)$ be a quasi-minimum for the functional* (3.1) *with condition* (3.2) *and* (3.3) *in $\Omega - E$, where E is a closed set of m-capacity zero. Then u is a quasi-minimum in Ω.*

Proof. — The assumption on E implies that there exists a sequence of C^∞ functions η_k with $0 \leq \eta_k \leq 1$, $\eta_k = 1$ on a neighbourhood of E, and such that meas (supp η_k) $\to 0$ and $\int |D\eta_k|^m dx \to 0$.

Let ϕ be any function with support in Ω and let

$$v = u + \phi$$
$$w_k = u + \phi - \phi\eta_k.$$

Moreover, let $S = \operatorname{supp} \phi$, $\Sigma_k = \operatorname{supp} \eta_k$ and $S_k = \operatorname{supp}(\phi - \phi\eta_k) \supset S - \Sigma_k$.
Since u is a Q-minimum in $\Omega - E$ we have

$$\mathscr{F}(u; S_k) \leq Q\mathscr{F}(w_k; S_k) = Q\{\mathscr{F}(v; S - \Sigma_k) + \mathscr{F}(w_k, \Sigma_k).$$

On the other hand

$$\mathscr{F}(v; S - \Sigma_k) \leq \mathscr{F}(v; S) + \int_{\Sigma_k} (b|v|^\gamma + g)dx$$

$$\mathscr{F}(w_k; \Sigma_k) \leq \int_{\Sigma_k} (\mu|Dw_k|^m + b|w_k|^\gamma + g)dx$$

$$\leq \int_{\Sigma_k} (\mu|Dv|^m + b|v|^\gamma + g)dx + c\int_{\Sigma_k} (\mu|D(\eta_k\phi)|^m + |\eta_k\phi|^\gamma)dx$$

and passing to the limit we get the conclusion of the theorem. q. e. d.

We remark that the assumption $u \in H^{1,m}(\Omega, \mathbb{R}^N)$ is quite strong, but seems difficult to release in general. When $N = 1$ we have much better results for solutions of elliptic equations (see e. g. [*23*]); however, in the case of quasi-minima we lack an essential tool, the comparison principle, even when $b = g = 0$. This can be easily seen from the example *c*) of the preceding section, since for two quasi-minima we may very well have $u \leq v$ at the end points of a segment and $v < u$ at some interior point.

4. REGULARITY OF QUASI-MINIMA, N = 1

As one may expect from the theory of elliptic differential equations and systems, the regularity results are by far more complete in the case of functionals depending on a single real-valued function u.

The main result have already been proved in [*12*] for minima. The proofs can be extended without difficulty to our situation.

THEOREM 4.1. — *Let $u \in H^{1,m}_{loc}(\Omega)$ be a quasi-minimum for the functional*

$$\mathscr{F}(u; \Omega) = \int_\Omega f(x, u, Du)dx \tag{4.1}$$

with conditions (1.2) *and* (1.3). *Suppose that* $N = 1$ *and* $g \in L^\sigma_{loc}(\Omega)$ *for some* $\sigma > \dfrac{n}{m}$. *Then u is locally bounded in* Ω.

THEOREM 4.2. — *Let the function $f(x, u, p)$ satisfy the growth condition*

$$|p|^m - g(x, M) \leq f(x, u, p) \leq \mu(M)|p|^m + g(x, M) \tag{4.2}$$

for every $x \in \Omega$, $p \in \mathbb{R}^n$ and $u \in \mathbb{R}$ with $|u| \leq M$. Let $u \in H^{1,m}_{loc}(\Omega)$ be a bounded *quasi-minimum for* $\mathscr{F}$, *and suppose that for every* M, $g(\cdot\,, \mathrm{M}) \in \mathrm{L}^{\sigma}_{\mathrm{loc}}(\Omega)$ *for some* $\sigma > \frac{n}{m}$. *Then u is Hölder continuous in* Ω.

Let us sketch the proof of theorem 4.2.

We can suppose that $|u| \leq \mathrm{M}$ in Ω. For $k > 0$ let

$$\mathrm{A}_k = \{ x \in \Omega : u(x) > k \}$$

and

$$\mathrm{A}_{k,s} = \mathrm{A}_k \cap \mathrm{B}_s$$

where $\mathrm{B}_s = \mathrm{B}_s(x_0)$ is the ball of center x_0 and radius s. Let $w = \max(u - k, 0)$ and let η be the usual cut-off function: $0 \leq \eta \leq 1, \eta = 1$ on B_t, supp $\eta \subset \mathrm{B}_s$, $|\mathrm{D}\eta| \leq 2(s-t)^{-1}$, $0 < t < s$.

We have $\mathscr{F}(u ; \mathrm{A}_{k,s}) \leq \mathrm{Q}\mathscr{F}(u - \eta w ; \mathrm{A}_{k,s})$ and therefore

$$\int_{\mathrm{A}_{k,s}} |\mathrm{D}u|^m dx \leq \gamma_1 \left\{ \int_{\mathrm{A}_{k,s}} (1-\eta)^m |\mathrm{D}u|^m dx + \int_{\mathrm{A}_{k,s}} w^m |\mathrm{D}\eta|^m dx + \left(\int_{\mathrm{A}_{k,s}} g^\sigma dx \right)^{\frac{1}{\sigma}} |\mathrm{A}_{k,s}|^{1-\frac{1}{\sigma}} \right\}.$$

Since $\eta = 1$ on B_t we get easily for $\mathrm{R} > s$:

$$\int_{\mathrm{A}_{k,t}} |\mathrm{D}u|^m dx \leq \gamma_2 \left\{ \int_{\mathrm{A}_{k,s}-\mathrm{A}_{k,t}} |\mathrm{D}u|^m dx + \frac{1}{(s-t)^m} \int_{\mathrm{A}_{k,\mathrm{R}}} |u-k|^m dx + \|g\|_\sigma |\mathrm{A}_{k,\mathrm{R}}|^{1-\frac{1}{\sigma}} \right\}.$$

We proceed now as in theorem 3.1, adding to both sides the left-hand term multiplied by γ_2. An application of lemma 3.2 gives at once

$$(4.3) \quad \int_{\mathrm{A}_{k,\rho}} |\mathrm{D}u|^m dx \leq \gamma_3 \left\{ \frac{1}{(\mathrm{R}-\rho)^m} \int_{\mathrm{A}_{k,\mathrm{R}}} |u-k|^m dx + \|g\|_\sigma |\mathrm{A}_{k,\mathrm{R}}|^{1-\frac{1}{\sigma}} \right\}$$

Since $-u$ is itself a quasi-minimum for the functional

$$\int_\Omega \bar{f}(x, v, \mathrm{D}v) dx$$

with $\bar{f}(x, v, p) = f(x, -v, -p)$ satisfying the same inequality (4.2), the estimate (4.3) holds as well for $-u$. A straightforward application of theorem 6.1 of Chapter II of [*16*] gives the result.

A similar method gives the boundary regularity for quasi-minima of $\mathscr{F}$ satisfying $u = \phi$ on $\partial\Omega$, provided $\partial\Omega$ is smooth enough and ϕ is Hölder-continuous. The obvious changes in the proof should be made in view of the application of theorem 7.1 of Chapter II of [*16*].

Theorem 4.2 contains the classical theorem by De Giorgi [*4*], as well as most of the Hölder-continuity results of chapter IV and V of [*16*].

We remark that the Hölder norm of u on compact subset of Ω will depend on the various constants and functions in (4.2), on Q, and on the $H^{1,m}$ and L^∞-norms of u, but not otherwise on u.

We pass now to « homogeneous » functionals; i. e. functionals (4.1) satisfying (4.2) with $g \equiv 0$. It will not be restrictive to consider only quasi-minima of

$$\mathscr{I}(u;\Omega) = \int_\Omega |\,Du\,|^m dx. \tag{4.4}$$

THEOREM 4.3. — (*Weak maximum principle*). *Let* $u \in H^{1,m}(\Omega)$ *be a quasi-minimum of the functional* (4.4). *Then*

$$\sup_\Omega u = \sup_{\partial\Omega} u$$

$$\inf_\Omega u = \inf_{\partial\Omega} u.$$

Proof. — It is sufficient to compare u *respectively with* $v = \min\{u, \sup_{\partial\Omega} u\}$ and with $w = \max\{u, \inf_{\partial\Omega} u\}$. q. e. d.

As we wrote in the introduction, we do not have a proof of the strong maximum principle for quasi-minima; let alone of Harnack's inequality. This is the reason why we only can prove a weak form of Liouville's theorem

THEOREM 4.4. — *Let* $u \in H^{1,m}_{loc}(\mathbb{R}^n)$ *be a quasi-minimum in* $\mathbb{R}^n$ *of*

$$\int |\,Du\,|^m dx$$

If u *is bounded,* $|\,u\,| \leq M$ *in* $\mathbb{R}^n$, *then* u *is constant.*

Proof. — Let u be a bounded Q-minimum, and let $u_k(x) = u(kx)$. It is easily seen that u_k is a Q-minimum of the same functional in $\mathbb{R}^n$. We have $|\,u_k\,| \leq M$; moreover from (3.6) with $b = g = 0$ we get for any ball B_R:

$$\int_{B_{R/2}} |\,Du_k\,|^m dx \leq Q_5 R^{-m} \int_{B_R} |\,u_k - u_{k,R}\,|^m \leq Q_6 M^m R^{n-m}$$

and hence the sequence $\{u_k\}$ is bounded in $H^{1,m}_{loc}(\mathbb{R}^n)$. By theorem 4.2 we can conclude that the functions u_k are equicontinuous and therefore, passing possibly to a subsequence, that they converge uniformly on compact subsets of $\mathbb{R}^n$ to some function v.

Suppose now that u is not constant. We have

$$\omega = \operatorname{osc}(u, \mathbb{R}^n) > 0$$

and therefore for some $R > 0$

$$\operatorname{osc}(u, B_R) > \frac{1}{2}\omega > 0.$$

On the other hand

$$\operatorname{osc}(u_k, B_{R/k}) = \operatorname{osc}(u, B_R) > \frac{1}{2}\omega$$

and therefore for every $\rho > 0$

$$\operatorname{osc}(v, B_\rho) \geq \frac{1}{2}\omega$$

contradicting the continuity of v. q. e. d.

5. QUASI-MINIMA AND Γ-CONVERGENCE

In this section we prove a stability result for Q-minima.

Let $\mathscr{F}_h(u;\Omega)$ be a sequence of functionals of the type (3.1), satisfying conditions (3.2), (3.3) uniformly with respect to $h \in \mathbb{N}$. We suppose that the sequence $\mathscr{F}_h$ converges to $\mathscr{F}_0$ in the following sense:

i) $\forall v \in H^{1,m}_{loc}(\Omega)$, $\forall A \subset\subset \Omega$, $\forall v_k \to v$ *weakly in* $H^{1,m}(A)$ *we have*

$$\mathscr{F}_0(v;A) \leq \liminf_{h\to\infty} \mathscr{F}_h(v_h;A).$$

ii) $\forall v \in H^{1,m}_{loc}(\Omega)$, $\forall A \subset\subset \Omega$, *there exists a sequence* $v_k \in H^{1,m}_{loc}(\Omega)$ *such that* $v_k = v$ *outside* A, $v_k \to v$ *weakly in* $H^{1,m}(A)$ *and*

$$\mathscr{F}_0(v;A) = \lim_{h\to\infty} \mathscr{F}_h(v_h;A).$$

We note that (*i*) (*ii*) are essentially equivalent to say that $\mathscr{F}_h$ Γ-converges (with respect to a suitable topology) in the sense of De Giorgi to $\mathscr{F}_0$ (see e. g. [*22*] [*5*]).

We have

THEOREM 5.1. — *Suppose that, for every $h \in \mathbb{N}$, u_h is a Q-minimum for $\mathscr{F}_h$, with Q independent of h, and that u_h converges weakly to u in $L^m_{loc}(\Omega)$. Then u is a Q-minimum for $\mathscr{F}_0$.*

Proof. — By (3.6) the sequence u_h is bounded in $H^{1,m}_{loc}$ (and therefore in L^γ_{loc}); from theorem 3.1 we deduce that $u_k \to u$ weakly in $H^{1,r}_{loc}(\Omega)$ for some $r > m$ and therefore strongly in L^γ_{loc}.

Let now $v \in H^{1,m}_{loc}(\Omega)$, with $v = u$ outside some open set $A \subset\subset \Omega$. Let A_1 be an open set with $A \subset\subset A_1 \subset\subset \Omega$, and let ϕ be a smooth function, $0 \leq \phi \leq 1$, $\phi = 0$ in A and $\phi = 1$ outside A_1. Let $\{v_k\}$ be the sequence in (*ii*) above. Setting

$$\tilde{v}_k = v_k + \phi(u_k - u)$$

we have $\tilde{v}_k = u_k$ outside A_1 and therefore

$$Q^{-1}\mathscr{F}_k(u_k;A_1) \leq \mathscr{F}_k(\tilde{v}_k;A_1) = \mathscr{F}_k(v_k;A) + \mathscr{F}_k(\tilde{v}_k;A_1 - A).$$

Passing to the limit as $k \to +\infty$ we get

$$(5.1) \qquad Q^{-1}\mathscr{F}_0(u;A_1) \le \mathscr{F}_0(v;A) + \limsup_{k\to\infty} \mathscr{F}_k(\tilde{v}_k;A_1 - A).$$

On the other hand, since $\tilde{v}_k = u + \phi(u_k - u)$ outside A, we have

$$\mathscr{F}_k(\tilde{v}_k;A_1-A) \le c\int_{A_1-A} \{ |Du|^m + |Du_k|^m + |D\phi|^m|u-u_k|^m + |u|^\gamma + \phi^\gamma|u-u_k|^\gamma + g \} dx .$$

From the boundedness of Du_k in L^r_{loc} we get

$$\int_{A_1-A} |Du_k|^m dx \le |A_1 - A|^{1-\frac{m}{r}} \left(\int_{A_1-A} |Du_k|^r dx \right)^{\frac{m}{r}} \le c\,|A_1 - A|^{1-\frac{m}{r}}$$

letting $k \to \infty$ we obtain

$$\limsup_{k\to\infty} \mathscr{F}_k(\tilde{v}_k, A_1-A) \le c\int_{A_1-A} \{ |Du|^m + |u|^\gamma + g \} dx + c|A_1-A|^{1-\frac{m}{r}}$$

inserting in (5.1) and letting $A_1 \to A$ we conclude that

$$\mathscr{F}_0(u;A) \le Q\mathscr{F}_0(v;A)$$

and therefore that u is a Q-minimum for $\mathscr{F}_0$. q. e. d.

In general it is not clear whether the limit functional $\mathscr{F}_0$ can be written as an integral. On the other hand, we have

$$\int_A (|Du|^m - b|u|^\gamma - g)dx \le \mathscr{F}_0(u;A) \le \int_A \{ \mu|Du|^m + b|u|^\gamma + g \} dx$$

and therefore every quasi-minimum for $\mathscr{F}_0$ is also a quasi-minimum for

$$\mathscr{I}(u;\Omega) = \int_\Omega (|Du|^m + b|u|^\gamma + g)dx .$$

In particular the regularity results of this paper hold for Q-minima of $\mathscr{F}_0$.

An interesting special situation is obtained when $\mathscr{F}_h = \mathscr{F}$; in this case $\mathscr{F}_0$ is the so-called lower semicontinuous envelope: namely the greatest lower-semicontinuous functional not exceeding $\mathscr{F}$. Finally, if $\mathscr{F}$ is lower semi-continuous in $H^{1,m}_{loc}$ we obtain that weak L^m limits of Q-minima are Q-minima.

6. QUASI-MINIMA AND QUASI-CONVEXITY

We describe here a result by P. Marcellini and C. Sbordone [*17*], that fits very well in the theory of quasi-minima.

We begin by recalling an abstract result due to I. Ekeland [*6*].

THEOREM 6.1. — *Let* (V, d) *be a complete metric space, and let* $\mathscr{F} : V \to [a, +\infty]$ *be a lower semicontinuous functional, not identically* $+\infty$. *Let* $\eta > 0$ *and* $v \in V$ *be such that*

$$\mathscr{F}(v) \leq \inf_{V} \mathscr{F} + \eta .$$

Then there exists $\tilde{u} \in V$, *with* $d(\tilde{u}, v) \leq 1$ *such that* $\mathscr{F}(\tilde{u}) \leq \mathscr{F}(v)$, *and moreover*

$$\mathscr{F}(\tilde{u}) \leq \mathscr{F}(w) + \eta d(\tilde{u}, w) \tag{6.1}$$

for every $w \in V$.

Inequality (5.1) means that $\tilde{u}$ minimizes the functional

$$\mathscr{F}(w) + \eta d(\tilde{u}, w)$$

Writing $\eta^{-\frac{1}{2}}d$ instead of d we can also conclude the existence of $u \in V$ with $d(u, v) < \eta^{\frac{1}{2}}$, minimizing

$$\mathscr{F}(w) + \eta^{\frac{1}{2}}d(u, w) .$$

In particular, if $\{ v_k \}$ is a minimizing sequence in V, the corresponding sequence $\{ u_k \}$ is also minimizing, and $d(u_k, v_k) \to 0$.

Suppose now that f is quasi-convex; namely that for every $x_0 \in \Omega$, $u_0 \in \mathbb{R}^N$, $p_0 \in \mathbb{R}^{nN}$ and for every $\phi \in C_0^1(\Omega, \mathbb{R}^N)$ we have

$$f(x_0, u_0, p_0) \leq \frac{}{}\!\!\!\!\!\!\int_{\Omega} f(x_0, u_0, p_0 + D\phi(x))dx . \tag{6.2}$$

For such functions we have the following semicontinuity result [*8*]:

THEOREM 6.2. — *Let* f *satisfy* (6.2) *and*

$$| f(x, u, p) | \leq \mu | p |^m + b | u |^\gamma + g(x) \tag{6.3}$$

with $m \geq 1$, $1 \leq \gamma < m^*$ *and* $g \in L^\sigma(\Omega)$, $\sigma > 1$.
Then the functional

$$\mathscr{F}(u ; \Omega) = \int_{\Omega} f(x, u, Du)dx$$

is sequentially lower semi-continuous in the weak topology of $H^{1,q}_{loc}(\Omega, \mathbb{R}^N)$ *for every* $q > m$.

We can now prove the existence of a minimum for the functional $\mathscr{F}$.

THEOREM 6.3. — *Let* f *be a quasi-convex function satisfying*

$$| p |^m - b | u |^\gamma - g(x) \leq f(x, u, p) \leq \mu | p |^m + b | u |^\gamma + g(x) \tag{6.4}$$

with $m > 1$, $1 \leq \gamma < m$ *and* $g \in L^\sigma(\Omega)$, $\sigma > 1$. *Let* $u_0 \in H^{1,m}(\Omega, \mathbb{R}^N)$. *The functional*

$$\mathscr{F}(u ; \Omega) = \int_{\Omega} f(x, u, Du)dx$$

attains its minimum in the class

$$V = \{ v \in H^{1,1}(\Omega, \mathbb{R}^N) : v - u_0 \in H_0^{1,1}(\Omega, \mathbb{R}^N) \}.$$

Moreover the minimiser $u \in H^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^N)$ *for some* $q > m$.

Proof. — The class V is a complete metric space with the distance

$$d(u, v) = \int_\Omega | Du - Dv | \, dx .$$

It is easily seen that the functional $\mathscr{F}$ is lower semicontinuous in V. Let $\{ v_k \}$ be a minimizing sequence, and let $\{ u_k \}$ be the corresponding (minimizing) sequence given by theorem 6.1. We have

$$(6.5) \qquad \int_\Omega | Du_k |^m dx \leq \mathscr{F}(u_k) + b \int_\Omega | u_k |^\gamma dx + \int_\Omega g dx ..$$

On the other hand, since $| u_k |^\gamma \leq c(| u_0 |^\gamma + | u_k - u_0 |^\gamma)$, we have, using the condition $\gamma < m$:

$$\int_\Omega | u_k |^\gamma dx \leq c_0 \int_\Omega | u_0 |^\gamma dx + c\varepsilon \int | u_k - u_0 |^m dx + c_1(\varepsilon)$$

and therefore

$$\int_\Omega | u_k |^\gamma dx \leqslant c_2 \cdot \varepsilon \int_\Omega | Du_k |^m dx + c_3 .$$

In conclusion, taking ε small enough, we get from (6.5)

$$\mathscr{F}(u_k) \geq \frac{1}{2} \int_\Omega | Du_k |^m dx - c_4$$

so that we can conclude that $\mathscr{F}$ is bounded from below and that the sequence $\{ u_k \}$ is bounded in $H^{1,m}(\Omega, \mathbb{R}^N)$. Since $m > 1$, we can suppose that u_k converges weakly to some functions $u \in H^{1,m}(\Omega, \mathbb{R}^N)$.

On the other hand, u_k satisfies for k large enough:

$$\mathscr{F}(u_k ; \Omega) \leq \mathscr{F}(w ; \Omega) + \int_\Omega |Du_k - Dw| \, dx \qquad \forall w \in V ;$$

if $K = \operatorname{supp} (u_k - w) \subset \Omega$ we have

$$\mathscr{F}(u_k ; K) \leq \mathscr{F}(w ; K) + \int_K | Du_k | \, dx + \int_K | Dw | \, dx .$$

Arguing as above, it is not difficult to show that u_k is a Q-minimum of the functional

$$\int_\Omega (| Du |^m + | u |^\gamma + g + 1) dx$$

with Q independent of k.

From theorem 3.1 we conclude that the sequence $\{ u_k \}$ is equibounded in $H^{1,q}_{loc}(\Omega, \mathbb{R}^N)$ for some $q > m$; and therefore u_k converges to u weakly in $H^{1,q}_{loc}$. From theorem 6.2 we conclude that u gives the required minimum.

q. e. d.

REFERENCES

[1] E. ACERBI, N. FUSCO, Semicontinuity Problems in the Calculus of Variations. *Arch. Rat. Mech. Anal.*, to appear.

[2] B. V. BOYARSKII, Homeomorphic solutions of Beltrami systems. *Dokl. Akad. Nauk SSR*, t. **102**, 1955, p. 661-664.

[3] B. V. BOYARSKII, Generalized solutions of a system of differential equations of the first order of elliptic type with discontinuous coefficients. *Mat. Sbornik*, t. **43**, 1957, p. 451-503.

[4] E. DE GIORGI, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. torino cl. Sci. Fis. Mat. Nat.*, (3), t. **3**, 1957, p. 25-43.

[5] E. DE GIORGI, Generalized limits in Calculus of Variations, in « Topics in Functional Analysis 1980-1981 ». *Quaderni Scuola Norm. Sup. Pisa*, 1981.

[6] I. EKELAND, Nonconvex minimization problems. *Bull. Amer. Math. Soc.*, t. **1**, 1979, p. 443-474.

[7] J. FREHSE, A discontinuous solution of a mildly nonlinear elliptic system. *Math. Z.*, t. **134**, 1973, p. 229-230.

[8] N. FUSCO, Quasi-convessità e semicontinuità per integrali multipli di ordine superiore. *Ricerche di Mat.*, t. **29**, 1980, p. 307-323.

[9] F. W. GEHRING, The L^p-integrability of the partial derivatives of a quasi conformal mapping. *Acta Math.*, t. **130**, 1973, p. 265-277.

[10] M. GIAQUINTA, Multiple integrals in the Calculus of Variations and Nonlinear Elliptic Systems. *Annals of Math. Studies*, t. **105**, Princeton University Press, 1983.

[11] M. GIAQUINTA, On the differentiability of the extremals of variational integrals, in « Nonlinear Analysis, Function Spaces and Application 2 », *Teubner Texte zur Mathem.*, Leipzig, 1982.

[12] M. GIAQUINTA, E. GIUSTI, On the regularity of the minima of variational integrals. *Acta Math.*, t. **148**, 1982, p. 31-46.

[13] M. GIAQUINTA, E. GIUSTI, Differentiability of minima of non-differentiable functionals. *Inventiones Math.*, t. **72**, 1983, p. 285-298.

[14] M. GIAQUINTA, G. MODICA, Regularity results for some classes of higher order nonlinear elliptic systems. *J. für reine u. angew. Math.*, t. **311-312**, 1979, p. 145-169.

[15] S. HILDEBRANDT, K.-O. WIDMAN, Some regularity results for quasilinear elliptic systems of second order. *Math. Z.*, t. **142**, 1975, p. 67-86.

[16] O. A. LADYZHENSKAYA, N. N. URAL'TSEVA, Linear and quasilinear elliptic equations. *Academic Press*, New York, 1968.

[17] P. MARCELLINI, C. SBORDONE, On the existence of minima of multiple integrals of the Calculus of Variations. *J. Math. pures et appl.*, t. **62**, 1983, p. 1-9.

[18] N. G. MEYERS, An L^p-estimate for the gradient of solutions of second order elliptic divergence equations. *Ann. Sc. Norm. Sup. Pisa*, (3), t. **17**, 1963, p. 189-206.

[19] N. G. MEYERS, A. ELCRAT, Some results on regularity for solutions of nonlinear elliptic systems and quasiregular functions. *Duke Math. J.*, t. **42**, 1975, p. 121-136.

[20] J. NECAS. Les Méthodes directes en théorie des équations elliptiques, Praha, Akademia, 1967.

[21] Y. G. RESHETNYAK, Extremal properties of mappings with bounded distortion. Siberian Math. J. *(transl. of Sibirsk Mat. Z.)*, t. **10**, 1969, p. 962-969.
[22] C. SBORDONE, Su alcune applicazioni di un tipo di convergenza variazionale. *Ann. Sc. Norm. Sup. Pisa*, (4), t. **2**, 1975, p. 617-638.
[23] J. SERRIN, Local behaviour of solutions of quasilinear equations. *Acta Math.*, t. **111**, 1964, p. 247-302.

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