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REMARKS ON t-TRANSFORMATIONS OF MEASURES AND CONVOLUTIONS ☆

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ABSTRACT. – A family of transformations of probability measures is constructed, and used to define transformations of convolutions. The relations between moments and cumulants of a measure and its transformation are presented. For transformed classical and free convolutions the central limit measures and the Poisson type limit measures are computed. Families of non-commutative random variables are constructed, which are associated to these central limit measures. They provide examples of "position operators" which act on the Interacting Fock Spaces. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Une famille de transformations de mesures de probabilités est construite et utilisée pour définir des transformations des convolutions. On présente les relations entre les moments et cumulants d'une mesure et de sa transformation. Pour les convolutions transformées classiques et libres les mesures limites du théorème limite central et les mesures limites du théorème limite Poisson sont calculées. On construit des familles de variables aléatoires non-commutatives qui sont associées à ces mesures limites. Elles sont une source d'exemples d'espaces de Fock interactifs. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

This paper is devoted to the study of a family of transformations of measures, which we call t-transformations, with $t > 0$. These transformations were introduced in [8], where we described some basic properties of them. Here our main focus is on finding realization of non-commutative random variables with distribution given by transformed free and classical central limit theorems. We also describe moments of these central limit measures, and compute the Poisson type limit theorems for both cases.

Through our transformation we obtain, for $t = 1 - \frac{1}{2N}$, the measures

$$\mathrm{d}\nu_N(x) = \frac{1}{\pi} \cdot \frac{\sqrt{2N - 1 - N^2x^2}}{1 - x^2}\,\mathrm{d}x$$

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with $|x| \leqslant \frac{\sqrt{2N-1}}{N}$, which were found by Kesten in [9] as the spectral measures of the simple random walks on free groups on $N \geqslant 2$ free generators.

The t-transformation of a probability measure can be also thought of as the tth Boolean convolution power of the measure, since it transforms the Boolean cumulants by multiplication by the positive number t. This is discussed in Section 4. We also exhibit, in Section 10, the relation between moments and cumulants for these convolutions.

In Section 6 we describe moments of the central limit measure for the t-transformed free convolution, which turn out to be polynomials in t with coefficients taken form the *Delaney's table*. For the moments of the central limit theorems for t-transformed classical convolution we show in Section 8 a combinatorial formula in which *outer connected components* of (general) partitions play a role.

Our interest is also focused on constructing non-commutative random variables with distributions given by the central limit measures. In the t-free case the random variables act on transformed full Fock space, and provide examples of the *Interacting Fock Spaces*, introduced by Accardi, Lu and Volovich in [2], and studied in particular in [1]. Recently Xu proved in [15] the condition for interacting Fock space equivalent to tracial property of the vacuum state on the von Neumann algebra generated by position operators. It turns out that our interacting Fock spaces do not satisfy this condition and, equivalently, the vacuum state is not tracial. The associated von Neumann algebras are studied elsewhere.

Finally, in Section 11, we study the Poisson type limit theorems for both types of convolutions.

2. Cauchy transform, continued fractions and orthogonal polynomials

For a given probability measure μ on the real line $\mathbb{R}$, its Cauchy transform G_μ is defined for $z \in \mathbb{C}^+ = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$, by:

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\mu(x)}{z-x} \tag{2.1}$$

and, if the measure μ has all moments, then by a theorem of Stieltjes (see [4]), it can be expressed as a continued fraction:

$$G_\mu(z) = \cfrac{1}{z - a_1 - \cfrac{b_1}{z - a_2 - \cfrac{b_2}{z - a_3 - \cfrac{b_3}{z - a_4 - \cfrac{b_4}{z - a_5 - \cfrac{b_5}{\ddots}}}}}} \tag{2.2}$$

The real numbers $a_1, a_2, \ldots$ and the non-negative numbers $b_1, b_2, b_3, \ldots$ come from the recurrence formula for the polynomials orthogonal with respect to the measure μ.

Namely, a finite part of the continued fraction is a rational function, thus a quotient of two polynomials:

$$\frac{Q_n(z)}{P_{n+1}(z)} = \cfrac{1}{z - a_1 - \cfrac{b_1}{z - a_2 - \cfrac{b_2}{z - a_3 - \cfrac{b_3}{z - a_4 - \ddots \cfrac{b_n}{z - a_{n+1}}}}}} \tag{2.3}$$

These polynomials satisfy the following recurrence relations:

$$(x - a_{n+1})P_n(x) = P_{n+1}(x) + b_n P_{n-1}(x)$$

and

$$(x - a_{n+2})Q_n(x) = Q_{n+1}(x) + b_{n+1} Q_{n-1}(x)$$

for $n \geqslant 1$, with $P_0(x) = 1 = Q_0(x)$, $P_1(x) = x - a_1$, $Q_1(x) = x - a_2$.

The polynomials $\{P_n \colon n \geqslant 0\}$ are orthonormal with respect to the measure μ, and the polynomials $\{Q_n \colon n \geqslant 0\}$ are associated ones. The most famous is the example of the Chebyshev polynomials of the first kind and of the second kind, where $\mu(\mathrm{d}x) = \frac{1}{2\pi}\frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x$, and $P_n(x) = \cos(n \cdot \arccos x)$, $Q_n(x) = \frac{1}{n+1}P'_{n+1}(x)$.

3. t-transformation of measures

We now define a family $\{U_t\}_{t\geqslant 0}$ of transformations, which act on probability measures on the real line. The definition is based on properties of the Cauchy transform. For the sake of completeness we recall here basic facts concerning this definition. More details can be found in [8].

Let t be a non-negative real number and let μ be a probability measure on the real line. Then the function $G_{\mu_t}(z)$ defined by the formula:

$$\frac{1}{G_{\mu_t}(z)} = \frac{t}{G_\mu} + (1-t)z \tag{3.4}$$

turns out to be the Cauchy transform of a probability measure denoted by $U_t(\mu) := \mu_t$. This is a consequence of the following:

THEOREM 3.1 (Nevanlinna). – *A function $F(z)$ is the reciprocal of the Cauchy transform of a probability measure on the real line if and only if there exists a positive measure ρ and a constant a such that for* $\mathrm{Im}(z) > 0$

$$F(z) = a + z + \int_{-\infty}^{+\infty} \frac{1 + xz}{x - z}\,\mathrm{d}\rho(x). \tag{3.5}$$

COROLLARY 3.2. – *For a pair of probability measures ρ and ν on the real line, and a real number $0 \leqslant t \leqslant 1$ there exists a probability measure μ such that*

$$\frac{1}{G_\mu(z)} = \frac{t}{G_\nu} + \frac{(1-t)}{G_\rho}. \tag{3.6}$$

This follows directly from the Nevanlinna's theorem. For our special choice of the measure $\rho = \delta_0$, we get a little more:

COROLLARY 3.3. – *For a given probability measure μ and a non-negative number $t \geqslant 0$, there exists a* (*unique*) *probability measure μ_t such that*

$$\frac{1}{G_{\mu_t}(z)} = \frac{t}{G_\mu} + (1-t)z. \tag{3.7}$$

Proof. – It follows from the Nevanlinna's theorem, that

$$\frac{t}{G_\mu} + (1-t)z = tF(z) + (1-t)z = ta + z + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z}\,\mathrm{d}(t\rho)(x) \tag{3.8}$$

is the reciprocal of the Cauchy transform of a probability measure, denoted by μ_t. $\square$

The t-transformation of a measure is best seen in terms of its continued fraction representation. It just multiplies the coefficients a_1 and b_1 by the positive number t:

$$G_{\mu_t}(z) = \cfrac{1}{z - t\cdot a_1 - \cfrac{t\cdot b_1}{z - a_2 - \cfrac{b_2}{z - a_3 - \cfrac{b_3}{z - a_4 - \cfrac{b_4}{z - a_5 - \cfrac{b_5}{\ddots}}}}}} \tag{3.9}$$

so only the "first level" is multiplied by t.

Example. – We shall show the t-transformation of a probability measure concentrated in two points. Let $\mu = p\delta_a + q\delta_b$ be the measure, with $a < b$ and $p + q = 1$, $p, q \geqslant 0$. The t-transformation of μ is the measure $\mu_t = P\delta_A + Q\delta_B$ where

$$A = \frac{1}{2}\left\{(a+b) - (b-a)\sqrt{1-4(1-t)pq}\right\},$$
$$B = \frac{1}{2}\left\{(a+b) + (b-a)\sqrt{1-4(1-t)pq}\right\}, \qquad P = \frac{pb+qa-A}{a+b-2A}.$$

It follows from these formulas, that $a + b = A + B$, and $a < A < B < b$, if $t \neq 1$. It should also be understood that these formulas are trivialized for $t = 1$.

Now we describe the change of moments, which is done by the t-transformation. Namely, we exhibit the relation between moments $a_n = \int x^n \, \mathrm{d}\mu(x)$ of a measure μ and moments $b_n = \int x^n \, \mathrm{d}\mu_t(x)$ of its t-transformation. We have the following

PROPOSITION 3.4. – *Let $t, s > 0$ be inverse of each other, $st = 1$, then for any positive integer n we have*

(1)

$$b_n = ta_n + (t-1) \cdot \sum_{k=1}^{n-1} a_k b_{n-k},$$

(2)

$$a_n = sb_n + (s-1) \cdot \sum_{k=1}^{n-1} a_k b_{n-k},$$

(3)

$$b_n = \frac{t}{t-1} \cdot \sum_{k=1}^{n} \sum_{j_1+\cdots+j_k=n} a_{j_1} \cdot a_{j_2} \cdot \cdots \cdot a_{j_k} \cdot (t-1)^k,$$

(4)

$$b_n = \frac{t}{t-1} \cdot \sum_{\mathcal{V} \in \mathcal{NC}_n^0} a(\mathcal{V}) \cdot (t-1)^{\sharp blocks(\mathcal{V})},$$

where $\mathcal{NC}_n^0$ is the set of these non-crossing partitions of the set $\{1, 2, \ldots, n\}$, which have only outer blocks (those are sometimes called “Boolean”partitions). If $\mathcal{V} = \{B_1, B_2, \ldots, B_k\}$ is such partition, consisting of k (outer) blocks, with $\sharp(B_s) = j_s$, then in our notation $a(\mathcal{V}) = a_{j_1} \cdot \cdots \cdot a_{j_k}$.

Proof. – First two formulas are equivalent in an obvious way. To prove any of them one may use relation between Cauchy transformation of a measure and the generating function of its moments:

$$G(z) = \frac{1}{z} M\left(\frac{1}{z}\right).$$

Generating function $M_t(z)$ of moments of μ_t is then related to the generating function $M(z)$ of moments of μ by

$$M(z) = tM_t(z) + (1-t)M(z) \cdot M_t(z).$$

Leibniz’s formula for multiple differentiation gives then for $z = 0$

$$\frac{M^{(n)}(0)}{n!} = t\frac{M_t^{(n)}(0)}{n!} + (1-t)\sum_{k=0}^{n} \frac{M^{(k)}(0)}{k!} \cdot \frac{M_t^{(n-k)}(0)}{(n-k)!}$$

which implies the desired formula. □

4. t-transformation as the Boolean convolution power

In this section we shall show that our t-transformation can be identified with convolution power of the Boolean convolution.

A comprehensive study of Boolean convolution was done by Speicher and Woroudi in [14]. For our purpose let us recall only that the Boolean convolution cumulants' generating function $\mathcal{K}_\nu(z) = \sum_{n=1}^{\infty} K_\nu(n) \cdot z^n$, of a measure ν, is related to the Cauchy transform $G_\nu(z)$ by the formula

$$G_\nu(z) = \frac{1}{z - \mathcal{K}_\nu(z)}. \tag{4.10}$$

As the general idea of a cumulant function is that it transforms convolution of measures into addition of their cumulants, an nth convolution power of a measure is transformed into multiplication of its cumulants' sequence by n. Therefore, we interpret multiplication of the Boolean cumulant sequence $(K_\nu(n))_{n=1}^{\infty}$ by a positive real number t as taking the tth Boolean convolution power of the measure ν.

For a probability measure μ let us consider its t-transformation μ_t. Then, for the Boolean cumulants we may write the formula:

$$G_{\mu_t}(z) = \frac{1}{z - \mathcal{K}_{\mu_t}(z)}. \tag{4.11}$$

Comparing this with the definition of t-transformation we obtain

$$tz - \frac{t}{G_\mu(z)} = \mathcal{K}_{\mu_t}(z). \tag{4.12}$$

However, the left-hand side of the above equation is, by definition of the Boolean cumulants, equal $t \cdot \mathcal{K}_\mu(z)$. Hence we obtain

$$t \cdot \mathcal{K}_\mu(z) = \mathcal{K}_{\mu_t}(z). \tag{4.13}$$

Thus the Boolean cumulants of t-transformed measure are obtained by multiplying by t the Boolean cumulants of the original measure. As we observed above, this multiplication may be interpreted as taking tth Boolean convolution power of the original measure. Hence our t-transformation turns out to be the same as tth Boolean convolution power.

5. t-transformation of convolutions

The t-convolution $\oplus_t$ is defined in the following way. Given two probability measures μ and ν on the real line, a non-negative number t, and a convolution $\oplus$ (for which the classical convolution, free Voiculescu convolutions, Boolean free convolution, and other convolutions may serve) one defines:

$$\mu \oplus_t \nu = (\mu_t \oplus \nu_t)_{1/t} = \mathcal{U}_{\frac{1}{t}}\big(\mathcal{U}_t(\mu) \oplus \mathcal{U}_t(\nu)\big). \tag{5.14}$$

If a given convolution is associative, then also its t-transformation is associative. In [8] we proved central limit theorems for t-transformed convolutions, and showed what was the limit in the cases of classical, free and Boolean convolutions. Since we shall be especially interested in the t-transformed free and classical convolutions, we recall the form of the limit measures in both cases.

Example. – In the case of t-transformed classical convolution the central limit measure has the following Cauchy transform:

$$G_{\mu^{(t)}}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{2t}{z - \cfrac{3t}{z - \cfrac{4t}{z - \cfrac{5t}{\ddots}}}}}} \tag{5.15}$$

and in the case of t-transformed free convolution the Cauchy transform of the central limit measure is:

$$G_{\nu^{(t)}}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{t}{z - \cfrac{t}{z - \cfrac{t}{z - \cfrac{t}{\ddots}}}}}} \tag{5.16}$$

The second measure was first found by Kesten [9] for $t = 1 - \frac{1}{2N}$, as the spectral measure for the simple random walk on free group on N free generators. In our case

$$G_{\nu^{(t)}}(z) = \frac{z(\frac{1}{2} - t) + \frac{1}{2}\sqrt{z^2 - 4t}}{z^2(1-t) - 1} \tag{5.17}$$

and the measure $\nu^{(t)}$ has absolutely continuous part of the form $\frac{1}{2\pi} \cdot \frac{\sqrt{4t - x^2}}{1-(1-t)x^2}\,\mathrm{d}x$ and is supported on the interval $[-2\sqrt{t}, 2\sqrt{t}]$. For $t < \frac{1}{2}$ the measure has atoms in $\pm\frac{1}{\sqrt{1-t}}$.

More general formula was also computed in [5] for the central limit theorem for conditionally free convolution, where our case is obtained for $\alpha = 1$, $\beta = \sqrt{t}$.

6. Moments of the central limit measure for the t-transformed free convolution

A general result by Bożejko describes moments of a measure by means of the coefficients in its Cauchy transform. The description involves the notion of "depth" of blocks in non-crossing partitions. The formula may be found, for example, in [1]. It follows from this result that in the case we consider, where $a_n = 0$ and $b_n = t$ for $n \geqslant 2$, $b_1 = 1$, one obtains the following formula for the moments:

$$m_{2n} = \mathcal{C}_n(t) = \sum_{\mathcal{V} \in \mathcal{NC}_2(2n)} t^{\sharp in(\mathcal{V})}, \tag{6.18}$$

where $\sharp in(\mathcal{V})$ is the number of inner blocks of a partition $\mathcal{V}$ and $\mathcal{NC}_2(2n)$ is the set of all non-crossing 2-partitions with two-element blocks only. For a non-crossing partition $\mathcal{V} \in \mathcal{NC}(n)$ with blocks $\mathcal{V} = \{B_1, B_2, \ldots, B_k\}$ a block $B_j = \{s_1, s_2, \ldots, s_{i_j}\}$ is called *outer* if there are no numbers $1 \leqslant p < q \leqslant n$, both in some other block of $\mathcal{V}$, such that $p < s_1, s_2, \ldots, s_{i_j} < q$. Blocks which do not enjoy this property are called *inner*. For example, the block which contains the number 1 is always outer, as well as the block which contains the number n. So there is always at least one outer block in a non-crossing partition. The trivial partition $\mathcal{V} = \{B\}$ for which $B = \{1, 2, \ldots, n\}$ consists of one outer block only. On the other hand, if a partition contains the outer block $B = \{1, n\}$, then all other blocks are inner.

Since the considered measure is symmetric, all odd moments are zero, and the $2n$th moment is a polynomial in t of degree $n-1$. These polynomials, denoted by $\mathcal{C}_n(t)$, are described in the following

PROPOSITION 6.1. – *The polynomials $\mathcal{C}_n(t)$ are given by $\mathcal{C}_1(t) = 1$ and*

$$\mathcal{C}_n(t) = 1 + \sum_{k=0}^{n-2} t^{k+1} \cdot \left\{ \binom{n+k}{k+1} - \binom{n+k}{k} \right\}. \tag{6.19}$$

In particular, for $t = 1$, we obtain Catalan numbers:

$$\mathcal{C}_n(1) = c_n = \frac{1}{n+1} \binom{2n}{n}.$$

Proof. – The proof is based on the following recurrence formula:

$$\mathcal{C}_n(t) = \sum_{k=1}^{n} c_{k-1} \cdot t^{k-1} \cdot \mathcal{C}_{n-k}(t),$$

where the kth Catalan number $c_k = \sharp \mathcal{NC}_2(2k)$ is the number of non-crossing pair partitions on a $2k$-element set. The formula can be seen by considering sums over all 2-partitions containing the block $\{1, 2k\}$, for $k = 1, 2, \ldots, n$ separately. Each such partition contains $k-1$ inner blocks inside the block $\{1, 2k\}$ and there is c_{k-1} such partitions.

This formula allows to compute the following generating function of the sequence $\mathcal{C}_n(t)$:

$$\mathcal{C}(x, t) = \sum_{n=0}^{\infty} \mathcal{C}_n(t) \cdot x^n = \frac{1}{1 - x \cdot \mathcal{S}(tx)} = \frac{2t}{2t - 1 + \sqrt{1 - 4tx}},$$

where $\mathcal{S}$ is the generating function of the Catalan numbers.

By expanding the function $\mathcal{C}$ into Taylor series in the variable x one gets the coefficients as stated in the proposition. $\square$

The coefficients

$$\binom{n+k}{k+1} - \binom{n+k}{k},$$

with $n \geqslant 1,\ 0 \leqslant k \leqslant n-2$, of the polynomials $\mathcal{C}_n(t)$ form an interesting triangular table, called *Delaney's table*:

$$\begin{array}{ccccccc}
1 \\
1 & 1 \\
1 & 2 & 2 \\
1 & 3 & 5 & 5 \\
1 & 4 & 9 & 14 & 14 \\
1 & 5 & 14 & 28 & 42 & 42 \\
1 & 6 & 20 & 48 & 90 & 132 & 132 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}$$

The rows are indexed by n, and the columns by k. Two most right numbers in each row are equal, and the right side of the "triangle" is formed by the sequence of Catalan numbers, since

$$\begin{aligned}
\binom{n+n-3}{n-2} - \binom{n+n-3}{n-3} &= \binom{n+n-2}{n-1} - \binom{n+n-2}{n-2} \\
&= \frac{1}{n}\binom{2n-2}{n-1} = c_{n-1}.
\end{aligned}$$

Moreover, the recurrence for computing the terms of the table is the following. Each number is the sum of the one above it and the one on the left-hand side next to it. Another property is that the sum of, say, k consecutive numbers in a given row, starting form (the most left number) 1 is equal to the number in the next row, standing just below the last number in the summation. In particular, the sum of all numbers in an nth row is equal to the $(n-1)$-st Catalan number c_{n-1}.

7. Model of t-free gaussian random variables

We are now going to construct a family of operators on a Hilbert space, and a state on a $*$-algebra generated by this family, with the distribution given by the central limit measure for the t-transformed free convolution. Our construction will depend on modification of a full Fock space and of creation and annihilation operators there.

For $t \geqslant 0$ and a given Hilbert space $\mathcal{H}$ with the scalar product $\langle \mid \rangle$, we consider the Fock space

$$\mathcal{F}_t(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n\geqslant 1} \mathcal{H}^{\otimes n}$$

completed with respect to the following scalar product:

$$(x_1 \otimes x_2 \otimes \cdots \otimes x_n \mid y_1 \otimes y_2 \otimes \cdots \otimes y_k)_t = \delta_{n,k} \cdot t^{n-1} \cdot \prod_{j=1}^{n} \langle x_j \mid y_j \rangle, \quad (\Omega \mid \Omega)_t = 1.$$

Now, given a vector $f \in \mathcal{H}$, we define creation operator $B(f)$ and annihilation operator $A(f)$ on $\mathcal{F}_t(\mathcal{H})$. For arbitrary $x_1, x_2, \ldots, x_n \in \mathcal{H}$ we put

$$B(f)x_1 \otimes x_2 \otimes \cdots \otimes x_n = f \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_n, \quad B(f)\Omega = f,$$

where $n \geqslant 1$, and

$$A(f)x_1 \otimes x_2 \otimes \cdots \otimes x_n = t \cdot \langle x_1 \mid f\rangle x_2 \otimes \cdots \otimes x_n,$$
$$A(f)\Omega = 0, \quad A(f)x_1 = \langle x_1 \mid f\rangle\Omega,$$

where $n \geqslant 2$. As one would expect, these operators are chosen to be adjoints of each other with respect to the scalar product.

PROPOSITION 7.1. – *In the Hilbert space $\mathcal{F}_t(\mathcal{H})$ with the scalar product $(\cdot \mid \cdot)_t$, for any $f \in \mathcal{H}$, and any $\xi, \eta \in \mathcal{F}(\mathcal{H})$*

$$\big(A(f)\xi \mid \eta\big)_t = \big(\xi \mid B(f)\eta\big)_t.$$

Proof. – First we check the case $\eta = \Omega$, $\xi = x \in \mathcal{H}$:

$$\big(A(f)x \mid \Omega\big)_t = \langle x \mid f\rangle, \qquad \big(x \mid B(f)\Omega\big)_t = (x \mid f)_t = \langle x \mid f\rangle.$$

It now suffices to show this property for all $\xi = x_1 \otimes x_2 \otimes \cdots \otimes x_n$ and $\eta = y_2 \otimes y_3 \otimes \cdots \otimes y_n$, with $n \geqslant 2$ and $x_1, \ldots, x_n, y_2, \ldots, y_n \in \mathcal{H}$.

We have then,

$$\begin{aligned}
&\big(A(f)x_1 \otimes x_2 \otimes \cdots \otimes x_n \mid y_2 \otimes \cdots \otimes y_n\big)_t \\
&\quad = t \cdot \langle x_1 \mid f\rangle \big(x_2 \otimes \cdots \otimes x_n \mid y_2 \otimes \cdots \otimes y_n\big)_t \\
&\quad = t^{n-1} \cdot \langle x_1 \mid f\rangle \cdot \prod_{j=2}^{n} \langle x_j \mid y_j\rangle
\end{aligned}$$

and

$$\begin{aligned}
(x_1 \otimes \cdots \otimes x_n \mid B(f)y_2 \otimes \cdots \otimes y_n)_t &= (x_1 \otimes \cdots \otimes x_n \mid f \otimes y_2 \otimes \cdots \otimes y_n)_t \\
&= t^{n-1} \cdot \langle x_1 \mid f\rangle \cdot \prod_{j=2}^{n} \langle x_j \mid y_j\rangle. \qquad \square
\end{aligned}$$

Remark 7.2. – Our construction is a concrete example of the general notion of INTERACTING FOCK SPACES, introduced by Accardi, Lu and Volovich in [2], and studied also in [1]. Namely, in the general construction we take the sequence $\lambda_n = t^{n-1}$ for $n \geqslant 1$ to obtain scalar product on our transformed Fock space, as well as the creation and annihilation operators.

Now let us define a vacuum state φ on all bounded operators T on the Fock space $\mathcal{F}_t(\mathcal{H})$:

$$\varphi(T) = (T\Omega \mid \Omega)_t$$

and position operators $G(f) = A(f) + B(f)$, where $f \in \mathcal{H}$.

THEOREM 7.3. – *For any vector $f \in \mathcal{H}$ with unit length, the position operator $G(f)$ has the distribution with respect to the vacuum state φ given by the central limit measure for the t-transformed free convolution, and hence the moments are expressed by the formula*:

$$\varphi\big(G(f)^{2n}\big) = \sum_{\mathcal{V}\in\mathcal{NC}_2(2n)} t^{\sharp inner(\mathcal{V})}. \tag{7.20}$$

Proof. – Let us fix a unit vector $f \in \mathcal{H}$, and write for short $G = G(f)$, $A^0 = A(f)$, and $A^1 = B(f)$. Since both creation and annihilation change the "parity" of a simple tensor, mapping $\mathcal{H}^{\otimes n}$ into $\mathcal{H}^{\otimes(n+1)}$ or $\mathcal{H}^{\otimes(n-1)}$, the odd moments of G are zero:

$$\varphi\big(G^{2n+1}\big) = \sum_{s_1,\dots,s_{2n+1}\in\{0,1\}} \big(A^{s_1}\cdots A^{s_{2n+1}}\Omega \mid \Omega\big)_t = 0.$$

To find the even moments we make use of the idea of Feller of associating with each nonzero term in the sum $\sum_{s_1,\dots,s_{2n}\in\{0,1\}}(A^{s_{2n}}\cdots A^{s_1}\Omega \mid \Omega)_t$ a path on the carthesian plane, which connects points $(0,0)$ and (n,n) through the lattice points (m,k) with $0 \leqslant m \leqslant k \leqslant n$. Let us recall that the path associated to $A^{s_{2n}}\cdots A^{s_1}$ consists of $2n$ intervals of length 1, where the kth interval is vertical if $s_k = 1$ and horizontal if $s_k = 0$. Thus the first interval is always vertical from $(0,0)$ to $(0,1)$, since only for $s_1 = 1$ we get a nonzero term; for $s_1 = 0$ there is $A^0\Omega = A\Omega = 0$. The second interval is vertical, if $s_2 = 1$, or horizontal, if $s_2 = 0$. It follows that the last interval connects $(n-1,n)$ with (n,n), because a nonzero term must have $s_{2n} = 0$ (by the symmetry in the scalar product). In this way to every sequence $(s_{2n},\dots,s_1)$ we associate an appropriate path, but a nonzero term in the considered sum has also factors t in some powers, because annihilations add them. In general, $A^{s_k}\cdots A^{s_1}\Omega$ is of the form $t^j \cdot f^{\otimes r}$ with $f^{\otimes r} = \Omega$ for $r = 0$, where $r = \sharp\{s_i = 1\} - \sharp\{s_i = 0\}$. Thus a nonzero term one gets' only for sequences $(s_{2n},\dots,s_1)$ with the same number of zeros and ones. Let us find out the power of t in such term. The crucial observation is that a factor t appears each time the annihilation A acts on a simple tensor $f^{\otimes k}$ with $k \geqslant 2$. This correspond to the part of the associated path, which is above the straight line from $(0,1)$ to $(n-1,n)$; the power of t is equal to the number of vertical segments above this line. Now we will show that there is a one-to-one correspondence between sequences $(s_{2n},\dots,s_2,s_1)$, which give nonzero terms, associated paths, and elements of the set $\mathcal{NC}_2(2n)$ of non-crossing 2-partitions of the set $\{1,2,\dots,2n\}$. Given such sequence we have $s_1 = \cdots = s_i = 1$ for a unique $1 \leqslant i \leqslant n$

with $s_{i+1} = 0$. Then we say that $\{i, i+1\}$ is a block of the corresponding 2-partition. To obtain other blocks, each of which must contain exactly 2 elements, we use induction on n, "deleting" s_{i+1} and s_i from the sequence, and repeating the same procedure for a shorter sequence of $2n-2$ elements. Then the induction works. Now let us observe that each block obtained in this way corresponds to a couple of segments in the associated path, and that if one of these segments has an end on the line form $(0,0)$ to (n,n), then the other segment also enjoys this property and, consequently, the corresponding block is outer. All other blocks of the associated 2-partition are inner. This shows that the number of inner blocks is equal to the number of vertical segments of the associated path, which are above the line from $(0,1)$ to $(n-1,n)$ (we allow segments which have one common point with this line, like the segment from $(1,2)$ to $(1,3)$). It follows that the power of t in a nonzero term $(A^{s_{2n}} \cdots A^{s_1}\Omega \mid \Omega)_t$ is exactly the same as the number of inner blocks of the corresponding 2-partition. □

8. Moments of the central limit measure for the t-transformed classical convolution

In this section we are going to prove, that the measure which appears in the central limit theorem for t-transformed classical convolution, and which has the Cauchy transform given by the following continued fraction:

$$G_{\mu^{(t)}}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{2t}{z - \cfrac{3t}{z - \cfrac{4t}{z - \cfrac{5t}{\ddots}}}}}} \tag{8.21}$$

has moments expressed by a formula similar to (6.18). Let us observe, that the odd moments of the measure are all zero, since the measure is symmetric.

We start with showing the relation between moments of the considered measure and moments of the gaussian measure.

LEMMA 8.1. – *Let g be the gaussian measure with density $\mathrm{d}g(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\,\mathrm{d}x$, and let μ be the central limit measure for t-transformed classical convolution. Let also G_g, G_μ be Cauchy transforms of these measures, and let M_g, M_μ be generating functions of moments of these measures. Then*

$$G_{U_t\mu}(z) = \frac{1}{\sqrt{t}} G_g\left(\frac{z}{\sqrt{t}}\right), \qquad M_{\mathcal{U}_t\mu}(z) = M_g\big(z\sqrt{t}\,\big). \tag{8.22}$$

Proof. – Let us observe that by applying $\mathcal{U}_t$ to the measure μ we obtain a probability measure with the following Cauchy transform:

$$G_{\mu^{(t)}}(z) = \cfrac{1}{z - \cfrac{1t}{z - \cfrac{2t}{z - \cfrac{3t}{z - \cfrac{4t}{z - \cfrac{5t}{\ddots}}}}}}$$

A simple calculation shows that this continued fraction is also equal to $\frac{1}{\sqrt{t}} G_g(\frac{z}{\sqrt{t}})$.

On the other hand, using the formula $G_{D_\lambda \nu}(z) = \frac{1}{\lambda} G_\nu(\frac{z}{\lambda})$ for the dilation $D_\lambda \nu$ of a measure ν by a (positive) number λ, one gets for $\lambda = \sqrt{t}$ that

$$G_{D_{\sqrt{t}} g}(z) = G_{\mathcal{U}_t \mu}(z)$$

which proves the lemma. □

From this lemma we get the following

COROLLARY 8.2. – *Let $M_t(n)$, $n = 0, 1, 2, \dots,$ be the sequence of moments of the measure $\mathcal{U}_t \mu$ and let $M(n)$ be the sequence of moments of the measure μ. Then odd moments are zero, i.e. for any integer $n \geqslant 0$ we have $M_t(2n+1) = M(2n+1) = 0$, and even moments are expressed by the following recurrence*:

$$M(2n) = s M_t(2n) + (s-1) \sum_{j=1}^{n-1} M(2j) \cdot M_t(2n-2j) \tag{8.23}$$

with $M_t(2n) = (2n-1)!! \cdot t^n$ and $s = 1/t$.

Proof. – Since moments of a measure dilated by a number are multiplied by powers of this number, we can write $M_t(2n) = M_g(2n) \cdot t^n = (2n-1)!! \cdot t^n$. Then, using the Proposition 3.4 one gets the recurrence formula. □

PROPOSITION 8.3. – *The even moments of the central limit measure for t-transformed classical convolution are given by*

$$M(2n) = \sum_{\mathcal{V} \in \mathcal{P}_2(2n)} t^{n - \sharp o(\mathcal{V})}. \tag{8.24}$$

Let us notice, that in this formula $\sharp o(\mathcal{V})$ is the number of *outer connected components* of a partition $\mathcal{V}$, and the partition runs over all pair partitions. In the case of partitions with crossings the number of outer connected components plus the number of its inner connected components may be less that n. For a partition without crossings this number is always equal n, since in this case each block is either inner or outer.

The definition of *inner* and *outer* connected components is slightly more complicated for partitions which might have crossings than for partitions without crossings. The picture is more clear if we consider the numbers $\{1, 2, \ldots, n\}$ as being placed on a circle in the natural order, and numbers which form a block are connected by lines. Then the partition can be regarded as a graph, and blocks as subgraphs. Each block is a connected subgraph, and some blocks may cross each other. Blocks which cross each other are contained in the same connected component. For example, the partition $\mathcal{V} = \{\{1, 3, 7\}, \{2, 8\}, \{4, 5, 6\}\}$ there are two connected components: $\{1, 2, 3, 7, 8\}$ and $\{4, 5, 6\}$. To be able to talk about inner and outer connected components we have to transform the above picture, remaining all the connections within blocks, onto linear one, in which the numbers $\{1, 2, \ldots, n\}$ are placed in the natural order on a line. It is important not to add any extra crossings while transforming the picture. Then the connected components are preserved, and one can talk about outer and inner components in the similar way as in the case of non-crossing partitions. In the partition $\mathcal{V} = \{\{1, 3, 7\}, \{2, 8\}, \{4, 5, 6\}\}$ the connected component $\{1, 2, 3, 7, 8\}$ is outer, and the connected component $\{4, 5, 6\}$ is inner.

The above proposition gives the following first five nonzero moments of our limit measure:

$$\begin{aligned} &M(0) = 1 = M(2), \quad M(4) = 2t + 1, \quad M(6) = 10t^2 + 4t + 1, \\ &M(8) = 74t^3 + 24t^2 + 6t + 1. \end{aligned} \tag{8.25}$$

Proof. – We shall prove this proposition by induction on n. We use the recurrence formula (2) from Proposition 3.4, which may be written as:

$$M(2n) = s \cdot \sum_{j=0}^{n-1} M(2j) \cdot M_t(2n - 2j) - \sum_{j=1}^{n-1} M(2j) \cdot M_t(2n - 2j).$$

In this formula we use the inductive assumption for $M(2j)$ and write also

$$M_t(2m) = t^m \cdot (2m - 1)!! = t^m \cdot \sum_{\mathcal{V} \in \mathcal{P}_2(2m)} 1^{\sharp o(\mathcal{V})}.$$

This gives

$$\begin{aligned} M(2n) = t^n \cdot \sum_{j=0}^{n-1} \sum_{\mathcal{V} \in \mathcal{P}_2(2j)} \sum_{\mathcal{W} \in \mathcal{P}_2(2n-2j)} s^{\sharp o(\mathcal{V})+1} \\ - t^n \cdot \sum_{j=1}^{n-1} \sum_{\mathcal{V}_1 \in \mathcal{P}_2(2j)} \sum_{\mathcal{W}_1 \in \mathcal{P}_2(2n-2j)} s^{\sharp o(\mathcal{V}_1)}. \end{aligned} \tag{8.26}$$

Let us observe that some terms $s^{\sharp o(\mathcal{V})+1} \cdot 1^{\sharp o(\mathcal{W})}$ in the first sum on the right-hand side cancel with some terms $s^{\sharp o(\mathcal{V}_1)+1} \cdot 1^{\sharp o(\mathcal{W}_1)}$ form the second sum. The necessary condition for this cancellation is $\sharp o(\mathcal{V}) + 1 = \sharp o(\mathcal{V}_1)$. Here we allow also $\mathcal{V} \in \mathcal{P}_2(0)$ to be empty partition.

For every $0 \leqslant j \leqslant n-1$ we treat in the natural way each pair of partitions $\mathcal{V} \in \mathcal{P}_2(2j)$, $\mathcal{W} \in \mathcal{P}_2(2n-2j)$ as a partition from $\mathcal{P}_2(2j) \times \mathcal{P}_2(2n-2j) \subset \mathcal{P}_2(2n)$. Similar situation is for $\mathcal{V}_1 \in \mathcal{P}_2(2j)$, $\mathcal{W}_1 \in \mathcal{P}_2(2n-2j)$ with $1 \leqslant j \leqslant n-1$.

For every $1 \leqslant j \leqslant n-1$ and every pair $(\mathcal{V}_1, \mathcal{W}_1) \in \mathcal{P}_2(2j) \times \mathcal{P}_2(2n-2j)$ we shall find exactly one pair $(\mathcal{V}, \mathcal{W})$, which will form the same element of $\mathcal{P}_2(2n)$, and which will satisfy $\sharp o(\mathcal{V}) + 1 = \sharp o(\mathcal{V}_1)$. Namely, if $\sharp o(\mathcal{V}_1) = 1$, then we take $\mathcal{V} = \emptyset$ and $\mathcal{W} = (\mathcal{V}_1, \mathcal{W}_1)$. If $\sharp o(\mathcal{V}_1) \geqslant 2$, then $\mathcal{V}_1 \in \mathcal{P}_2(2j)$ with $2 \leqslant j \leqslant n-1$; hence 1 and $2j$ are in two different outer connected components of $\mathcal{V}_1$. In this case the least number in the connected component which contains $2j$ is some odd number $2m+1$ with $0 \leqslant m \leqslant j-1$. Then the partition $\mathcal{V}_1$ is a direct product of a partition $\mathcal{V} \in \mathcal{P}_2(\{1, \ldots, 2m\})$ and a partition $\mathcal{Y} \in \mathcal{P}_2(\{2m+1, \ldots, 2j\})$. We define $\mathcal{W} = (\mathcal{Y}, \mathcal{W}_1)$, which is an element of $\mathcal{P}_2(\{2m+1, \ldots, 2n\})$.

This way we see, that the second summation in (8.26) will be cancelled with (a part of) the first summation. What will be left in the first summation? Our construction of $\mathcal{W}$ required it to have at least two outer connected components, because $\mathcal{W}_1$ is non-empty. Hence there will remain not cancelled these terms $s^{\sharp o(\mathcal{V})+1}$ in which $\mathcal{V} \in \mathcal{P}_2(2j)$ is arbitrary and $\mathcal{W} \in \mathcal{P}_2(2n-2j)$ satisfies $\sharp o(\mathcal{W}) = 1$. However, for such pair $(\mathcal{V}, \mathcal{W})$ we have $\sharp o(\mathcal{V}) + 1 = \sharp o(\mathcal{V}, \mathcal{W})$. Therefore the remaining terms form exactly the sum

$$\sum_{\mathcal{U} \in \mathcal{P}_2(2n)} s^{\sharp o(\mathcal{U})}$$

which is the desired expression for $M(2n)$. $\square$

9. Model of t-classical gaussian random variables

In a similar way as for the t-free gaussian random variables, we construct a model of non-commutative random variables, which have distribution given by the measure, which appears in the central limit theorem for t-transformed classical convolution. The random variables will be operators on a Hilbert space, and their distribution is meant with respect to a given state.

The Hilbert space is just the *symmetric* Fock space, with modified scalar product. The construction runs as follows.

For $t \geqslant 0$ and a Hilbert space $\mathcal{H}$, with a given scalar product $\langle \bullet \mid \bullet \rangle$ we take the symmetric Fock space $\mathcal{F}_t(H)_{sym}$, which is the space spanned by a distinguished vector Ω, called *vacuum*, and simple symmetric tensors $(x_1, x_2, \ldots, x_n)_{sym} := \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)}$, where S_n are the symmetric (permutation) groups. The scalar product $(\bullet \mid \bullet)_t$ is defined as follows: $(\Omega \mid \Omega)_t := 1$ and

$$\big((x_1, \ldots, x_n)_{sym} \mid (y_1, \ldots, y_n)_{sym}\big)_t := t^{n-1} \sum_{\sigma \in S_n} \langle x_1 \mid y_{\sigma(1)} \rangle \langle x_2 \mid y_{\sigma(2)} \rangle \cdots \langle x_n \mid y_{\sigma(n)} \rangle$$

for $n \geqslant 1$ and for $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{H}$.

For a vector $v \in \mathcal{H}$ we define the *creation* and *annihilation* operators $B(v)$ and $A(v)$ in the following way:

$$B(v)\Omega = v, \quad B(v)\big((x_1, \dots, x_n)_{sym}\big) = (v, x_1, \dots, x_n)_{sym}$$

and

$$\begin{aligned} &A(v)\Omega = 0, \quad A(v)x_1 = \langle v \mid x_1\rangle\Omega, \\ &A(v)(x_1, \dots, x_n)_{sym} = t \cdot \sum_{j=1}^{n} \langle v \mid x_j\rangle (x_1, \dots, \check{x}_j, \dots, x_n)_{sym}, \end{aligned}$$

where $\check{x}_j$ denotes removed element x_j. In this setting we have the following properties.

PROPOSITION 9.1. – *For any $v \in \mathcal{H}$ the operators $B(v)$ and $A(v)$ are unbounded and satisfy:*

$$(B(v)f \mid g)_t = (f \mid A(v)g)_t$$

for arbitrary vectors $f, g \in \mathcal{F}(H)$, which are finite combinations of simple symmetric tensors.

The proof goes by computation on simple symmetric tensors, in a similar manner as for t-free case, so we omit it.

Now we consider the vacuum state $\varphi = \varphi_{\Omega,t}$ on the $*$-algebra $\mathcal{M}$ generated by the *position* operators $G(v) = B(v) + A(v)$, our non-commutative (t-gaussian) random variables, where $v \in \mathcal{H}$:

$$\varphi(X) := (X\Omega \mid \Omega)_t.$$

With respect to this state each random variable $G(v)$, with $v \in \mathcal{H}$ a unit vector, has distribution given by the central limit measure for t-transformed classical convolution.

THEOREM 9.2. – *For any vector $v \in \mathcal{H}$ with unit length, the position operator $G(v)$ satisfies*:

$$\varphi\big(G(v)^{2n}\big) = \sum_{\mathcal{V} \in \mathcal{P}_2(2n)} t^{n - \sharp o(\mathcal{V})}. \tag{9.27}$$

Proof. – To simplify the notation we fix a unit vector $v \in \mathcal{H}$ and will denote $D^{-1} := A(v)$ and $D^1 := B(v)$. Then $G(v) = D^{-1} + D^1$ and

$$\varphi\big(G^{2n}(v)\big) = \sum_{\varepsilon_1, \dots, \varepsilon_{2n} \in \{-1,1\}^{2n}} \big(D^{\varepsilon_{2n}} \cdots D^{\varepsilon_1}\Omega \mid \Omega\big)_t. \tag{9.28}$$

Since

$$(D^1)^k\Omega = (\underbrace{v, \dots, v}_{k})_{sym} \quad \text{and} \quad D^{-1}(\underbrace{v, \dots, v}_{k})_{sym} = kt\,(\underbrace{v, \dots, v}_{k-1})_{sym},$$

for $k \geqslant 2$, nonzero terms in the above sum will appear only when there is the same number of creations and annihilations, and also if $\varepsilon_1 + \cdots + \varepsilon_j \geqslant 0$ for $1 \leqslant j \leqslant 2n$.

Application of each annihilation operator results in multiplication by the number kt, provided it acts onto (symmetric) vector

$$\underbrace{(v, \ldots, v)}_{k}{}_{sym}$$

of length $k \geqslant 2$. We are going to show, that $(D^\varepsilon\Omega \mid \Omega)_t = c_\varepsilon t^{n-s}$, where $\varepsilon = (\varepsilon_{2n}, \ldots, \varepsilon_1)$, $\varepsilon_1 + \cdots + \varepsilon_k \geqslant 0$ for $k = 1, \ldots n$ and $\varepsilon_1 + \cdots + \varepsilon_{2n} = 0$, is equal to the number c_ε of these 2-partitions from $\mathcal{P}_2(2n)$ which are "associated" with the sequence ε, multiplied by the number t^{n-s}, where s is the number of outer connected components of the partitions. By "associated" we mean a partition $\mathcal{V}$, the blocks of which are of the form $\{\varepsilon_j, \varepsilon_k\} = \{-1, 1\}$, with $j \neq k$ and $\varepsilon_j = 1, \varepsilon_k = -1$ if and only if $j > k$. In particular, a connected component of $\mathcal{V}$ is outer if and only if there is no block $\{j, k\}$ of $\mathcal{V}$ such that $j < i < k$ for all i which belong to this component. We know from previous sections, that there is exactly one non-crossing partition associated with a given sequence ε, but there are always other arbitrary partitions associated with the same sequence. For example, with the sequence $\varepsilon = (-1, -1, 1, 1)$ we associate the non-crossing partition $\mathcal{V}_0 = \{\{2, 3\}, \{1, 4\}\}$, and also the partition $\mathcal{V} = \{\{1, 3\}, \{2, 4\}\}$. The non-crossing partition associated with given ε has a given number of outer blocks, say s. We shall show that every 2-partition associated with the same sequence ε has the same number s of outer connected components.

LEMMA 9.3. – *Let $\varepsilon = (\varepsilon_{2n}, \ldots, \varepsilon_1)$ be a given sequence of -1's and 1's, satisfying $\varepsilon_1 + \cdots + \varepsilon_k \geqslant 0$ for $k = 1, \ldots 2n$ and $\varepsilon_1 + \cdots + \varepsilon_{2n} = 0$. Let also s be the number of times the sum $\varepsilon_1 + \cdots + \varepsilon_k$ equals 0, when k varies from 1 to $2n$. Let $\mathcal{V}_0$ be the non-crossing partition associated with ε and let $\mathcal{V}$ be an arbitrary partition associated with this sequence. Then the number of outer blocks of $\mathcal{V}_0$ and the number of outer connected components of $\mathcal{V}$ are both equal s. Moreover, the number of $\mathcal{V} \in \mathcal{P}_2(2n)$, associated with ε, is equal to the coefficient c_ε.*

Proof. – For the proof of the lemma let us observe that a block $\{j, k\}$, $j > k$, of $\mathcal{V}_0$ is outer if and only if $\varepsilon_j = 1, \varepsilon_k = -1$ and $0 = \varepsilon_1 + \cdots + \varepsilon_{j-1} = \varepsilon_j + \cdots + \varepsilon_k$. However, the same condition guarantees, and is equivalent to the fact, that the connected component of $\mathcal{V}$ which contains the block $\{j, k\}$, is outer. Therefore, $\mathcal{V}$ cannot have more outer components than s, since any outer connected component of it must satisfy the same condition.

Let us now assume that $s = 1$, then $\{1, 2n\}$ is the only outer block of $\mathcal{V}_0$. It follows that $\varepsilon_2 + \cdots + \varepsilon_{2n-1} = 0$, and that $\varepsilon_1 + \cdots + \varepsilon_k \geqslant 1$ for $1 \leqslant k \leqslant 2n - 1$. Therefore, $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_j = 1$, for some $2 \leqslant j \leqslant n$, and $\varepsilon_{j+1} = \cdots = \varepsilon_{j+r} = -1$ if $j + 1$ is the least number for which $\varepsilon_{j+1} = -1$, and $1 \leqslant r < j < n$ is such that $\varepsilon_{j+r+1} = 1$. In $\mathcal{V}$ each of the numbers $j + 1, \ldots, j + r$ forms a block with one of the numbers $1, \ldots, j$. This condition gives exactly $j(j - 1) \cdots (j - r)$ possible partitions $\mathcal{V} \in \mathcal{P}_2(2n)$. If we now forget about these blocks, we will find ourselves in the same situation as above, but now for partitions of $2n - 2r$ elements, for which induction can be applied.

Similar process appears when we consider the term $(D^\varepsilon\Omega \mid \Omega)_t$. Then

$$D^{\varepsilon_{j+r}}\cdots D^{\varepsilon_{j+1}}D^{\varepsilon_j}\cdots D^{\varepsilon_1}\Omega = D^{\varepsilon_{j+r}}\cdots D^{\varepsilon_{j+1}}(\underbrace{v,\ldots,v}_{j})_{sym} = j(j-1)\cdots(j-r)t^r$$

if $r > j$, and for $r = j = n$ we have

$$D^{\varepsilon_{2n}}\cdots D^{\varepsilon_{n+1}}D^{\varepsilon_n}\cdots D^{\varepsilon_1}\Omega = D^{\varepsilon_{2n}}\cdots D^{\varepsilon_{n+1}}(\underbrace{v,\ldots,v}_{n})_{sym} = n!\cdot t^{n-1}.$$

Since for $s = 1$ we have $\varepsilon_1 + \cdots + \varepsilon_j \geqslant 1$ if $1 \leqslant j \leqslant 2n-1$, application of each D^{ε_j} with $\varepsilon_j = -1$ results in multiplication by t for such j. Therefore $(D^\varepsilon\Omega \mid \Omega)_t = j(j-1)\cdots(j-r)t^r(D^{\varepsilon_{2n}}\cdots D^{\varepsilon_{j-r+1}}D^{\varepsilon_{j-r-1}}\cdots D^{\varepsilon_1}\Omega \mid \Omega)_t$, and the induction can be also applied here. For completeness of the proof let us also observe, that the power $n-s$, in which t appears in $(D^\varepsilon\Omega \mid \Omega)_t$, can be computed in the following easy way. Since D^{-1} does not multiply by t only when acting on tensors of length 1, it follows that if $\varepsilon_1+\cdots+\varepsilon_j \geqslant 1$ for $1 \leqslant j \leqslant k$ and $\varepsilon_1+\cdots+\varepsilon_k = 0$, then $D^{\varepsilon_k}\cdots D^{\varepsilon_1}\Omega = c_{(\varepsilon_k,\ldots,\varepsilon_1)}t^{k-1}\Omega$. On the other hand, $\{1,k\}$ is then an outer block of $\mathcal{V}_0$. Hence, by induction on s, or performing the same reduction for each successive outer block, the statement about the power of t follows. This shows that the lemma is true. □

Using this lemma we can now conclude the proof of the theorem. The non-zero terms in the summation (9.28) appear only for these sequences $\varepsilon = (\varepsilon_{2n},\ldots,\varepsilon_1)$ which are associated with non-crossing partitions. Hence, with each such term also a number of arbitrary 2-partitions is associated. It follows form the lemma, that this number is equal c_ε, and that each of these partitions has s outer connected components, when $(D^\varepsilon\Omega \mid \Omega) = c_\varepsilon t^{n-s}$. This finishes the proof of the theorem. □

10. Cumulants for t-classical and t-free convolutions

For a given convolution $\bullet$ we would like to define a mapping $\mu \mapsto \{R_n(\mu)\}_{n\geqslant 1}$, which associates a sequence of numbers to a given measure μ. The mapping should posses the following “linearization” property:

$$R_n(\mu\bullet\nu) = R_n(\mu) + R_n(\nu) \tag{10.29}$$

for pairs of measures μ and ν with moments. If such transformation exists for a convolution $\bullet$, then it is called the cumulant transformation and, for a measure μ, the sequence $\{R_n(\mu)\}_{n\geqslant 1}$ is called the cumulant sequence.

It turns out that if, for a given convolution $\bullet$ there exist cumulants, then its t-transformation also has cumulants. The idea is to define t-transformed cumulants of a measure by the given cumulants of the t-transformed measure.

PROPOSITION 10.1. – *Let $\bullet$ be a convolution and let $\mathcal{R}$ be the cumulant transformation for it. For any $t \geqslant 0$ and any positive integer n define $\mathcal{R}_n^t(\mu) := \mathcal{R}_n(\mathcal{U}_t\mu)$. Then $\mathcal{R}_n^t(\mu\bullet_t\nu) = \mathcal{R}_n^t(\mu) + \mathcal{R}_n^t(\nu)$.*

Proof. – The proof is and easy computation, since the definition is the right one:

$$\begin{aligned}\mathcal{R}_n^t(\mu \bullet_t \nu) &= \mathcal{R}_n^t(\mathcal{U}_{t^{-1}}(\mathcal{U}_t\mu \bullet \mathcal{U}_t\nu) = \mathcal{R}_n(\mathcal{U}_t\mu \bullet \mathcal{U}_t\nu)\\ &= \mathcal{R}_n(\mathcal{U}_t\mu) + \mathcal{R}_n(\mathcal{U}_t\nu) = \mathcal{R}_n^t(\mu) + \mathcal{R}_n^t(\nu). \qquad \square\end{aligned}$$

Remark 10.2. – Cumulant t-transformation is not uniquely defined, since for any sequence $\lambda = (\lambda_n)_{n\geqslant 1}$ one can define also $\mathcal{R}_{n,\lambda}^t(\mu) := \lambda_n \cdot \mathcal{R}_n(\mathcal{U}_t\mu)$ and obtain the required linearization property.

The two fundamental examples of existence of cumulants are known in the cases of classical convolution $*$ and free convolution $\oplus$. In both cases there is the relation between moments and cumulants of a measure.

Example. – For the classical convolution the relation between moments $\mathcal{M}_n(\mu)$ and cumulants $\mathcal{R}_n(\mu)$ of a measure μ is provided by the following formula:

$$\mathcal{M}_n(\mu) = \sum_{\mathcal{V}\in\mathcal{P}(n)} \mathcal{R}(\mathcal{V}), \tag{10.30}$$

where $\mathcal{P}(n)$ is the set of all partitions of $\{1, 2, \ldots, n\}$, and for a partition $\mathcal{V} \in \mathcal{P}(n)$,which consists of blocks $\mathcal{V} = \{B_1, B_2, \ldots, B_k\}$ we define

$$\mathcal{R}(\mathcal{V}) = \mathcal{R}_{i_1} \cdot \mathcal{R}_{i_2} \cdot \cdots \cdot \mathcal{R}_{i_k}$$

if $\sharp B_1 = i_1, \ldots, \sharp B_k = i_k$ are numbers of elements in blocks.

Example. – For the free convolution the relation between moments $\mathcal{M}_n(\mu)$ and cumulants $\mathcal{R}_n(\mu)$ of a measure was discovered by Speicher [13], and is given by the following formula:

$$\mathcal{M}_n(\mu) = \sum_{\mathcal{V}\in\mathcal{NC}(n)} \mathcal{R}(\mathcal{V}) \tag{10.31}$$

with the notation as in the previous example.

As in the classical case the summation is over all partitions, in the free case the summation is over non-crossing partitions. Similar situation is for the t-transformed classical and free convolutions, with the modification of the appearance of outer blocks in the t-free case, and outer connected components in the t-classical case.

Now we can describe the relation between moments and cumulants in the t-transformed classical and free convolutions.

PROPOSITION 10.3. – *Let $(\mathcal{M}_n(\mu))_{n\geqslant 0}$ be the sequence of moments of a measure μ and let $(\mathcal{R}_n(\mu))_{n\geqslant 1}$ be the sequence of free cumulants of the measure. Then for the t-free convolution the relation between moments and t-free cumulants is given by*:

$$\mathcal{M}_n(\mu) = \sum_{\mathcal{V}\in\mathcal{NC}(n)} \mathcal{R}(\mathcal{V}) \cdot t^{-\sharp o(\mathcal{V})}, \tag{10.32}$$

where for $\mathcal{V} \in \mathcal{NC}(n)$ we define $\sharp o(\mathcal{V})$ = number of outer blocks of the partition $\mathcal{V}$.

Proof. – The proof is inductive and uses the recursion formula (2) of Proposition 3.4 for the relation between moments of a measure and its t-transformation. The formula can be written as

$$a_n = s \cdot \sum_{k=0}^{n-1} a_k b_{n-k} - \sum_{k=1}^{n-1} a_k b_{n-k},$$

where $a_n = \int x^n \, \mathrm{d}\mu(x)$ and $b_n = \int x^n \, \mathrm{d}\mu_t(x)$. By the inductive hypothesis and the general formula relating moments and free cumulants the above may be rewritten as

$$\begin{aligned}
a_n = &\sum_{k=0}^{n-1} \sum_{\mathcal{V}_1 \in \mathcal{NC}(k)} \mathcal{R}(\mathcal{V}_1) s^{\sharp o(\mathcal{V}_1)+1} \sum_{\mathcal{V}_2 \in \mathcal{NC}(n-k)} \mathcal{R}(\mathcal{V}_2) \\
&- \sum_{l=1}^{n-1} \sum_{\mathcal{W}_1 \in \mathcal{NC}(l)} \mathcal{R}(\mathcal{W}_1) s^{\sharp o(\mathcal{W}_1)} \sum_{\mathcal{W}_2 \in \mathcal{NC}(n-l)} \mathcal{R}(\mathcal{W}_2) \\
= &\sum_{k=0}^{n-1} \sum_{\mathcal{V}_1 \in \mathcal{NC}(k)} \sum_{\mathcal{V}_2 \in \mathcal{NC}(n-k)} \mathcal{R}(\mathcal{V}_1)\mathcal{R}(\mathcal{V}_2) s^{\sharp o(\mathcal{V}_1)+1} \\
&- \sum_{l=1}^{n-1} \sum_{\mathcal{W}_1 \in \mathcal{NC}(l)} \sum_{\mathcal{W}_2 \in \mathcal{NC}(n-l)} \mathcal{R}(\mathcal{W}_1)\mathcal{R}(\mathcal{W}_2) s^{\sharp o(\mathcal{W}_1)}.
\end{aligned}$$

Let us observe that in both sums (over k and l) some terms are equal and may be cancelled. For this purpose let us consider when the equality

$$\mathcal{R}(\mathcal{V}_1)\mathcal{R}(\mathcal{V}_2) s^{\sharp o(\mathcal{V}_1)+1} = \mathcal{R}(\mathcal{W}_1)\mathcal{R}(\mathcal{W}_2) s^{\sharp o(\mathcal{W}_1)} \tag{10.33}$$

holds. In the first summation (over k) we have $0 \leqslant k \leqslant n-1$, which allows the partition $\mathcal{V}_1$ to be empty (if $k = 0$). In the second summation (over l) we have $1 \leqslant l, n-l \leqslant n-1$, so both partitions $\mathcal{W}_1$ and $\mathcal{W}_2$ are nonempty. Now, the picture will become more clear if we think of $\mathcal{V}_1 \in \mathcal{NC}(\{1, 2, \ldots, k\})$ and $\mathcal{V}_2 \in \mathcal{NC}(\{k+1, k+2, \ldots, n\}$, and in the same manner $\mathcal{W}_1 \in \mathcal{NC}(\{1, 2, \ldots, l\})$ and $\mathcal{W}_2 \in \mathcal{NC}(\{l+1, l+2, \ldots, n\}$. In general, a pair of partitions of two sets may be treated as a partition of a disjoint sum of these sets.

For fixed $l \in \{1, 2, \ldots, n-1\}$ consider a pair $\mathcal{W}_1 \in \mathcal{NC}(\{1, 2, \ldots, l\})$ and $\mathcal{W}_2 \in \mathcal{NC}(\{l+1, l+2, \ldots, n\}$. We are going to construct a pair $\mathcal{V}_1, \mathcal{V}_2$ of partitions, which appear in the first sum, and for which the equality

$$\mathcal{R}(\mathcal{V}_1)\mathcal{R}(\mathcal{V}_2) s^{\sharp o(\mathcal{V}_1)+1} = \mathcal{R}(\mathcal{W}_1)\mathcal{R}(\mathcal{W}_2) s^{\sharp o(\mathcal{W}_1)}$$

holds. In the partition $\mathcal{W}_1$ the block containing l is outer, and has a minimal element, say k, with $1 \leqslant k \leqslant l$. Let $\mathcal{U}$ be equal to $\mathcal{W}_1$ restricted to the set $\{k, \ldots, l\}$. It consists of one outer block and, possibly but not necessarily, some inner blocks. Let us define $\mathcal{V}_1$ to be equal to the rest of $\mathcal{W}_1$, which is the restriction of $\mathcal{W}_1$ to the set $\{1, \ldots, k-1\}$ (if $k = 1$, then, by definition, $\mathcal{V}_1$ is empty). Now let us define $\mathcal{V}_2 \in \mathcal{NC}(\{k, k+1, \ldots, n\})$ to be equal $\mathcal{U}$ on $\{k, \ldots, l\}$ and equal $\mathcal{W}_2$ on $\{l+1, \ldots, n\}$. It is clear that $\mathcal{V}_1$ has exactly one outer block less than $\mathcal{W}_1$, and that blocks of the disjoint sum of $\mathcal{W}_1$ and $\mathcal{W}_2$ are exactly the same as blocks of the disjoint sum of $\mathcal{V}_1$ and $\mathcal{V}_2$. Therefore the required equality holds.

This way we can observe that all the terms in the summation over l will be cancelled by equal terms in the summation over k. What we need to find out yet is what terms will remain in the first summation. In the above construction the partition $\mathcal{V}_2$ has always at least two outer blocks: one which is from $\mathcal{U}$, and at least one from $\mathcal{W}_2$. This means that we cannot obtain pairs $(\mathcal{V}_1, \mathcal{V}_2) \in \mathcal{NC}(\{1, \dots, k\}) \times \mathcal{NC}(\{k+1, \dots, n\}$ in which $\mathcal{V}_2$ has exactly one outer block. This outer block obviously contains both n and $k+1$. If we identify (disjoint sum of) such $\mathcal{V}_1$ and $\mathcal{V}_2$ with a partition $\mathcal{V} \in \mathcal{NC}(\{1, \dots, n\}$, then

$$\mathcal{R}(\mathcal{V}_1) \cdot \mathcal{R}(\mathcal{V}_2) = \mathcal{R}(\mathcal{V}) \quad \text{and} \quad \sharp o(\mathcal{V}_1) + 1 = \sharp o(\mathcal{V}).$$

Hence, after cancellation of equal terms and the identification, we obtain the formula given in the proposition. □

In a similar manner, one proves the following proposition for the t-classical convolution

PROPOSITION 10.4. – *Let $(\mathcal{M}_n(\mu))_{n\geqslant 0}$ be the sequence of moments of a measure μ and let $(\mathcal{R}_n(\mu))_{n\geqslant 1}$ be the sequence of classical cumulants of the measure. Then for the t-classical convolution the relation between moments and t-classical cumulants is given by*:

$$\mathcal{M}_n(\mu) = \sum_{\mathcal{V}\in\mathcal{P}(n)} \mathcal{R}(\mathcal{V}) \cdot t^{-\sharp o(\mathcal{V})}, \tag{10.34}$$

where for $\mathcal{V} \in \mathcal{P}(n)$ we define $\sharp o(\mathcal{V})$ =number of outer connected components of the partition $\mathcal{V}$.

11. Poisson type limit theorems

In this section we will study the Poisson type limit theorems for t-transformed free and classical convolutions.

The setup for both is the following. For a given number $0 \leqslant \alpha \leqslant 1$ we consider a sequence of measures

$$\mu_N = \left(1 - \frac{\alpha}{N}\right)\delta_0 + \frac{\alpha}{N}\delta_1,$$

which are convex combinations of point mass measures concentrated at 0 and 1. The problem is to find the limit (if it exists)

$$\lim_{N\to\infty} \mu_N \bullet_t \cdots \bullet_t \mu_N = \mu, \tag{11.35}$$

where $\bullet$ is either of the two considered convolutions.

In our cases we can use appropriate cumulants $\mathcal{R}^t$ to compute

$$\mathcal{R}_n^t(\mu_N \bullet_t \cdots \bullet_t \mu_N) = N \cdot \mathcal{R}_n^t(\mu_N) = N \cdot \mathcal{R}_n(\mathcal{U}_t \mu_N)$$

and then the estimation $\mathcal{R}_n(\mu) = \mathcal{M}_n(\mu) + \mathrm{O}(\frac{1}{N^2})$ to get the limit.

Let us recall, that the t-transform of the measure μ_N is

$$\mathcal{U}_t \mu_N = P_N \delta_{A_N} + Q_N \delta_{B_N}, \tag{11.36}$$

where

$$A_N = \frac{1}{2}(1-\gamma_N), \qquad B_N = \frac{1}{2}(1+\gamma_N), \qquad \gamma_N = \sqrt{1-4(1-t)\frac{\alpha}{N}\left(1-\frac{\alpha}{N}\right)},$$

$$P_N = \frac{B_N - \frac{\alpha}{N}}{\gamma_N}, \qquad Q_N = \frac{\frac{\alpha}{N} - A_N}{\gamma_N}.$$

Because of the different tools used, we will treat both cases separately.

Case 1. t-transformed free convolution.

In the case of t-transformed free convolution we compute directly t-transformed free cumulants of the limit measure μ:

$$\mathcal{R}_n^t(\mu) = \mathcal{R}_n(\mathcal{U}_t\mu) = t\alpha.$$

The generating function $\mathcal{R}^t(z)$ of the cumulants is then given by the series:

$$\mathcal{R}^t(z) = \frac{1}{z}\sum_{n=1}^{\infty} \mathcal{R}_n(\mathcal{U}_t\mu) z^n = \frac{t\alpha}{1-z}.$$

We use the relation between $\mathcal{R}^t$ and the Cauchy transform g^t of the measure μ_t, discovered in [13], to get

$$g^t(z) = \frac{z + (1-t\alpha) - \sqrt{(z-(1+t\alpha))^2 - 4t\alpha}}{2z}.$$

Let us observe, that this formula, for $\beta = t\alpha$ appears in [5]. Since for the Cauchy transform G_μ of the limit measure μ we have (with $s = \frac{1}{t}$):

$$\frac{1}{G_\mu(z)} = \frac{s}{g^t(z)} + (1-s)z$$

we can easily get the following formula for G_μ

$$G_\mu(z) = \frac{(2t-1)tz + t(1-t\alpha) - t\sqrt{(z-(1+t\alpha))^2 - 4t\alpha}}{2t(t-1)z^2 + (1+t\alpha)z - 2t\alpha}, \tag{11.37}$$

or

$$G_\mu(z) = \frac{(2t-1)z + (1-t\alpha) - \sqrt{(z-(1+t\alpha))^2 - 4t\alpha}}{2(t-1)z^2 + s(1+t\alpha)z - 2\alpha}. \tag{11.38}$$

This formula may be transformed into the following continued fraction:

$$G_\mu(z) = \cfrac{1}{z - \frac{1}{t}\cdot\alpha - \cfrac{\frac{1}{t}\cdot\alpha}{z-(1+\alpha) - \cfrac{t\cdot\alpha}{z-(1+\alpha)-\cfrac{t\cdot\alpha}{z-(1+\alpha)-\cfrac{t\cdot\alpha}{\ddots}}}}} \tag{11.39}$$

or

$$G_\mu(z) = \cfrac{1}{z - s\cdot\alpha - \cfrac{s\cdot\alpha}{z-(1+\alpha) - \cfrac{t\cdot\alpha}{z-(1+\alpha)-\cfrac{t\cdot\alpha}{z-(1+\alpha)-\cfrac{t\cdot\alpha}{\ddots}}}}} \tag{11.40}$$

with the help of the well-known formula:

$$\sqrt{a^2+b} = a + \cfrac{b}{2a + \cfrac{b}{2a+\cfrac{b}{\ddots}}}$$

For $t = 1$ one gets the Poisson limit measure for the free convolution.

Case 2. t-transformed classical convolution.

In the case of t-transformed classical convolution $*_t$ we use Fourier transform. For each positive integer N let ρ_N be the measure defined as $\rho_N = \mu_N *_t \cdots *_t \mu_N$. We are going to show, that this sequence of probability measures has a limit ρ. For this purpose we first find the Fourier transform of the t-transformation of ρ_N, and then the limit of it. It is clear that

$$(\mathcal{U}_t\rho_N)^\wedge(x) = \left[(\mathcal{U}_t\mu_N)^\wedge(x)\right]^N,$$

so it suffices to compute the Fourier transform of the measure $\mathcal{U}_t\mu_N$, which is

$$(\mathcal{U}_t\mu_N)^\wedge(x) = P_N \mathrm{e}^{-\mathrm{i}xA_N} + Q_N\mathrm{e}^{-\mathrm{i}xB_N}.$$

Since we study the limit, as N tends to infinity, of the Nth power of the Fourier transform, we may take the Taylor expansion of the exponents above, and only the terms containing $\frac{1}{N}$ in powers 0 or 1 will contribute to the limit. So let us consider the Taylor expansion:

$$(\mathcal{U}_t\mu_N)^\wedge(x) = \sum_{n=0}^{\infty}\frac{(-\mathrm{i}x)^n}{n!}\cdot\left(P_N A_N^n + Q_N B_N^n\right)$$

and check what contribution comes from each term of this series. Let us start with the observation that $A_N B_N = (1-t)\frac{\alpha}{N}(1-\frac{\alpha}{N}) = \mathrm{O}(\frac{1}{N})$. Using this we get the following:

LEMMA 11.1. – *For each positive integer n the equality $A_N^n + B_N^n = 1 + \mathrm{O}(\frac{1}{N})$ holds.*

Proof. – For $n = 1$ we have $A_N + B_N = 1$, and for $n \geqslant 2$

$$1 = (A_N + B_N)^n = A_N^n + B_N^n + A_N B_N \sum_{k=1}^{n-1} \binom{n}{k} A_N^k B_N^{n-1-k} = A_N^n + B_N^n + \mathrm{O}\left(\frac{1}{N}\right). \qquad \square$$

The following lemma shows what contribution of each term of the considered series should we take into account.

LEMMA 11.2. – *For each positive integer $n \geqslant 2$ we have $P_N A_N^n + Q_N B_N^n = \frac{t\alpha}{N} + \mathrm{O}(\frac{1}{N^2})$, and for $n = 1$ we get $P_N A_N + Q_N B_N = \frac{\alpha}{N}$.*

Proof. – Using the definition of all terms on the left-hand side of the above equality, and that $\gamma_N = B_N - A_N$, we may write for $n \geqslant 3$:

$$\begin{aligned}
P_N A_N^n + Q_N B_N^n &= \frac{1}{\gamma_N}\left[\left(B - \frac{\alpha}{N}\right)A_N^n + \left(\frac{\alpha}{N} - A_N\right)B_N^n\right] \\
&= \frac{\alpha}{N}\left(B_N^{n-1} + B_N^{n-2}A_N + \cdots + B_N A_N^{n-2} + A_N^{n-1}\right) \\
&\quad - B_N A_N\left(B_N^{n-2} + B_N^{n-3}A_N + \cdots + B_N A_N^{n-3} + A_N^{n-2}\right) \\
&= \frac{\alpha}{N}\left(B_N^{n-1} + A_N^{n-1}\right) - (1-t)\frac{\alpha}{N}\left(B_N^{n-2} + A_N^{n-2}\right) + \mathrm{O}\left(\frac{1}{N^2}\right) \\
&= \frac{\alpha}{N}\left(-A_N B_N^{n-2} - B_N A_N^{n-2}\right) + \frac{t\alpha}{N}\left(B_N^{n-2} + A_N^{n-2}\right) + \mathrm{O}\left(\frac{1}{N^2}\right)r \\
&= \frac{t\alpha}{N} + \mathrm{O}\left(\frac{1}{N^2}\right).
\end{aligned}$$

For $n = 2$ one easily computes $P_N A_N^2 + Q_N B_N^2 = \frac{t\alpha}{N} + \frac{(1-t)\alpha^2}{N^2}$. $\square$

Using these two lemmas we can now compute the Fourier transform of the t-transformation $\mathcal{U}_t\rho$ of the limit measure.

PROPOSITION 11.3. – *The Fourier transform of the t-transformation of the limit measure is given by the formula*

$$(\mathcal{U}_t\rho)^\wedge(x) = \exp\left[t\alpha \mathrm{e}^{-\mathrm{i}x} - t\alpha - \mathrm{i}x(1-t)\alpha\right] = \mathrm{e}^{t\alpha(\mathrm{e}^{-\mathrm{i}x}-1)} \cdot \mathrm{e}^{-\mathrm{i}x(1-t)\alpha}.$$

Proof. – It follows from the lemmas above that

$$\begin{aligned}
(\mathcal{U}_t\mu_N)^\wedge(x) &= 1 + \frac{\alpha}{N}(-\mathrm{i}x) + \sum_{n=2}^{\infty} \frac{(-\mathrm{i}x)^n}{n!}\left(\frac{t\alpha}{N} + \mathrm{O}\left(\frac{1}{N^2}\right)\right) \\
&= 1 + \frac{\alpha}{N}(-\mathrm{i}x) + \frac{t\alpha}{N}\sum_{n=2}^{\infty} \frac{(-\mathrm{i}x)^n}{n!} + \mathrm{O}\left(\frac{1}{N^2}\right).
\end{aligned}$$

Hence

$$\lim_N (\mathcal{U}_t \rho_N)^\wedge(x) = \lim_N \big[(\mathcal{U}_t \mu_N)^\wedge(x)\big]^N$$
$$= \lim_N \left[1 + \frac{\alpha}{N}(-\mathrm{i}x) + \frac{t\alpha}{N}\sum_{n=2}^{\infty}\frac{(-\mathrm{i}x)^n}{n!}\right] = \mathrm{e}^{t\alpha(\mathrm{e}^{-\mathrm{i}x}-1)} \cdot \mathrm{e}^{-\mathrm{i}x(1-t)\alpha}. \qquad \square$$

By direct computation one can now check the following

COROLLARY 11.4. – *The t-transformation of the Poisson limit measure ρ is*

$$\rho_t = \mathrm{e}^{-t\alpha}\sum_{n=0}^{\infty}\frac{(t\alpha)^n}{n!}\delta_{n+(1-t)\alpha}.$$

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