

PROBABILITY STRUCTURE PRESERVING AND ABSOLUTE CONTINUITY [☆]

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ABSTRACT. – The concept of *probability structure preserving mapping* is introduced. The idea is applied to define stochastic integral for fractional Brownian motion (fBm) and to obtain an anticipative Girsanov theorem for fBm. © 2002 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Un concept d'application préservant la probabilité est introduit. L'idée est appliquée à la définition de l'intégrale stochastique pour le mouvement brownien fractionnaire et à la définition d'une formule de Girsanov anticipante. © 2002 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Fractional Brownian motion (fBm) has received a great deal of attention in recent years. Various authors have developed stochastic calculus applicable to fBm (see for example, [1,3–5,10] for recent development).

This paper is motivated by the problem of absolute continuity for fBm with respect to its translation. Let Ω be the space of real valued continuous functions $\omega(t)$, $t \in \mathbb{R}$, with $\omega(0) = 0$. Define

$$D(\omega_1, \omega_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{-n \leq t \leq n} |\omega_1(t) - \omega_2(t)|}{1 + \sup_{-n \leq t \leq n} |\omega_1(t) - \omega_2(t)|}. \quad (1.1)$$

Then it is easy to see that D is a metric on Ω and (Ω, D) is a polish space (see [20]). Let \mathcal{F} be the Borel σ -algebra of Ω . For any given number $H \in (0, 1)$, there is a probability

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measure P^H on (Ω, \mathcal{F}) such that $B_t^H : \Omega \rightarrow \mathbb{R}, t \in \mathbb{R}$, defined by $B_t^H(\omega) = \omega(t), t \in \mathbb{R}$, is a *fractional Brownian motion with Hurst parameter H* . Namely, $(B_t^H, t \in \mathbb{R})$ is a Gaussian process with mean 0, and covariance

$$\mathbb{E}(B_t^H B_s^H) = c_H (|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

where $c_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-H)}$. We shall call $(\Omega, \mathcal{F}, P^H)$ the canonical fractional Wiener space with Hurst parameter H . The expectation on this probability space is denoted by \mathbb{E}^H . It is customary to denote the element of Ω by B^H . When $H = 1/2$, we obtain the (usual) canonical Wiener space and we will omit the dependence on H when $H = 1/2$.

Now let Λ be a transform from $(\Omega, \mathcal{F}, P^H)$ to $(\Omega, \mathcal{F}, P^H)$ defined by

$$B^H \rightarrow B^H + \int_0^\cdot g(s) ds, \tag{1.2}$$

where g is an anticipative stochastic process. Under suitable conditions, Λ induces another probability measure $P^H \circ \Lambda^{-1}$ on (Ω, \mathcal{F}) given by

$$P^H \circ \Lambda^{-1}(A) = P^H(\Lambda^{-1}(A)) = P^H(\{\omega, \Lambda(\omega) \in A\}), \quad \forall A \in \mathcal{F}.$$

We are interested in the problem of absolute continuity of $P^H \circ \Lambda^{-1}$ with respect to P^H .

In the classical Brownian motion case (i.e., when $H = 1/2$) this problem has been studied by many authors. We refer to [2,21] and in particular the references therein.

In the general fractional Brownian motion case ($H \neq 1/2$), this problem has been studied extensively when g is a deterministic function (see [11] and the references therein).

This paper studies the general (anticipative) case. We obtain a general theorem about the absolute continuity and a general formula for the Radon–Nikodym density.

Presumably, we may utilize the general formula of Ramer–Kusuoka and compute the Carleman–Fredholm determinant appeared in the Ramer–Kusuoka formula. This has been the idea for example in [2,21] in the classical Brownian motion case. However, this paper develops another method which makes use of the known results in the classical Brownian motion case.

By a theorem (see Theorem 2.1 below) in the framework of measure theory, it is known that if we can find a measure-preserving one-to-one mapping T from (Ω, \mathcal{F}, P) to $(\Omega, \mathcal{F}, P^H)$ and a measurable mapping Γ from Ω to Ω such that $\Lambda \circ T = T \circ \Gamma$. Then

$$\frac{dP^H \circ \Lambda^{-1}}{dP^H} = \frac{dP \circ \Gamma^{-1}}{dP} \circ T^{-1} \tag{1.3}$$

as long as we know that $\frac{dP \circ \Gamma^{-1}}{dP}$ exists.

It is interesting to directly construct a mapping T from (Ω, \mathcal{F}, P) to $(\Omega, \mathcal{F}, P^H)$ and a mapping Γ from (Ω, \mathcal{F}, P) to itself such that $\Lambda \circ T = T \circ \Gamma$. We may then obtain the Radon–Nikodym derivative by using (1.3) since there is an extensive study on the computation of $\frac{dP \circ \Gamma^{-1}}{dP}$ in the classical Brownian motion case.

However, we will extend (1.3) to a more general case. We introduce the concept of *probability structure preserving mapping* V from the set of measurable functions on (Ω, \mathcal{F}, P) to the set of measurable functions on $(\Omega, \mathcal{F}, P^H)$. Γ (and Λ) can be considered as transform which maps a function on (Ω, \mathcal{F}, P) (and on $(\Omega, \mathcal{F}, P^H)$) to another function on (Ω, \mathcal{F}, P) (and on $(\Omega, \mathcal{F}, P^H)$). We shall prove that if $\Lambda \circ V = V \circ \Gamma$, then

$$\frac{dP^H \circ \Lambda^{-1}}{dP^H} = V \circ \frac{dP \circ \Gamma^{-1}}{dP}. \quad (1.4)$$

This is an extension of (1.3). From this result and a result on the Radon–Nikodym derivative for the classical Brownian motion, we obtain a formula for the Radon–Nikodym derivative for the fractional Brownian motion.

The idea of the probability structure preserving mapping may carry many other established results on one probability space to another probability space. We do not claim that all results on classical Brownian motion may be extended to fractional Brownian motion by using this idea. However, many other results on the classical Brownian motion may also be extended to the fractional Brownian motion case through this probability structure preserving mapping.

The probability structure preserving mapping V introduced in this paper is similar to a correspondence introduced by the author and his adviser, Prof. P.A. Meyer in [7–9]. However, in that correspondence the multiple Stratonovich integrals play an important role. In the mapping considered in this paper, the similar role is played by multiple Itô type integrals.

In Section 2, we give the definition of the probability structure preserving mapping V and prove that this mapping may be defined for any measurable function. We also present a general way to construct this mapping for abstract Wiener space case.

In Section 3, we introduce a particular probability structure preserving mapping V between Brownian motion and fractional Brownian motion with Hurst parameter H and establish some useful properties of V . We use this mapping and the definition of stochastic integral for Brownian motion to define stochastic integral for fractional Brownian motion.

In Section 4, we apply the probability structure preserving mapping V introduced in Section 3 to obtain a Girsanov type theorem for fractional Brownian motion.

2. Probability structure preserving

Let $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ be two measurable spaces. Let T be an invertible measurable mapping from Ω_1 to Ω_2 such that

$$\int_{\Omega_1} F \circ T(\omega_1) P_1(d\omega_1) = \int_{\Omega_2} F(\omega_2) P_2(d\omega_2)$$

for all bounded measurable function $F : \Omega \rightarrow \mathbb{R}$. Namely, T is a measure-preserving one to one transform.

THEOREM 2.1. – *Let Γ be a measurable transformation from Ω_1 to itself and let Λ be a measurable transformation from Ω_2 to itself. Assume that the following diagram*

$$\begin{array}{ccc}
 \Omega_1 & \xrightarrow{T} & \Omega_2 \\
 \downarrow \Gamma & & \downarrow \Lambda \\
 \Omega_1 & \xrightarrow{T} & \Omega_2
 \end{array} \tag{2.1}$$

commutes. Then

$$\frac{dP_2 \circ \Lambda^{-1}}{dP_2} = \frac{dP_1 \circ \Gamma^{-1}}{dP_1} \circ T^{-1}. \tag{2.2}$$

Proof. – This result may be known. However, I could not find it in a standard reference. I will sketch a simple proof.

Let $A \in \mathcal{F}_2$ and $B = T^{-1}(A) \in \mathcal{F}_1$. Then

$$\begin{aligned}
 P_2 \Lambda^{-1}(A) &= P_2(\Lambda^{-1}T(B)) = P_2(T\Gamma^{-1}(B)) = P_1(\Gamma^{-1}(B)) \\
 &= \int_B \frac{dP_1 \circ \Gamma^{-1}}{dP_1}(x) P_1(dx) = \int_A \frac{dP_1 \circ \Gamma^{-1}}{dP_1}(T^{-1}y) P_2(dy).
 \end{aligned}$$

This proves (2.2). \square

Let (Ω, H, P) be an abstract Wiener space, where H is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_H$. There is no ambiguity to use H both for the Hurst parameter and for a Hilbert space. For any $h \in H$, there is a Gaussian random variable, denoted by $\langle \omega, h \rangle$, such that it has mean 0 and covariance

$$\mathbb{E}(\langle \cdot, h_1 \rangle \langle \cdot, h_2 \rangle) = \langle h_1, h_2 \rangle_H.$$

Define \mathcal{F} the smallest σ -algebra such that for all $h \in H$, $\langle \omega, h \rangle$ is \mathcal{F} -measurable. Let $h \in H$. Then $F(\omega) = \exp(\langle \omega, h \rangle)$ is called an *exponential functional* or an *exponential vector*. Let $\mathcal{E} = \mathcal{E}(\Omega, \mathcal{F}, P)$ be the space of finite linear combination of the exponential functionals (abbreviated as FLICEF). Then \mathcal{E} is a linear space which is closed with respect to multiplication, i.e., if $F, G \in \mathcal{E}$, then $FG \in \mathcal{E}$. \mathcal{E} is also dense in $L^p(\Omega, \mathcal{F}, P)$ for any $p \in [1, \infty)$.

Let $(\tilde{\Omega}, \tilde{H}, \tilde{P})$ be another abstract Wiener space. Let $\tilde{\mathcal{E}}$ be the space of FLICEF on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

DEFINITION 2.2. – *A mapping V from \mathcal{E} to $\tilde{\mathcal{E}}$ is called probability structure preserving mapping if*

$$V(F + G) = V(F) + V(G), \quad \forall F, G \in \mathcal{E}, \tag{2.3}$$

$$V(FG) = V(F)V(G), \quad \forall F, G \in \mathcal{E}, \tag{2.4}$$

$$\mathbb{E}[F] = \tilde{\mathbb{E}}[VF], \quad \forall F \in \mathcal{E}, \tag{2.5}$$

where \mathbb{E} and $\tilde{\mathbb{E}}$ denote the expectations on the probability spaces (Ω, \mathcal{F}, P) and $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{P})$, respectively.

The correspondence introduced in [7–9] satisfies the properties (2.3)–(2.4). However, it does not satisfy (2.5).

Example 2.3. – If T is an invertible measurable mapping from Ω to $\tilde{\Omega}$ and if $\tilde{P} = P \circ T^{-1}$, then $VF := F \circ T^{-1}$ (where $F \in \mathcal{E}$) is a the probability structure preserving mapping.

It is clear that when V is given by an invertible measurable mapping then V is defined for all measurable function F and $V(FG) = (VF)(VG)$.

Now we will show that in general case it is also true, i.e., V can be defined for all measurable function F and $V(FG) = (VF)(VG)$.

LEMMA 2.4. – *If $F \in L^2(\Omega, \mathcal{F}, P)$, then VF is well-defined.*

Proof. – Let $\{F_n, n \geq 1\}$ be a sequence in \mathcal{E} with the property that $F_n \rightarrow F$ in $L^2(\Omega, \mathcal{F}, P)$. Then for any $n, m \geq 1$, we have

$$\begin{aligned} \tilde{\mathbb{E}}|VF_n - VF_m|^2 &= \tilde{\mathbb{E}}[(VF_n)^2 + (VF_m)^2 - 2VF_nVF_m] \\ &= \tilde{\mathbb{E}}[V(F_n)^2 + V(F_m)^2 - 2V(F_nF_m)] \\ &= \tilde{\mathbb{E}}[V(F_n^2 + F_m^2 - 2F_nF_m)] \\ &= \tilde{\mathbb{E}}[(F_n^2 + F_m^2 - 2F_nF_m)] \\ &= \mathbb{E}(F_n - F_m)^2. \end{aligned}$$

Thus $\{VF_n, n \geq 1\}$ is a Cauchy sequence in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. It is also easy to see that the limit is independent of the choice of $\{VF_n, n \geq 1\}$. Therefore the limit of the sequence $\{VF_n, n \geq 1\}$ can be defined as VF . \square

LEMMA 2.5. – *If $F, G \in L^\infty(\Omega, \mathcal{F}, P)$, then*

$$V(FG) = (VF)(VG).$$

Proof. – The assumption of the lemma implies that F and G are in $L^2(\Omega, \mathcal{F}, P)$. Thus $V(F)$ and $V(G)$ are well-defined by Lemma 2.4. Since F and G are in $L^4(\Omega, \mathcal{F}, P)$, there are sequences $\{F_n, n \geq 1\} \subset \mathcal{E}$ and $\{G_n, n \geq 1\} \subset \mathcal{E}$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E}|F_n - F|^4 = \lim_{n \rightarrow \infty} \mathbb{E}|G_n - G|^4 = 0.$$

Similar to the proof of Lemma 2.4, we obtain

$$\begin{aligned} \tilde{\mathbb{E}}|V(F_nG_n) - V(FG)|^2 &= \mathbb{E}|F_nG_n - FG|^2 \\ &\leq (\mathbb{E}|F_n|^4)^{1/2}(\mathbb{E}|G_n - G|^4)^{1/2} + (\mathbb{E}|G|^4)^{1/2}(\mathbb{E}|F_n - F|^4)^{1/2} \\ &\rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Thus there is a subsequence (without loss of generality we may choose the sequence itself) such that $V(F_nG_n)$ converges to $V(FG)$ almost surely as $n \rightarrow \infty$. On the other hand, since $V(F_n) \rightarrow V(F)$ and $V(G_n) \rightarrow V(G)$ in $L^2(\Omega, \mathcal{F}, \tilde{P})$ we can find a subsequence such that $V(F_{n_k}) \rightarrow V(F)$ and $V(G_{n_k}) \rightarrow V(G)$ almost surely.

Then $V(F_n G_n) = V(F_{n_k})V(G_{n_k}) \rightarrow V(F)V(G)$ almost surely. Consequently, we have $V(FG) = V(F)V(G)$. \square

LEMMA 2.6. – *If $F \in L^\infty(\Omega, \mathcal{F}, P)$ and if $f \rightarrow \mathbb{R}$ is a continuous function, then*

$$Vf(F) = f(VF).$$

Proof. – Let $M > 0$ be such that $|F| \leq M$ a.s. By the Weierstrass approximation theorem, there is a sequence of polynomials $\{P_n(x), n \geq 1\}$ such that P_n converges uniformly to $f(x)$ on $[-M, M]$. By Lemma 2.5 it is easy to see that $V P_n(F) = P_n(VF)$. Since P_n converges to f , $V P_n(F) = P_n(VF)$ converges to $f(VF)$ a.s. on the event $\{|VF| \leq K\}$ for any $K > 0$. Letting $K \rightarrow \infty$, we see that $V P_n(F)$ converges to $f(VF)$ a.s. On the other hand,

$$\mathbb{E}|V(P_n(F)) - V(f(F))|^2 = \mathbb{E}|P_n(F) - f(F)|^2 \rightarrow 0 \quad (n \rightarrow \infty).$$

This proves the lemma easily. \square

LEMMA 2.7. – *Let $F \in L^\infty(\Omega, \mathcal{F}, P)$ and let f be a continuous function. Then for any $\varepsilon \in \mathbb{R}$,*

$$\tilde{P}(f(VF) \geq \varepsilon) = P(f(F) \geq \varepsilon). \tag{2.6}$$

Proof. – Notice that $g_{n,\varepsilon} := e^{-n(\varepsilon - f(x))^+}$ is a continuous function of x , where a^+ denotes the positive part of a and that

$$\lim_{n \rightarrow \infty} g_{n,\varepsilon}(x) = I_{\{f(x) \geq \varepsilon\}} \quad \forall x \in \mathbb{R},$$

where I denotes the indicate function. Then

$$\begin{aligned} \tilde{P}(f(VF) \geq \varepsilon) &= \tilde{\mathbb{E}} I_{\{f(VF) \geq \varepsilon\}} = \tilde{\mathbb{E}} \lim_{n \rightarrow \infty} e^{-n(\varepsilon - f(VF))^+} \\ &= \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} e^{-n(\varepsilon - f(VF))^+} = \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} V e^{-n(\varepsilon - f(F))^+} \\ &= \lim_{n \rightarrow \infty} \mathbb{E} e^{-n(\varepsilon - f(F))^+} = \mathbb{E} \lim_{n \rightarrow \infty} e^{-n(\varepsilon - f(F))^+} \\ &= \mathbb{E} I_{\{f(F) \geq \varepsilon\}} = P(f(F) \geq \varepsilon). \end{aligned}$$

This proves the lemma. \square

Now we are ready to state and prove the main theorem of this section.

THEOREM 2.8. – *Let $F : \Omega \rightarrow \mathbb{R}$ be measurable and $F < \infty$ a.s.*

(i) *If $F_n \rightarrow F$ in probability, where $F_n \in L^\infty(\Omega, \mathcal{F}, P)$, then $V F_n$ converges in probability. The limit in probability of $V F_n$ is defined as $V F$.*

(ii) *If f is continuous, then*

$$Vf(F) = f(VF). \tag{2.7}$$

Proof. – (i) From Lemma 2.7, it follows that

$$\tilde{P}(|V F_m - V F_m| \geq \varepsilon) = \tilde{P}(|V(F_m - F_m)| \geq \varepsilon) = P(|F_m - F_m| \geq \varepsilon).$$

This shows that $\{VF_n, n \geq 1\}$ is a Cauchy sequence with respect to the convergence in probability. The unique limit of this sequence is independent of the choice of $\{VF_n, n \geq 1\}$. This limit is defined as VF .

(ii) Now let $F_n = FI_{\{|F| \leq n\}}$. Since $VF_n \rightarrow VF$ in probability, then there is a subsequence n_k such that $VF_{n_k} \rightarrow VF$ a.s. Thus

$$V(f(F_{n_k})) = f(VF_{n_k}) \rightarrow f(VF) \text{ a.s.} \tag{2.8}$$

Let $K > 0$ be a given arbitrary number. Since f is uniformly continuous on $[-K, K]$, there is a $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ for all $x, y \in [-K, K]$ with $|x - y| < \delta$. By Lemma 2.7, it follows that

$$\begin{aligned} \tilde{P}(|V(f(F_n)) - V(f(F))| \geq \varepsilon) &= P(|f(F_n) - f(F)| \geq \varepsilon) \\ &= P(\{|f(F_n) - f(F)| \geq \varepsilon\} \cap \{|F| \leq K\}) \\ &\quad + P(\{|f(F_n) - f(F)| \geq \varepsilon\} \cap \{|F| > K\}) \\ &\leq P(\{|F_n - F| \geq \delta\}) + P(\{|F| > K\}). \end{aligned} \tag{2.9}$$

From this inequality it follows that $V(f(F_n)) \rightarrow V(f(F))$ in probability. When combined with (2.8), this implies the second part of the theorem. \square

Remark 2.9. – By Theorem 2.8, we shall call V a probability structure preserving mapping from (Ω, \mathcal{F}, P) to $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

COROLLARY 2.10. – V is injective.

Proof. – Let F be measurable such that $VF = 0$ a.s. Then

$$P(|F| > 0) = \tilde{P}(|VF| > 0) = 0.$$

This implies that $F = 0$ a.s. \square

It is interesting to know the general conditions under which a probability structure preserving mapping is given by a measurable transform T , i.e., $V \circ F = F \circ T$.

Now let Γ be an invertible measurable mapping from Ω to itself and let Λ be an invertible measurable mapping from $\tilde{\Omega}$ to itself. Γ induces a mapping from \mathcal{E} to another functional space \mathcal{B} . This mapping is still denoted by Γ . Namely, $(\Gamma F)(\omega) = F \circ \Gamma(\omega)$, for all $F \in \mathcal{E}$ and $\omega \in \Omega$. Let Λ be the corresponding induced mapping from $\tilde{\mathcal{E}}$ to $\tilde{\mathcal{B}}$. We are going to establish

THEOREM 2.11. – *Let V be a probability structure preserving mapping from (Ω, \mathcal{F}, P) to $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. Let Γ (and Λ) be invertible measurable mapping from Ω (and $\tilde{\Omega}$) to themselves. Assume that the following diagram*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{V} & \tilde{\mathcal{E}} \\ \Gamma \downarrow & & \downarrow \Lambda \\ \mathcal{B} & \xrightarrow{V} & \tilde{\mathcal{B}} \end{array} \tag{2.10}$$

commutes. If $P \circ \Gamma^{-1}$ is absolutely continuous with respect to P , then $\tilde{P} \circ \Lambda^{-1}$ is absolutely continuous with respect to \tilde{P} . Moreover, the following identity holds:

$$\frac{d\tilde{P} \circ \Lambda^{-1}}{d\tilde{P}} = V \circ \left(\frac{dP \circ \Gamma^{-1}}{dP} \right). \tag{2.11}$$

Proof. – Assume that $P \circ \Gamma^{-1}$ is absolutely continuous with respect to P . Let $G \in \tilde{\mathcal{E}}$ and denote $F = V^{-1}G$. By the commutativity of the diagram (2.10) we obtain that

$$G \circ \Lambda = V \circ (F \circ \Gamma).$$

Denote

$$R_1(x) = \frac{d\tilde{P} \circ \Gamma^{-1}}{d\tilde{P}}(x), \quad x \in \Omega.$$

From the probability structure preserving property of V , it follows that

$$\begin{aligned} \int_{\tilde{\Omega}} G \circ \Lambda(y) \tilde{P}(dy) &= \int_{\Omega} F \circ \Gamma(x) P(dx) = \int_{\Omega} F(x) R_1(x) P(dx) \\ &= \int_{\tilde{\Omega}} V(FR_1)(y) \tilde{P}(dy) = \int_{\tilde{\Omega}} (VF)(y) (VR_1)(y) \tilde{P}(dy) \\ &= \int_{\tilde{\Omega}} G(y) (VR_1)(y) \tilde{P}(dy). \end{aligned}$$

This yields that

$$\frac{d\tilde{P} \circ \Lambda^{-1}}{d\tilde{P}}(y) = (VR_1)(y), \quad \text{for a.s. } y \in \tilde{\Omega},$$

proving the theorem. \square

Multiple stochastic integral over a Wiener space is well-defined (see [6] and the references therein). Let us recall that any element of $L^2(\Omega, \mathcal{F}, P)$ can be represented by its chaos expansion:

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f_n), \quad \text{with } \sum_{n=0}^{\infty} \frac{1}{n!} \|f_n\|_{H^{\otimes n}}^2 < \infty$$

where $f_n \in H^{\otimes n}$ (the symmetric tensor product Hilbert space over H) and I_n is the multiple Itô type multiple stochastic integral. It is known that

$$\mathbb{E}(F^2) = \sum_{n=0}^{\infty} \frac{1}{n!} \|f_n\|_{H^{\otimes n}}^2 < \infty.$$

The Fock space over H is a Hilbert space defined by

$$\Phi(H) = \{f = (f_0, f_1, \dots, f_n, \dots), f_n \in H^{\otimes n}\}$$

with the Hilbert norm

$$\|f\|_{\Phi(H)}^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \|f_n\|_{H^{\otimes n}}^2.$$

Thus we have an isometry between $L^2(\Omega, \mathcal{F}, P)$ and the Fock space $\Phi(H)$ over the Hilbert space H (see [6,12,16], and the references therein for more detail).

Let ρ be a one-to-one mapping from H to \tilde{H} such that

$$\langle \rho(h_1), \rho(h_2) \rangle_{\tilde{H}} = \langle h_1, h_2 \rangle_H.$$

Thus ρ induces a mapping from $H^{\otimes n}$ to $\tilde{H}^{\otimes n}$ in the following way: Let e_1, \dots, e_k, \dots be orthonormal system of H . Then $\tilde{e}_k = \rho(e_k)$, $k = 1, 2, \dots$, is orthonormal system of \tilde{H} . If $f = \sum a_{i_1 \dots i_n} e_{i_1} \otimes \dots \otimes e_{i_n}$, then we define

$$\rho^{\otimes n} f = \sum a_{i_1 \dots i_n} \tilde{e}_{i_1} \otimes \dots \otimes \tilde{e}_{i_n}.$$

It is easy to see that for any $f, g \in H^{\otimes n}$,

$$\langle \rho^{\otimes n} f, \rho^{\otimes n} g \rangle_{\tilde{H}^{\otimes n}} = \langle f, g \rangle_{H^{\otimes n}}.$$

For any element $f = (f_0, f_1, \dots, f_n, \dots)$ in $\Phi(H)$ we define

$$\Phi(\rho) f = (f_0, \rho(f_1), \dots, \rho^{\otimes n}(f_n), \dots).$$

Then it is easy to check that $\Phi(\rho)$ is an isometry between the Fock spaces $\Phi(H)$ and $\Phi(\tilde{H})$.

Since there is an isometry between $L^2(\Omega, \mathcal{F}, P)$ and $\Phi(H)$, we obtain an isometry between $L^2(\Omega, \mathcal{F}, P)$ and $L^2(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{P})$. The explicit form of this isometry may be described as follows: For any $F \in L^2(\Omega, \mathcal{F}, P)$,

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f_n).$$

Then $\tilde{F} = \Phi(\rho)F$ is given by

$$\tilde{F} = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\rho^{\otimes n} f_n).$$

THEOREM 2.12. – *Let ρ be an isometry between H and \tilde{H} . Then $\Phi(\rho)$ is a probability structure preserving mapping from (Ω, \mathcal{F}, P) to $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{P})$.*

Proof. – Recall that \mathcal{E} is the set of finite linear combinations of $\varepsilon(h) := \exp(\langle \cdot, h \rangle - \|h\|_H^2)$, $h \in H$ and that \mathcal{E} is an algebra and a dense subset of $L^2(\Omega, \mathcal{F}, P)$. It is easy to verify that $\mathbb{E}(F) = \tilde{\mathbb{E}}(\Phi(\rho)F)$ and $\Phi(\rho)(F + G) = \Phi(\rho)F + \Phi(\rho)G$. We need to prove

the product preserving property (2.4). It is easy to see that

$$\varepsilon(h) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(h^{\otimes n}).$$

Thus

$$\Phi(\rho)\varepsilon(h) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\rho(h)^{\otimes n}) = \varepsilon(\rho(h)).$$

On the other hand, it is easy to verify that

$$\begin{aligned} \varepsilon(h_1)\varepsilon(h_2) &= \exp[\langle \cdot, h_1 + h_2 \rangle - \|h_1\|_H^2 - \|h_2\|_H^2] \\ &= \varepsilon(h_1 + h_2) \exp(\langle h_1, h_2 \rangle_H). \end{aligned}$$

Thus

$$\begin{aligned} \Phi(\rho)[\varepsilon(h_1)\varepsilon(h_2)] &= \exp[\langle h_1, h_2 \rangle_H] \Phi(\rho)[\varepsilon(h_1 + h_2)] \\ &= \exp[\langle h_1, h_2 \rangle_H] \varepsilon(\rho(h_1 + h_2)) \\ &= \exp[\langle \rho(h_1), \rho(h_2) \rangle_H] \varepsilon(\rho(h_1) + \rho(h_2)) \\ &= \varepsilon(\rho(h_1))\varepsilon(\rho(h_2)) \\ &= [\Phi(\rho)\varepsilon(h_1)][\Phi(\rho)\varepsilon(h_2)]. \end{aligned}$$

From this the theorem follows. \square

Remark 2.13. – We shall call $\Phi(\rho)$ the probability structure preserving mapping induced by ρ .

LEMMA 2.14. – *If ρ is an isometry from the Hilbert space H to the Hilbert space \tilde{H} and if $V = \Phi(\rho)$ is the probability structure preserving mapping induced by ρ , then for any $F, G \in \mathcal{E}$,*

$$V(F \diamond G) = (VF) \diamond (VG), \tag{2.12}$$

where diamond denotes the Wick product (see [5]).

Proof. – Let $F = \varepsilon(h_1)$ and $G = \varepsilon(h_2)$, where $h_1, h_2 \in H$. We have

$$F \diamond G = \varepsilon(h_1 + h_2).$$

Hence

$$V(F \diamond G) = \varepsilon(\rho(h_1 + h_2)) = \varepsilon(\rho(h_1)) \diamond \varepsilon(\rho(h_2)) = (VF) \diamond (VG).$$

This proves the lemma through a linearity argument. \square

3. Stochastic integral for fractional Brownian motions

Let us recall some results from [19] and [11]. From now on (Ω, \mathcal{F}, P) will be the classical canonical Wiener space and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) = (\Omega, \mathcal{F}, P^H)$ will be the canonical space for fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

Fractional integrals of order $\alpha \in (0, 1)$ (of Riemann–Liouville type) of a function f on \mathbb{R} are defined as

$$I_{\pm}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} f(x \pm t) dt, \tag{3.1}$$

where $\Gamma(x)$ is the gamma function. They are also called fractional integral of Weyl type in [14]. When $\alpha = -\beta$ is negative, I_{\pm}^{α} will be the fractional derivatives (of Marchaud type)

$$I_{\pm}^{\alpha} f(x) = D_{\pm}^{\beta} f(x) = \frac{\beta}{\Gamma(1-\beta)} \int_0^{\infty} \frac{f(x) - f(x \mp t)}{t^{1+\beta}} dt. \tag{3.2}$$

We denote by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} e^{ix\xi} f(x) dx, \quad \xi \in \mathbb{R}$$

the Fourier transform of a function f . The following lemma is from [19], Theorem 7.1.

LEMMA 3.1. – *If $0 < \alpha < 1$ and $f \in L^1(\mathbb{R})$, then the Fourier transform of $I_{\pm}^{\alpha} f$ is given by*

$$\mathcal{F}(I_{\pm}^{\alpha} f)(\xi) = (\mp i\xi)^{-\alpha} \hat{f}(\xi), \quad \xi \in \mathbb{R}, \tag{3.3}$$

where

$$(\mp i\xi)^{-\alpha} = |\xi|^{-\alpha} e^{\mp \frac{\alpha\pi}{2} \text{sign}\xi}.$$

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions.

From the definition of $I_{\pm}^{\alpha} f$ and from this lemma it follows easily that if $f \in \mathcal{S}(\mathbb{R})$, then (3.3) holds for any real number α .

The following equation holds ([19], Eqs. (5.16) and (5.17)):

$$\int_{\mathbb{R}} g(x) I_{\pm}^{\alpha} f(x) dx = \int_{\mathbb{R}} f(x) I_{\mp}^{\alpha} g(x) dx \tag{3.4}$$

for all $f, g \in \mathcal{S}(\mathbb{R})$.

If $f \in \mathcal{S}(\mathbb{R})$ and $I_{-}^{\alpha}(f) = 0$, then by (3.3) we see that

$$(i\xi)^{\alpha} \hat{f}(\xi) = 0.$$

Thus $\hat{f}(\xi) = 0$ for almost all $\xi \in \mathbb{R}$. We have then $f = 0$. Denote

$$\mathcal{S}_{-}^{\alpha}(\mathbb{R}) = I_{-}^{\alpha}(\mathcal{S}(\mathbb{R})) = \{I_{-}^{\alpha}(f), f \in \mathcal{S}(\mathbb{R})\}.$$

Then I_{-}^{α} is a bijective linear transformation from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}_{-}^{\alpha}(\mathbb{R})$.

For any two elements f and g of $\mathcal{S}_{-}^{\alpha}(\mathbb{R})$, define

$$\langle f, g \rangle_{\Theta_H} := \frac{1}{2\pi} \int_{\mathbb{R}} |\xi|^{1-2H} \hat{f}(\xi) \hat{g}(\xi) d\xi, \quad \forall f, g \in \mathcal{S}(\mathbb{R}).$$

It is easy to verify that $\langle \cdot, \cdot \rangle_{\Theta_H}$ is an inner product on $\mathcal{S}^\alpha_-(\mathbb{R})$. Then $\mathcal{S}^\alpha_-(\mathbb{R})$ is a pre-Hilbert space with respect to the Hilbert norm induced by this inner product:

$$\|f\|_{\Theta_H} = \sqrt{\langle f, g \rangle_{\Theta_H}} = \sqrt{\int_{\mathbb{R}} |\xi|^{1-2H} |\hat{f}(\xi)|^2 d\xi}.$$

Let Θ_H denote its completion with respect to the Hilbert norm $\|\cdot\|_{\Theta_H}$. Therefore, Θ_H is a Hilbert space. For any $f \in \mathcal{S}(\mathbb{R})$ denote $g(x) = (I^{\frac{1}{2}-H}_- f)(x)$, $x \in \mathbb{R}$. Then by (3.3)

$$\hat{g}(\xi) = (i\xi)^{H-\frac{1}{2}} \hat{f}(\xi).$$

From the definition of the norm $\|\cdot\|_{\Theta_H}$ and Parserval identity

$$\begin{aligned} \|g\|_{\Theta_H}^2 &= \frac{1}{2\pi} \int_{\mathbb{R}} |\xi|^{1-2H} |\hat{g}(\xi)|^2 d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} |\xi|^{1-2H} |(i\xi)^{H-\frac{1}{2}} \hat{f}(\xi)|^2 d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} |f(x)|^2 dx = \|f\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

This means that

$$\|I^{\frac{1}{2}-H}_- f\|_{\Theta_H} = \|f\|_{L^2(\mathbb{R})}, \quad \forall f \in \mathcal{S}(\mathbb{R}).$$

Since $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and $\mathcal{S}^\alpha_-(\mathbb{R})$ is dense in Θ_H (by the definition of Θ_H) we can extend $I^{\frac{1}{2}-H}_-$ to an isometry between $L^2(\mathbb{R})$ and Θ_H .

Namely, we have

LEMMA 3.2. – $I^{1/2-H}_-$ can be extended to an isometry from $L^2(\mathbb{R})$ to Θ_H .

It is interesting to say more about the Hilbert space Θ_H .

PROPOSITION 3.3. – $\mathcal{S}(\mathbb{R})$ is a dense subset of Θ_H .

Proof. – Let g be an element of $\mathcal{S}(\mathbb{R})$ and define $f = I^{H-\frac{1}{2}}_- g$. Then

$$\hat{f}(\xi) = (i\xi)^{\frac{1}{2}-H} \hat{g}(\xi).$$

It is clear that

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} |\xi|^{1-2H} |\hat{g}(\xi)|^2 d\xi < \infty$$

since $g \in \mathcal{S}(\mathbb{R})$ and $H < 1$. Thus $f \in L^2(\mathbb{R})$. But

$$\mathcal{F}(I^{1/2-H}_- f)(\xi) = (i\xi)^{H-1/2} (i\xi)^{1/2-H} \hat{g}(\xi) = \hat{g}(\xi),$$

where $\mathcal{F}(g)$ denotes the Fourier transform of g . This means that $I^{1/2-H}_- f = g$. This implies that $\mathcal{S}(\mathbb{R})$ is a subset of Θ_H .

Now we need to show that $\mathcal{S}(\mathbb{R})$ is dense in Θ_H . First let us assume that $H < 1/2$. Let $\phi \in C^\infty(\mathbb{R})$ be a positive smooth function with compact support and such that

$$\int_{\mathbb{R}} \phi(x) dx = 1.$$

Denote

$$h(t) = \begin{cases} 0 & t > 0, \\ |t|^{-H-\frac{1}{2}}/\Gamma(\frac{1}{2}-H) & t < 0, \end{cases}$$

and

$$\phi_\varepsilon(x) = \frac{\phi(x/\varepsilon)}{\varepsilon}, \quad \varepsilon > 0, x \in \mathbb{R}.$$

Therefore ϕ_ε is C^∞ and with compact support (hence $\phi_\varepsilon \in \mathcal{S}(\mathbb{R})$). Consider

$$h_\varepsilon(t) = \phi_\varepsilon * h := \int_{\mathbb{R}} \phi_\varepsilon(x)h(t-x) dx,$$

where and in what follows $*$ denote the convolution. Then it is easy to see that h_ε is an element of $\mathcal{S}(\mathbb{R})$. For any function $f \in \mathcal{S}(\mathbb{R})$, define

$$f_\varepsilon(x) = (h_\varepsilon * f)(x).$$

Then f_ε is an element of $\mathcal{S}(\mathbb{R})$. On the other hand we have

$$f_\varepsilon = h_\varepsilon * f = (\phi_\varepsilon * h) * f = \phi_\varepsilon * (h * f).$$

By Eq. (5.4) of [19], we see that

$$h * f = I^{1/2-H} f.$$

Namely,

$$f_\varepsilon = \phi_\varepsilon * (I^{1/2-H} f).$$

Applying the Fourier transformation, we obtain

$$\hat{f}_\varepsilon(\xi) = \hat{\phi}_\varepsilon(\xi)(i\xi)^{H-1/2} \hat{f}(\xi).$$

It is easy to verify that

$$\hat{\phi}_\varepsilon(\xi) \rightarrow 1 \quad \text{and} \quad |\hat{\phi}_\varepsilon(\xi)| \leq 1$$

for almost all ξ . Thus by Lebesgue's dominate convergence theorem, we see that

$$\hat{f}_\varepsilon(\xi) = \hat{\phi}_\varepsilon(\xi)(i\xi)^{H-1/2} \hat{f}(\xi)$$

converges to $(i\xi)^{H-1/2} \hat{f}(\xi)$ in $L^2(\mathbb{R})$ as $\varepsilon \rightarrow 0$. Hence f_ε converges to $I^{1/2-H} f$ in Θ_H as $\varepsilon \rightarrow 0$. This implies that $\mathcal{S}(\mathbb{R})$ is dense in Θ_H . When $H > 1/2$, we need to use $I^{1/2-H} = I^{3/2-H} \frac{d}{dx}$. In this case we define

$$h(t) = \begin{cases} 0 & t > 0, \\ |t|^{1/2-H} / \Gamma(\frac{3}{2} - H) & t < 0, \end{cases}$$

and

$$h_\varepsilon(t) = \phi_\varepsilon * h = \int_{\mathbb{R}} \phi_\varepsilon(x) h(t - x) dx,$$

which is an element of $\mathcal{S}(\mathbb{R})$, where ϕ_ε is defined as above. For any function $f \in \mathcal{S}(\mathbb{R})$, define

$$f_\varepsilon(x) = (h_\varepsilon * f')(x),$$

where f' is the derivative of f . Then f_ε is an element of $\mathcal{S}(\mathbb{R})$. On the other hand we have

$$f_\varepsilon = h_\varepsilon * f = (\phi_\varepsilon * h) * f' = \phi_\varepsilon * (h * f').$$

By Eq. (5.4) of [19], we see that

$$h * f = I^{3/2-H} f' = I^{1/2-H} f.$$

Namely,

$$f_\varepsilon = \phi_\varepsilon * (I^{1/2-H} f).$$

A similar argument can be applied to show that f_ε converges to $I^{1/2-H} f$ in Θ_H as $\varepsilon \rightarrow 0$. This implies that $\mathcal{S}(\mathbb{R})$ is dense in Θ_H in the case $H > 1/2$. \square

From the above argument, we also conclude that the isometry $I_-^{1/2-H} : L^2(\mathbb{R}) \rightarrow \Theta_H$ has the inverse which coincides with $I_-^{H-1/2}$ on $\mathcal{S}(\mathbb{R})$. We denote this inverse by $I_-^{H-1/2}$, which is an isometry from Θ_H to $L^2(\mathbb{R})$.

In [18], some other types of space were introduced. For example, it is denoted that

$$\begin{aligned} \Lambda_H &= \left\{ f: \int_{\mathbb{R}} [(I_-^{H-1/2} f)(s)]^2 ds < \infty \right\} \\ &= \left\{ f: \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(u) (u-s)_+^{H-3/2} \right]^2 ds < \infty \right\}, \end{aligned}$$

when $H > 1/2$ and

$$\Lambda_H = \{ f: \exists \phi \in L^2(\mathbb{R}) \text{ such that } f = I_-^{1/2-H} \phi \}$$

when $H < 1/2$. The property of these and other relevant spaces are studied also. It is clear from Theorem 3.2 and Theorem 3.3 of [18] that Λ_H is a subset of Θ_H . However, Λ_H is not appropriate in our paper since it is not a Hilbert space.

In recent years, there has been a boom of study of a particular family of Gaussian processes, called *fractional Brownian motions*. Fractional Brownian motion with *Hurst parameter* $H \in (0, 1)$ has been introduced by Mandelbrot and Van Ness as the *fractional derivative* of Brownian motion. More precisely, fBm $B^H = (B_t^H, t \in \mathbb{R}_+)$ with Hurst parameter $H \in (0, 1)$ is defined as

$$\begin{aligned}
 B_t^H &= \frac{1}{\Gamma(H + 1/2)} \int_{-\infty}^0 [(t - s)^{H-1/2} - (-s)^{H-1/2}] dB_s \\
 &\quad + \int_0^t (t - s)^{H-1/2} dB_s, \quad t \in \mathbb{R}_+,
 \end{aligned}
 \tag{3.5}$$

where $B = (B_s, s \in \mathbb{R})$ is a Wiener process on some probability space (Ω, \mathcal{F}, P) . This probability space will be fixed. The expectation on (Ω, \mathcal{F}, P) is denoted by \mathbb{E} . The fractional Brownian motion satisfies

$$\mathbb{E}(B_t^H B_s^H) = c_H (|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

where $c_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-H)}$.

As in [11], heuristically we may write (3.5) as

$$B_t^H = I_+^{H+1/2}(\dot{B})(t) - I_+^{H+1/2}(\dot{B})(0).$$

[Each of these two terms may not be well-defined.] Thus we have

$$\dot{B}_t^H = \frac{d}{dt} I_+^{H+1/2}(\dot{B})(t) = I_+^{H-1/2}(\dot{B})(t).$$

Hence formally, \dot{B}^H is the fractional integral of order $H - 1/2$ of the white noise \dot{B} when $H > 1/2$ and \dot{B}^H is the fractional derivative of order $1/2 - H$ of the white noise \dot{B} when $H < 1/2$.

Using the adjoint operator (i.e., $I_+^{H-1/2}$) of $I_+^{H+1/2}$, we know that if $f \in \Theta_H$, then $\int_{\mathbb{R}} f(t) dB_t^H$ may be well defined by

$$\int_{\mathbb{R}} f(t) dB_t^H = \int_{\mathbb{R}} I_-^{H-1/2}(f)(t) dB_t.
 \tag{3.6}$$

It is easy to see that

$$\mathbb{E}\left(\int_{\mathbb{R}} f(t) dB_t^H\right) = 0, \quad \mathbb{E}\left(\int_{\mathbb{R}} f(t) dB_t^H \int_{\mathbb{R}} g(t) dB_t^H\right) = \langle f, g \rangle_{\Theta_H}.$$

(See [11] for more discussion.)

From now on we denote by V the probability structure preserving mapping between $L^2(\Omega, \mathcal{F}, P)$ and $L^2(\Omega, \mathcal{F}, P^H)$ induced by $I_-^{1/2-H}$.

Eq. (3.6) can also be used to define the stochastic integral for general random kernel.

DEFINITION 3.4. – Let $f : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ be a stochastic process not necessarily adapted. Assume that for almost every $\omega \in \Omega$, $f(\cdot, \omega) \in \Theta_H$. If $\int_{\mathbb{R}} I_-^{1/2-H} V^{-1} f(t) dB_t$ exists as a random variable, then we say that $\int_{\mathbb{R}} f(t) dB_t^H$ exists and define the anticipative stochastic integral by

$$\int_{\mathbb{R}} f(t) dB_t^H = V \left(\int_{\mathbb{R}} I_-^{H-1/2} (V^{-1} f)(t) dB_t \right). \tag{3.7}$$

PROPOSITION 3.5. – If $H > 1/2$ and if $\int_{\mathbb{R}} f(t) dB_t^H$ is well-defined in the sense of [5], then this definition (3.7) coincides with the definition introduced in [5].

Proof. – Let F be an exponential function on $(\Omega, \mathcal{F}, P^H)$. Consider $f(t) = \chi_{(a,b)}(t) F$, $t \in \mathbb{R}$, where $-\infty < a < b < \infty$ are given numbers. Then by the definition of [5], we have $\int_{\mathbb{R}} f(t) d\tilde{B}_t^H = F \diamond (B_b^H - B_a^H)$, where \diamond denotes the Wick product. Here we temporarily denote the stochastic integral in [5] by $\int_{\mathbb{R}} f(t) d\tilde{B}_t^H$. On the other hand, by (3.7),

$$\begin{aligned} \int_{\mathbb{R}} f(t) dB_t^H &= V \left(\int_{\mathbb{R}} (V^{-1} F) [I_-^{H-1/2} \chi_{(a,b)}](t) dB_t \right) \\ &= V \left(V^{-1} F \diamond \int_{\mathbb{R}} [I_-^{H-1/2} \chi_{(a,b)}](t) dB_t \right) \\ &= F \diamond \int_{\mathbb{R}} \chi_{(a,b)}(t) dB_t^H = F \diamond (B_b^H - B_a^H). \end{aligned}$$

Hence $\int_{\mathbb{R}} f(t) d\tilde{B}_t^H = \int_{\mathbb{R}} f(t) dB_t^H$ for step functions. The proposition follows from linearity and a limiting argument. \square

Let e_1, \dots, e_k, \dots be an ONB of $L^2(\mathbb{R})$ such that $e_k \in \mathcal{S}(\mathbb{R})$, $k = 1, 2, \dots$. Then $\{\tilde{e}_1, \tilde{e}_2, \dots\}$ is an ONB of Θ_H . The Malliavin derivative D_s of a smooth functional $F = f(\int_{\mathbb{R}} e_1(t) dB_t, \dots, \int_{\mathbb{R}} e_k(t) dB_t)$ is defined as

$$D_s F = \sum_{n=1}^k \frac{\partial f}{\partial x_n} \left(\int_{\mathbb{R}} e_1(t) dB_t, \dots, \int_{\mathbb{R}} e_k(t) dB_t \right) e_n(s).$$

Now let $G = g(\int_{\mathbb{R}} \tilde{e}_1(t) dB_t^H, \dots, \int_{\mathbb{R}} \tilde{e}_k(t) dB_t^H)$. We define the derivative D_s^H by

$$D_s^H G = \sum_{n=1}^k \frac{\partial f}{\partial x_n} \left(\int_{\mathbb{R}} \tilde{e}_1(t) dB_t^H, \dots, \int_{\mathbb{R}} \tilde{e}_k(t) dB_t^H \right) \tilde{e}_n(s).$$

In the case of no ambiguity (as it usually is) we omit the dependence on H in D_t^H .

As in [15] (see also [13,17]) we denote by $\mathbb{L}^{1,2}(\Omega, \mathcal{F}, P)$ the class of processes $u \in L^2(T \times \Omega)$ on the probability space (Ω, \mathcal{F}, P) such that $u(t) \in D^{1,2}$ for almost all t , and that

$$\int_{\mathbb{R}^2} (D_s u(t))^2 ds dt < \infty.$$

This means that

$$\mathbb{L}^{1,2}(\Omega, \mathcal{F}, P) = \left\{ f : \int_{\mathbb{R}} \mathbb{E}|f(t)|^2 dt + \int_{\mathbb{R}^2} \mathbb{E}|D_s f(t)|^2 ds dt < \infty \right\}.$$

The following proposition will be useful.

PROPOSITION 3.6. – *Let $f \in \mathbb{L}^{1,2}(\Omega, \mathcal{F}, P)$. Then*

$$V \left(\int_{\mathbb{R}} f(t) dB_t \right) = \int_{\mathbb{R}} [I_-^{1/2-H}(Vf)](t) dB_t^H. \tag{3.8}$$

Proof. – Since for a.a. $\omega \in \Omega$, $f(\cdot, \omega) \in L^2(\mathbb{R})$. $Vf(\cdot, \omega) \in L^2(\mathbb{R})$. Thus

$$g(\cdot, \omega) := I_-^{\frac{1}{2}-H}(Vf)(\cdot, \omega) \in \Theta_H$$

and

$$I_-^{H-\frac{1}{2}}g(\cdot, \omega) = (Vf)(\cdot, \omega).$$

Consequently,

$$I_-^{H-\frac{1}{2}}(V^{-1}g) = f.$$

By definition (3.4) we obtain

$$\begin{aligned} \int_{\mathbb{R}} (I_-^{1/2-H})(Vf)(t) dB_t^H &= \int_{\mathbb{R}} g(t) dB_t^H = V \left(\int_{\mathbb{R}} I_-^{H-1/2}(V^{-1}g)(t) dB_t \right) \\ &= V \left(\int_{\mathbb{R}} f(t) dB_t \right). \end{aligned}$$

This proves the proposition. \square

LEMMA 3.7. – *Let $F \in \mathbb{L}^{1,2}(\Omega, \mathcal{F}, P)$. Then*

$$VD_s F = I_-^{H-1/2}(D)_s(VF), \tag{3.9}$$

where $I_-^{H-1/2}(D^H)_s G$ denotes the application of $I_-^{H-1/2}$ to $D_s^H G$ (as a function of s).

Proof. – Let $F = \exp(\int_{\mathbb{R}} h(s) dB_s)$. Then

$$D_s F = h(s)F.$$

Thus

$$VD_s F = h(s) \exp \left(\int_{\mathbb{R}} I_-^{1/2-H} h(s) dB_s^H \right).$$

On the other hand, we have

$$VF = \exp\left(\int_{\mathbb{R}} I_-^{1/2-H} h(s) dB_s^H\right).$$

Hence

$$D_s^H VF = I_-^{1/2-H} h(s) \exp\left(\int_{\mathbb{R}} I_-^{1/2-H} h(s) dB_s^H\right).$$

This proves that $VD_s F = I_-^{H-1/2}(D^H)_s VF$ for exponential functions. By linearity of V and D_s , we can conclude that (3.9) is true for all $F \in \mathcal{E}$. The theorem is proved by a limiting argument. \square

LEMMA 3.8. – *Let $\Gamma : \Omega \rightarrow \Omega$ and $\Lambda : \tilde{\Omega} \rightarrow \tilde{\Omega}$ be measurable mappings such that the diagram (2.10) commutes. Let $V(F \circ \Gamma) = (VF) \circ \Lambda$ for any $F \in \mathcal{E}$. Then for all $F \in \mathcal{E}$,*

$$V(F \circ \Gamma^{-1}) = (VF) \circ \Lambda^{-1}. \tag{3.10}$$

Proof. – From $F = F \circ \Gamma^{-1} \circ \Gamma$, it follows that

$$V \circ F = V \circ (F \circ \Gamma^{-1} \circ \Gamma) = V \circ (F \circ \Gamma^{-1}) \circ \Lambda.$$

Thus $V \circ (F \circ \Gamma^{-1}) = (V \circ F) \circ \Lambda^{-1}$. \square

LEMMA 3.9. – *Let the assumptions of Lemma 3.8 be satisfied and let Γ be differentiable in the sense that for any $F \in \mathbb{L}^{1,2}$, $F \circ \Gamma$ is in $\mathbb{L}^{1,2}$. Then*

$$V(D_s(F \circ \Gamma^{-1}) \circ \Gamma) = I_-^{H-1/2}(D^H)_s [(VF) \circ \Lambda^{-1}] \circ \Lambda. \tag{3.11}$$

Proof. – From Lemmas 3.7 and 3.8, it follows that

$$\begin{aligned} V(D_s(F \circ \Gamma^{-1}) \circ \Gamma) &= [VD_s(F \circ \Gamma^{-1})] \circ \Lambda \\ &= I_-^{H-1/2}(D^H)_s (V(F \circ \Gamma^{-1})) \circ \Lambda \\ &= I_-^{H-1/2}(D^H)_s [(VF) \circ \Lambda^{-1}] \circ \Lambda. \end{aligned}$$

This proves the lemma. \square

4. Application to absolute continuity

Now consider an (anticipative) translation of B^H

$$\Lambda: B^H + \int_0^\cdot f(s, B^H) ds. \tag{4.1}$$

Define an anticipative translation of B by

$$\Gamma: B + \int_0^\cdot g(s, B) ds, \tag{4.2}$$

where $g = I_+^{1/2-H} V^{-1} f$. Thus f can also be computed from g by $f = I_+^{H-1/2} V g$.

LEMMA 4.1. – *The following diagram*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{V} & \tilde{\mathcal{E}} \\ \Gamma \downarrow & & \downarrow \Lambda \\ \mathcal{B} & \xrightarrow{V} & \tilde{\mathcal{B}} \end{array} \tag{4.3}$$

commutes, where \mathcal{E} is the set of finite linear combinations of exponential functionals on (Ω, \mathcal{F}, P) and $\tilde{\mathcal{E}}$ is the set of finite linear combinations of exponential functionals on $(\Omega, \mathcal{F}, P^H)$ and $\mathcal{B} = \Gamma \mathcal{E}$ and $\tilde{\mathcal{B}} = \Lambda \tilde{\mathcal{E}}$.

Proof. – Denote $\tilde{g} = V^{-1} f$. Let $F = \exp(\int_{\mathbb{R}} h(s) dB_s)$, where $h \in L^2(\mathbb{R})$. As discussed before we denote $T = I_-^{1/2-H}$. Then

$$\begin{aligned} \Gamma F &= \exp \left[\int_{\mathbb{R}} h(s) dB_s + \int_{\mathbb{R}} h(s) g(s) ds \right] \\ &= \exp \left[\int_{\mathbb{R}} h(s) dB_s + \int_{\mathbb{R}} h(s) I_+^{1/2-H} \tilde{g}(s) ds \right] \\ &= \exp \left[\int_{\mathbb{R}} h(s) dB_s + \int_{\mathbb{R}} (Th)(s) \tilde{g}(s) ds \right]. \end{aligned}$$

Thus

$$\begin{aligned} V \Gamma F &= \exp \left[\int_{\mathbb{R}} (Th)(s) dB_s^H + \int_{\mathbb{R}} (Th)(s) V \tilde{g}(s) ds \right] \\ &= \exp \left[\int_{\mathbb{R}} (Th)(s) dB_s^H + \int_{\mathbb{R}} (Th)(s) f(s) ds \right]. \end{aligned}$$

On the other hand,

$$V F = \exp \left[\int_{\mathbb{R}} (Tf)(s) dB_s^H \right].$$

Therefore, we have

$$\Lambda V F = \exp \left[\int_{\mathbb{R}} (Th)(s) dB_s^H + \int_{\mathbb{R}} (Th)(s) f(s) ds \right].$$

This shows that the diagram commute for exponential functionals. The lemma is completed by a linearity argument. \square

LEMMA 4.2. – *Let $\Gamma : \Omega \rightarrow \Omega$ and $\Lambda : \tilde{\Omega} \rightarrow \tilde{\Omega}$ be defined by (4.1) and (4.2). Then for all measurable F ,*

$$V(F \circ \Gamma) = (VF) \circ \Lambda. \tag{4.4}$$

Proof. – Let $F = \exp\{\int h(s) dB_s\}$. Then

$$\begin{aligned} F \circ \Gamma &= \exp\left\{\int h(s) dB_s + \int h(s) I_+^{1/2-H} V^{-1} f(s) ds\right\} \\ &= \exp\left\{\int h(s) dB_s + \int (I_-^{1/2-H} h)(s) V^{-1} f(s) ds\right\}. \end{aligned}$$

Consequently,

$$V(F \circ \Gamma) = \exp\left\{\int (I_-^{1/2-H} h)(s) dB_s^H + \int (I_-^{1/2-H} h)(s) f(s) ds\right\} = (VF) \circ \Lambda.$$

This proves the lemma for exponential functional. The lemma follows from a linearity argument. \square

Let us recall a result on Radon–Nikodym derivative. The following results can be found in [2] when the interval is $[0, 1]$.

THEOREM 4.3. – *Let f and g be as in (4.1)–(4.2) such that f and g are in $\mathbb{L}^{1,2}$. Let the following conditions be satisfied*

(i) *There is a positive number $\gamma \in (0, 1)$ such that*

$$\int_{\mathbb{R}^2} |D_t g(s)|^2 ds dt \text{ is bounded by } \gamma. \tag{4.5}$$

(ii) *There is a positive number $q > 1$ with*

$$\mathbb{E}\left[\exp\left\{\frac{q}{2} \int_{\mathbb{R}} g^2(s) ds\right\}\right] < \infty. \tag{4.6}$$

Then Γ is invertible and $P \circ \Gamma^{-1}$ is absolutely continuous with respect to P . Moreover, the following identity is true:

$$\frac{dP \circ \Gamma^{-1}}{dP} = \kappa \exp\left\{-\int_{\mathbb{R}} g(s) dB_s - \frac{1}{2} \int_{\mathbb{R}} g^2(s) ds\right\}, \tag{4.7}$$

with

$$\kappa = \exp\left\{-\int_{\mathbb{R}} \int_0^s D_s g(r) D_r [g(s, \Gamma_s^{-1})] \circ \Gamma_s\right\}, \tag{4.8}$$

where Γ_s is defined by

$$\Gamma_s : B \rightarrow B + \int_0^{s \wedge \cdot} g(u) du$$

and Γ_s^{-1} is the inverse of Γ_s .

To obtain a Radon–Nikodym derivative for fractional Brownian motions, we define

$$h(u) := I_+^{H-1/2} (I_{[0,s]} I_+^{1/2-H} f)(u), \quad u \in \mathbb{R} \tag{4.9}$$

and

$$\Lambda_s : B^H \rightarrow B^H + \int_0^\cdot h(u) du. \tag{4.10}$$

The main theorem of this section is

THEOREM 4.4. – *Let the following conditions be satisfied*

(i) *There is a positive number $\gamma \in (0, 1)$ such that*

$$\int_{\mathbb{R}^2} |I_-^{H-1/2} (D^H)_s I_+^{1/2-H} (f)(s)|^2 ds dt \text{ is bounded by } \gamma. \tag{4.11}$$

(ii) *There is a positive number $q > 1$ with*

$$\mathbb{E} \left[\exp \left\{ \frac{q}{2} \|f\|_{\Theta^H}^2 \right\} \right] < \infty. \tag{4.12}$$

Then Γ is invertible and $P^H \circ \Lambda^{-1}$ is absolutely continuous with respect to P^H . Moreover, the following identity is true:

$$\frac{dP^H \circ \Lambda^{-1}}{dP^H} = \mathcal{L} = \tilde{\kappa} \exp \left\{ - \int_{\mathbb{R}} I_-^{1/2-H} I_+^{1/2-H} f(s) dB_s^H - \frac{1}{2} \|f\|_{\Theta^H}^2 \right\}, \tag{4.13}$$

where

$$\begin{aligned} \tilde{\kappa} = \exp \left\{ - \int_{\mathbb{R}} \int_0^s I_-^{H-1/2} (D^H)_s I_+^{1/2-H} (f)(r) I_-^{H-1/2} (D^H)_r \right. \\ \left. \times (I_+^{1/2-H} f)(s, \Lambda_s) \circ \Lambda_s^{-1} ds dr \right\}, \end{aligned} \tag{4.14}$$

where Λ_s is defined by

$$\Lambda_s : B^H \rightarrow B^H + \int_0^\cdot h(u) du$$

and

$$h(u) := I_+^{H-1/2} (I_{[0,s]} I_+^{1/2-H} f)(u), \quad u \in \mathbb{R},$$

and Λ_s^{-1} is the inverse of Λ_s .

Proof. – By Lemma 3.7, we have

$$\begin{aligned} V \left\{ \int_{\mathbb{R}^2} |D_t g(s)|^2 ds dt \right\} &= V \int_{\mathbb{R}^2} |D_t I_+^{1/2-H} V^{-1} f|^2 ds dt \\ &= \int_{\mathbb{R}^2} |V D_t I_+^{1/2-H} V^{-1} f|^2 ds dt \\ &= \int_{\mathbb{R}^2} |I_-^{H-1/2} (D^H)_t I_+^{1/2-H} f|^2 ds dt. \end{aligned}$$

Thus (4.11) implies that (4.5) holds. By the probability structure preserving property of V , we obtain

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ \frac{q}{2} \int_{\mathbb{R}} g^2(s) ds \right\} \right] &= \mathbb{E} \left[V \exp \left\{ \frac{q}{2} \int_{\mathbb{R}} g^2(s) ds \right\} \right] \\ &= \mathbb{E} \left[\exp \left\{ \frac{q}{2} \int_{\mathbb{R}} |I_-^{1/2-H} f(s)|^2 ds \right\} \right] \\ &= \mathbb{E} \left[\exp \left\{ \frac{q}{2} \|f\|_{\Theta_H}^2 \right\} \right]. \end{aligned}$$

Hence (4.12) implies that (4.6) is true. Therefore under the assumptions of Theorem 4.4, $\frac{dP \circ \Gamma^{-1}}{dP}$ exists. It suffices to compute $V \circ \frac{dP \circ \Lambda^{-1}}{dP}$. First let’s make the following computation.

$$\begin{aligned} &V \exp \left\{ - \int_{\mathbb{R}} g(s) dB_s - \frac{1}{2} \int_{\mathbb{R}} g^2(s) ds \right\} \\ &= \exp \left\{ - \int_{\mathbb{R}} I_-^{1/2-H} V g(s) dB_s^H - \frac{1}{2} \int_{\mathbb{R}} V g^2(s) ds \right\} \\ &= \exp \left\{ - \int_{\mathbb{R}} I_-^{1/2-H} I_+^{1/2-H} f(s) dB_s^H - \frac{1}{2} \int_{\mathbb{R}} |I_+^{1/2-H} f|^2(s) ds \right\} \\ &= \exp \left\{ - \int_{\mathbb{R}} I_-^{1/2-H} I_+^{1/2-H} f(s) dB_s^H - \frac{1}{2} \|f\|_{\Theta_H}^2 \right\}. \end{aligned}$$

Next we have to evaluate $\tilde{\kappa} = V\kappa$. Similar to Lemmas 4.1 and 4.2, we can prove that the following diagram

$$\begin{array}{ccc}
 \mathcal{E} & \xrightarrow{V} & \tilde{\mathcal{E}} \\
 \downarrow \Gamma_s & & \downarrow \Lambda_s \\
 \mathcal{B} & \xrightarrow{V} & \tilde{\mathcal{B}}
 \end{array} \tag{4.15}$$

commutes and that $V(F \circ \Gamma_s) = (VF) \circ \Lambda_s$ holds. By Lemma 3.7,

$$V(D_s g(r)) = I_-^{H-1/2} (D^H)_s [Vg(r)] = I_-^{H-1/2} (D^H)_s I_+^{1/2-H} (f)(r).$$

From Lemma 3.9 it follows that

$$\begin{aligned}
 V(D_r(g(s, \Gamma_s^{-1})) \circ \Gamma_s) &= I_-^{H-1/2} (D^H)_r [(Vg) \circ \Lambda_s^{-1}] \circ \Lambda_s \\
 &= I_-^{H-1/2} (D^H)_r (I_+^{1/2-H} f)(s, \Lambda_s^{-1}) \circ \Lambda_s.
 \end{aligned}$$

Thus

$$\begin{aligned}
 V\kappa = \exp \left\{ - \int_{\mathbb{R}} \int_0^s I_-^{H-1/2} (D)_s I_+^{1/2-H} (f)(r) I_-^{H-1/2} (D)_r \right. \\
 \left. \times (I_+^{1/2-H} f)(s, \Lambda_s) \circ \Lambda_s^{-1} ds dr \right\}. \tag{4.16}
 \end{aligned}$$

This proves the theorem. \square

Remark 4.5. – (a) If f is deterministic, then $D_s f = 0$. Thus $\tilde{\kappa} \equiv 1$. Therefore if $f \in \Theta_h$ is deterministic, then

$$\frac{dP^H \circ \Lambda^{-1}}{dP^H} = \exp \left\{ - \int_{\mathbb{R}} I_-^{1/2-H} I_+^{1/2-H} f(s) dB_s^H - \frac{1}{2} \|f\|_{\Theta_H}^2 \right\}. \tag{4.17}$$

By elementary results from the fractional calculus, we see that this formula coincides with the formula in [11]. This is in fact Cameron–Martin formula. See [16] for general Gaussian process case.

(b) To obtain the Girsanov formula for finite interval, one needs to compute the conditional expectation as indicated in [11].

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