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FREE DIFFUSIONS, FREE ENTROPY AND FREE FISHER INFORMATION

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ABSTRACT. – Motivated by the stochastic quantization approach to large N matrix models, we study solutions to free stochastic differential equations $dX_t = dS_t - \frac{1}{2} f(X_t)\, dt$ where S_t is a free brownian motion. We show existence, uniqueness and Markov property of solutions. We define a relative free entropy as well as a relative free Fisher information, and show that these quantities behave as in the classical case. Finally we show that, in contrast with classical diffusions, in general the asymptotic distribution of the free diffusion does not converge, as $t \to \infty$, towards the master field (i.e., the Gibbs state). © 2001 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Nous étudions des équations différentielles stochastiques du type $dX_t = dS_t - \frac{1}{2} f(X_t)\, dt$ où S_t est un mouvement brownien libre, suggérées par la quantification stochastique des modèles matriciels de grande taille. Nous établissons l'existence et l'unicité des solutions, ainsi que leur caractère Markovien. Nous définissons une entropie libre relative et une information de Fisher relative, adaptées à ces diffusions et montrons que ces quantités se comportent comme leurs analogues classiques. Enfin nous montrons qu'en général, contrairement à ce qui se passe dans le cas classique, la distribution de la diffusion ne converge pas vers l'état de Gibbs. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

The purpose of this paper is to start the study of diffusion equations where the driving noise is a free brownian motion. Reasons for considering such equations will be explained in the next sections of this introduction.

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1.1. Gibbs states and diffusion theory

Let V be a C^2 function on $\mathbb{R}^d$, with

$$Z = \int_{\mathbb{R}^d} \mathrm{e}^{-V(x)}\,dx < \infty.$$

The probability measure

$$\mu(dx) = \frac{1}{Z}\mathrm{e}^{-V(x)}\,dx \tag{1.1.1}$$

is called the Gibbs state associated with the potential V. A well known way of obtaining the Gibbs state (useful for example in a Monte-Carlo simulation), is to construct the diffusion process on $\mathbb{R}^d$, with drift $-\frac{1}{2}\nabla V$, which is the solution to the stochastic differential equation

$$dX_t = dB_t - \frac{1}{2}\nabla V(X_t)\,dt. \tag{1.1.2}$$

Here B_t is a brownian motion on $\mathbb{R}^d$. Then, for any initial distribution of the diffusion, the distribution of X_t converges, as $t \to \infty$ to the Gibbs state. More precise statements can be given if one introduces the following quantities. For two mutually absolutely continuous probability measures let

$$H(\nu \mid \mu) = \int_{\mathbb{R}} \log\left(\frac{\nu(dx)}{\mu(dx)}\right)\nu(dx) \tag{1.1.3}$$

be the relative entropy of ν with respect to μ. Recall that $H(\nu \mid \mu) \geqslant 0$ with equality only if $\mu = \nu$. If the density $p = \frac{\nu(dx)}{\mu(dx)}$ is differentiable, let

$$I(\nu \mid \mu) = \int_{\mathbb{R}} \left|\frac{\nabla p(x)}{p(x)}\right|^2 \nu(dx) \tag{1.1.4}$$

be the relative free Fisher information. When μ is the Gibbs state (1.1.1), and $\nu(dx) = q(x)\,dx$, one has

$$\begin{aligned} H(\nu \mid \mu) &= \int_{\mathbb{R}} q(x)\log q(x)\,dx + \int_{\mathbb{R}} V(x)q(x)\,dx + \log Z \\ &= H(\nu) + \int_{\mathbb{R}} V(x)q(x)\,dx + \log Z, \end{aligned} \tag{1.1.5}$$

$H(\nu)$ being the entropy of ν, and

$$I(\nu \mid \mu) = \int_{\mathbb{R}} \left|\frac{\nabla q(x)}{q(x)} + \nabla V(x)\right|^2 \mu(dx). \tag{1.1.6}$$

For $\mu_t(dx)$, the distribution of X at time t, one has

$$\frac{\partial}{\partial t} H(\mu_t \mid \mu) = -\frac{1}{2} I(\mu_t \mid \mu)$$

so that the relative entropy is nonincreasing. Furthermore one has

$$H(\mu_t \mid \mu) \to 0 \quad \text{as } t \to \infty.$$

Exponential rate of convergence to 0 is obtained when μ satisfies a Logarithmic Sobolev Inequality, i.e.

$$H(\nu \mid \mu) \leqslant \frac{1}{2\rho} I(\nu \mid \mu)$$

for some positive constant ρ, and for all measures ν such that the right hand side is finite, see, e.g., [15] for a discussion.

1.2. Matrix models

In these models one considers the limit, as $N \to \infty$, of the quantities

$$\frac{1}{Z_N} \int\limits_{(\mathcal{H}_N)^k} \frac{1}{N} tr_N\big(Q(M)\big) \mathrm{e}^{-N tr_N(P(M))}\, dM, \tag{1.2.1}$$

where P and Q are non-commutative polynomials in k non-commuting indeterminates, tr_N denotes the trace over $N \times N$ matrices, the integral is over the set $(\mathcal{H}_N)^k$ of k-tuples $M = (M_1, \ldots, M_k)$ of Hermitian $N \times N$ matrices and

$$Z_N = \int\limits_{(\mathcal{H}_N)^k} \mathrm{e}^{-N tr_N(P(M))}\, dM.$$

For fixed P, if the limit exists for all Q, then it can be put (via the GNS construction) in the form

$$\lim_{N\to\infty} \frac{1}{Z_N} \int\limits_{(\mathcal{H}_N)^k} \frac{1}{N} tr_N\big(Q(M)\big) \mathrm{e}^{-N tr_N(P(M))}\, dM = \tau_P\big(Q(X)\big), \tag{1.2.2}$$

where $X = (X_1, \ldots, X_k)$ is a k-tuple of self-adjoint elements in some von Neumann algebra $\mathcal{A}_P$, equipped with a normal tracial state τ_P. The k-tuple of operators X is called the master field, see [7,11]. Proving its existence seems to be a difficult problem, with implications in the physics of quantum fields. In one dimension (i.e., for a one-matrix model), the situation is well understood, and the master field is then characterized by its distribution, which is the probability measure on the real line whose moments are given by $\tau_P(X^n)$; $n \geqslant 0$. This probability measure achieves the unique global maximum of the functional

$$\Sigma_P(\nu) = \iint\limits_{\mathbb{R}} \log|x-y|\, \nu(dx)\, \nu(dy) - \int\limits_{\mathbb{R}} P(x)\, \nu(dx)$$

on probability measures ν on $\mathbb{R}$, and thus it is uniquely determined by the polynomial P. It exists if and only if the polynomial P is non-constant and bounded below on $\mathbb{R}$. In fact potentials more general than polynomials can be considered, see [16] where such maximization problems are thoroughly considered. In the multi-matrix case, very little is known unless the polynomial P splits as a sum

$$P(M_1, \ldots, M_k) = P_1(M_1) + \cdots + P_k(M_k)$$

for some one-variable polynomials $P_1, \ldots, P_k$. Then the master field $(X_1, \ldots, X_k)$ is known to consist of free random variables in the sense of Voiculescu [22]. The distribution of each of the variables is obtained by resolving the corresponding one-matrix model, while their joint distribution, i.e., the computation of all moments $\tau_P(Q(X))$ is obtained by Voiculescu's freeness prescription. As soon as there is a nontrivial interaction between the components of the matrix model, we do not know any way to prove the existence of the master field, and no explicit formula for the joint moments.

1.3. Free stochastic quantization

A stochastic quantization approach to the master field has been proposed in the physical literature, see [12,6,7]. At the level of the matrix models this means looking, as in (1.1.2), at the solution to the diffusion equation (also called Langevin equation)

$$dM_t = dB_t - \frac{1}{2} N \nabla (tr_N P)(M_t)\, dt, \tag{1.3.1}$$

where B denotes a brownian motion on $(\mathcal{H}_N)^k$, normalized so that $E[tr_N(B_i(t)^2)] = N^2 t$. In terms of the components $M_1, \ldots, M_k$ this gives

$$dM_i(t) = dB_i(t) - \frac{1}{2} N \partial_i P\big(M_i(t)\big)\, dt, \tag{1.3.2}$$

where ∂_i is the ith partial cyclic derivative on polynomials in k non-commuting indeterminates $(X_1, \ldots, X_k)$, given by

$$\partial_i a_1 X_i a_2 X_i \cdots a_{n-1} X_i a_n = \sum_{k=2}^{n} a_k X_i a_{k+1} \cdots X_i a_n a_1 X_i \cdots X_i a_{k-1},$$

when $a_1, \ldots, a_n$ are polynomials in the other variables. Remark that if Π_P is the map

$$(M_1, \ldots, M_n) \mapsto tr_N\big(P(M_1, \ldots, M_n)\big)$$

then the cyclic derivative satisfies

$$\nabla_i \Pi_P(M).H = tr_N\big(H \partial_i P(M)\big).$$

By the finite dimensional Gibbs state result, we know that the measure

$$\frac{1}{Z_N}\mathrm{e}^{-Ntr_N(P(M))}\,dM$$

is the large t limit of the distribution of M_t. As explained above, the master field should be obtained by taking the large N limit of this Gibbs state. Free diffusions arise when one tries exchanging the large t and large N limits, so that one considers first the large N limit of (1.3.1). In taking the large N limit, one first rescales the time by $1/N$, so that the brownian motion $B_{t/N}$ converges as $N \to \infty$ towards the free brownian motion (see, e.g., [1]), namely one looks at the equation

$$dM_i(t) = dB_i(t/N) - \frac{1}{2}\partial_i P\big(M(t)\big)\,dt$$

and when $N \to \infty$ the equation becomes

$$dX_i(t) = dS_i(t) - \frac{1}{2}\partial_i P(X_t)\,dt, \tag{1.3.3}$$

where $X_i(t)$ are the unknown non-commutative random variables, and $S_i(t)$ are free brownian motions. The problem is to understand the large t limit of the solution to Eq. (1.3.3). One hopes that when $t \to \infty$ the $X_i(t)$ will converge to the master field. As we shall see, in general this procedure fails to recover the master fields, but in an interesting way. We shall give a rigourous mathematical treatment of Eq. (1.3.3), with special emphasis on the case of the one-matrix models. In this case, we shall consider the equation

$$dX(t) = dS(t) - \frac{1}{2}f(X_t)\,dt \tag{1.3.4}$$

for some class of drift f, and prove under regularity assumptions on f that Eq. (1.3.4) with given initial value admits a unique solution. We shall define quantities analogous to (1.1.3) and (1.1.4) which play the same role for free diffusions. We shall then show that in general the distribution of the solution X_t fails to converge to the master field distribution as $t \to \infty$.

1.4. Large matrix heuristics

In this section we shall give a quick heuristic derivation, based on the large matrix approximation, of some of the results we prove rigorously below. Let $f:\mathbb{R} \to \mathbb{R}$ be a smooth function, and let us consider the stochastic differential equation on $N \times N$ Hermitian matrices

$$dM_t = dB_{t/N} - \frac{1}{2}f(M_t)\,dt,$$

where f is meant to act by functional calculus for Hermitian operators. Because of the time scaling, the brownian motion $B_{t/N}$ is associated with the rescaled Hilbert–Schmidt scalar product $\frac{1}{N}tr_N(AB^*)$. It follows from the expression of the Laplace operator in polar coordinates on $\mathcal{H}_N$ (see, e.g., [8]) that the eigenvalues $(\lambda_1(t),\dots,\lambda_N(t))$ of the matrix M_t satisfy a stochastic differential equation

$$d\lambda_i(t)=\frac{1}{\sqrt{N}}\,d\beta_i(t)+\frac{1}{N}\sum_{\substack{1\leqslant j\leqslant N\\ i\neq j}}\frac{1}{\lambda_i-\lambda_j}\,dt-\frac{1}{2}f\big(\lambda_i(t)\big)\,dt,$$

where the β_i are independent one dimensional brownian motions. The generator of the eigenvalue diffusion process is thus the sum of the diffusive term $\frac{1}{2\sqrt{N}}\Delta$ (where Δ is the usual Laplace operator), an entropic term

$$\frac{1}{N}\sum_{i=1}^{N}\bigg(\sum_{\substack{1\leqslant j\leqslant N\\ i\neq j}}\frac{1}{\lambda_i-\lambda_j}\bigg)\frac{\partial}{\partial\lambda_i},$$

and the drift term $\sum_{i=1}^{N}\frac{1}{2}f(\lambda_i)\frac{\partial}{\partial\lambda_i}$. We now argue that when N is large, the brownian term in this equation becomes small in front of the other terms, and in finite time, the process behaves like a dynamical system with a small random perturbation (see, e.g., [9] for the theory of such dynamical systems). In the large N limit, the trajectory of the eigenvalue vector behaves as that of the flow of the deterministic vector field

$$\sum_{i=1}^{N}\bigg(\frac{1}{N}\sum_{\substack{1\leqslant j\leqslant N\\ i\neq j}}\frac{1}{\lambda_i-\lambda_j}-\frac{1}{2}f(\lambda_i)\bigg)\frac{\partial}{\partial\lambda_i}. \tag{1.4.1}$$

Assume that the empirical distribution $\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i(t)}$ converges, as $N\to\infty$, towards some limit distribution $\lambda_t(dx)=p_t(x)\,dx$, then for any smooth test function g one has

$$\frac{\partial}{\partial t}\frac{1}{N}\sum_{i=1}^{N}g\big(\lambda_i(t)\big)=\frac{1}{N^2}\sum_{i=1}^{N}g'\big(\lambda_i(t)\big)\bigg(\sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}-\frac{1}{2}f\big(\lambda_i(t)\big)\bigg)$$

and taking the large N limit, one gets

$$\frac{\partial}{\partial t}\int g(x)p_t(x)\,dx=\int_{\mathbb{R}}g'(x)\Big(Hp_t(x)-\frac{1}{2}f(x)\Big)p_t(x)\,dx,$$

where

$$Hu(x):=p.v.\int_{\mathbb{R}}\frac{u(y)}{x-y}\,dy$$

is (up to a multiplicative constant) the Hilbert transform. The flow equation gives then the following “free Fokker–Planck equation” for p_t

$$\frac{\partial p_t}{\partial t}=-\frac{\partial}{\partial x}\Big(p_t\Big(Hp_t-\frac{1}{2}f\Big)\Big). \tag{1.4.2}$$

On the other hand, the vector field (1.4.1) is $\frac{1}{2}\nabla\Phi_N$ for the function

$$\Phi_N(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\sum_{i\neq j}\log|\lambda_i - \lambda_j| - \sum_{i=1}^{N} F(\lambda_i),$$

where F is a primitive of f. It follows that along the trajectories of the flow, the quantity $\Phi_N(\lambda(t))$ is increasing with derivative $\frac{1}{2}|\nabla\Phi_N(\lambda(t))|^2$. One has

$$\lim_{N\to\infty}\frac{1}{N}\Phi_N\big(\lambda(t)\big) = \int_{\mathbb{R}}\log|x-y|\,p_t(x)p_t(y)\,dx\,dy - \int_{\mathbb{R}} F(x)p(x)\,dx := \Sigma_F(p_t) \tag{1.4.3}$$

and

$$\lim_{N\to\infty}\frac{1}{N}\big|\nabla\Phi_N\big(\lambda(t)\big)\big|^2 = 4\int_{\mathbb{R}}\left(Hp_t(x) - \frac{1}{2}f(x)\right)^2 p_t(x)\,dx := I_F(p_t). \tag{1.4.4}$$

We obtain from this the differential relation

$$\frac{\partial}{\partial t}\Sigma_F(p_t) = \frac{1}{2}I_F(p_t) \tag{1.4.5}$$

which, as one can check at least formally, follows from (1.4.2). The two expressions (1.4.3) and (1.4.4) are closely related to the free entropy and the free Fisher information measure defined by Voiculescu [18] (which correspond to the case where $f = 0$). We shall prove, using the free Ito's formula, that the distribution of the solution of the free stochastic differential equation (1.3.4) indeed satisfies, in a weak sense, the free Fokker–Planck equation (1.4.2), and that (1.4.5) holds. As we are dealing with gradient flows, we know that in general the trajectory of such a gradient flow converges to a local maximum of the function, but depending on the initial distribution, and may not converge towards the global maximum. We shall see that such a phenomenon occurs for the free stochastic differential equation, and this explains why we should not expect, in general, to get the master field as the large t limit distribution of the free diffusion.

It should be possible to make our arguments rigourous and prove the formulas above for the free diffusions, starting from the matricial approximations. However such an approach, which relies on an understanding of the eigenvalues of the approximating matrices is restricted to the case of one matrix models, whereas we want to derive methods applicable to multimatrix models. Therefore we shall follow another route and work directly on the limiting objects, i.e., semi-circular systems.

The stationary (i.e., $\frac{\partial p}{\partial t} = 0$) Fokker–Planck equation reads

$$p\left(Hp - \frac{1}{2}f\right) = 0. \tag{1.4.6}$$

This is the Schwinger–Dyson equation of the matrix model (cf. [7,11]). As we see, this does only give us the equation for a local maximum of the relative free entropy (1.4.3), and so it may have solutions which are not given by the master field.

Although the preceding arguments give us some insight into the free diffusions, they do not provide a complete picture. Indeed the one dimensional distributions of the free diffusion are not enough to determine the process, in particular as we shall see the free diffusion is a non-commutative Markov process with probability transition functions satisfying a linearized version of the free Fokker–Planck equation. This cannot be seen if one only looks at the behaviour of the eigenvalues of the matrix models, indeed the Markov property really comes from the behaviour of the eigenspaces of the diffusing matrix.

1.5.

This paper is organized as follows: in Section 2 we recall some preliminary facts from free probability theory. In Section 3 we introduce the free diffusion equation and obtain existence and uniqueness results, with bounds on the solutions. We also establish the Markov property of the solution. We specialize to the one-dimensional case in Section 4, where we derive the Fokker–Planck equation, and discuss more thoroughly the Markov property of the free diffusion. An Euler scheme for the approximation of the solution is described in Section 5, and used to prove regularity properties of the distribution of the free diffusion. These regularity properties are then used in Section 6 to prove the relation between relative free entropy and free Fisher information. Finally in Section 7 we consider the asymptotic behaviour of the free diffusion.

2. Preliminaries and notations

2.1. Non-commutative probability spaces and free random variables

In this paper we will consider non-commutative probability spaces which are von Neumann algebras with a faithful, normal tracial state. We refer to [22,20,14,3], for further information on the basics of free probability. We shall recall some facts about freeness with amalgamation. Let $(\mathcal{A},\tau)$ be a non-commutative probability space, and let $\mathcal{B}$ be a unital weakly closed subalgebra of $\mathcal{A}$, then we denote by $\tau(.\mid\mathcal{B})$ the conditional expectation onto $\mathcal{B}$. One defines $\mathcal{B}$-free independence of subalgebras of $\mathcal{A}$ containing $\mathcal{B}$, in a similar way as free independence, using the conditional expectation $\tau(.\mid\mathcal{B})$ in place of the state τ, see, e.g., [14].

LEMMA 2.1. – *Let $(\mathcal{A},\tau)$ be a von Neumann non-commutative probability space, let $\mathcal{B}_1,\mathcal{B}_2\subset\mathcal{A}$ be free von Neumann-subalgebras, and let $X=X^*\in\mathcal{B}_1$, then the algebras $(\mathcal{B}_2\cup\{X\})''$ and $\mathcal{B}_1$ are $\{X\}''$-free, and for any $Y\in(\mathcal{B}_2\cup X)''$ one has $\tau(Y\mid\mathcal{B}_1)=\tau(Y\mid X)$. Furthermore $\tau(.\mid X)$ maps $C^*(\mathcal{B}_2,X)$ onto $C^*(X)$.*

Proof. – Let $b_1,\ldots,b_n\in\mathcal{B}_2$, then it follows from the moment cumulant formula [14], that for any $b\in\mathcal{B}_1$ the expression $\tau(b_1Xb_2X\cdots b_{n-1}Xb_nb)$ can be expressed as a linear combination of products of the form

$$\left(\prod_{r=1}^{k}\tau\left(\prod_{i\in I_r}b_i\right)\right)\left(\prod_{s=1}^{v}\tau\left(X^{l_s}\right)\right)\tau\left(X^lb\right),$$

where $\{1,\dots,n\} = I_1 \cup \cdots \cup I_k$, $n = l + l_1 + \cdots + l_v$. We deduce that the conditional expectation of $b_1 X \cdots b_{n-1} X b_n$ onto $\mathcal{B}_1$ is of the form $P(X)$ for some polynomial P. The assertions in the lemma follow easily from this observation. □

2.2. Free brownian motion

Let $(\mathcal{A},\tau)$ be a von Neumann non-commutative probability space. We shall assume that $\mathcal{A}$ is filtered, so that there exists a family $(\mathcal{A}_t)_{t\in\mathbb{R}_+}$ of unital, weakly closed $*$-subalgebras of $\mathcal{A}$, such that $\mathcal{A}_s \subset \mathcal{A}_t$ for all s,t with $s \leqslant t$. Further we shall assume that there exists an $(\mathcal{A}_t)_{t\in\mathbb{R}_+}$-free brownian motion $(S_t)_{t\in\mathbb{R}_+}$, i.e., each S_t is a self adjoint element of $\mathcal{A}$ with semi-circular distribution of mean zero and variance t, one has $X_t \in \mathcal{A}_t$ for all $t \geqslant 0$, and for all s,t with $s \leqslant t$, the element $S_t - S_s$ is free with $\mathcal{A}_s$, and has semi-circular distribution of mean zero and variance $t-s$. Once this brownian motion exists, one can define stochastic integrals of biprocesses with respect to S, as in [5]. The main results about stochastic integrals that we shall use are the free Burkholder–Gundy inequality (Theorem 3.2.1 of [5]), and the free Itô's formula (Theorem 4.1.2 of [5], or the functional form, see Section 4.3).

2.3. Operator Lipschitz function

Let $f:\mathbb{R}\to\mathbb{C}$ be a locally bounded measurable function, it is called an operator Lipschitz function if there exists a constant $K>0$ such that

$$\big\|f(X)-f(Y)\big\| \leqslant K\|X-Y\| \tag{2.3.1}$$

for all self-adjoint operators X,Y on a Hilbert space. The function f is called locally operator Lipschitz if for every $A>0$ there exists a constant $K_A>0$ such that (2.3.1) holds for all self-adjoint operators X,Y, of norm less than A. Clearly, an operator Lipschitz function is a Lipschitz function, but the converse is not true, and in fact being a C^1 function does not insure that the function is locally operator Lipschitz. Examples of operator Lipschitz functions are functions of the form

$$f(x) = \int_{\mathbb{R}} \mathrm{e}^{ixy}\mu(dy),$$

where μ is a bounded complex measure such that $\int_{\mathbb{R}} |x|\,|\mu|(dx) < \infty$, (this follows from Duhamel's formula, see, e.g., Section 1.2 in [5]). From this one can infer easily that C^2 functions are locally operator Lipschitz. More precise description of the classes of operator Lipschitz and locally operator Lipschitz functions can be given in terms of Besov spaces, see, e.g., Peller [13].

If $(\mathcal{A},\tau)$ is a non-commutative probability space, we shall also consider more generally functions $Q:\mathcal{A}_{sa}^k \to \mathcal{A}_{sa}$ (where $\mathcal{A}_{sa}$ is the self-adjoint part of $\mathcal{A}$), and call such functions locally Lipschitz if there exists constants $C(K)$; $K>0$ such that for all X_i, Y_i with norms $\leqslant K$, one has

$$\big\|Q(X_1,\dots,X_k) - Q(Y_1,\dots,Y_k)\big\| \leqslant C(K)\left(\sum_{i=1}^k \|X_i - Y_i\|\right).$$

Examples of such functions include self-adjoint polynomials in non-commuting indeterminates, but also many functions not given by functional calculus.

3. Existence of multidimensional free diffusions

Let $(\mathcal{A},\tau)$ be a filtered non-commutative probability space, as in Section 2.2, in which $S_1(t),\ldots,S_k(t)$ $(t\geqslant 0,)$ a k-dimensional free brownian motion, is defined. Each $S_i(t)$ is an $\mathcal{A}_t$-free brownian motion, and $\{S_1(t)\mid t\geqslant 0\},\ldots,\{S_k(t)\mid t\geqslant 0\}$ are free in $(\mathcal{A},\tau)$. Let $Q_1,\ldots,Q_k:\mathcal{A}_{sa}^k\to\mathcal{A}_{sa}$ be k locally Lipschitz functions, such that each Q_i maps $\mathcal{A}_{s,sa}^k$ to $\mathcal{A}_{s,sa}$ for all $s\geqslant 0$. Consider the system of stochastic differential equations

$$dX_i(t)=Q_i\big(X_1(t),\ldots,X_k(t)\big)\,dt+dS_i(t)\quad(1\leqslant i\leqslant k)\tag{3.1.1}$$

which means that we are looking for maps $t\mapsto X_i(t)$ with values in $\mathcal{A}$, such that

$$X_i(t)=X_i(0)+S_i(t)+\int_0^t Q_i\big(X_1(s),\ldots,X_k(s)\big)\,ds\quad\text{for all } t>0,\tag{3.1.2}$$

where $X_i(0)$ are the initial data.

THEOREM 3.1. – *Assume the following condition is satisfied for some constants $a\in\mathbb{R}$ and $b\geqslant 0$:*

$$\sum_{i=1}^k\big(Q_i(X_1,\ldots,X_k)X_i+X_iQ_i(X_1,\ldots,X_k)+1\big)\leqslant a\sum_{i=1}^k X_i^2+b\tag{3.1.3}$$

for all $X_1,\ldots,X_k\in\mathcal{A}_{sa}$. Then, given arbitrary initial conditions $X_i(0)=X_i^0\in\mathcal{A}_0$ $(i=1,\ldots,k)$, the system (3.1.1) *of stochastic differential equations has a unique solution $(X_1(t),\ldots,X_k(t))$ for all $t\geqslant 0$. Furthermore, we have $X_i(t)\in\mathcal{A}_t$ for all $i=1,\ldots,k$ and all $t\geqslant 0$, and the maps $t\mapsto X_i(t)$ are norm continuous.*

Proof. – We will construct the solution by the Picard iteration method. In order to keep the Lipschitz constants bounded we have to truncate the polynomials for big norms. This will be done by a function $h:[0,\infty)\to[0,1]$ which has the following properties: there exists $R>0$ such that h is identically 1 on $[0,R]$ and identically 0 on $[2R,\infty]$; h is continuous, $0\leqslant h(t)\leqslant 1$ for all $t\geqslant 0$, and h has a finite Lipschitz constant, i.e., there exists a $C>0$ such that

$$\big|h(t)-h(s)\big|\leqslant C|t-s|\quad\text{for all } t,s\geqslant 0.$$

Then we truncate a given locally Lipschitz function Q by going over to

$$f(X_1,\ldots,X_k):=Q(X_1,\ldots,X_k)h\left(\sum_{i=1}^k\|X_i\|\right).$$

LEMMA 3.2. – *There exists a constant* $c > 0$ *such that for* $X_1, \ldots, X_k, Y_1, \ldots, Y_k \in \mathcal{A}$, *we have the estimate*:

$$\left\| f(X_1, \ldots, X_k) - f(Y_1, \ldots, Y_k) \right\| \leqslant c \sum_{i=1}^{k} \|X_i - Y_i\|.$$

Proof. – Put

$$m_X := \max\bigl\{\|X_1\|, \ldots, \|X_k\|\bigr\} \quad \text{and} \quad m_Y := \max\bigl\{\|Y_1\|, \ldots, \|Y_k\|\bigr\}.$$

Note that $m_X \geqslant 2R$ implies $f(X_1, \ldots, X_k) = 0$. Thus the estimate is trivially satisfied in the case $m_X \geqslant 2R$ and $m_Y \geqslant 2R$.

Consider now the case $m_X \geqslant 2R$ and $m_Y < 2R$. Then we have

$$\begin{aligned} \left\| f(X_1, \ldots, X_k) - f(Y_1, \ldots, Y_k) \right\| &= \left\| f(Y_1, \ldots, Y_k) \right\| \\ &= \left\| Q(Y_1, \ldots, Y_k) \right\| \cdot \left| h\left(\sum_{i=1}^{k} \|Y_i\| \right) \right|. \end{aligned}$$

Now note that $Q(Y_1, \ldots, Y_k)$ is bounded on the set given by $m_Y < 2R$ and that

$$\begin{aligned} \left| h\left(\sum_{i=1}^{k} \|Y_i\| \right) \right| &= \left| h\left(\sum_{i=1}^{k} \|Y_i\| \right) - h\left(\sum_{i=1}^{k} \|X_i\| \right) \right| \\ &\leqslant C \left| \sum_{i=1}^{k} \|Y_i\| - \sum_{i=1}^{k} \|X_i\| \right| \leqslant C \sum_{i=1}^{k} \|Y_i - X_i\|. \end{aligned}$$

The case $m_X < 2R$ and $m_Y \geqslant 2R$ is analogous. So assume finally that both $m_X < 2R$ and $m_Y < 2R$. Then we have

$$\begin{aligned} &\left\| f(X_1, \ldots, X_k) - f(Y_1, \ldots, Y_k) \right\| \\ &\quad \leqslant \left\| Q(X_1, \ldots, X_k) \right\| \cdot \left| h\left(\sum_{i=1}^{k} \|X_i\| \right) - h\left(\sum_{i=1}^{k} \|Y_i\| \right) \right| \\ &\qquad + \left\| Q(X_1, \ldots, X_k) - Q(Y_1, \ldots, Y_k) \right\| \cdot \left| h\left(\sum_{i=1}^{k} \|Y_i\| \right) \right| \\ &\quad \leqslant \left\| Q(X_1, \ldots, X_k) \right\| \cdot C \sum_{i=1}^{k} \|Y_i - X_i\| + \left\| Q(X_1, \ldots, X_k) - Q(Y_1, \ldots, Y_k) \right\|. \end{aligned}$$

The assertion follows now by noticing that $Q(X_1, \ldots, X_k)$ remains bounded on the set given by $m_X < 2R$ and that we have on the set given by $m_X < 2R$ and $m_Y < 2R$ an estimate of the form

$$\left\| Q(X_1, \ldots, X_k) - P(Y_1, \ldots, Y_k) \right\| \leqslant \tilde{c} \sum_{i=1}^{k} \|X_i - Y_i\|$$

for some constant $\tilde{c}$. $\square$

We can now approximate the solution of the system (3.1.1) by replacing the functions Q_i by their truncated versions

$$f_i(X_1,\dots,X_k)=Q_i(X_1,\dots,X_k)h\left(\sum_{j=1}^k\|X_j\|\right)$$

for a function h as above with some fixed $R>0$. Thus we consider now the system of stochastic differential equations

$$dX_i(t)=f_i\bigl(X_1(t),\dots,X_k(t)\bigr)\,dt+dS_i(t)\quad(1\leqslant i\leqslant k)\tag{3.1.4}$$

with the initial conditions

$$X_i(0)=X_i^0.$$

This can be solved by Picard iteration applied to the integral equations

$$X_i(t)=X_i^0+\int_0^t f_i\bigl(X_1(s),\dots,X_k(s)\bigr)\,ds+S_i(t),$$

which gives in the usual way the existence and uniqueness of the solution of the system (3.1.4). Since the whole construction above depends via the function h on the parameter R we will denote this solution by $(X_1^R(t),\dots,X_k^R(t))$. It is also clear that as long as $\sum_{i=1}^k\|X_i^R(t)\|\leqslant R$, this solution $(X_1^R(t),\dots,X_k^R(t))$ is also a solution of the original problem (3.1.1). To ensure that for $R\to\infty$ we get a solution of (3.1.1), we thus need an argument ensuring that the norms of our solutions do not explode in finite time.

To see this let us consider the function

$$Z(t):=\sum_{i=1}^k X_i(t)^2.$$

An application of the free Ito's formula yields

$$\begin{aligned}d\bigl(\mathrm{e}^{-a\tau}Z(\tau)\bigr)&=-a\mathrm{e}^{-a\tau}\left(\sum_{i=1}^k X_i(\tau)^2\right)d\tau+\mathrm{e}^{-a\tau}\sum_{i=1}^k d\bigl(X_i(\tau)^2\bigr)\\
&=-a\mathrm{e}^{-a\tau}\left(\sum_{i=1}^k X_i(\tau)^2\right)d\tau\\
&\quad+\mathrm{e}^{-a\tau}\sum_{i=1}^k\Bigl(f_i\bigl(X_1(t),\dots,X_k(t)\bigr)X_i(\tau)+X_i(\tau)f_i\bigl(X_1(t),\dots,X_k(t)\bigr)+1\Bigr)\,d\tau\\
&\quad+\mathrm{e}^{-a\tau}\sum_{i=1}^k\bigl(dS_i(\tau)X_i(\tau)+X_i(\tau)\,dS_i(\tau)\bigr).\end{aligned}$$

By the norm continuity of stochastic integrals with respect to their upper bounds, the norms $\|X_i(t)^R\|$ are continuous functions of the time t. Let us consider

$T_R = \inf\{t \mid \sum_{i=1}^k \|X_i^R(t)\| > R\}$, then $T_R > 0$ and for $t \leqslant T_R$, by the hypothesis on the functions Q_i, we have

$$Z(t) \leqslant Z(0) + \mathrm{e}^{at} b \int_0^t \mathrm{e}^{-a\tau}\, d\tau + \mathrm{e}^{at} \sum_{i=1}^k \left(\int_0^t dS_i(\tau) X_i(\tau) \mathrm{e}^{-a\tau} + \int_0^t X_i(\tau) \mathrm{e}^{-a\tau}\, dS_i(\tau) \right).$$

By using the Burkholder–Gundy inequality in operator norm for stochastic integrals with respect to free brownian motion [5] we obtain

$$\begin{aligned}
\|Z(t)\| &\leqslant \|Z(0)\| + \frac{b}{a}\big(\mathrm{e}^{at} - 1\big) \\
&\quad + \mathrm{e}^{at} \sum_{i=1}^k \left(\left\| \int_0^t dS_i(\tau)\, X_i(\tau) \mathrm{e}^{-a\tau} \right\| + \left\| \int_0^t X_i(\tau) \mathrm{e}^{-a\tau}\, dS_i(\tau) \right\| \right) \\
&\leqslant \|Z(0)\| + \frac{b}{a}\big(\mathrm{e}^{at} - 1\big) + \sum_{i=1}^k 2 \cdot 2\sqrt{2} \left(\int_0^t \|X_i(\tau)^2\| \mathrm{e}^{-2a\tau}\, d\tau \right)^{1/2} \mathrm{e}^{at} \\
&\leqslant \|Z(0)\| + \frac{b}{a}\big(\mathrm{e}^{at} - 1\big) + k \cdot 4\sqrt{2} \max_{0 \leqslant s \leqslant t} \sqrt{\|Z(s)\|} \left(\int_0^t \mathrm{e}^{-2a\tau}\, d\tau \right)^{1/2} \mathrm{e}^{at} \\
&\leqslant \|Z(0)\| + \frac{b}{a}\big(\mathrm{e}^{at} - 1\big) + k \cdot 4 \max_{0 \leqslant s \leqslant t} \sqrt{\|Z(s)\|} \sqrt{\frac{\mathrm{e}^{2at} - 1}{a}}.
\end{aligned}$$

Put now

$$\varphi(t) := \max_{0 \leqslant s \leqslant t} \|Z(s)\|.$$

Then we have, for all $t \leqslant T_R$

$$\varphi(t) \leqslant \varphi(0) + \frac{b}{a}\big(\mathrm{e}^{at} - 1\big) + 4k\sqrt{\varphi(t)\big(\mathrm{e}^{2at} - 1\big)/a}.$$

At time T_R one has $\max \|X_i(T_R)\| \geqslant R/k$, therefore $R^2/k^2 \leqslant \varphi(T_R) \leqslant R^2$, hence

$$R^2/k^2 \leqslant \varphi(0) + \frac{b}{a}\big(\mathrm{e}^{aT_R} - 1\big) + 4kR\sqrt{\big(\mathrm{e}^{2aT_R} - 1\big)/a}.$$

From this we deduce a lower bound for T_R, so that $T_R \to \infty$ as $R \to \infty$ if $a > 0$, and $T_R = \infty$ for R large enough if $a < 0$. Thus for fixed t, one has $X_t^R = X_t$ for all $R \geqslant R_0$ and if $a < 0$, then there exists a uniform bound on the solution for all times, i.e., $\max_{i,t \geqslant 0} \|X_i(t)\| < \infty$. $\square$

3.2. Markov property of the free diffusion

Define for $t \geqslant 0$ the following C^* and von Neumann subalgebras of $\mathcal{A}$;

$$\mathcal{F}_t^0 = C^*\big(\{X_0;\ S_i(s);\ i = 1, \ldots, k;\ s \leqslant t\}\big); \quad \mathcal{X}_t^0 = C^*\big(X_i(t); i = 1, \ldots, k\big);$$

$$\mathcal{G}_t^0 = C^*\big(\{X_i(t);\ S_i(s) - S_i(t);\ i = 1, \dots, k;\ s \geqslant t\}\big),$$

$$\mathcal{F}_t = \big(\mathcal{F}_t^0\big)'';\quad \mathcal{X}_t = \big(\mathcal{X}_t^0\big)'';\quad \mathcal{G}_t = \big(\mathcal{G}_t^0\big)''$$

which represent respectively the past, present and future of the process. Observe that $\mathcal{X}_t^0 \subset \mathcal{F}_t^0 \cap \mathcal{G}_t^0$. The following proposition is an immediate consequence of the uniqueness of the solution of (3.1.1), and of Lemma 2.1.

PROPOSITION 3.3. – *The algebras $\mathcal{F}_t$ and $\mathcal{G}_t$ are $\mathcal{X}_t$-free, furthermore the conditional expectation $\tau(.\mid\mathcal{F}_t)$ maps $\mathcal{G}_t^0$ onto $\mathcal{X}_t^0$.*

This means that the process X_t is a free Markov process with respect to the filtrations $\mathcal{F}$ and $\mathcal{G}$. It follows that for all times $s < t$ one can define an operator $P_{s,t} : \mathcal{X}_t^0 \mapsto \mathcal{X}_s^0$ by the formula

$$\tau\big(Y \mid \mathcal{F}_s^0\big) = \tau\big(Y \mid \mathcal{X}_s^0\big) = P_{s,t}Y.$$

This operator is a completely positive map, and is a non-commutative analogue of a probability transition function.

4. One dimensional free diffusions and the Fokker–Planck equation

4.1. The free Fokker–Planck equation

Let $f : \mathbb{R} \to \mathbb{C}$ be a locally operator Lipschitz function, such that

$$-xf(x) \leqslant ax^2 + b \quad \text{for all } x \in \mathbb{R}, \tag{4.1.1}$$

and let X be the solution to

$$dX_t = dS_t - \frac{1}{2} f(X_t)\,dt \tag{4.1.2}$$

which exists according to Theorem 3.1. Let φ be a C^2 function on $\mathbb{R}$, then by the free Itô's formula ([5, Section 4.3]), one has

$$\varphi(X_t) = \varphi(X_0) + \int_0^t \partial\varphi(X_s)\sharp dS_s - \frac{1}{2}\int_0^t \varphi'(X_s) f(X_s)\,ds + \frac{1}{2}\int_0^t \Delta_s\varphi(X_s)\,ds,$$

where one defines

$$\Delta_s g(x) = 2\frac{\partial}{\partial x}\left(\int_{\mathbb{R}} \frac{g(x) - g(y)}{x - y}\,\mu_s(dy)\right)$$

for a C^2 function g, μ_s being the distribution of X_s. Note that the factor 2 in the definition of Δ_s above was overlooked in [5]. Taking the trace of both sides of the equation, one has, since the stochastic integral has expectation 0,

$$\tau\big(\varphi(X_t)\big) = \tau\big(\varphi(X_0)\big) - \frac{1}{2}\int_0^t \tau\big(\varphi'(X_s) f(X_s)\big)\, ds + \frac{1}{2}\int_0^t \tau\big(\Delta_s \varphi(X_s)\big)\, ds$$

or

$$\int_{\mathbb{R}} \varphi(x)\, \mu_t(dx) = \int_{\mathbb{R}} \varphi(x)\, \mu_0(dx) - \frac{1}{2}\int_0^t \int_{\mathbb{R}} \varphi'(x) f(x)\, \mu_s(dx)\, ds$$
$$+ \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial x}\bigg(\int_{\mathbb{R}} \frac{\varphi(x)-\varphi(y)}{x-y}\mu_s(dy)\bigg)\, \mu_s(dx)\, ds. \qquad (4.1.3)$$

As we shall see later, the distribution of X_t has a bounded (non smooth in general) density $p_t(x)$, for all $t > 0$. Making formal integration by parts in this formula, Eq. (4.1.3) means that $\mu_t(dx) = p_t(x)\, dx$ is a weak solution of the free Fokker–Planck equation (1.4.2). It is known that even for a smooth initial distribution, if $f = 0$, then the distribution of X_t can develop singularities, i.e., for t large enough, there will exist points where the density p_t is not differentiable, hence in general Eq. (1.4.2) cannot be taken in a pointwise sense.

4.2. Free Markov property of the diffusion

In the one-dimensional case, the results from Section 3.2 yield an operator $P_{s,t} : \mathcal{X}_t^0 \mapsto \mathcal{X}_s^0$ which is a Markov operator and is given by a Feller kernel of probability measure $P_{s,t}(x, dy)$ on $Spec(X_s) \times Spec(X_t)$. The free Markov property of the diffusion now implies that for all times $t_1 < t_2 < \cdots < t_n$, the following time ordered moments can be computed

$$\tau\big(f_1(X_{t_1}) \cdots f_n(X_{t_n})\big)$$
$$= \int \cdots \int_{\mathbb{R}} f_1(x_1) \cdots f_n(x_n)\, \mu_{t_1}(dx_1) P_{t_1,t_2}(x_1, dx_2) \cdots P_{t_{n-1},t_n}(x_{n-1}, dx_n)$$

where $f_1, \ldots, f_n$ are bounded Borel functions on $\mathbb{R}$. In particular, they coincide with the time ordered moments of the classical Markov process with transition probabilities given by $P_{s,t}$ and one dimensional distributions μ_t. Taking conditional expectations in Itô's formula yields that $P_{s,t} = p_{s,t}(x, y)\, dy$ is a weak solution to the linearized, non time homogeneous, free Fokker–Planck equation

$$\frac{\partial}{\partial t} p_{s,t}(x_0, x) = -Hp_t(x) \frac{\partial}{\partial x} p_{s,t}(x_0, x) - p_t(x) \frac{\partial}{\partial x}\big(Hp_{s,t}(x_0, x)\big)$$
$$+ \frac{\partial}{\partial x}\big(p_{s,t}(x_0, x) f(x)\big).$$

This is to be compared with the Markov property for processes with free increments (see [4]).

4.3. The stationary case

Suppose that the initial distribution of X_t is chosen so that the distribution $\mu_t(dx) = p_t(x)\,dx$ is constant. In this case one has, from the free Fokker–Planck equation, that $Hp = \frac{1}{2}f$ on the support of $p(x)\,dx$. As we shall see, this can be achieved by taking X_0 to be distributed as the measure maximizing the relative free entropy. In this case, the Markov operators $P_{s,t}$ become time homogeneous, indeed they depend only on $t-s$. As it is easy to check, these correspond to the classical Markov transition operator associated with the Dirichlet form

$$Q(\varphi) = \frac{1}{2}\iint_{\mathbb{R}} \left|\frac{\varphi(x)-\varphi(y)}{x-y}\right|^2 \mu(dx)\,\mu(dy).$$

See, e.g., [10]. Observe that this Dirichlet form can be put in the form

$$Q(\varphi) = \frac{1}{2}\mu\otimes\mu\big(|\partial\varphi|^2\big),$$

where, e.g., on polynomials, $\partial : \mathbb{C}[X] \to \mathbb{C}[X]\otimes\mathbb{C}[X]$ is the non-commutative derivation (for the natural $\mathbb{C}[X]$ bimodule structures of $\mathbb{C}[X]$ and $\mathbb{C}[X]\otimes\mathbb{C}[X]$) such that $\partial X = 1\otimes 1$. More generally ∂ can be defined on functions by

$$\partial\varphi(x,y) = \frac{\varphi(x)-\varphi(y)}{x-y}.$$

This is to be compared with the expression of the stationary diffusion (1.1.2), which is associated with the Dirichlet form on $L^2(p(x)\,dx)$

$$Q(\varphi) = \frac{1}{2}\int_{\mathbb{R}} \left|\nabla\varphi(x)\right|^2 \mu(dx).$$

Thus, going from the classical diffusions to the free diffusions means replacing the gradient by its non-commutative analogue.

5. Euler scheme for the solution of the free diffusion equation

5.1. Convergence of the Euler scheme

In order to obtain regularity results on the free diffusion, we shall develop in this section an Euler scheme for the solution of our diffusion equation. This Euler scheme could be defined in the multidimensional case, in the setting of Theorem 3.1, and the arguments presented below would imply its convergence without difficulty, however for notational simplicity, and since we shall only use the one dimensional case, we stick to this case here. We continue with the hypotheses of Section 4.1. Let us fix $n\in\mathbb{N}$ and define a process $X^{(n)}$, first in the interval $[0,1/n]$ by

$$
\begin{aligned}
X_0^{(n)} &= X_0,\\
X_u^{(n)} &= X_0 + S_u \quad \text{for } \leqslant u < \frac{1}{n},\\
X_{\frac{1}{n}}^{(n)} &= X_0 + S_{\frac{1}{n}} - \frac{1}{2n} f(X_0 + S_{\frac{1}{n}}),
\end{aligned}
$$

and then continue by induction on k, assuming $X_u^{(n)}$ has been defined for $u \leqslant \frac{k}{n}$, define

$$
\begin{aligned}
X_u^{(n)} &= X_{\frac{k}{n}}^{(n)} + S_u - S_{\frac{k}{n}} \quad \text{for } \frac{k}{n} \leqslant u < \frac{k+1}{n}, \quad \text{and}\\
X_{\frac{k+1}{n}}^{(n)} &= X_{\frac{k}{n}}^{(n)} + S_{\frac{k+1}{n}} - S_{\frac{k}{n}} - \frac{1}{2n} f\left(X_{\frac{k}{n}}^{(n)} + S_{\frac{k+1}{n}} - S_{\frac{k}{n}}\right).
\end{aligned}
$$

This will give an Euler approximation to the solution of Eq. (4.1.2).

THEOREM 5.1. – *Let $t > 0$, then there exists a positive constant C (depending on f, t and $\|X_0\|$), such that*

$$
\sup_{s \leqslant t} \left\| X_s^{(n)} - X_s \right\| \leqslant \frac{C}{\sqrt{n}} \quad \textit{for all } n \geqslant 0. \tag{5.1.1}
$$

Proof. – We know that $\sup_{s \leqslant t} \|X_s\| < \infty$, hence we can always assume that $\|f\|_\infty < \infty$, and f is operator Lipschitz with some constant K (depending on t). Define for $k \geqslant 1$,

$$
a_k^{(n)} = \left\| X_{\frac{k}{n}}^{(n)} - X_{\frac{k}{n}} \right\| \quad \text{and} \quad u_k^{(n)} = \left\| X_{\frac{k}{n}}^{(n)} - X_{\frac{k}{n}} - \left(X_{\frac{k-1}{n}}^{(n)} - X_{\frac{k-1}{n}} \right) \right\|,
$$

then, clearly

$$
a_0^{(n)} = 0, \quad a_k^{(n)} \leqslant a_{k-1}^{(n)} + u_k^{(n)} \quad \text{for } k \geqslant 1
$$

and

$$
\left\| X_t^{(n)} - X_t \right\| \leqslant a_{k-1}^{(n)} + \frac{1}{2n} \|f\|_\infty
$$

for $\frac{k-1}{n} \leqslant t < \frac{k}{n}$, therefore it is enough to prove estimate (5.1.1) for the times t of the form $\frac{k}{n}$. One has

$$
\begin{aligned}
u_k^{(n)} &= \left\| \frac{1}{2n} f\left(X_{\frac{k-1}{n}}^{(n)} + S_{\frac{k}{n}} - S_{\frac{k-1}{n}} \right) - \frac{1}{2} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(X_s)\, ds \right\|\\
&\leqslant \frac{K}{2} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\left\| X_s - X_{\frac{k-1}{n}}^{(n)} - S_{\frac{k}{n}} + S_{\frac{k-1}{n}} \right\| \right) ds\\
&\leqslant \frac{K}{2} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\left\| X_s - X_{\frac{k-1}{n}} - S_{\frac{k}{n}} + S_{\frac{k-1}{n}} \right\| + \left\| X_{\frac{k-1}{n}} - X_{\frac{k-1}{n}}^{(n)} \right\| \right) ds
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{K}{2}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\left\|S_{\frac{k}{n}}-S_s-\frac{1}{2}\int_{\frac{k-1}{n}}^{s} f(X_u)\,du\right\|+a_{k-1}^{(n)}\right)ds\\
&\leqslant \frac{K}{2}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(2\sqrt{\frac{k}{n}-s}+\frac{1}{2}\|f\|_\infty\left(s-\frac{k-1}{n}\right)+a_{k-1}^{(n)}\right)ds\\
&\leqslant \frac{K}{2}\left(a_{k-1}^{(n)}n^{-1}+2n^{-3/2}+\frac{1}{4}\|f\|_\infty n^{-2}\right)\\
&\leqslant \frac{K}{2n}a_{k-1}^{(n)}+\chi n^{-3/2}
\end{aligned}
$$

for some constant χ. Finally we get

$$a_k^{(n)} \leqslant a_{k-1}^{(n)}\left(1+\frac{K}{2n}\right)+\chi n^{-3/2}.$$

Let $b_k^{(n)} = a_k^{(n)}(1+\frac{K}{2n})^{-k}$, then one has

$$b_k^{(n)} \leqslant b_{k-1}^{(n)}+\chi\left(1+\frac{K}{2n}\right)^k n^{-3/2} \leqslant b_{k-1}^{(n)}+\chi \mathrm{e}^{K\frac{k}{2n}}n^{-3/2}.$$

Summing over k gives

$$a_k^{(n)} \leqslant b_k^{(n)}\mathrm{e}^{K\frac{k}{2n}} \leqslant \chi\frac{k}{n}\mathrm{e}^{K\frac{k}{n}}n^{-1/2}.$$

Estimation (5.1.1) follows from this. $\square$

5.2. Regularity of the free diffusion

Let us denote by $D^{1/2}g$ the half derivative of a function g, then $\|D^{1/2}g\|^2 = \int_{\mathbb{R}}|x|\hat{g}(x)|\,|^2\,dx$, where $\hat{g}$ is its Fourier transform. We assume that f has a derivative, and we are in the situation of Theorem 3.1, with either f a Lipschitz function, or f locally Lipschitz, satisfying (4.1.1), with $a<0$. In these latter case, since X_t remains uniformly bounded, we can as well assume that f is in fact a Lipschitz operator function, and let $2K$ be a Lipschitz constant for f.

THEOREM 5.2. – *There exist constants K_1, K_2 depending only on f, such that the distribution of X_t has a density satisfying*

$$\|p_t\|_\infty \leqslant K_1/\sqrt{t}+K_2 \quad \textit{and} \quad \left\|D^{1/2}p_t\right\|_2 \leqslant \frac{K_1}{t}+K_2 \quad \textit{for all } t>0.$$

Proof. – We shall prove only the first estimate, since this is the only one we shall use in the sequel. The second one can be obtained along similar lines, using the results of [19]. Let us fix t, then by Theorem 5.1, we know that the Euler scheme approximation $X_t^{(n)}$ converges in norm towards X_t. In particular, this implies that the distribution of $X_t^{(n)}$

converges weakly towards that of X_t, hence it suffices to prove that for n large enough, $X_t^{(n)}$ has a density satisfying the estimates of Theorem 5.2, for some constants K_1, K_2, independent of n and t. Let us prove first that $X_t^{(n)}$ has a density for all $t > 0$. For $0 < t < 1/n$, the distribution of $X_t^{(n)}$ is obtained from that of X_0 by a convolution with a semi circular distribution of variance t, hence we know from [2] that the distribution of $X_t^{(n)}$ has a density $p_t^{(n)}$, which is continuous, analytic on the set where it is > 0 and bounded by $1/(\pi\sqrt{t})$. For n large enough, the map $x \mapsto x - \frac{1}{2n} f(x)$ is a diffeomorphism, whose inverse has a derivative bounded above by $(1 - \frac{1}{2n}\|f'\|_\infty)^{-1}$, hence the density of $X_{1/n}^{(n)}$ is bounded above by $\sqrt{n}/(\pi(1 - \frac{K}{n}))$. By induction on k, we see that for all t with $\frac{k}{n} \leqslant t < \frac{k+1}{n}$ the distribution of $X_t^{(n)}$ has a bounded density, and the maximum of this density is smaller than the maximum of the density of $X_{\frac{k}{n}}^{(n)}$, since it is obtained by a free convolution (see, e.g., [2]). It is therefore enough to prove the bounds for the times $t = \frac{k}{n}$. Passing from $X_{\frac{k-1}{n}}^{(n)}$ to $X_{\frac{k}{n}}^{(n)}$ consists in convolving freely with a semi-circular distribution of variance n^{-1} and then applying a diffeomorphism of derivative bounded above by $1 - \frac{K}{n}$. Let v_k be the supremum of the density of $X_{\frac{k}{n}}^{(n)}$, then one has, by [2], Lemma 6 and Proposition 5,

$$v_k \leqslant \frac{v_{k-1}}{(1 + \frac{2}{\pi}\arctan(2\pi v_{k-1}^2/n))(1 - \frac{K}{n})}.$$

Let

$$\rho_n(x) = \frac{x}{(1 + \frac{2}{\pi}\arctan(2\pi x^2/n))(1 - \frac{K}{n})},$$

then the function ρ_n satisfies the following properties, for n large enough:

(1) ρ_n is increasing on $[0, +\infty[$.

(2) $\rho_n(x) > x$ for $0 < x < x_n = \sqrt{\frac{n}{2\pi}\tan(\frac{\pi}{2}\frac{K}{n-K})}$.

(3) $\rho_n(x) < x$ for $x > x_n$.

It follows that one has $v_k \leqslant u_k$ where u_k is the recursive sequence

$$u_1 = \sqrt{n}/\pi, \qquad u_{k+1} = \rho_n(u_k),$$

This sequence is decreasing and $u_k \to x_n$ as $k \to \infty$. One has

$$u_{k+1}^{-2} - u_k^{-2} = u_k^{-2}\left((1 - K/n)^2\left(1 + \frac{2}{\pi}\arctan\left(2\pi u_k^2/n\right)\right)^2 - 1\right).$$

Since the function arctan is concave on $[0, +\infty[$ and $u_k^2/n \leqslant u_1^2/n = \pi^{-2}$ one has for all $k \geqslant 1$, $\frac{2}{\pi}\arctan(2\pi u_k^2/n) > cu_k^2/n$ for some universal constant c. Therefore

$$\begin{aligned}(1 - K/n)\left(1 + \frac{2}{\pi}\arctan\left(2\pi u_k^2/n\right)\right) &\geqslant (1 - K/n)\left(1 + cu_k^2/n\right)\\ &\geqslant 1 - K/n + cu_k^2/n - \left(cu_1^2/n\right)(K/n)\\ &\geqslant 1 - K/n + cu_k^2/n - \pi^{-2}cK/n.\end{aligned}$$

Let $K_2 = \sqrt{K(\frac{1}{\pi^2} + \frac{1}{c})}$, then there exists a constant α such that if $u_k > K_2$ then one has

$$u_l^{-2}\left((1 - K/n)^2\left(1 + \frac{2}{\pi}\arctan\left(2\pi u_l^2/n\right)\right)^2 - 1\right) \geqslant \alpha/n$$

for all $1 \leqslant l \leqslant k$. In this case one has

$$u_k^{-2} = u_1^{-2} + \sum_{l=1}^{k-1}\left(u_{l+1}^{-2} - u_l^{-2}\right) \geqslant \alpha k/n,$$

hence $v_k \leqslant u_k \leqslant \sqrt{\frac{n}{\alpha k}}$. Therefore for all k one has $v_k \leqslant \sqrt{\frac{n}{\alpha k}} + K_2$. This yields the required estimate. $\square$

The continuity in L^p of the Hilbert transform yields the following

COROLLARY 5.3. – *The densities $p_t^{(n)}$ belong to all L^p spaces as well as their Hilbert transforms for $1 < p < \infty$, and one has $p_t^{(n)} \to p_t$ and $Hp_t^{(n)} \to Hp_t$ in L^p for every $1 < p < \infty$. The map $t \mapsto p_t$ is continuous on $]0, \infty[$, in L^p for every $1 < p < \infty$. Furthermore, the logarithmic energy*

$$\int \log|x - y|\, p_t(x)\, p_t(y)\, dx\, dy$$

is defined and continuous in t and one has $\int \log|x - y|\, p_t^{(n)}(x)\, p_t(y)^{(n)}\, dx\, dy \to \int \log|x - y|\, p_t(x)\, p_t(y)\, dx\, dy$ as $n \to \infty$.

6. Connection with free entropy and free Fisher information

6.1. Relative free entropy and free Fisher information

Let F be a locally bounded Borel function on $\mathbb{R}$, let us introduce the following quantity

$$\Sigma_F(\mu) := \iint_{\mathbb{R}^2} \log|x - y|\, \mu(dx)\, \mu(dy) - \int_{\mathbb{R}} F(x)\, \mu(dx) \tag{6.1.1}$$

for a probability measure μ, with compact support in $\mathbb{R}$. This quantity always makes sense in $[-\infty, +\infty[$. We shall also need the following relative version of free information, defined for differentiable F with $F' = f$ by

$$I_F(\mu) = 4\int_{\mathbb{R}} \left(Hp(x) - \frac{1}{2}f(x)\right)^2 p(x)\, dx \tag{6.1.2}$$

if μ has a density $p \in L^3(\mathbb{R})$, and $I_F(\mu) = \infty$ if not. We shall see that the quantities above are the right analogues of the relative entropy given by (1.1.5), in the classical case and of the relative Fisher information (1.1.6). The limit eigenvalue distribution of the matrix model with potential F is the unique measure μ which maximizes the quantity $\Sigma_F(\mu)$ (see [16] for some information about such maximization problems).

6.2. Nonincrease of the relative free entropy for the free diffusion

Let us consider again Eq. (4.1.2), where we assume now that f is a C^2 function. As we shall see now, as in the case of classical diffusions, the quantity $\Sigma_F(\mu_t)$ is nondecreasing with t, so that it converges to some value as $t \to \infty$, but in general this limit value is strictly smaller than the maximum value.

PROPOSITION 6.1. – *Let $\mu_t(dx) = p_t(x)\,dx$ be the distribution of the solution to* (4.1.2)*, then one has for $t > 0$*

$$\frac{d}{dt}\Sigma_F(\mu_t) = 2\int_{\mathbb{R}} \left(Hp_t(x) - \frac{1}{2}f(x)\right)^2 p_t(x)\,dx = \frac{1}{2}I_F(\mu_t). \tag{6.2.1}$$

Proof. – We shall prove that for all $s < t$ one has

$$\Sigma_F(\mu_t) - \Sigma_F(\mu_s) = 2\int_s^t \left[\int_{\mathbb{R}} \left(Hp_u(x) - \frac{1}{2}f(x)\right)^2 p_u(x)\,dx\right] du.$$

Since, by Corollary 5.3 the quantity integrated is continuous in t, this will prove the claim. Choose the nearest integers k, l such that $\frac{k}{n} \leqslant s \leqslant t \leqslant \frac{l}{n}$. Denote by $\mu_t^{(n)}$ the distribution of $X_t^{(n)}$ and by $\mu_{\frac{k}{n}-}^{(n)}$ the distribution of $X_{\frac{k-1}{n}}^{(n)} + S_{\frac{k}{n}} - S_{\frac{k-1}{n}}$. Since one obtains $\mu_{\frac{k+1}{n}-}^{(n)}$ from $\mu_{\frac{k}{n}}^{(n)}$ by free convolution with a semi-circular distribution, one has, for all $k \geqslant 1$,

$$\Sigma_0\big(\mu_{\frac{k+1}{n}-}^{(n)}\big) - \Sigma_0\big(\mu_{\frac{k}{n}}^{(n)}\big) = 2\int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(\int_{\mathbb{R}} p_s^{(n)}(x)Hp_s^{(n)}(x)^2\,dx\right) ds$$

cf. [18], and from Eq. (4.1.3), applied to the case $dX_t = dS_t$,

$$\int_{\mathbb{R}} F(x)\,\mu_{\frac{k+1}{n}-}^{(n)}(dx) - \int_{\mathbb{R}} F(x)\,\mu_{\frac{k}{n}}^{(n)}(dx) = -\int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(\int_{\mathbb{R}} f(x)p_s^{(n)}(x)Hp_s^{(n)}(x)\,dx\right) ds,$$

hence

$$\begin{aligned}
&\Sigma_F\big(\mu_{\frac{k+1}{n}-}^{(n)}\big) - \Sigma_F\big(\mu_{\frac{k}{n}}^{(n)}\big)\\
&\quad= \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(\int_{\mathbb{R}} p_s^{(n)}(x)\big(2Hp_s^{(n)}(x)^2 - f(x)Hp_s^{(n)}(x)\big)\,dx\right) ds.
\end{aligned} \tag{6.2.2}$$

The measure $\mu^{(n)}_{\frac{k}{n}}$ is the image of $\mu^{(n)}_{\frac{k}{n}-}$ by the map $x \mapsto x - \frac{1}{2n} f(x)$, consequently one has

$$\begin{aligned}
\Sigma_F\big(\mu^{(n)}_{\frac{k+1}{n}}\big) - \Sigma_F\big(\mu^{(n)}_{\frac{k+1}{n}-}\big) &= \iint_{\mathbb{R}} \left(\log\left|x - y - \frac{1}{2n} f(x) + \frac{1}{2n} f(y)\right| - \log|x-y|\right) \\
&\quad \times p^{(n)}_{\frac{k+1}{n}-}(x) p^{(n)}_{\frac{k+1}{n}-}(y)\, dx\, dy \\
&\quad - \int_{\mathbb{R}} \left(F\left(x - \frac{1}{2n} f(x)\right) - F(x)\right) p^{(n)}_{\frac{k+1}{n}-}(x)\, dx \\
&= \iint_{\mathbb{R}} \frac{1}{2n} \frac{f(x) - f(y)}{x - y} p^{(n)}_{\frac{k+1}{n}-}(x) p^{(n)}_{\frac{k+1}{n}-}(y)\, dx\, dy \\
&\quad + \frac{1}{2} \int_{\mathbb{R}} \frac{1}{n} f^2(x) p^{(n)}_{\frac{k+1}{n}-}(x)\, dx + \mathrm{O}\left(\frac{1}{n^2}\right),
\end{aligned}$$

where the O is uniform in k. Moreover one has $\|X^{(n)}_{\frac{k+1}{n}} - X_s\| = \mathrm{O}(n^{-1/2})$ for $\frac{k}{n} < s < \frac{k+1}{n}$, uniformly over x and s, therefore, since $\frac{f(x)-f(y)}{x-y}$ is Lipschitz as a function of two variables, one has

$$\begin{aligned}
&\iint_{\mathbb{R}^2} \frac{1}{n} \frac{f(x) - f(y)}{x - y} p^{(n)}_{\frac{k+1}{n}-}(x) p^{(n)}_{\frac{k+1}{n}-}(y)\, dx\, dy \\
&\quad = \int_{\frac{k}{n}}^{\frac{k+1}{n}} \iint_{\mathbb{R}^2} \frac{f(x) - f(y)}{x - y} p^{(n)}_s(x) p^{(n)}_s(y)\, dx\, dy\, ds + \mathrm{O}\left(\frac{1}{n^{3/2}}\right) \\
&\quad = 2 \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{\mathbb{R}} f(x) p^{(n)}_s(x)\, H p^{(n)}_s(x)\, dx\, ds + \mathrm{O}\left(\frac{1}{n^{3/2}}\right). \qquad (6.2.3)
\end{aligned}$$

Comparing (6.2.2) and (6.2.3), we get the convergence result. □

7. Asymptotic behaviour of the free diffusion

7.1. Nonconvergence towards the master field

We shall now give examples of potentials F for which there exist initial distributions such that the distribution of X_t does not converge towards that of the master field. The examples we give have several potential wells, and if these wells are deep enough then no mass can escape from them. Let us consider a function f such that f has isolated zeros at $x_1, \ldots, x_n \in \mathbb{R}$, and there exist positive constants a, b such that $f' > a$ on the interval $]x_j - b, x_j + b[$, and these intervals are two by two disjoint. Then there exists a nonnegative C^2 function G such that $G(x) = (x - x_j)^2$ on the interval $]x_j - b, x_j + b[$, and $J = \bigcup_{j=1}^n]x_j - b, x_j + b[$ is exactly the set where $G(x) < b^2$. Let us apply Ito's formula to $\mathrm{e}^{2at} G(X_t)$, we get

$$\mathrm{e}^{2at}G(X_t) = G(X_0) + \int_0^t \mathrm{e}^{2as}\partial G(X_s)\,\sharp dS_s - \int_0^t \mathrm{e}^{2as}G'(X_s)f(X_s)\,ds$$
$$+ 2a\int_0^t \mathrm{e}^{2as}G(X_s)\,ds + \frac{1}{2}\int_0^t \mathrm{e}^{2as}\Delta_s G(X_s)\,ds.$$

Suppose that $\|G(X_0)\| < b^2$ and let $T = \inf\{t \,|\, \|G(X_t)\| \geqslant b^2\}$. Then the support of the distribution of X_t is included in J for $t \leqslant T$, and for such t one has

$$\big\|\partial G(X_t)\big\|_{L^\infty(\tau\otimes\tau^{op})} \leqslant \sup_{x,y\in J}\left|\frac{G(x)-G(y)}{x-y}\right| \leqslant 2b,$$

$$-G'(X_t)f(X_t) + 2aG(X_t) \leqslant 0 \quad \text{as self adjoint operators},$$

$$\big\|\Delta_t G(X_t)\big\| \leqslant \sup_{x,y\in J}\frac{\partial}{\partial x}\left(\frac{G(x)-G(y)}{x-y}\right) \leqslant 2,$$

so that

$$\mathrm{e}^{2at}G(X_t) \leqslant G(X_0) + \int_0^t \mathrm{e}^{2as}\partial G(X_s)\,\sharp dS_s + \frac{1}{2}\int_0^t \mathrm{e}^{2as}\Delta_s G(X_s)\,ds,$$

therefore, using the free Burkholder–Gundy inequality

$$\big\|\mathrm{e}^{2at}G(X_t)\big\| \leqslant \|G(X_0)\| + 4b\sqrt{2\big(\mathrm{e}^{4at}-1\big)/4a} + 2\big(\mathrm{e}^{2at}-1\big)/2a.$$

Assume that

$$\|G(X_0)\| + 4b/\sqrt{2a} + 1/(2a) < b^2,$$

then we see that $t < T$ for all time $t > 0$, i.e., $T = \infty$, and thus the support of the distribution of X_t always remains in J. Since X_t is norm continuous, it follows that the mass put by the distribution of X_t on each of the intervals $]x_j - b, x_j + b[$ remains constant in time. Therefore, the distribution of X_t cannot converge towards the master field, unless one puts the right masses in the wells at the initial distribution.

Let us consider the quartic model with $P(X) = \frac{1}{2}X^2 + \frac{g}{4}X^4$. Then the limit distribution of the matrix model has a Cauchy transform given by

$$G(z) = \frac{1}{2}\big(z + gz^3\big) - \frac{1}{2}\big(1 + 2ga^2 + gz^2\big)\sqrt{z^2 - 4a^2},$$

where $3ga^4 + a^2 = 1$ (see, e.g., [7, Eq. (5.3)]). In particular, this solution exhibits an analytic continuation for negative g, with $g > -1/12$, although then one has $Z_N = \infty$ for all N. These solutions are interpreted as coming from the local minimum at 0 of the potential P. Indeed if we look at the free diffusion equation

$$dX_t = dS_t - \frac{1}{2}\big(X_t + gX_t^3\big)\,dt$$

then starting at $X_0 = 0$, and using the same technique as above with the free Burkholder–Gundy inequality, one can see that for g negative, but close enough to zero there is indeed a solution defined for all times t, which moreover remains uniformly bounded in norm. This suggests that the distribution of X_t converges towards the analytic continuation of the solution, although we cannot prove this. The advantage of considering free diffusion equations is that we can study with the same techniques equations involving more than one variable, indeed for models such as the AB model where

$$P(A, B) = \alpha A^4 + \beta B^4 + A^2 + B^2 + \gamma(AB + BA)$$

one can prove that for $|\gamma| < 1$ and for negative values of α and β close to zero the solution starting from 0 again exists and remains bounded over all times, suggesting that an analytic continuation of the model exists for such α and β. Of course this argument works for much more general models, but we do not have any rigorous argument for the convergence of the solution towards some limit distribution.

7.2. The case of the free Ornstein–Uhlenbeck process

On the other hand in the case where $f(x)$ is a linear function, then we know that the distribution of X_t always converges to the semicircular distribution. We even have an exponential rate of convergence for the relative free entropy, thanks to the free analogue of the log Sobolev inequality due to Voiculescu.

Let $F(x) = \lambda x^2$ for a positive $\lambda > 0$. Then we have $f(x) = 2\lambda x$ and the diffusion equation $dX_t = dS_t - \lambda X_t\, dt$ has the solution

$$X_t = \mathrm{e}^{-\lambda t} X_0 + \int_0^t \mathrm{e}^{-\lambda(t-\tau)}\, dS_\tau.$$

The distribution of X_t is given by the free convolution of the distribution of $\mathrm{e}^{-\lambda t} X_0$ with a semi-circle of variance $(1 - \mathrm{e}^{2\lambda t})/(2\lambda)$. In particular, for $t \to \infty$ the distribution of X_t converges towards a semi-circle of variance $\frac{1}{2\lambda}$, i.e., towards the distribution of $\frac{1}{\sqrt{2\lambda}} S$, where S is a semi-circular of variance 1. In this case we have

$$\begin{aligned}\Sigma(X_\infty) - \Sigma(X) &= \chi(X_\infty) - \lambda\varphi\left(X_\infty^2\right) - \chi(X) + \lambda\tau\left(X^2\right)\\ &= \chi(S) - \frac{1}{2}\log(2\chi) - \frac{1}{2} - \chi(X) + \lambda\tau\left(X^2\right)\end{aligned}$$

and

$$I(X) = \Phi(X) - 4\lambda + 4\lambda^2\tau\left(X^2\right),$$

where χ and Φ are the free entropy and the free Fisher information, respectively. We claim now that we have a corresponding free logarithmic Sobolev-inequality for $\rho = 2\lambda$, i.e.,

$$\frac{1}{2\rho} I(X) \geqslant \Sigma(X_\infty) - \Sigma(X)$$

or

$$\Phi(X) \geqslant 4\lambda\chi(S) - 2\lambda\log(2\lambda) + 2\lambda - 4\lambda\chi(X).$$

To prove this it suffices to have this inequality for that value of λ which maximizes the right hand side. This value of λ is determined by

$$\log(2\lambda) = 2\chi(S) - 2\chi(X),$$

and the free logarithmic Sobolev inequality for the Ornstein–Uhlenbeck process is thus equivalent to the statement

$$2\chi(X) \geqslant 2\chi(S) - \log\Phi(X) = \log\frac{2\pi\mathrm{e}}{\Phi(X)}.$$

But this inequality was proved by Voiculescu in Proposition 7.9 of [21].

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