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T. FUNAKI

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# Free boundary problem from stochastic lattice gas model

by

**T. FUNAKI** [1]

Graduate School of Mathematical Sciences, University of Tokyo,
3-8-1 Komaba Meguro-ku, Tokyo 153-8914, Japan

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ABSTRACT. – We consider a system consisting of two types of particles called "water" and "ice" on $d$-dimensional periodic lattices. The water particles perform excluded interacting random walks (stochastic lattice gases), while the ice particles are immobile. When a water particle touches an ice particle, it immediately dies. On the other hand, the ice particle disappears after receiving the $\ell$th visit from water particles. This interaction models the melting of a solid with latent heat. We derive the nonlinear one-phase Stefan free boundary problem in a hydrodynamic scaling limit. Derivation of two-phase Stefan problem is also discussed. © Elsevier, Paris

RÉSUMÉ. – Nous considérons un système comportant deux types de particules appelés "eau" et "glace" sur un réseau périodique en dimension "$d$". Les particules d'eau suivent des marches aléatoires interagissant avec exclusion (gaz stochastiques sur réseau), alors que les particules de glace sont immobiles. Quand une particule d'eau touche une particule de glace, elle meurt immédiatement. D'autre part, une particule de glace disparaît quand elle a reçu une $\ell$ième visite de particules d'eau. Cette interaction modélise la fusion d'un solide avec chaleur latente. Nous obtenons le problème des frontières libres non linéaire de Stefan

[1] E-mail: funaki@ms.u-tokyo.ac.jp.

à une phase dans la limite hydrodynamique. Nous discutons aussi la dérivation d'un problème de Stefan à deux phases. 

## 1. INTRODUCTION

In the context of hydrodynamic limits, various types of nonlinear partial differential equations are derived from the underlying microscopic particle models with stochastic dynamics via a suitably taken long-time and large-space scaling limit, see [16]. The equations obtained in the limit describe the evolution law of local density of conserved quantities at the macroscopic level. When a medium is accompanied by a change of phases, the process of diffusion or heat conduction in it are mathematically formulated as the Stefan problem, see, e.g., [2,5,13,14]. Typical example is a liquid-solid system. The sharp transition from one phase to the other gives rise to idealized interfaces called free boundaries whose locations are not a priori known. The aim of this article is to derive the free boundary problem from certain microscopic particle systems.

Now we broadly mention the microscopic model which we shall explore. We consider a liquid-solid system, which is sometimes symbolically called water–ice system. To model such system, two types of particles, called "water" and "ice" located on a square lattice, are introduced. The dynamics of water particles are the exclusion processes with speed change, in other words, the stochastic lattice gases. We shall discuss two kinds of models. In the first model, we assume that ice particles never move. Then, at a microscopic level, two large regions, respectively, consisting of one of the two types of particles, are formed and separated by transition regions which vaguely look surfaces. The interactions between distinct types of particles occur only through such surfaces. We study the system with melting of a solid; namely, when a water particle jumps onto the site where an ice particle already occupies, the water particle disappears at once, while the ice particle disappears and the site simultaneously becomes vacant right after having the $\ell$th visit from water particles.

This kind of model was first investigated by Chayes and Swindle [3]. In their case, the water particles perform a simple exclusion process (i.e., Kawasaki dynamics) and accordingly the equation obtained in the limit was a linear heat equation in the liquid region. They studied the system of melting (with $\ell = 1$) or freezing of solid in one-dimension and derived the free boundary Stefan condition which characterizes the motion of macroscopic phase separation point. In this paper, we shall generalize their results for the exclusion processes with speed change in case of melting of solid in higher dimensions taking the effect of latent heat of fusion into account. We assume the gradient and the detailed balance conditions for the jump rates of water particles (and ice particles in the second model in which ice particles are also active). The hydrodynamic limit for such stochastic dynamics for water particles without the presence of ice particles was studied by [6]. A nonlinear diffusion equation is derived in the limit and its diffusion coefficient is expressed in terms of a thermodynamic function. Thus, one can easily imagine that the hydrodynamic limit of our liquid-solid system might be described by the Stefan problem in which the same nonlinear diffusion equation as in [6] governs the evolution law of the density in the liquid region. Since the particles are immobile in the solid region, the so-called one-phase Stefan problem will be obtained.

The basic method widely known to be effective for establishing the hydrodynamic behavior was exploited by Guo, Papanicolaou and Varadhan [10] and uses several estimates based on entropy and entropy production. Such method is, however, powerful only if the invariant measures for the dynamics restricted on finite domains are mutually absolutely continuous. In our case, the dynamics have two distinct types of invariant measures; one is concentrated on liquid region and the other on solid region. The invariant measures of different types have therefore disjoint supports so that it seems hopeless to apply the method of [10] for our system straightforwardly. The method for the proof requires some modifications. The entropy arguments actually deduce the convergence rate of the system to the equilibrium states; however, such precise estimates are unnecessary since the system discussed in this paper is of gradient type.

The paper is organized as follows. After the model is described, the main result is stated in Section 2. The proof of the main result begins in Section 3. To complete it, a local ergodic theorem is established in Section 4 and arguments based on Young measures are developed in Section 5. The local ergodic theorem enables us to replace the sample mean of microscopic variables with their average under the equilibrium (Gibbs) measure having a microscopically defined sample density as its density-parameter. The arguments in Section 5 are required to replace such microscopically defined sample density further with a macroscopic one. If ice particles also move with jump rates or velocities different from water particles, one can derive two-phase Stefan problem. Section 6 is devoted to the study of such model.

The Stefan problem was derived by [11] from one-dimensional symmetric simple exclusion process added one particle which feels, in contrast to the other particles, a constant external driving force. [1] studied annihilating particles' system, called "$A+B \to 0$", consisting of two types $A$ and $B$. Another approach was taken in [8] to the motion of interfaces starting from the microscopic models. We assume the gradient condition. Without such condition, the hydrodynamic limit for systems with single type of particles was proved by [7] assuming that the reversible measures of dynamics are Bernoulli. Generalization of our result to such non-gradient system looks rather hard at this moment.

## 2. MODEL AND MAIN RESULT

We consider a particle system on a $d$-dimensional periodic lattice $\Gamma_N := (\mathbb{Z}/N\mathbb{Z})^d$ represented by $\{1, 2, \dots, N\}^d$. Since the lattice size $N$ changes and eventually goes to infinity, the jump rates of water particles are defined on the whole lattice $\mathbb{Z}^d$. The configuration space of water particles on $\mathbb{Z}^d$ is denoted by $\mathcal{X}^+ := \{0, 1\}^{\mathbb{Z}^d}$. For $\xi = \{\xi_x; x \in \mathbb{Z}^d\} \in \mathcal{X}^+$, $\xi_x = 0$ or $1$ indicate that the site $x$ is vacant or occupied by a water particle, respectively. We denote by $\xi^{x,y} \in \mathcal{X}^+$ the configuration obtained from $\xi$ by exchanging its states at two sites $x$ and $y \in \mathbb{Z}^d$; i.e., $(\xi^{x,y})_x = \xi_y$, $(\xi^{x,y})_y = \xi_x$ and $(\xi^{x,y})_z = \xi_z$ for $z \neq x, y$. Let $\tau_x, x \in \mathbb{Z}^d$, be shift operators acting on $\mathcal{X}^+$ by $(\tau_x\xi)_y = \xi_{y+x}, y \in \mathbb{Z}^d$. They also act on functions $f$ on $\mathcal{X}^+$ by $\tau_x f(\xi) = f(\tau_x\xi)$. We denote by $\mathcal{D}^+$ the class of all local functions $f$ on $\mathcal{X}^+$, where $f$ is called local if it depends only on $\xi_{\Lambda_R} := \{\xi_x; x \in \Lambda_R\}$, $\Lambda_R = \{x \in \mathbb{Z}^d; |x| \leqslant R\}$ for some nonnegative integer $R$. The smallest number among such $R$'s is written by $R(f)$. The jump rates of water particles are then specified by nonnegative functions $c_{x,y}(\xi)$ defined for $\xi \in \mathcal{X}^+$ and $x, y \in \mathbb{Z}^d$, $|x-y| = 1$, which satisfy the following conditions (a)–(f):

(a) $c_{x,y}(\xi) = c_{y,x}(\xi)$.

(b) (spatial homogeneity) $c_{x,y} = \tau_x c_{0,y-x}$.

(c) (locality) $c_{x,y} \in \mathcal{D}^+$.

(d) (positivity) $c_{x,y}(\xi) > 0$ if $\xi_x \neq \xi_y$.

(e) (detailed balance condition, uniform mixing property). There exists a set function $\Phi$ on $\mathbb{Z}^d$, which is translation-invariant (i.e., $\Phi(A) = \Phi(A+x)$ for every $A \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$) and has a finite range $R$ (i.e., $\Phi(A)=0$ if the diameter of $A$ is larger than $R$), such that for every $\xi\in\mathcal{X}^+$,

$$c_{x,y}(\xi)\exp\bigl[-E^{\Phi}_{\{x,y\}}(\xi)\bigr]=c_{x,y}(\xi^{x,y})\exp\bigl[-E^{\Phi}_{\{x,y\}}(\xi^{x,y})\bigr].$$

Here, for a finite subset $\Lambda$ of $\mathbb{Z}^d$, $E^{\Phi}_{\Lambda}(\xi)$ denotes a function on $\mathcal{X}^+$ defined by

$$E^{\Phi}_{\Lambda}(\xi):=\sum_{A\subset\mathbb{Z}^d:\ A\cap\Lambda\neq\emptyset}\Phi(A)\prod_{x\in A}\xi_x.$$

We assume that an extremal canonical Gibbs measure, denoted by $\nu_\rho$, corresponding to the Hamiltonian $E^{\Phi}_{\Lambda}$ with density $\rho$ (i.e., $E^{\nu_\rho}[\eta_0]=\rho$) uniquely exists for every $\rho\in[0,1]$ and the law of large numbers holds in $L^2(\nu_\rho)$-sense uniformly in $\rho$:

$$\lim_{K\to\infty}\sup_{\rho\in[0,1]}E^{\nu_\rho}\bigl[\bigl|\bar{f}_{0,K}(\eta)-\langle f\rangle(\bar{\eta}_{0,K})\bigr|^2\bigr]=0,\quad f\in\mathcal{D}^+,$$

see (4.1) and (4.2) below for $\bar{f}_{0,K}(\eta)$ and $\langle f\rangle(\rho)$, respectively.

(f) (gradient condition). There exist $h_1,\dots,h_d\in\mathcal{D}^+$ such that the currents have the forms:

$$c_{0,e_i}(\xi)(\xi_{e_i}-\xi_0)=\tau_{e_i}h_i(\xi)-h_i(\xi),\quad 1\leqslant i\leqslant d,\ \xi\in\mathcal{X}^+,$$

where $e_i\in\mathbb{Z}^d$, $|e_i|=1$, stands for the unit vector to the direction $i$. We assume that the equilibrium means $P^+(\rho):=E^{\nu_\rho}[h_i]$, $\rho\in[0,1]$, are independent of $i$.

Throughout the paper, replacing $h_i(\xi)$ with $h_i(\xi)-P^+(0)$ if necessary, we shall always normalize $h_i$ in such a way that

$$h_i(\xi)=0\quad\text{if } \xi\equiv 0 \text{ on } \Lambda_{\overline{R}} \text{ for every } 1\leqslant i\leqslant d,\tag{2.1}$$

where $\overline{R}=\max_{1\leqslant i\leqslant d}R(h_i)$. Note that $P^+(0)=0$ under this normalization, and also note that $P^+\in C([0,1])$ holds by the uniqueness of extremal canonical Gibbs measure for each $\rho$. An example of jump rates satisfying all these conditions will be given at the end of this section. See [6] for some explanations of these conditions. In particular, $P^+(\rho)$ is nondecreasing in $\rho$.

Now, let us describe the microscopic dynamics on $\Gamma_N$ corresponding to the liquid-solid system consisting of two types of particles. In order to record the number of times hit by water particles, we label the ice particles by $-\ell, \dots, -1$ ($\ell \in \mathbb{N}$) and regard as different microscopic states for ice. The ice particles melt and disappear after they are hit $\ell$ times by water particles. Thus the configuration space is $\mathcal{X}_N := \{-\ell, \dots, 1\}^{\Gamma_N}$. For $\eta = \{\eta_x; x \in \Gamma_N\} \in \mathcal{X}_N$, $\eta_x = 1$ and 0 mean that the site $x$ is occupied by a water particle or vacant, respectively, and $\eta_x = -i$ ($1 \leqslant i \leqslant \ell$) means that the site $x$ is occupied by an ice particle which was hit $\ell - i$ times by water particles in the past. To determine the dynamics of our particle system, let us consider the following Markov generator:

$$L_N f(\eta) = \sum_{x,y \in \Gamma_N:\ |x-y|=1} c_{x,y}(\eta^+) \pi_{x;y} f(\eta), \tag{2.2}$$

for functions $f$ on $\mathcal{X}_N$, where $\eta^+ := \eta \vee 0$, i.e., $\eta_x^+ = \max\{\eta_x, 0\}$, $x \in \Gamma_N$, and it is identified with its periodic extension to $\mathcal{X}^+$. The operator $\pi_{x;y}$ is defined by

$$\pi_{x;y} f(\eta) = f(\eta^{x;y}) - f(\eta), \tag{2.3}$$

where the configuration $\eta^{x;y}$ is obtained from $\eta$ after a water particle jumps from the site $x$ to $y$ under the following rule: If another water particle already occupies the site $y$, the jump is suppressed (by exclusion rule). If there is no particle at $y$, or if there is an ice particle at $y$, the water particle at $x$ jumps to $y$ and the site $x$ becomes vacant. In the latter case, the state of the ice particle increases by 1 counting the number of times that it was hit by water particles. Namely, if $\eta_x = 1$ and $\eta_y^+ = 0$,

$$(\eta^{x;y})_z = \begin{cases} 0, & \text{if } z = x, \\ \eta_y + 1, & \text{if } z = y, \\ \eta_z, & \text{if } z \neq x, y, \end{cases} \tag{2.4}$$

and $\eta^{x;y} = \eta$ otherwise. The configuration $\eta^+$ is obtained from $\eta$ by disregarding ice particles and thus the jump rate $c_{x,y}(\eta^+)$ in the definition of $L_N$ in (2.2) means that the water particles move around without feeling the presence of the ice particles. We investigate the Markov process $\eta^N(t) = \{\eta_x^N(t); x \in \Gamma_N\}$, $t \geqslant 0$ on $\mathcal{X}_N$ with an infinitesimal generator $N^2 L_N$.

The goal is to study the asymptotic behavior as $N \to \infty$ of the macroscopic empirical-mass distribution of $\eta^N(t)$ defined by

$$\alpha_t^N(d\theta) := \frac{1}{N^d} \sum_{x \in \Gamma_N} \eta_x^N(t) \delta_{x/N}(d\theta), \quad \theta \in \mathbb{T}^d, \tag{2.5}$$

where $\mathbb{T}^d$ is the $d$-dimensional torus identified with $[0,1)^d$ and $\delta_\theta$ stands for the $\delta$-measure at $\theta$. We assume that the random initial distribution $\alpha_0^N$ converges in probability to some nonrandom measure $\alpha_0 = a(0,\theta)d\theta \in \mathcal{M}(\mathbb{T}^d)$ which has a density as $N \to \infty$. Here $\mathcal{M}(\mathbb{T}^d)$ denotes the set of all signed measures $\alpha$ on $\mathbb{T}^d$ satisfying $-\ell \leqslant \alpha(A) \leqslant 1$ for every Borel set $A$ of $\mathbb{T}^d$. The main result of this paper can now be stated.

THEOREM 2.1. – *For every $t > 0$, $\alpha_t^N$ converges to $a(t,\theta)d\theta$ in probability. The limit function $a(t,\theta) \in [-\ell, 1]$ is nonrandom and a unique solution of the equation*:

$$\langle a(t), J\rangle = \langle a(0), J\rangle + \int_0^t \langle P(a(s)), \Delta J\rangle \, ds, \tag{2.6}$$

*for every $J \in C^\infty(\mathbb{T}^d)$, where $\langle a, J\rangle = \int_{\mathbb{T}^d} a(\theta)J(\theta)\,d\theta$. The function $P$ on $[-\ell, 1]$ is defined by $P(a) = P^+(a)$ for $a \in [0,1]$ and $P(a) = 0$ for $a \in [-\ell, 0]$.*

*Remark* 2.1. – Eq. (2.6) is the weak or enthalpy formulation of the following one-phase Stefan problem for the density $\rho(t,\theta) \in [0,1]$ of water:

$$\begin{aligned} \frac{\partial\rho}{\partial t} &= \Delta P^+(\rho) && \text{on } \mathcal{L}(t) := \{\theta \in \mathbb{T}^d;\ \rho(t,\theta) > 0\},\\ \rho(t,\theta) &= 0 && \text{on } \Sigma(t) := \partial\mathcal{L}(t),\\ \ell V &= -\mathbf{n}\cdot\nabla P^+(\rho) && \text{on } \Sigma(t), \end{aligned}$$

where $\mathbf{n}$ denotes the unit normal vector on $\Sigma(t)$ directed to $\mathcal{L}(t)$, $V$ is the velocity of $\Sigma(t)$ to the direction $\mathbf{n}$ and $\nabla P^+(\rho)$ is the limit of the gradient of $P^+(\rho)$ at $\theta \in \Sigma(t)$ when approached from $\mathcal{L}(t)$. Indeed, if the interface $\Sigma(t)$ separating the liquid region $\mathcal{L}(t)$ and the solid region $\mathcal{S}(t) := \mathbb{T}^d \setminus \{\mathcal{L}(t) \cup \Sigma(t)\}$ is sufficiently smooth, one can easily derive (2.6) for the enthalpy function defined by $a(t,\theta) := \rho(t,\theta)$, $\theta \in \mathcal{L}(t)$, and $:= -\ell$, $\theta \in \mathcal{S}(t)$, from these classical equations by means of the integration by parts formula, see [2,13]. The speeds of loosing masses of water and ice at $\Sigma(t)$ are given by $\mathbf{n}\cdot\nabla P^+(\rho)$ and $-V$, respectively. Since the loosing speed for water is $\ell$ times faster than that for ice, we have the last condition, which is called the free boundary Stefan condition, as a result of the balance between these two melting speeds.

The thickness of the interface $\Sigma(t)$, which microscopically corresponds to the states $-\ell+1,\dots,-1$, is macroscopically negligible.

We finally give an example of jump rates which satisfy all conditions (a)–(f) listed above. The simple exclusion (i.e., $c_{x,y}\equiv 1$) is a trivial example. See [6] for other examples in one-dimension.

*Example* 2.1. – For $\alpha>-1/2$, set

$$c_{0,e_i}(\xi)=1+\alpha(\xi_{-e_i}+\xi_{2e_i}).$$

Then the gradient condition (f) is satisfied with

$$h_i(\xi)=(1+\alpha)\xi_0+\alpha(\xi_{-e_i}-\xi_0)(\xi_0-\xi_{e_i}).$$

The detailed balance condition (e) holds for $\Phi\equiv 0$ and therefore the canonical Gibbs measures are Bernoulli in this case. In particular, we have $P^+(\rho)=\rho+\alpha\rho^2$.

## 3. RATE OF CHANGE OF EMPIRICAL MASS-DISTRIBUTION

To study the limit of $\langle\alpha_t^N, J\rangle$, we rewrite it by using Itô's formula into

$$\langle\alpha_t^N, J\rangle=\langle\alpha_0^N, J\rangle+\int_0^t b^N(\eta^N(s))\,ds+M^N(t). \tag{3.1}$$

Here, the drift term has a form

$$b^N(\eta)=N^2L_N\alpha_J(\eta),$$

where

$$\alpha_J(\eta)=N^{-d}\sum_{z\in\Gamma_N}\eta_z J(z/N),\qquad \eta\in\mathcal{X}_N,$$

and the last term $M^N(t)$ is a martingale having a quadratic variational process given by

$$\langle M^N\rangle_t=\int_0^t\gamma^N(\eta^N(s))\,ds,$$

with

$$\gamma^N(\eta)=N^2L_N\alpha_J^2(\eta)-2\alpha_J(\eta)N^2L_N\alpha_J(\eta).$$

The functions $b^N$ and $\gamma^N$ on $\mathcal{X}_N$ are explicitly computable as in the next proposition. We shall denote

$$\Delta_i^N J(z/N) := N^2\{J((z+e_i)/N) + J((z-e_i)/N) - 2J(z/N)\}. \quad (3.2)$$

PROPOSITION 3.1. –

$$b^N(\eta) = N^{-d} \sum_{z\in\Gamma_N} \sum_{i=1}^{d} \tau_z h_i(\eta^+) \Delta_i^N J(z/N), \quad (3.3)$$

$$\gamma^N(\eta) = \frac{1}{2} N^{2-2d} \sum_{z_1,z_2\in\Gamma_N:\ |z_1-z_2|=1} c_{z_1,z_2}(\eta^+)\left(\eta_{z_1}^+ - \eta_{z_2}^+\right)^2 \times \left\{J(z_1/N) - J(z_2/N)\right\}^2. \quad (3.4)$$

*Proof of (3.3).* – We prepare a lemma.

LEMMA 3.1. – *For all $z \in \Gamma_N$,*

$$L_N \eta_z = \sum_{x:\ |x-z|=1} c_{x,z}(\eta^+)\left(\eta_x^+ - \eta_z^+\right). \quad (3.5)$$

*Proof.* – Since $\pi_{x;y}\eta_z \neq 0$ implies $x = z$ or $y = z$, recalling the condition (a), we have

$$L_N \eta_z = \sum_{x:\ |x-z|=1} c_{x,z}(\eta^+)(\pi_{z;x}\eta_z + \pi_{x;z}\eta_z).$$

However, one can readily see

$$\pi_{z;x}\eta_z = \eta_z^+\left(\eta_x^+ - \eta_z^+\right), \qquad \pi_{x;z}\eta_z = \eta_x^+\left(\eta_x^+ - \eta_z^+\right), \quad (3.6)$$

and these identities complete the proof of the lemma. □

Using this lemma and the gradient condition (f), one can rewrite $b^N(\eta)$ into

$$b^N(\eta) = N^{2-d} \sum_{z\in\Gamma_N} J(z/N) \times \sum_{i=1}^{d} \left\{\tau_{z+e_i} h_i(\eta^+) + \tau_{z-e_i} h_i(\eta^+) - 2\tau_z h_i(\eta^+)\right\}.$$

The equality (3.3) follows by the summation by parts.

*Proof of (3.4).* – First let us compute $L_N(\eta_{z_1}\eta_{z_2})$. We shall denote $\eta_z^- := -\min\{\eta_z, 0\}$.

LEMMA 3.2. – (i) *If* $z_1 = z_2 = z$,

$$L_N \eta_z^2 = (1 - 2\eta_z^-) \sum_{x:\, |x-z|=1} c_{x,z}(\eta^+)(\eta_x^+ - \eta_z^+). \tag{3.7}$$

(ii) *If* $z_1 \neq z_2$,

$$\begin{aligned} L_N(\eta_{z_1}\eta_{z_2}) = \eta_{z_2} & \sum_{x:\, |x-z_1|=1} c_{x,z_1}(\eta^+)(\eta_x^+ - \eta_{z_1}^+) \\ & + \eta_{z_1} \sum_{x:\, |x-z_2|=1} c_{x,z_2}(\eta^+)(\eta_x^+ - \eta_{z_2}^+) \\ & - 1_{\{|z_1-z_2|=1\}} c_{z_1,z_2}(\eta^+)(\eta_{z_1}^+ - \eta_{z_2}^+)^2. \end{aligned} \tag{3.8}$$

*Proof.* – The assertion (i) follows from the identities

$$\pi_{z;x}\eta_z^2 = \eta_z^+(\eta_x^+ - \eta_z^+), \qquad \pi_{x;z}\eta_z^2 = \eta_x^+(\eta_x^+ - \eta_z^+ - 2\eta_z^-).$$

To prove the assertion (ii), we first see that

$$\begin{aligned} L_N(\eta_{z_1}\eta_{z_2}) = {} & \eta_{z_2} L_N \eta_{z_1} + \eta_{z_1} L_N \eta_{z_2} \\ & + 1_{\{|z_1-z_2|=1\}} c_{z_1,z_2}(\eta^+) R_{z_1,z_2}(\eta), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} R_{z_1,z_2}(\eta) := {} & \pi_{z_1;z_2}(\eta_{z_1}\eta_{z_2}) + \pi_{z_2;z_1}(\eta_{z_1}\eta_{z_2}) - \eta_{z_2}\pi_{z_1;z_2}\eta_{z_1} \\ & - \eta_{z_2}\pi_{z_2;z_1}\eta_{z_1} - \eta_{z_1}\pi_{z_2;z_1}\eta_{z_2} - \eta_{z_1}\pi_{z_1;z_2}\eta_{z_2}. \end{aligned}$$

In fact, this is obvious if $|z_1 - z_2| \geqslant 2$. When $|z_1 - z_2| = 1$, the sum of the first two terms in the right hand side of (3.9) counts several terms duplicately so that we need the third term for correction. However, since $\pi_{z_1;z_2}(\eta_{z_1}\eta_{z_2}) = \eta_{z_1}^+\eta_{z_2}^-$, using (3.6), we obtain that $R_{z_1,z_2}(\eta) = -(\eta_{z_1}^+ - \eta_{z_2}^+)^2$. This proves equality (3.8). □

We can now complete the proof of (3.4). From Lemma 3.2, we have

$$\begin{aligned} N^2 L_N \alpha_J^2 &= N^{2-2d} \sum_{z_1,z_2 \in \Gamma_N} J(z_1/N) J(z_2/N) L_N(\eta_{z_1}\eta_{z_2}) \\ &= N^{2-2d} \sum_{z_1,z_2 \in \Gamma_N} J(z_1/N) J(z_2/N) \\ &\qquad \times 2\eta_{z_2} \sum_{x:\, |x-z_1|=1} c_{x,z_1}(\eta^+)(\eta_x^+ - \eta_{z_1}^+) \end{aligned}$$

$$- N^{2-2d} \sum_{\substack{z_1, z_2 \in \Gamma_N: \\ |z_1 - z_2| = 1}} J(z_1/N) J(z_2/N) c_{z_1,z_2}(\eta^+) (\eta_{z_1}^+ - \eta_{z_2}^+)^2$$

$$+ N^{2-2d} \sum_{z \in \Gamma_N} J(z/N)^2 \times R_z(\eta), \tag{3.10}$$

where

$$R_z(\eta) := \sum_{x:\, |x-z|=1} c_{x,z}(\eta^+) \{ (1 - 2\eta_z^-)(\eta_x^+ - \eta_z^+) - 2\eta_z(\eta_x^+ - \eta_z^+) \}$$

$$= \sum_{x:\, |x-z|=1} c_{x,z}(\eta^+)(\eta_x^+ - \eta_z^+)^2. \tag{3.11}$$

Since the first term in the right hand side of (3.10) coincides with $2\alpha_J N^2 L_N \alpha_J$, $\gamma^N(\eta)$ is the sum of the second and the third terms and this implies equality (3.4). The proof of Proposition 3.1 is also complete. □

We conclude this section by showing the tightness of the measure-valued processes $\{\alpha_t^N\}_{N\in\mathbb{N}}$ and some properties of the limit. Let $Q^N$ be the distribution of $\alpha_\cdot^N$ on the space $D([0,T], \mathcal{M}(\mathbb{T}^d))$.

PROPOSITION 3.2. – (i) *$\{Q^N\}_{N\in\mathbb{N}}$ is tight.*
(ii) *Let $Q$ be an arbitrary limit of $\{Q^N\}_{N\in\mathbb{N}}$. Then we have*

$$Q\{\alpha_\cdot \in C([0,T], \mathcal{M}(\mathbb{T}^d))\} = 1,$$

$$Q\{\alpha_t(d\theta) = a(t,\theta)\, d\theta \text{ for some } a(t,\theta) \in [-\ell, 1]\} = 1.$$

*Proof.* – Since Proposition 3.1 shows $|b^N(\eta)| \leqslant C$ and

$$E[\{M^N(t) - M^N(s)\}^2 \,|\, \mathcal{F}_s^N] \leqslant CN^{-d}(t-s), \quad 0 \leqslant s \leqslant t,$$

we see

$$E[\{\langle \alpha_t^N, J\rangle - \langle \alpha_s^N, J\rangle\}^2 \,|\, \mathcal{F}_s^N] \leqslant C(t-s), \quad 0 \leqslant s \leqslant t \quad (t-s \leqslant 1),$$

for every $J \in C^\infty(\mathbb{T}^d)$, where $\mathcal{F}_s^N := \sigma\{\eta^N(u); u \leqslant s\}$. This proves the tightness of $\{Q^N\}$, see [4]. Since the sizes of jumps of $\langle \alpha_t^N, J\rangle$ are bounded by $C||J||_\infty N^{-d}$ which tends to 0 as $N \to \infty$, we see that every limit $Q$ of $\{Q^N\}$ is a measure on $C([0,T], \mathcal{M}(\mathbb{T}^d))$. Moreover, since

$$-\ell N^{-d} \sum_x J(x/N) \leqslant \langle \alpha_t^N, J\rangle \leqslant N^{-d} \sum_x J(x/N)$$

for $J \geqslant 0$, we see

$$-\ell ||J||_{L^1(\mathbb{T}^d)} \leqslant \langle \alpha_t, J\rangle \leqslant ||J||_{L^1(\mathbb{T}^d)}, \quad Q\text{-a.s.}$$

This implies the second assertion in (ii). □

## 4. LOCAL ERGODIC THEOREM

Let $\mu_t^N \in \mathcal{P}(\mathcal{X}_N)$ be the probability distribution of $\eta^N(t)$ on $\mathcal{X}_N$ and let $\widetilde{\mu}^N$ be the space-time average of $\{\mu_t^N\}_{0\leqslant t\leqslant T}$ defined by

$$\widetilde{\mu}^N = \frac{1}{TN^d} \sum_{x\in\Gamma_N} \int_0^T \mu_t^N \circ \tau_x^{-1}\, dt.$$

We sometimes regard $\widetilde{\mu}^N \in \mathcal{P}(\mathcal{X})$, $\mathcal{X} := \{-\ell, \dots, 1\}^{\mathbb{Z}^d}$ by periodically extending the configurations. Here, the family of all probability measures on the space $\mathcal{E}$ is generally denoted by $\mathcal{P}(\mathcal{E})$. We also denote by $\mathcal{D}$ the class of all local functions on $\mathcal{X}$. For $f \in \mathcal{D}$, we set

$$\bar{f}_{x,K} = \frac{1}{|\Lambda_K|} \sum_{y\in x+\Lambda_K} \tau_y f, \quad x \in \mathbb{Z}^d,\ K \in \mathbb{N}, \tag{4.1}$$

$$\langle f\rangle(\rho) = E^{\nu_\rho}[f], \quad \rho \in [0,1], \tag{4.2}$$

recall $\Lambda_K = \{x \in \mathbb{Z}^d;\ |x| \leqslant K\}$ and that $\nu_\rho$ is a unique extremal canonical Gibbs measure with density $\rho$. We shall simply write $\overline{\eta}_{x,K}$ for $\overline{(\eta_0)}_{x,K}$. The following theorem formulates the local ergodic theorem in the liquid region. Similar idea was employed by [12] or for the proof of Lemma 4 in [17].

THEOREM 4.1. – *For every $f \in \mathcal{D}$,*

$$\lim_{K\to\infty} \limsup_{N\to\infty} E^{\widetilde{\mu}^N}\left[\left|\bar{f}_{0,K}(\eta) - \langle f\rangle(\overline{\eta}_{0,K}^+)\right|^2 \cdot 1_{\mathbb{A}_K}\right] = 0, \tag{4.3}$$

*where $\overline{\eta}_{0,K}^+ = \overline{\eta}_{0,K} \vee 0$ and $\{\mathbb{A}_K\}_K$ is a sequence of $\Lambda_K$-measurable subsets of $\mathcal{X}$ given by*

$$\mathbb{A}_K = \{\eta \in \mathcal{X};\ \eta_x = 1 \textit{ for some } x \in \Lambda_K\}.$$

*Proof.* – Let $L$ be an operator on $\mathcal{D}$ defined by

$$Lf(\eta) = \sum_{x,y\in\mathbb{Z}^d:\ |x-y|=1} c_{x,y}(\eta^+)\pi_{x;y}f(\eta), \quad f \in \mathcal{D}.$$

Then we have

$$\frac{d}{dt}\mu_t^N(f) = \mu_t^N(N^2L_Nf) = N^2\mu_t^N(Lf), \quad f \in \mathcal{D},$$

for sufficiently large $N$; we occasionally denote $E^\mu[f]$ by $\mu(f)$. Integrate this equality in $t \in [0, T]$ with $f$ replaced by $f \circ \tau_x$, and sum in $x \in \Gamma_N$. Thus you have

$$\frac{1}{TN^{d+2}} \sum_{x\in\Gamma_N} \{\mu_T^N(f\circ\tau_x) - \mu_0^N(f\circ\tau_x)\} = \widetilde{\mu}^N(Lf).$$

However, since the absolute value of the left hand side is bounded by $2\|f\|_\infty/(N^2T)$, we see $\lim_{N\to\infty}\widetilde{\mu}^N(Lf) = 0$. Noting that $\{\widetilde{\mu}^N\}_N$ is tight in $\mathcal{P}(\mathcal{X})$, take an arbitrary limit $\mu \in \mathcal{P}(\mathcal{X})$ as $N \to \infty$. Then $\mu(Lf) = 0$ holds for every $f \in \mathcal{D}$; in other words, $\mu$ is an $L$-stationary measure. Moreover, by definition, $\mu$ is invariant under spatial translations.

At this point we need the following lemma.

LEMMA 4.1 (Characterization of translation-invariant $L$-stationary measures). – *The $L$-stationary measure $\mu \in \mathcal{P}(\mathcal{X})$ invariant under spatial translations has the following decomposition*

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2, \tag{4.4}$$

*for some $\lambda \in [0,1]$, $\mu_1 \in \mathcal{P}(\{-\ell,\dots,0\}^{\mathbb{Z}^d})$ invariant under spatial translations and $\mu_2 \in \mathcal{P}(\{0,1\}^{\mathbb{Z}^d})$ having the form*

$$\mu_2 = \int_0^1 \nu_\rho\, d\beta(\rho),$$

*with some $\beta \in \mathcal{P}([0,1])$. In particular, different types of configurations* 1 *and $\{-\ell,\dots,-1\}$ can not coexist in the support of $\mu$:*

$$\mu(\eta_x \leqslant -1 \text{ and } \eta_y = 1 \text{ for some } x, y \in \mathbb{Z}^d) = 0. \tag{4.5}$$

*Proof.* – Let us first prove (4.5). In fact, the $L$-stationarity of $\mu$ and its translation-invariance imply

$$0=\mu(L\eta_z^+)=\sum_{x:\ |x-z|=1} E^\mu\big[c_{x,z}(\eta^+)\{1_{\{\eta_x=1,\eta_z=0\}}-1_{\{\eta_x\leqslant 0,\eta_z=1\}}\}\big]$$
$$=-\sum_{x:\ |x-z|=1} E^\mu\big[c_{x,z}(\eta^+)1_{\{\eta_x\leqslant -1,\eta_z=1\}}\big],$$

and, by the positivity of $c_{x,z}$, this shows

$$\mu(\eta_x\leqslant -1,\eta_z=1)=0, \tag{4.6}$$

if $|x-z|=1$. Now assume that (4.5) does not hold. Then,

$$\mu(\eta_x\leqslant -1,\eta_y=1)>0$$

holds for some $x, y\in\mathbb{Z}^d$ and hence,

$$\mu(\eta_{x_1}\leqslant -1,\eta_{x_2}=\cdots=\eta_{x_{n-1}}=0,\eta_{x_n}=1)>0$$

for some $x_1,\ldots,x_n\in\mathbb{Z}^d$, $n\geqslant 3$ satisfying $|x_i-x_{i+1}|=1$ $(1\leqslant i\leqslant n-1)$. Consider a process $\eta(t)=(\eta_x(t))_{x\in\mathbb{Z}^d}$ having the infinitesimal generator $L$. Then, for this process, the probability that the configurations at sites $x_1,\ldots,x_{n-2}$ does not change and those at $x_{n-1}$ and $x_n$ interchange during a sufficiently small time interval $[0,\varepsilon]$ is positive. Therefore, by the $L$-stationarity of $\mu$, we have

$$\mu(\eta_{x_1}\leqslant -1,\eta_{x_2}=\cdots=\eta_{x_{n-2}}=0,\eta_{x_{n-1}}=1)>0.$$

Repeating the same arguments, we finally get

$$\mu(\eta_{x_1}\leqslant -1,\eta_{x_2}=1)>0,$$

but this contradicts with (4.6). Thus (4.5) is shown; in other words, support $(\mu)\subset\{-\ell,\ldots,0\}^{\mathbb{Z}^d}\cup\{0,1\}^{\mathbb{Z}^d}$. Decompose $\mu$ by restricting it on these two sets. Then, both measures are $L$-stationary and translation-invariant. However, it is known that the translation-invariant $L$-stationary measure on $\{0,1\}^{\mathbb{Z}^d}$ is a superposition of canonical Gibbs measures, see [9], Corollary (3.44), while all measures on $\{-\ell,\ldots,0\}^{\mathbb{Z}^d}$ are stationary for $L$ since the ice particles are immobile. Thus we get (4.4). □

We continue the proof of Theorem 4.1. By Lemma 4.1, we see

$$\limsup_{N\to\infty} E^{\tilde\mu^N}\big[\big|\bar f_{0,K}(\eta)-\langle f\rangle(\overline{\eta}^+_{0,K})\big|^2\cdot 1_{\mathbb{A}_K}\big]$$
$$\leqslant \sup{}^* E^{\mu}\big[\big|\bar f_{0,K}(\eta)-\langle f\rangle(\overline{\eta}^+_{0,K})\big|^2\cdot 1_{\mathbb{A}_K}\big], \tag{4.7}$$

where $\sup^*$ is taken over all $\mu\in\mathcal{P}(\mathcal{X})$ with the form (4.4). Since $\mu_1(\mathbb{A}_K)=0$ for such $\mu$, (4.7) is further bounded by

$$\sup_{\rho\in[0,1]} E^{\nu_\rho}\big[\big|\bar f_{0,K}(\eta)-\langle f\rangle(\overline{\eta}_{0,K})\big|^2\big]$$

and this supremum converges to 0 as $K\to\infty$ by the condition (e). □

## 5. YOUNG MEASURES

In this section, we shall prove that the sample density defined microscopically can be replaced in the limit with the density function $a(t,\theta)$ obtained by Proposition 3.2 in the liquid region, see Theorem 5.1 below. Then, combining with the results in Section 4, the proof of the main theorem will be concluded. The basic idea for the approach of this section comes from Varadhan [20] and uses Young measures. We shall compute correlations of $a(\in[-\ell,1])$ and $P(a)$ under the Young measures and deduce from such computations the triviality of the Young measures in the limit, see Proposition 5.1(iii).

For a function $G=G(x/N)$ on $N^{-1}\Gamma_N$, consider

$$\langle\alpha, G*\alpha\rangle=\frac{1}{N^{2d}}\sum_{z_1,z_2\in\Gamma_N} G\big((z_1-z_2)/N\big)\eta_{z_1}\eta_{z_2}, \tag{5.1}$$

where $\alpha\equiv\alpha(\eta):=N^{-d}\sum_{x\in\Gamma_N}\eta_x\delta_{x/N}$.

LEMMA 5.1. – *Assume $G$ is symmetric, $G(x/N)=G(-x/N)$. Then*

$$N^2L_N\langle\alpha,G*\alpha\rangle=\Psi_1^N(\eta;G)+\Psi_2^N(\eta;G), \tag{5.2}$$

*where*

$$\Psi_1^N(\eta;G)=2N^{-2d}\sum_{z_1,z_2\in\Gamma_N}\sum_{i=1}^d\Delta_i^NG((z_1-z_2)/N)\tau_{z_1}h_i(\eta^+)\eta_{z_2},$$

$$\Psi_2^N(\eta;G)=-N^{-2d}\sum_{z\in\Gamma_N}\sum_{i=1}^d\Delta_i^NG(0)c_{z,z+e_i}(\eta^+)\big(\eta_z^+-\eta_{z+e_i}^+\big)^2.$$

*Proof.* – Using Lemma 3.2,

$$
\begin{aligned}
&N^2 L_N\langle \alpha, G*\alpha\rangle \\
&\quad = N^{2-2d} \sum_{z_1,z_2\in\Gamma_N} G((z_1-z_2)/N) \\
&\qquad\qquad \times \Big\{ \eta_{z_2} \sum_{x:\,|x-z_1|=1} c_{x,z_1}(\eta^+)(\eta_x^+ - \eta_{z_1}^+) \\
&\qquad\qquad\quad + \eta_{z_1} \sum_{x:\,|x-z_2|=1} c_{x,z_2}(\eta^+)(\eta_x^+ - \eta_{z_2}^+) \Big\} \\
&\qquad - N^{2-2d} \sum_{z_1,z_2\in\Gamma_N:\,|z_1-z_2|=1} G((z_1-z_2)/N) c_{z_1,z_2}(\eta^+)(\eta_{z_1}^+ - \eta_{z_2}^+)^2 \\
&\qquad + N^{2-2d} \sum_{z\in\Gamma_N} G(0) R_z(\eta),
\end{aligned}
$$

where $R_z(\eta)$ is defined in (3.11). By the symmetry of $G$, the first term in the right hand side is

$$
2N^{2-2d} \sum_{z_1,z_2\in\Gamma_N} G((z_1-z_2)/N)\eta_{z_2} \sum_{x:\,|x-z_1|=1} c_{x,z_1}(\eta^+)(\eta_x^+ - \eta_{z_1}^+).
$$

Then, using the gradient condition (f), this can be rewritten as $\Psi_1^N(\eta; G)$. On the other hand, the sum of the second and third terms is rewritten as

$$
-N^{2-2d} \sum_{z_1,z_2\in\Gamma_N:\,|z_1-z_2|=1} \{G((z_1-z_2)/N) - G(0)\} c_{z_1,z_2}(\eta^+)(\eta_{z_1}^+ - \eta_{z_2}^+)^2,
$$

and this coincides with $\Psi_2^N(\eta; G)$, since

$$
G(\pm e_i/N) - G(0) = \Delta_i^N G(0)/2N^2
$$

for symmetric $G$. □

Now we state the main result of this section. Remind that $Q^N$ denotes the distribution of the process $\alpha_\cdot^N$ and $\alpha_t(d\theta) = {}^\exists a(t,\theta)\,d\theta$, $Q$-a.s. for an arbitrary limit $Q$ of $\{Q^N\}$: $Q = \lim_{N'\to\infty} Q^{N'}$, see Proposition 3.2. For $\theta \in \mathbb{T}^d$, set

$$
a_K^N(t,\theta) := \frac{1}{|\Lambda_K|} \sum_{x\in[N\theta]+\Lambda_K} \eta_x^N(t),
$$

where $[N\theta] \in \mathbb{Z}^d$ stands for the integral part of $N\theta$ taken componentwisely.

THEOREM 5.1. – *For every* $g = g(\alpha, u) \in C_b(\mathcal{M}(\mathbb{T}^d) \times \mathbb{R})$ *and* $F = F(t, \theta, p) \in C_b([0, T] \times \mathbb{T}^d \times [0, P(1)])$,

$$\lim_{K\to\infty}\lim_{N'\to\infty} E\left[g\left(\alpha_T^{N'}, \int_0^T dt \int_{\mathbb{T}^d} F\left(t,\theta, P\left(a_K^{N'}(t,\theta)\right)\right) d\theta\right)\right]$$

$$= E^Q\left[g\left(\alpha_T, \int_0^T dt \int_{\mathbb{T}^d} F\left(t,\theta, P\left(a(t,\theta)\right)\right) d\theta\right)\right],$$

*where* $a(t,\theta)$ *is the density function obtained by Proposition* 3.2.

*Proof.* – Let $q_r^N(x/N)$, $r \geqslant 0$, $x \in \Gamma_N$ be the fundamental solution of the discrete heat equation on $N^{-1}\Gamma_N (\subset \mathbb{T}^d)$:

$$\begin{cases} \dfrac{\partial q_r^N}{\partial r} = \dfrac{1}{2}\Delta^N q_r^N, & r > 0, \\ q_0^N = \delta_0^N, \end{cases} \tag{5.3}$$

where $\Delta^N := \sum_{i=1}^d \Delta_i^N$ is the discrete Laplacian on $N^{-1}\Gamma_N$ and $\delta_0^N$ is defined by $\delta_0^N(x/N) = N^d$ $(x = 0)$, $= 0$ $(x \neq 0)$. The function $q_r^N$ has the following expression:

$$q_r^N(x/N) = \sum_{\mathbf{n}} \mathrm{e}^{-\lambda_{\mathbf{n}}^N r/2} \xi_{\mathbf{n}}^N(x/N)\xi_{\mathbf{n}}^N(0). \tag{5.4}$$

Here, $\mathbf{n} = (n_1, \dots, n_d) \in \mathcal{N}_N := \{0, 1, \dots, N-1\}^d$ are multi-indices and the sum is taken over all $\mathbf{n} \in \mathcal{N}_N$. To define $\lambda_{\mathbf{n}}^N$ and $\xi_{\mathbf{n}}^N$, first for $n \in \{0, 1, \dots, N-1\}$ and $x \in \{1, 2, \dots, N\} = \mathbb{Z}/N\mathbb{Z}$, set $\lambda_n^N = 4N^2 \sin^2 n\pi/N$ and

$$\xi_n^N(x/N) = \begin{cases} \sqrt{2}\beta_n \cos 2n\pi x/N, & 0 \leqslant n \leqslant N/2, \\ \sqrt{2}\sin 2(N-n)\pi x/N, & N/2 < n \leqslant N-1, \end{cases}$$

where $\beta_n = 1/\sqrt{2}$ if $n = 0$ or if $N$ is even and $n = N/2$, and $\beta_n = 1$ otherwise. Then, for $\mathbf{n} \in \mathcal{N}_N$ and $x = (x_1, \dots, x_d) \in \Gamma_N$,

$$\lambda_{\mathbf{n}}^N = \sum_{i=1}^d \lambda_{n_i}^N, \qquad \xi_{\mathbf{n}}^N(x/N) = \prod_{i=1}^d \xi_{n_i}^N(x_i/N).$$

Note that $\{-\lambda_n^N, \xi_n^N\}_{0\leqslant n\leqslant N-1}$ are the eigenvalues and the corresponding eigenfunctions of the operator $\Delta^N$ in one-dimension; moreover $\{\xi_n^N\}$ is orthonormal, see [15], pp. 54–58. Accordingly, we see $\Delta_i^N \xi_{\mathbf{n}}^N = -\lambda_{n_i}^N \xi_{\mathbf{n}}^N$, $\Delta^N \xi_{\mathbf{n}}^N = -\lambda_{\mathbf{n}}^N \xi_{\mathbf{n}}^N$ and that $\{\xi_{\mathbf{n}}^N\}$ is orthonormal:

$$(\xi_{\mathbf{n}}^N, \xi_{\mathbf{m}}^N) := N^{-d} \sum_{x \in \Gamma_N} \xi_{\mathbf{n}}^N(x/N) \xi_{\mathbf{m}}^N(x/N) = \delta_{\mathbf{n},\mathbf{m}}, \quad \mathbf{n}, \mathbf{m} \in \mathcal{N}_N.$$

Now, let

$$\overline{\alpha}_K \equiv \overline{\alpha}_K(\eta) := N^{-d} \sum_{x \in \Gamma_N} \overline{\eta}_{x,K} \delta_{x/N} \in \mathcal{M}(\mathbb{T}^d),$$

and consider

$$\overline{\alpha}_{t,K}^N := \overline{\alpha}_K(\eta^N(t)).$$

Note that $\langle \overline{\alpha}_K, G * \overline{\alpha}_K \rangle = \langle \alpha, \overline{\overline{G}}_K * \alpha \rangle$ holds with

$$\overline{\overline{G}}_K(x/N) := \frac{1}{|\Lambda_K|^2} \sum_{y_1, y_2 \in \Lambda_K} G((x + y_1 + y_2)/N).$$

Then, taking $G = \overline{\overline{(q_r^N)}}_K$ in Lemma 5.1, we have

$$\begin{aligned} &E[\langle \overline{\alpha}_{T,K}^N, q_r^N * \overline{\alpha}_{T,K}^N \rangle] - E[\langle \overline{\alpha}_{0,K}^N, q_r^N * \overline{\alpha}_{0,K}^N \rangle] \\ &= \int_0^T \left\{ E^{\mu_t^N}\left[\Psi_1^N(\eta; \overline{\overline{(q_r^N)}}_K)\right] + E^{\mu_t^N}\left[\Psi_2^N(\eta; \overline{\overline{(q_r^N)}}_K)\right] \right\} dt. \end{aligned} \tag{5.5}$$

We shall integrate both sides over the interval $[\tau/N^2, \delta]$ in $r$ for $\tau, \delta > 0$. Then, for the second term, we have the following.

LEMMA 5.2. –

$$\lim_{\tau \to \infty} \sup_{N,K} \sup_{\delta > 0} \int_{\tau/N^2}^{\delta} dr \int_0^T E^{\mu_t^N}\left[\left|\Psi_2^N(\eta; \overline{\overline{(q_r^N)}}_K)\right|\right] dt = 0.$$

·*Proof.* – From the expression (5.4) of $q_r^N$ and noting that $|\xi_{\mathbf{n}}^N| \leqslant 2^{d/2}$, we see

$$\left|\Delta_i^N \overline{\overline{(q_r^N)}}_K(0)\right| \leqslant 2^d \sum_{\mathbf{n}} e^{-\lambda_{\mathbf{n}}^N r/2} \lambda_{n_i}^N.$$

Therefore,

$$\int_{\tau/N^2}^{\delta} dr \int_0^T E^{\mu_t^N}\left[\left|\Psi_2^N(\eta; \overline{\overline{(q_r^N)}}_K)\right|\right] dt \leqslant d2^d \|c\|_\infty \frac{T}{N^d} \sum_{\mathbf{n}} \lambda_{n_i}^N$$

$$\times \int_{\tau/N^2}^{\delta} e^{-\lambda_{\mathbf{n}}^N r/2}\, dr \leqslant d2^{d+1} \|c\|_\infty \frac{T}{N^d} \sum_{\mathbf{n}} \exp\left\{-2\tau \sum_{i=1}^d \sin^2 n_i \pi/N\right\},$$

and the last term goes to 0 as $\tau \to \infty$ uniformly in $N$. □

We rewrite and decompose the integrand of the first term in the right hand side of (5.5) into

$$\begin{aligned}
&\Psi_1^N\left(\eta; \overline{\overline{(q_r^N)}}_K\right)\\
&= 2N^{-2d} \sum_{z_1, z_2 \in \Gamma_N} \sum_{i=1}^d \Delta_i^N q_r^N\big((z_1 - z_2)/N\big) \overline{(h_i(\eta^+))}_{z_1,K} \overline{\eta}_{z_2,K}\\
&= 2\sum_{i=1}^d \Psi_{1,i,K}^{N,(1)}(r, \eta) + 2\Psi_{1,K}^{N,(2)}(r, \eta),
\end{aligned}$$

where

$$\begin{aligned}
\Psi_{1,i,K}^{N,(1)}(r, \eta) &:= N^{-2d} \sum_{z_1, z_2 \in \Gamma_N} \Delta_i^N q_r^N\big((z_1 - z_2)/N\big)\\
&\qquad \times \left\{\overline{(h_i(\eta^+))}_{z_1,K} - P\left(\overline{\eta}_{z_1,K}\right)\right\} \overline{\eta}_{z_2,K},\\
\Psi_{1,K}^{N,(2)}(r, \eta) &:= N^{-2d} \sum_{z_1, z_2 \in \Gamma_N} \Delta^N q_r^N\big((z_1 - z_2)/N\big) P\left(\overline{\eta}_{z_1,K}\right) \overline{\eta}_{z_2,K}.
\end{aligned}$$

Then, as an application of the local ergodic theorem, we have the next lemma.

LEMMA 5.3. –

$$\lim_{K\to\infty} \limsup_{N\to\infty} \sup_{\delta,\tau>0} \left| \int_{\tau/N^2}^{\delta} dr \int_0^T E^{\mu_t^N}\left[\Psi_{1,i,K}^{N,(1)}(r, \eta)\right] dt \right| = 0.$$

*Proof.* – The square of the above integral is rewritten as

$$\left| T E^{\tilde{\mu}^N}\left[ N^{-d} \sum_{z_1 \in \Gamma_N} \left\{\overline{(h_i(\eta^+))}_{z_1,K} - P\left(\overline{\eta}_{z_1,K}\right)\right\} \times N^{-d} \sum_{z_2 \in \Gamma_N} \overline{\eta}_{z_2,K} \int_{\tau/N^2}^{\delta} \Delta_i^N q_r^N\big((z_1 - z_2)/N\big)\, dr \right] \right|^2$$

and, by Schwarz's inequality, this is bounded by

$$T^2 E^{\tilde{\mu}^N}\left[\left|\overline{(h_i(\eta^+))}_{0,K} - P(\overline{\eta}_{0,K})\right|^2\right] \times \sup_{\eta\in\mathcal{X}_N} \varphi^N_{K;\delta,\tau}(\eta), \tag{5.6}$$

where

$$\varphi^N_{K;\delta,\tau}(\eta) = N^{-d}\sum_{z_1\in\Gamma_N}\left\{N^{-d}\sum_{z_2\in\Gamma_N}\overline{\eta}_{z_2,K}\int_{\tau/N^2}^{\delta}\Delta_i^N q_r^N((z_1-z_2)/N)\,dr\right\}^2.$$

Therefore, once we can show that

$$\sup_{\delta,\tau>0}\sup_{N,K}\sup_{\eta}\varphi^N_{K;\delta,\tau}(\eta) < \infty, \tag{5.7}$$

the conclusion follows from the local ergodic theorem. In fact, noting that $P(a) = P^+(a^+)$, the expectation in (5.6) can be decomposed as

$$\begin{aligned}&E^{\tilde{\mu}^N}\left[\left|\overline{(h_i(\eta^+))}_{0,K} - P^+(\overline{\eta}^+_{0,K})\right|^2\cdot 1_{\mathbb{A}_K}\right]\\ &\quad + E^{\tilde{\mu}^N}\left[\left|\overline{(h_i(\eta^+))}_{0,K} - P^+(\overline{\eta}^+_{0,K})\right|^2\cdot 1_{\mathbb{A}_K^c}\right],\end{aligned}$$

where $\mathbb{A}_K$ is the set introduced in Theorem 4.1. Since $\eta\in\mathbb{A}_K^c$ implies $\overline{\eta}^+_{0,K}=0$ and $\tau_y h_i(\eta^+)=0$, $y\in\Lambda_{K-\overline{R}}$, ($\overline{R}$ is the constant in (2.1)), the second term vanishes as $K\to\infty$ uniformly in $N$. The first term also vanishes as $N\to\infty$ and then $K\to\infty$ by Theorem 4.1. Recall that $P^+(\rho) = \langle h_i\rangle(\rho)$ is independent of $i$ by the condition (f).

To prove (5.7), rewrite the sum in $z_2\in\Gamma_N$ in $\varphi^N_{K;\delta,\tau}(\eta)$ into

$$\begin{aligned}&\sum_{z_2\in\Gamma_N}(\Delta^N)^{-1}\Delta_i^N\overline{\eta}_{z_2,K}\int_{\tau/N^2}^{\delta}\Delta^N q_r^N((z_1-z_2)/N)\,dr\\ &\quad = 2\sum_{z_2\in\Gamma_N} f(z_2/N)\{q_\delta^N((z_1-z_2)/N) - q^N_{\tau/N^2}((z_1-z_2)/N)\},\end{aligned}$$

where $f(z/N) := (\Delta^N)^{-1}\Delta_i^N\overline{\eta}_{z,K}$; we introduce a slight abuse of notation, $\Delta_i^N$ acts on $\overline{\eta}_{z,K}$ thought as a function on $N^{-1}\Gamma_N$. Note that the operation $(\Delta^N)^{-1}$ is well-defined, since $(\Delta_i^N\overline{\eta}_{z,K}, 1) = 0$. However, for every $r>0$,

$$N^{-d}\sum_{z_1\in\Gamma_N}\left\{N^{-d}\sum_{z_2\in\Gamma_N} f(z_2/N)q_r^N((z_1-z_2)/N)\right\}^2$$
$$\leqslant N^{-d}\sum_{z\in\Gamma_N} f^2(z/N)\leqslant \ell^2.$$

The first inequality is established by Young's inequality and

$$N^{-d}\sum_{z\in\Gamma_N} q_r^N(z/N)=1,$$

and the second one follows by expanding $f$ in terms of the orthonormal system $\{\xi_{\mathbf{n}}^N\}_{\mathbf{n}\in\mathcal{N}_N}$. Thus (5.7) is shown. □

Since $|\langle\overline{\alpha}_{t,K}^N, q_r^N * \overline{\alpha}_{t,K}^N\rangle|\leqslant \ell^2$ for every $t\geqslant 0$, (5.5) and Lemmas 5.2, 5.3 imply

$$\lim_{\delta\downarrow 0}\limsup_{\tau\to\infty}\limsup_{K\to\infty}\limsup_{N\to\infty}\left|\int_{\tau/N^2}^{\delta} dr\int_0^T E^{\mu_t^N}\left[\Psi_{1,K}^{N,(2)}(r,\eta)\right]dt\right|=0$$

and thus

$$A_1\leqslant A_2, \tag{5.8}$$

where

$$A_1:=\liminf_{\tau\to\infty}\ \liminf_{K=K^*\to\infty}\ \liminf_{N=N^*\to\infty} E^{\tilde{\mu}^N}\left[N^{-d}\sum_{z\in\Gamma_N} q_{\tau/N^2}^N(z/N)P(\overline{\eta}_{0,K})\overline{\eta}_{z,K}\right],$$

$$A_2:=\limsup_{\delta\downarrow 0}\ \limsup_{K=K^*\to\infty}\ \limsup_{N=N^*\to\infty} E^{\tilde{\mu}^N}\left[N^{-d}\sum_{z\in\Gamma_N} q_\delta^N(z/N)P(\overline{\eta}_{0,K})\overline{\eta}_{z,K}\right],$$

for arbitrary subsequences $\{K^*\}$ and $\{N^*\}$.

We introduce Young measures: Set $\mathcal{M}=\mathcal{M}([0,T]\times\mathbb{T}^d\times[-\ell,1])$ and consider $\mathcal{M}$-valued random variables

$$\pi_K^N(dt\,d\theta\,da):=dt\,d\theta\,\delta_{a_K^N(t,\theta)}(da).$$

We denote the joint distributions of $(\alpha_\cdot^N,\pi_K^N)$ on the space $D([0,T],\mathcal{M}(\mathbb{T}^d))\times\mathcal{M}$ by $\widehat{Q}^{N,K}$. Then, the family $\{\widehat{Q}^{N,K}\}_{N,K}$ is tight. Indeed, the first marginal of $\widehat{Q}^{N,K}$ is $Q^N$ which makes a tight family by Proposition 3.2 and the compactness of the space $\mathcal{M}$ implies the tightness of $\{\pi_K^N\}_{N,K}$ in the second coordinate. Therefore, based on the diagonal argument, from an arbitrary subsequence $\{N''\}$ of the sequence $\{N'\}$ given just before the statement of Theorem 5.1, one can find a further subsequence $\{N'''\}$ such that $\widehat{Q}^{N''',K}$ weakly converges to some $\widehat{Q}^K$ for all $K$. Note that the first marginal of the limit measure $\widehat{Q}^K$ is $Q$ for all $K$.

Denote by $\pi^P \in \mathcal{M}^P := \mathcal{M}([0,T] \times \mathbb{T}^d \times [0, P(1)])$ the image measure of $\pi \in \mathcal{M}$ under the map

$$(t,\theta,a) \in [0,T] \times \mathbb{T}^d \times [-\ell, 1] \\ \mapsto \big(t,\theta,P(a)\big) \in [0,T] \times \mathbb{T}^d \times \big[0, P(1)\big],$$

and by $\widehat{Q}^{K,P}$ the image measure of $\widehat{Q}^K$ under the map $(\alpha.,\pi) \mapsto (\alpha.,\pi^P)$. Then, Theorem 5.1 readily follows from the following proposition:

PROPOSITION 5.1. –
(i) $\widehat{Q}^{K,P}$ *weakly converges to some* $\widehat{Q}^P$ *as* $K \to \infty$. *The first marginal of* $\widehat{Q}^P$ *is* $Q$.
(ii) $\widehat{Q}^P\{(\alpha.,\pi);\, \alpha_t(d\theta) = a(t,\theta)\, d\theta$ *for some* $a(t,\theta) \in [-\ell,1]\} = 1$.
(iii) $\widehat{Q}^P\{(\alpha.,\pi);\, \pi^P(dt\, d\theta\, dp) = dt\, d\theta \delta_{P(a(t,\theta))}(dp)\} = 1$.

The assertion (i) of this proposition is a consequence of the rest of assertions, since $\{\widehat{Q}^{K,P}\}_K$ is tight and its limit $\widehat{Q}^P$ is uniquely characterized by (ii) and (iii). The assertion (ii) is a restatement of Proposition 3.2(ii). Therefore, the proof of (iii) is only left. To complete it, we prepare a lemma. Since $\{\widehat{Q}^K\}_K$ is tight, from an arbitrary subsequence $\{K'\}$ of $\{K\}$, one can find a further subsequence $\{K''\}$ such that $\widehat{Q}^{K''}$ weakly converges to some $\widehat{Q}$ as $K'' \to \infty$. Note that the image measure of $\widehat{Q}$ under the map $(\alpha.,\pi) \mapsto (\alpha.,\pi^P)$ is certainly $\widehat{Q}^P$.

LEMMA 5.4. –
(i) $\widehat{Q}\{(\alpha.,\pi);\, \pi(dt\, d\theta\, da) = dt\, d\theta \pi_{t,\theta}(da)$ *for some* $\pi_{t,\theta}\} = 1$.
(ii) $\widehat{Q}\{(\alpha.,\pi);\, \alpha_t(d\theta) = a(t,\theta)\, d\theta$ *and* $a(t,\theta) = \int_{-\ell}^1 a\pi_{t,\theta}(da)\} = 1$.

*Proof.* – The limits are taken twice along $N''' \to \infty$ and then $K'' \to \infty$. The relation (i) is shown first for $\widehat{Q}^K$ by noting that $\pi_K^{N'''}$ enjoys the property

$$\int_{[0,T]\times\mathbb{T}^d\times[-\ell,1]} J(t,\theta)\, \pi_K^{N'''}(dt\, d\theta\, da) = \int_{[0,T]\times\mathbb{T}^d} J(t,\theta)\, dt\, d\theta$$

for every bounded measurable function $J(t,\theta)$. Then, (i) is established for $\widehat{Q}$ by repeating a similar argument. To prove (ii), we note that

$$\int_{\mathbb{T}^d\times[-\ell,1]} J(\theta)a\,\pi_{K''}^{N'''}(dt\,d\theta\,da) = \int_{\mathbb{T}^d} J(\theta)a_{K''}^{N'''}(t,\theta)\,dt\,d\theta$$
$$= \langle \alpha_t^{N'''}, J\rangle dt + \mathrm{o}(1)$$

as $N''' \to \infty$, $K'' \to \infty$ for every $J \in C^\infty(\mathbb{T}^d)$. Then, recall Proposition 3.2(ii). The proof is essentially due to Varadhan [20], Lemma 7.8; see also [18], Lemma 7.4. □

We are now at the position to give the proof of Proposition 5.1(iii). To this end, we continue the computations of $A_1$ and $A_2$ taking with $N^* = N'''$ and $K^* = K''$.

LEMMA 5.5. –

$$A_1 = \frac{1}{T}E^{\widehat{Q}}\left[\int_0^T dt\int_{\mathbb{T}^d} d\theta\int_{-\ell}^1 aP(a)\pi_{t,\theta}(da)\right].$$

*Proof.* – Set $A_1(\eta) \equiv A_{1;N,K,\tau}(\eta)$ the inside of the expectation in the definition of $A_1$:

$$A_1(\eta) := N^{-d}\sum_{z\in\Gamma_N} q_{\tau/N^2}^N(z/N)P(\overline{\eta}_{0,K})\overline{\eta}_{z,K}$$
$$= P(\overline{\eta}_{0,K})\sum_{z\in\Gamma_N}\overline{\eta}_{z,K}\widetilde{q}_\tau^N(z),$$

where $\widetilde{q}_\tau^N(z) := N^{-d}q_{\tau/N^2}^N(z/N)$ is the heat kernel on $\Gamma_N$. Let $p_\tau(z)$ be the heat kernel on the whole lattice $\mathbb{Z}^d$ and define a local function $\widetilde{A}_1$ by

$$\widetilde{A}_1(\eta) \equiv \widetilde{A}_{1;K,\tau}(\eta) := P(\overline{\eta}_{0,K})\sum_{z\in\mathbb{Z}^d:\ |z|\leqslant\tau}\overline{\eta}_{z,K}\,p_\tau(z).$$

Then, if $1 < \tau < N$,

$$\left|A_1(\eta) - \widetilde{A}_1(\eta)\right| \leqslant ||P||_\infty \ell\sum_{z\in\Gamma_N}\left|\widetilde{q}_\tau^N(z) - p_\tau(z)\right|$$
$$+ ||P||_\infty \ell\sum_{z\in\mathbb{Z}^d:\ |z|>\tau}|p_\tau(z)|.$$

The first term tends to 0 as $N \to \infty$ for fixed $\tau$, while the second term also tends to 0 as $\tau \to \infty$. Hence, it is sufficient to study the limit of $E^{\tilde{\mu}^N}[\widetilde{A}_1(\eta)]$.

Since $E^{\tilde{\mu}^N}[\widetilde{A}_1(\eta)] = E^{\tilde{\mu}^N}[\overline{(\widetilde{A}_1)_{0,M}}(\eta)]$ for every $0 < M < N$, using the local ergodic theorem, we have

$$\lim_{M\to\infty} \limsup_{N\to\infty} \left| E^{\tilde{\mu}^N}[\widetilde{A}_1(\eta)] - E^{\tilde{\mu}^N}[\langle \widetilde{A}_1\rangle(\overline{\eta}_{0,M}^+)\cdot 1_{\mathbb{A}_M}]\right| = 0,$$

where $\mathbb{A}_M$ is the set given in Theorem 4.1; note that $\eta \in \mathbb{A}_M^c$ implies $\tau_y \widetilde{A}_1(\eta) = 0$ for $M > K$ and $y \in \Lambda_{M-K}$. Set $F(a) = aP(a)$ for $a \in [0, 1]$. Then,

$$\begin{aligned}|\langle \widetilde{A}_1\rangle(a) - F(a)| \leqslant{}& \left| E^{\nu_a}\left[\{P(\overline{\eta}_{0,K}) - P(a)\} \sum_{z\in\mathbb{Z}^d:\ |z|\leqslant\tau} \overline{\eta}_{z,K}\, p_\tau(z)\right]\right| \\ &+ \left| P(a) E^{\nu_a}\left[\sum_{z\in\mathbb{Z}^d:\ |z|>\tau} \overline{\eta}_{z,K}\, p_\tau(z)\right]\right|,\end{aligned}$$

which tends to 0 as $K \to \infty$ and then $\tau \to \infty$. Thus we have proved

$$\lim_{\tau\to\infty} \limsup_{K\to\infty} \limsup_{M\to\infty} \limsup_{N\to\infty} \left| E^{\tilde{\mu}^N}[\widetilde{A}_1(\eta)] - E^{\tilde{\mu}^N}[F(\overline{\eta}_{0,M}^+)]\right| = 0,$$

note that $\eta \in \mathbb{A}_M^c$ implies $F(\overline{\eta}_{0,M}^+) = 0$ and therefore

$$E^{\tilde{\mu}^N}[F(\overline{\eta}_{0,M}^+)\cdot 1_{\mathbb{A}_M}] = E^{\tilde{\mu}^N}[F(\overline{\eta}_{0,M}^+)].$$

However,

$$\begin{aligned}E^{\tilde{\mu}^N}[F(\overline{\eta}_{0,M}^+)] &= \frac{1}{T} E^{\widehat{Q}^{N,M}}\left[\int_0^T \int_{\mathbb{T}^d} \int_{-\ell}^1 F(a^+)\pi(dt\,d\theta\,da)\right] \\ &\to \frac{1}{T} E^{\widehat{Q}}\left[\int_0^T dt \int_{\mathbb{T}^d} d\theta \int_{-\ell}^1 aP(a)\pi_{t,\theta}(da)\right]\end{aligned}$$

along the sequences $N = N''' \to \infty$ and then $M = M'' \to \infty$ by taking $M''$ the same sequence as $K''$ given just before Lemma 5.4. This completes the proof of lemma. □

LEMMA 5.6. –

$$A_2 = \frac{1}{T} E^{\widehat{Q}}\left[\int_0^T dt \int_{\mathbb{T}^d} d\theta \int_{-\ell}^1 P(a)\pi_{t,\theta}(da) \int_{-\ell}^1 a'\pi_{t,\theta}(da')\right].$$

*Proof.* – The expectation in the definition of $A_2$ can be rewritten as

$$\frac{1}{T}\int_0^T dt N^{-d} \sum_{x\in\Gamma_N} E\left[P\left(\overline{\eta}_{x,K}(t)\right) \times N^{-d} \sum_{z\in\Gamma_N} \overline{\eta}_{x+z,K}(t) q_\delta^N(z/N)\right]$$

$$= \frac{1}{T}\int_0^T dt E\left[\int_{\mathbb{T}^d} P\left(a_K^N(t,\theta)\right) d\theta \int_{\mathbb{T}^d} a_K^N(t,\theta') q_\delta^N\left([\theta'] - [\theta]\right) d\theta'\right],$$

where $[\theta] := [N\theta]/N$. Letting $N = N''' \to \infty, K = K'' \to \infty$, this quantity converges to

$$\frac{1}{T} E^{\widehat{Q}}\left[\int_0^T dt \int_{\mathbb{T}^d} d\theta \int_{-\ell}^1 P(a)\pi_{t,\theta}(da) \int_{\mathbb{T}^d} a(t,\theta') q_\delta(\theta'-\theta)\, d\theta'\right],$$

where $q_r(\theta), r > 0, \theta \in \mathbb{T}^d$ is the heat kernel on $\mathbb{T}^d$. However, since

$$a(t,\theta') = \int_{-\ell}^1 a'\, \pi_{t,\theta'}(da'), \qquad \widehat{Q}\text{-a.s.}$$

by Lemma 5.4(ii), this is further rewritten as

$$\frac{1}{T} E^{\widehat{Q}}\left[\int_0^T dt \int_{\mathbb{T}^d\times\mathbb{T}^d} q_\delta(\theta'-\theta)\, d\theta\, d\theta' \int_{-\ell}^1 P(a)\pi_{t,\theta}(da) \int_{-\ell}^1 a'\pi_{t,\theta'}(da')\right]$$

which converges to the desired quantity as $\delta \downarrow 0$, cf. [18]. □

Since $P$ is nondecreasing and $\pi_{t,\theta}$ is a probability measure on $[-\ell, 1]$, it is obvious that

$$\int_{-\ell}^1 P(a)\pi_{t,\theta}(da) \int_{-\ell}^1 a'\pi_{t,\theta}(da') \leqslant \int_{-\ell}^1 aP(a)\pi_{t,\theta}(da). \tag{5.9}$$

However, since $A_1 \leqslant A_2$, Lemmas 5.5 and 5.6 imply the converse inequality to (5.9) when it is integrated in $d\widehat{Q}\, dt\, d\theta$. Hence, we see that (5.9) holds in equality for $d\widehat{Q}\, dt\, d\theta$-a.e. Moreover, if (5.9) holds in equality, again noting that $P$ is nondecreasing, we see that the distribution $\pi_{t,\theta} \circ P^{-1}(dp)$ (the image measure of $\pi_{t,\theta}$ under the map

$P:[-\ell,1]\to[0,P(1)]$) concentrates on a single point; in other words, we have $\pi_{t,\theta}\circ P^{-1}(dp)=\delta_{P(a(t,\theta))}(dp)$. This completes the proof of Proposition 5.1(iii) and consequently that of Theorem 5.1. □

*Completion of the proof of Theorem 2.1.* – Since (3.4) implies that $0\leqslant\gamma^N(\eta)\leqslant CN^{-d}$, we have $\lim_{N\to\infty}E[(M^N(t))^2]=0$ and thus the martingale term in (3.1) vanishes in the limit. On the other hand, we have from (3.3)

$$b^N(\eta)=N^{-d}\sum_{z\in\Gamma_N}\sum_{i=1}^{d}\overline{(h_i(\eta^+))}_{z,K}(\eta)\frac{\partial^2 J}{\partial\theta_i^2}(z/N)+R_K^N(\eta),$$

with an error term satisfying $\lim_{N\to\infty}\sup_\eta|R_K^N(\eta)|=0$ for each $K>0$. Set

$$b_K^N(\eta):=N^{-d}\sum_{z\in\Gamma_N}P(\overline{\eta}_{z,K})\Delta J(z/N).$$

Then, the same argument developed in the proof of Lemma 5.3 based on the local ergodic theorem proves

$$\lim_{K\to\infty}\limsup_{N\to\infty}E\left[\left|\int_0^T\{b^N(\eta^N(t))-b_K^N(\eta^N(t))\}\,dt\right|\right]=0.$$

Finally, from Theorem 5.1, we see

$$\lim_{K\to\infty}\lim_{N'\to\infty}E\left[g\left(\alpha_T^{N'},\int_0^T b_K^{N'}(\eta^{N'}(t))\,dt\right)\right]$$
$$=E^Q\left[g\left(\alpha_T,\int_0^T dt\int_{\mathbb{T}^d}P(a(t,\theta))\Delta J(\theta)\,d\theta\right)\right]$$

for all $g=g(\alpha,u)$. Therefore, $a(t,\theta)$ satisfies the Eq. (2.6) under $Q$. The conclusion follows by the uniqueness of its solutions [19]; note that the function $P(a)$ is nondecreasing and the initial measure $\alpha_0$ has a bounded density $a(0,\theta)$. □

## 6. TWO-PHASE STEFAN PROBLEM

So far we have considered the model in which ice particles are immobile. In this section, we shall make ice particles with label $-\ell$

active. They perform stochastic lattice gases with jump rates different from those for water particles. The particles with labels $-\ell+1,\dots,-1$ remain immobile and are regarded as those in intermediate states between ice and water.

Let two nonnegative functions $c^+_{x,y}(\xi)$ and $c^-_{x,y}(\xi)$ defined for $\xi\in\mathcal{X}^+$ and $x,y\in\mathbb{Z}^d$, $|x-y|=1$ be given and both satisfy the conditions (a)–(f) listed in Section 2. The canonical Gibbs measures $\{\nu_\rho;\rho\in[0,1]\}$ and functions $\{h_1,\dots,h_d\}$ appearing in the conditions (e) and (f) may be different for $c^+_{x,y}$ and $c^-_{x,y}$, and are denoted by $\nu^+_\rho$, $\nu^-_\rho$ and $\{h^+_1,\dots,h^+_d\}$, $\{h^-_1,\dots,h^-_d\}$, respectively. We write $P^+(\rho)=E^{\nu^+_\rho}[h^+_i]$ and $P^-(\rho)=E^{\nu^-_\rho}[h^-_i]$, $\rho\in[0,1]$. Consider the Markov generator $L_N$ defined by

$$L_N f(\eta)=\sum_{x,y\in\Gamma_N:\ |x-y|=1}\{c^+_{x,y}(\eta^+)1_{\{\eta_x=1\}}+c^-_{x,y}(\eta^{(-\ell)})1_{\{\eta^{(-\ell)}_x=1\}}\}\pi_{x;y}f(\eta)$$

for functions $f$ on the configuration space $\mathcal{X}_N=\{-\ell,\dots,1\}^{\Gamma_N}$, where $\eta^{(-\ell)}:=-((\eta+\ell-1)\wedge 0)$, i.e., $\eta^{(-\ell)}_x=-\min\{\eta_x+\ell-1,0\}$, $x\in\Gamma_N$. The operator $\pi_{x;y}$ is defined by (2.3), but the definition (2.4) of $\eta^{x;y}$ should be modified as follows. If $\eta_x=1$ and $\eta^+_y=0$, or if $\eta^{(-\ell)}_x=1$ and $\eta^{(-\ell)}_y=0$,

$$(\eta^{x;y})_z=\begin{cases}\eta_x-\eta^+_x+\eta^{(-\ell)}_x, & \text{if } z=x,\\ \eta_y+\eta^+_x-\eta^{(-\ell)}_x, & \text{if } z=y,\\ \eta_z, & \text{if } z\neq x,y,\end{cases}$$

and $\eta^{x;y}=\eta$ otherwise; in particular, the conservation law $\eta_x+\eta_y=(\eta^{x;y})_x+(\eta^{x;y})_y$ holds. The configuration $\eta^{x;y}$ is obtained from $\eta$ after a (water or ice) particle located at $x$ jumps to $y$. A water particle can jump to neighboring vacant sites with rates $c^+_{x,y}(\eta^+)$. When a water particle jumps to the site where an ice particle or a particle in an intermediate state occupies or when an ice particle jumps to the site where a water particle occupies, the water particle dies while the label of the ice particle or the particle in an intermediate state is increased by 1. An ice particle at site $x$ can jump with rates $c^-_{x,y}(\eta^{(-\ell)})$ to a vacant site or to the site $y$ where a particle in an intermediate state occupies. In this case, the label becomes $-\ell+1$ at site $x$ and the label at $y$ decreases by 1. Let $\eta^N(t)=\{\eta^N_x(t);x\in\Gamma_N\}$, $t\geqslant 0$ be the Markov process on $\mathcal{X}_N$ with an infinitesimal generator $N^2L_N$. The macroscopic empirical-mass distribution of $\eta^N(t)$

is similarly defined by (2.5). Then, under the same condition on the initial distribution, we have the following theorem.

THEOREM 6.1. – *For every $t > 0$, $\alpha_t^N$ converges to $a(t,\theta)d\theta$ in probability. The limit function $a(t,\theta) \in [-\ell, 1]$ is a unique solution of Eq.* (2.6). *The function $P$ on $[-\ell, 1]$ is different from that in Theorem* 2.1 *and now defined by $P(a) = P^+(a)$ for $a \in [0,1]$, $P(a) = 0$ for $a \in [-\ell+1, 0]$ and $P(a) = -P^-(-a-\ell+1)$ for $a \in [-\ell, -\ell+1]$; note that $P^+(0) = P^-(0) = 0$ by the convention* (2.1) *for $h_i^{\pm}$.*

*Remark* 6.1. – With this definition of $P(a)$, Eq. (2.6) is the weak formulation of the following two-phase Stefan problem for the density $\rho^+(t,\theta) \in [0,1]$ of water and $\rho^-(t,\theta) \in [0,1]$ of ice:

$$\frac{\partial \rho^+}{\partial t} = \Delta P^+(\rho^+) \qquad \text{on } \mathcal{L}(t),$$

$$\frac{\partial \rho^-}{\partial t} = \Delta P^-(\rho^-) \qquad \text{on } \mathcal{S}(t),$$

$$\rho^{\pm}(t,\theta) = 0 \qquad \text{on } \Sigma(t) := \partial\mathcal{L}(t) = \partial\mathcal{S}(t),$$

$$(\ell - 1)V = -\mathbf{n}\cdot\left(\nabla P^+(\rho^+) - \nabla P^-(\rho^-)\right) \quad \text{on } \Sigma(t),$$

where $\mathbf{n}$ denotes the unit normal vector on $\Sigma(t)$ directed to $\mathcal{L}(t)$, $V$ is the velocity of $\Sigma(t)$ to the direction $\mathbf{n}$ and $\nabla P^+(\rho^+)$ (respectively $\nabla P^-(\rho^-)$) is the limit of the gradient of $P^+(\rho^+)$ (respectively $P^-(\rho^-)$) at $\theta \in \Sigma(t)$ when approached from $\mathcal{L}(t)$ (respectively $\mathcal{S}(t)$). The enthalpy function defined by $a(t,\theta) := \rho^+(t,\theta)$, $\theta \in \mathcal{L}(t)$, and $:= -\rho^-(t,\theta) - \ell + 1$, $\theta \in \mathcal{S}(t)$, gives the solution of (2.6).

*Outline of the proof of Theorem 6.1.* – The proof goes quite similarly to that of Theorem 2.1. So we shall only indicate the necessary modifications. In the present situation the formula (3.1) for $\langle \alpha_t^N, J\rangle$ holds with $b^N(\eta)$ and $\gamma^N(\eta)$ replaced as in the next lemma:

LEMMA 6.1. –

$$b^N(\eta) = N^{-d} \sum_{z\in\Gamma_N} \sum_{i=1}^{d} \tau_z\{h_i^+(\eta^+) - h_i^-(\eta^{(-\ell)})\}\Delta_i^N J(z/N), \tag{6.1}$$

$$\gamma^N(\eta) = \frac{1}{2}N^{2-2d} \sum_{z_1,z_2\in\Gamma_N:\ |z_1-z_2|=1} \{J(z_1/N) - J(z_2/N)\}^2 \times \left\{c_{z_1,z_2}^+(\eta^+)\left(\eta_{z_1}^+ - \eta_{z_2}^+\right)^2 + c_{z_1,z_2}^-(\eta^{(-\ell)})\left(\eta_{z_1}^{(-\ell)} - \eta_{z_2}^{(-\ell)}\right)^2\right\}. \tag{6.2}$$

*Proof.* – Since $\pi_{x;y}\eta_z \neq 0$ implies $x = z$ or $y = z$, we have

$$L_N\eta_z = \sum_{x:\,|x-z|=1} c^+_{x,z}(\eta^+)\{1_{\{\eta_z=1\}}\pi_{z;x}\eta_z + 1_{\{\eta_x=1\}}\pi_{x;z}\eta_z\}$$
$$+ \sum_{x:\,|x-z|=1} c^-_{x,z}(\eta^{(-\ell)})\{1_{\{\eta_z^{(-\ell)}=1\}}\pi_{z;x}\eta_z + 1_{\{\eta_x^{(-\ell)}=1\}}\pi_{x;z}\eta_z\}.$$

However, it is not difficult to see

$$\pi_{z;x}\eta_z = \eta_z^+(\eta_x^+ - 1) + \eta_z^{(-\ell)}(1 - \eta_x^{(-\ell)}),$$
$$\pi_{x;z}\eta_z = \eta_x^+(1 - \eta_z^+) + \eta_x^{(-\ell)}(\eta_z^{(-\ell)} - 1),$$

and these identities show

$$L_N\eta_z = \sum_{x:\,|x-z|=1} \{c^+_{x,z}(\eta^+)(\eta_x^+ - \eta_z^+)$$
$$- c^-_{x,z}(\eta^{(-\ell)})(\eta_x^{(-\ell)} - \eta_z^{(-\ell)})\}, \quad z \in \Gamma_N.$$

The equality (6.1) is proved from this formula by using the gradient conditions and then rearranging the sum. The formula (6.2) follows from the next lemma which is a replacement of Lemma 3.2. □

LEMMA 6.2. – (i) *If* $z_1 = z_2 = z$,

$$L_N\eta_z^2 = (1 - 2\eta_z^-) \sum_{x:\,|x-z|=1} c^+_{x,z}(\eta^+)(\eta_x^+ - \eta_z^+)$$
$$+ (1 - 2\eta_z - 2\eta_z^{(-\ell)}) \sum_{x:\,|x-z|=1} c^-_{x,z}(\eta^{(-\ell)})(\eta_x^{(-\ell)} - \eta_z^{(-\ell)}). \quad (6.3)$$

(ii) *If* $z_1 \neq z_2$,

$$L_N(\eta_{z_1}\eta_{z_2})$$
$$= \eta_{z_2} \sum_{x:\,|x-z_1|=1} \{c^+_{x,z_1}(\eta^+)(\eta_x^+ - \eta_{z_1}^+) - c^-_{x,z_1}(\eta^{(-\ell)})(\eta_x^{(-\ell)} - \eta_{z_1}^{(-\ell)})\}$$
$$+ \eta_{z_1} \sum_{x:\,|x-z_2|=1} \{c^+_{x,z_2}(\eta^+)(\eta_x^+ - \eta_{z_2}^+) - c^-_{x,z_2}(\eta^{(-\ell)})(\eta_x^{(-\ell)} - \eta_{z_2}^{(-\ell)})\}$$
$$- 1_{\{|z_1-z_2|=1\}}\{c^+_{z_1,z_2}(\eta^+)(\eta_{z_1}^+ - \eta_{z_2}^+)^2$$
$$+ c^-_{z_1,z_2}(\eta^{(-\ell)})(\eta_{z_1}^{(-\ell)} - \eta_{z_2}^{(-\ell)})^2\}. \quad (6.4)$$

*Proof.* – The assertion (i) follows from the following identities:

$$\pi_{z;x}\eta_z^2 = \eta_z^+(\eta_x^+ - 1) + \eta_z^{(-\ell)}(2\ell - 1)(\eta_x^{(-\ell)} - 1),$$
$$\pi_{x;z}\eta_z^2 = \eta_x^+(1 - \eta_z^+ - 2\eta_z^-) + \eta_x^{(-\ell)}(1 - 2\eta_z - (2\ell + 1)\eta_z^{(-\ell)}).$$

To prove the assertion (ii), we use these identities and

$$\pi_{z_1;z_2}(\eta_{z_1}\eta_{z_2}) = \eta^+_{z_1}(\eta^+_{z_2} - \eta_{z_2}) + \eta^{(-\ell)}_{z_1}(\ell - 1 + \eta_{z_2} + \eta^{(-\ell)}_{z_2}). \qquad \square$$

Set $\nu_a = \nu^+_a$ ($a \in [0,1]$) and $\nu_a = \nu^-_{-a-\ell+1} \circ \mathrm{r}_\ell^{-1}$ ($a \in [-\ell, -\ell+1]$), where $\mathrm{r}_\ell : \mathcal{X}^+ \to \mathcal{X}$ is a map defined by $(\mathrm{r}_\ell\eta)_x = -\eta_x - \ell + 1$, $x \in \mathbb{Z}^d$. For $a \in [-\ell, 1]$, we denote $E^{\nu_a}[f]$ by $\langle f\rangle(a)$ ($a \in [-\ell, -\ell+1] \cup [0,1]$) for $f \in \mathcal{D}$, the class of all local functions on $\mathcal{X}$. Just for simplifying the notation, we set $E^{\nu_a}[f] = \langle f\rangle(a) = 0$ for $a \in (-\ell+1, 0)$. Then, Theorem 4.1 (the local ergodic theorem) can be replaced with

LEMMA 6.3. – *For every $f \in \mathcal{D}$,*

$$\lim_{K\to\infty} \limsup_{N\to\infty} E^{\tilde{\mu}^N}\left[\left|\bar{f}_{0,K}(\eta) - \langle f\rangle(\overline{\eta}_{0,K})\right|^2 \cdot 1_{\mathbb{A}_K}\right] = 0, \tag{6.5}$$

*where $\mathbb{A}_K = \{\eta \in \mathcal{X};\, \eta_x = 1$ for some $x \in \Lambda_K\}$ or $\mathbb{A}_K = \{\eta \in \mathcal{X};\, \eta_x = -\ell$ for some $x \in \Lambda_K\}$.*

For the proof of this lemma, note that three types of configurations 1, $\{-\ell+2, \ldots, -1\}$ and $-\ell$ cannot coexist in the support of translation-invariant stationary measures for the corresponding dynamics in infinite region.

The formula (5.2) in Lemma 5.1 remains true, if we change $h_i(\eta^+)$ with $h^+_i(\eta^+) - h^-_i(\eta^{(-\ell)})$ in the definition of $\Psi^N_1(\eta; G)$ and $c_{z,z+e_i}(\eta^+) \times (\eta^+_z - \eta^+_{z+e_i})^2$ with $c^+_{z,z+e_i}(\eta^+)(\eta^+_z - \eta^+_{z+e_i})^2 + c^-_{z,z+e_i}(\eta^{(-\ell)})(\eta^{(-\ell)}_z - \eta^{(-\ell)}_{z+e_i})^2$ in that of $\Psi^N_2(\eta; G)$, respectively. Then, Theorem 5.1 holds exactly in the same form; note that the definition of the function $P$ is now given as in the statement of Theorem 6.1 and different from that in Theorem 5.1. Its proof goes quite similarly; $h_i(\eta^+)$ should be replaced with $h^+_i(\eta^+) - h^-_i(\eta^{(-\ell)})$ as indicated above. Note that

$$E^{\nu_a}\left[h^+_i(\eta^+) - h^-_i(\eta^{(-\ell)})\right] = P(a), \quad a \in [-\ell, 1].$$

After these preparations, the proof of Theorem 6.1 can be concluded. $\square$

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## REFERENCES

[1] M. BRAMSON and J.L. LEBOWITZ, Asymptotic behavior of densities for two-particle annihilating random walks, *J. Statis. Phys.* 62 (1991) 297–372.

[2] G. CAGINALP, An analysis of a phase field model of a free boundary, *Arch. Rat. Mech. Anal.* 92 (1986) 205–245.

[3] L. CHAYES and G. SWINDLE, Hydrodynamic limits for one-dimensional particle systems with moving boundaries, *Ann. Probab.* 24 (1996) 559–598.

[4] W.H. FLEMING and M. VIOT, Some measure-valued Markov processes in population genetics theory, *Indiana Univ. Math. J.* 28 (1979) 817–943.

[5] A. FRIEDMAN, *Variational Principles and Free-Boundary Problems*, Wiley, New York, 1982.

[6] T. FUNAKI, K. HANDA and K. UCHIYAMA, Hydrodynamic limit of one-dimensional exclusion processes with speed change, *Ann. Probab.* 19 (1991) 245–265.

[7] T. FUNAKI, K. UCHIYAMA and H.T. YAU, Hydrodynamic limit for lattice gas reversible under Bernoulli measures, in: T. Funaki and W.A. Woyczynski (Eds.), *Nonlinear Stochastic PDE's: Hydrodynamic Limit and Burgers' Turbulence*, IMA, Vol. 77, Univ. Minnesota, Springer, 1995, pp. 1–40.

[8] T. FUNAKI and H. SPOHN, Motion by mean curvature from the Ginzburg–Landau $\nabla\phi$-interface model, *Commun. Math. Phys.* 185 (1997) 1–36.

[9] H.O. GEORGII, *Canonical Gibbs Measures*, Lect. Notes in Math., Vol. 760, Springer, 1979.

[10] M.Z. GUO, G.C. PAPANICOLAOU and S.R.S. VARADHAN, Nonlinear diffusion limit for a system with nearest neighbor interactions, *Commun. Math. Phys.* 118 (1988) 31–59.

[11] C. LANDIM, S. OLLA and S. VOLCHAN, Driven tracer particle in one dimensional symmetric simple exclusion, *Commun. Math. Phys.* 192 (1998) 287–307.

[12] F. REZAKHANLOU, Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$, *Commun. Math. Phys.* 140 (1991) 417–448.

[13] J.-F. RODRIGUES, Variational methods in the Stefan problem, in: A. Visintin (Ed.), *Phase Transitions and Hysteresis*, Lect. Notes in Math., Vol. 1584, Springer, 1994, pp. 147–212.

[14] L.I. RUBINSTEIN, *The Stefan Problem*, Amer. Math. Soc. Transl. Monogr. 27, Providence, 1971.

[15] A.A. SAMARSKII and E.S. NIKOLAEV, *Numerical Methods for Grid Equations*, Vol. I, Birkhäuser, 1989.

[16] H. SPOHN, *Large Scale Dynamics of Interacting Particles*, Springer, 1991.

[17] H. SPOHN, Interface motion in models with stochastic dynamics, *J. Statis. Phys.* 71 (1993) 1081–1132.

[18] Y. SUZUKI and K. UCHIYAMA, Hydrodynamic limit for a spin system on a multidimensional lattice, *Probab. Theory Related Fields* 95 (1993) 47–74.

[19] K. UCHIYAMA, Scaling limits of interacting diffusions with arbitrary initial distributions, *Probab. Theory Related Fields* 99 (1994) 97–110.

[20] S.R.S. VARADHAN, Scaling limits for interacting diffusions, *Commun. Math. Phys.* 135 (1991) 313–353.