Annales de l'I. H. P., section B

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Annales de l'I. H. P., section B, tome 34, n° 5 (1998), p. 687-704 http://www.numdam.org/item?id=AIHPB 1998 34 5 687 0>

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Large and moderate deviations for the local time of a recurrent Markov chain on \mathbb{Z}^2

by

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ABSTRACT. – Let (X_n) be a recurrent Markov chain on \mathbb{Z}^2 with $X_0=(0,0)$ such that for some constant C, $P[X_k=(0,0)]\leq \frac{C}{k}$, and whose truncated Green function is slowly varying at infinity. Let L_n^0 denote the local time at zero of such a Markov chain. We prove various moderate and large deviation statements and limit laws for rescaled versions of L_n^0 , including functional versions of these. A version of Strassen's functional law of the iterated logarithm, recently discovered by E. Csáki, P. Révész and J. Rosen, can be derived as a corollary. © Elsevier, Paris

Key words: Local time, Markov chain, large deviations, Strassen's law.

RÉSUMÉ. – Soit (X_n) une chaîne de Markov récurrente sur \mathbb{Z}^2 , avec $X_0=(0,0)$, telle que pour une constante C, $P[X_k=(0,0)] \leq \frac{C}{k}$, et telle que la fonction de Green est de variation lente à l'infini. Avec L_n^0 le temps local de (X_n) à zero, nous démontrons des résultats de grandes déviations

AMS Classifications: 60J10, 60J55, 60F10.

^{*} This research was carried out while visiting the Department of Electrical Engineering, Technion Haifa, and was supported by the Swiss National Science foundation under grant 8220–046518.

[†] Partially supported by a US-Israel BSF grant and by grant NCR 94-22513.

et de déviations modérées pour certains changements d'échelle de L_n^0 , ainsi qu'une version fonctionnelle. Comme corollaire, on note un théorème du logarithme itéré fonctionnel de type Strassen, demontré récemment par E. Csáki, P. Révész, et J. Rosen. © Elsevier, Paris

1. INTRODUCTION AND STATEMENT OF RESULTS

Let (X_n) be a recurrent Markov chain on \mathbb{Z}^2 with $X_0=(0,0)$, and let $g(n):=\sum_{k=0}^n P[X_k=(0,0)]$ be the truncated Green function. We can extend g to a continuous, increasing function $g(t), t \geq 0$. Since (X_n) is recurrent, $g(t) \to \infty$ for $t \to \infty$.

We will assume throughout that, for some positive constant C,

$$P[X_k = (0,0)] \le \frac{C}{k}$$
, (1)

hence $g(n) \leq C \log n$. We will also assume throughout that

$$g$$
 is slowly varying at ∞ , (2)

that is $g(tx)/g(t) \underset{t \to \infty}{\longrightarrow} 1$ for any x > 0. Note that (1) is satisfied for symmetric random walks on \mathbb{Z}^2 , i.e. if $P[X_1 = (y,z)] = P[X_1 = -(y,z)]$, see [6], Proposition 2.14. Since our results depend only on (1) and (2), they might also apply to symmetric recurrent random walks on \mathbb{Z} in the domain of attraction of a Cauchy random variable.

We denote by L_n^0 the local time of X at (0,0), i.e. $L_n^0:=|\{0 \le k \le n: X_k=(0,0)\}|$, and $L_0^0=0$. Let $\rho_0=0$, $\rho_k=\min\{j: j>\rho_{k-1}, X_j=(0,0)\}$, $k=1,2,3,\cdots$. It is known, see [6], (and will follow from the proof of Theorem 1), that $L_n^0/g(n)$ converges in distribution to an exponential distribution, i.e.

$$P\left[\frac{L_n^0}{g(n)} \ge y\right] \xrightarrow[n \to \infty]{} e^{-y} \text{ for } y \ge 0.$$
 (3)

Our goal is to investigate the fluctuations of L_n^0 , and associated functional laws.

Theorem 1 (Moderate Deviations). – Let $\psi(n)$ be a positive, non-decreasing function such that

$$\gamma_n := \frac{n}{\psi(n)g(n)} \xrightarrow[n \to \infty]{} \infty.$$

Then $L_n^0/g(n)\psi(n)$ satisfies a large deviation principle with speed $\psi(n)g(n)/g(\gamma_n)$ and rate function y.

We refer to [2] for the definition of a large deviation principle. Here, it will be enough to show that

$$\frac{g(\gamma_n)}{\psi(n)g(n)}\log P\left[\frac{L_n^0}{g(n)\psi(n)} \ge y\right] \xrightarrow[n\to\infty]{} -y. \tag{4}$$

Theorem 1 is a moderate deviation principle since the speed can vary without changing the rate function. Further, the rate function does not depend on the distribution of ρ_1 .

The next theorem gives a large deviation principle for the distributions of L_n^0/n , with rate function which does depend on the distribution of ρ_1 .

Theorem 2 (Large Deviations). – Let $\Lambda^*(y) = \sup_{\lambda \leq 0} (\lambda y - \log E[e^{\lambda \rho_1}])$ and

$$J(y) = \begin{cases} y \Lambda^* \left(\frac{1}{y}\right), & 0 < y \le 1 \\ \\ 0, & y = 0 \\ \\ +\infty, & \textit{otherwise} \end{cases}$$

Then the distributions of L_n^0/n satisfy a LDP with speed n and rate function J.

Remarks

- 1. Comparing with Theorem 1, the large deviation principle holds for $\psi(n) = \frac{n}{g(n)}$. In this case, $\gamma_n = 1$ and Theorem 1 does not apply. Considering the proof of Theorem 2, it is easy to show that we have a LDP whenever $\gamma_n \underset{n \to \infty}{\longrightarrow} \alpha$, $0 < \alpha < 1$.
- 2. Let $p_0 := P[X_1 = (0,0)]$. Then we have $J(1) = -\log p_0$ if $p_0 > 0$ and $J(1) = \infty$ otherwise.
- 3. Let $L^0(\cdot)$ be the linear interpolation of L^0 between integer points. We believe (but have not checked the details) that the standard argument (see e.g. [2], Section 5.1) allows one to conclude that the distributions of $(\frac{L^0(nt)}{n})_{0 \le t \le 1}$ satisfy a large deviation principle (in C[0,1]) with

rate function

$$\tilde{J}(f) = \begin{cases} \int_0^1 J(f'(s))ds, & f \text{ absolutely continuous with derivative } f' \\ +\infty, & \text{otherwise.} \end{cases}$$

As usual, we can derive an Erdös-Renyi law from the large deviation principle:

Corollary 1. – Let c>0 and $\eta_{n,j}:=\frac{1}{c\log g(n)}(L^0_{j+\lfloor c\log g(n)\rfloor}-L^0_j),\ j=0,1,2,\cdots,n-\lfloor c\log g(n)\rfloor.$ Then $\lim_{n\to\infty}\sup_{j=0,1,\cdots,n-\lfloor c\log g(n)\rfloor}\eta_{n,j}=d_c$, a.s., where $d_c=\inf\big\{y:\ J(y)\geq \frac{1}{c}\big\}.$

For a random walk on \mathbb{Z} , this complements results of [5].

We next turn to the appropriate functional statements. Let $\psi(n)$ and γ_n be as in the statement of Theorem 1, and let t(n,x) be a sequence of positive, increasing (in n,x) functions satisfying, for any $x \in]0,1]$,

$$\lim_{n \to \infty} \frac{g(\frac{t(n,x)}{g(n)\psi(n)})}{g(\gamma_n)} = x > 0.$$
 (5)

For example, if $g(n) \sim C \log n$, and $\frac{\log \psi(n)}{\log n} \underset{n \to \infty}{\longrightarrow} 0$, we can take $t(n,x) = n^x$. If $g(n) \sim C \log n$ and $\psi(n) = n^\beta$, $(0 < \beta < 1)$, we can take $t(n,x) = n^{x(1-\beta)+\beta}$. If $g(n) \sim C \log_2 n$ and $\frac{\log \psi(n)}{\log n} \underset{n \to \infty}{\longrightarrow} 0$, we can take $t(n,x) = e^{(\log n)^x}$ (here and throughout, $\log_k n$ denotes the k-th iterated logarithm function). If $g(n) \sim C \log_2 n$ and $\psi(n) = n^\beta$, $(0 < \beta < 1)$, we can take $t(n,x) = n^\beta e^{(\log n)^x}$.

It is straightforward to check, using (5), that for $0 \le x_1 < x_2 \le 1$, we have

$$\frac{t(n,x_1)}{t(n,x_2)} \xrightarrow[n \to \infty]{} 0. \tag{6}$$

Let

$$\overline{L}_n(x) := \frac{L_{t(n,x)}^o}{g(n)\psi(n)}, \quad x \in [0,1].$$

Note that $\overline{L}_n(x) \in M_+$, the space of non-negative Borel measures on [0,1]. Equip M_+ with the topology of weak convergence. Our main functional statement is the following:

Theorem 3 (Functional Moderate Deviations). – $\overline{L}_n(x)$ satisfies in M_+ a large deviation principle with speed $g(n)\psi(n)/g(\gamma_n)$ and rate function

$$I(m) = \begin{cases} \int_0^1 \frac{1}{x} m(dx), & \frac{1}{x} \in L_1(m) \\ & \infty, & \text{otherwise.} \end{cases}$$

As in the one-dimensional case, we can deduce convergence in distribution from our large deviation bounds, taking $\psi(n) \equiv 1$.

Theorem 4 (Functional Limit Law). – Let t(n,x) be such that $g(t(n,x)) \sim xg(n)$, $x \in [0,1]$. The distributions of $(\frac{L_{t(n,x)}^0}{g(n)})_{0 \leq x \leq 1}$ converge weakly to $\mu \in M_1(M_+)$, the distribution of the process $(Z_x)_{0 \leq x \leq 1}$ with increasing paths and independent increments given by

$$P[Z_{x_2} - Z_{x_1} \in B] = \frac{x_1}{x_2} \delta_o(B) + \left(1 - \frac{x_1}{x_2}\right) \int_B \frac{1}{x_2} e^{-\frac{1}{x_2}u} du, \quad (7)$$

for any $0 \le x_1 < x_2 \le 1$, B Borel subset of $[0, \infty[$.

J. Bertoin kindly pointed out to us that in fact the process $(Z_x)_{0 \leq x \leq 1}$ in Theorem 4 is a pure jump process which can be constructed from an inhomogeneous Poisson point process. Indeed, one may construct a Poisson point process N(x,z) on $[0,1] \times \mathbb{R}_+$ with intensity $n(x,z)dxdz = x^{-2}\exp(-z/x)dxdz$ and define $Y_x = \int_0^\infty z d_z N(x,z)$. Obviously, $(Y_x)_{0 \leq x \leq 1}$ possesses increasing paths and independent increments. Moreover, it is not hard to check, using the identity valid for any $\alpha, \beta > 0$,

$$\lim_{\epsilon \to 0} \left(\int_{\epsilon}^{\infty} \frac{e^{-\alpha z}}{z} dz - \int_{\epsilon}^{\infty} \frac{e^{-\beta z}}{z} dz \right) = \log \beta - \log \alpha ,$$

that for any $\lambda \geq 0$,

$$E(\exp\left(-\lambda(Y_{x+y}-Y_x)\right)) = \frac{1+\lambda x}{1+\lambda(x+y)} = E(\exp\left(-\lambda(Z_{x+y}-Z_x)\right)),$$

proving that the processes $(Z_x)_{0 \le x \le 1}$ and $(Y_x)_{0 \le x \le 1}$ have the same law.

We close this section by mentioning that the functional moderate deviations of Theorem 3 are strong enough to derive by standard arguments the following Strassen law of the iterated logarithm presented in [1], Theorem 5. Obtaining such a derivation was actually the original motivation

for this work. Since the arguments are standard, see [3], Theorem 1.4.1, we do not provide a proof.

Theorem 5 (E. Csáki, P. Révész and J. Rosen). — Let t(n,x) be such that $g(t(n,x)) \sim xg(n)$, $x \in [0,1]$. The set $(\frac{L^0_{t(n,x)}}{g(n)\log_2 g(n)})_{0 \leq x \leq 1}$, n large enough, is relatively compact in M_+ with limit points K, where $K = \{m : I(m) \leq 1\}$.

2. PROOFS

We begin by stating some simple bounds on g(n).

Lemma 1. – We have

$$\lim_{n \to \infty} \frac{g(n)}{g(ng(n))} = 1, \tag{8}$$

and

$$\lim_{n \to \infty} \frac{g(n)}{g(n/g(n))} = 1. \tag{9}$$

Proof of Lemma 1. - We have

$$g(ng(n)) - g(n) \le \sum_{j=n}^{\lceil ng(n) \rceil} P[X_j = (0,0)]$$

$$\le C \sum_{j=n}^{\lceil ng(n) \rceil} \frac{1}{j} \le C' \log g(n),$$

where C' is some (fixed, depending on C) constant. The limit (8) follows by dividing by g(ng(n)) and using the monotonicity of $g(\cdot)$. The proof of (9) is analogous.

Lemma 1 is needed for the following crucial estimate for the tail of the distribution of the excursion ρ_1 . For a more precise statement, which we do not need here, see [6].

Proposition 1

$$P[\rho_1 > n] \le \frac{1}{g(n)}$$

and

$$P[\rho_1 > n] \sim \frac{1}{g(n)}$$

i.e.
$$g(n)P[\rho_1 > n] \xrightarrow[n \to \infty]{} 1$$
.

Proof of Proposition 1

1. A last exit decomposition gives

$$\sum_{k=0}^{n} P[X_k = (0,0)] P[L_{n-k}^0 = 0] = 1.$$

Since $P[L^0_{n-k}=0]\geq P[L^0_n=0], k=0,1,...,n,$ this implies $g(n)P[L^0_n=0]\leq 1,$ hence

$$P[\rho_1 > n] = P[L_n^0 = 0] \le \frac{1}{q(n)}$$
.

2. In the same way,

$$1 \le \sum_{j=0}^{k} P[X_j = (0,0)] P[L_{n-k}^0 = 0] + \sum_{j=k+1}^{n} P[X_j = (0,0)]$$

hence $1 \le g(k)P[L_{n-k}^0 = 0] + g(n) - g(k)$, so

$$g(k)P[L_{n-k}^0 = 0] \ge 1 - (g(n) - g(k)).$$
 (10)

Choose $k = k(n) = \left| n - \frac{n}{g(n)} \right|$, and note that, for some C', C'' > 0,

$$g(n) - g(k) = \sum_{j=k}^{n} P[X_j = (0,0)] \le C \sum_{j=k}^{n} \frac{1}{j}$$

$$\le C'(\log n - \log k) \le C'' \log(1 - \frac{1}{g(n)}) \underset{n \to \infty}{\longrightarrow} 0.$$

This, together with (9) of Lemma 1, yields the proposition. \Box

Proof of Theorem 1. – We begin with a quick proof of the lower bound in (4). Let Y_1, Y_2, \ldots be i.i.d. with the same distribution as ρ_1 . Then

$$P[L_n^0 \ge \psi(n)g(n)y] \ge P\left[\sum_{i=1}^{\lceil g(n)\psi(n)y \rceil} Y_i \le n\right]$$

$$\ge P\left[\max_{1 \le i \le \lceil g(n)\psi(n)y \rceil} Y_i \le \frac{n}{\lceil g(n)\psi(n)y \rceil}\right]$$

$$= \left(1 - P\left[\rho_1 > \frac{n}{\lceil g(n)\psi(n)y \rceil}\right]\right)^{\lceil g(n)\psi(n)y \rceil}$$

Now apply Proposition 1 and the fact that $g(\cdot)$ is slowly varying to get

$$\liminf_{n \to \infty} \frac{g(\frac{n}{\psi(n)g(n)})}{g(n)\psi(n)} \log P[L_n^0 \ge \psi(n)g(n)y] \ge -y.$$

We next turn to the proof of the upper bound. We follow the standard strategy to apply Chebycheff's inequality and to optimize over the parameter. Due to Chebycheff's inequality,

$$P[L_n^0 \ge g(n)\psi(n)y] \le P\left[\sum_{i=1}^{\lfloor g(n)\psi(n)y\rfloor} Y_i \le n\right]$$

$$\le E\left[e^{-\lambda_n Y_1}\right]^{\lfloor g(n)\psi(n)y\rfloor} e^{\lambda_n n} \tag{11}$$

for each $\lambda_n>0$. Recall $\gamma_n=\frac{n}{\psi(n)g(n)}$. Taking logarithms and dividing by $\frac{g(n)\psi(n)}{g(\gamma_n)}$, (11) yields

$$\frac{g(\gamma_n)}{g(n)\psi(n)} \log P[L_n^0 \ge g(n)\psi(n)y]
\le g(\gamma_n)y \frac{\lfloor g(n)\psi(n)y \rfloor}{g(n)\psi(n)y} \log E[e^{-\lambda_n Y_1}] + \frac{g(\gamma_n)\lambda_n n}{\psi(n)g(n)}$$
(12)

Next we show that for each $\delta > 0$, and $C_n > 0$ large enough, we have

$$\log E[e^{-\lambda_n Y_1}] \le \frac{1-\delta}{g(C_n)} (e^{-\lambda_n C_n} - 1). \tag{13}$$

Indeed, observe that

$$\begin{split} \log E[e^{-\lambda_n Y_1}] &= \log E[e^{-\lambda_n \rho_1}] \le E[e^{-\lambda_n \rho_1}] - 1 \\ &\le e^{-\lambda_n C_n} P[\rho_1 \ge C_n] + P[\rho_1 < C_n] - 1 \\ &= P[\rho_1 \ge C_n] (e^{-\lambda_n C_n} - 1) \le \frac{1 - \delta}{g(C_n)} (e^{-\lambda_n C_n} - 1) \end{split}$$

where we used Proposition 1 in the last inequality.

Substituting this estimate in (12), we get

$$\frac{g(\gamma_n)}{\psi(n)g(n)} \log P[L_n^0 \ge g(n)\psi(n)y]
\le y(1-\delta)\frac{g(\gamma_n)}{g(C_n)} (e^{-\lambda_n C_n} - 1) + \frac{g(\gamma_n)\gamma_n}{C_n} \lambda_n C_n.$$
(14)

Choose $C_n = K\gamma_n g(\gamma_n)$, $\lambda_n = \frac{K'}{C_n}$ with K, K' > 0. Then the r.h.s. of (14) is

$$y(1-\delta)\frac{g(\gamma_n)}{g(K\gamma_n g(\gamma_n))}(e^{-K'}-1) + \frac{g(\gamma_n)}{Kg(\gamma_n)}K'.$$
 (15)

Due to Lemma 1 and the fact that $g(\cdot)$ is slowly varying, $\frac{g(\gamma_n)}{g(K\gamma_ng(\gamma_n))} \underset{n \to \infty}{\longrightarrow} 1$. Hence (14) and (15) yield

$$\limsup_{n\to\infty}\frac{g(\gamma_n)}{\psi(n)g(n)}\log P[L_n^0\geq g(n)\psi(n)y]\leq y(1-\delta)(e^{-K'}-1)+\frac{K'}{K}$$

and the upper bound follows by letting $\delta \to 0, K' \to \infty, \frac{K'}{K} \to 0.$

Remark. – In particular, taking in the proof of the upper and the lower bound $\psi(n) \equiv 1$, we have

$$\frac{g\left(\frac{n}{g(n)}\right)}{g(n)}\log P\left[\frac{L_n^0}{g(n)} \geq y\right] \underset{n \to \infty}{\longrightarrow} -y \,.$$

Together with (9) in Lemma 1, this implies that for $y \ge 0$,

$$P\left[\frac{L_n^0}{g(n)} \ge y\right] \xrightarrow[n \to \infty]{} e^{-y},$$

as noted in (3).

Proof of Theorem 2. – Note first that $P[L_n^0 \ge ny] = 0$ if y > 1. As in the proof of Theorem 1, we have

$$P\left[\sum_{i=1}^{\lceil ny\rceil}Y_i\leq n\right]\leq P[L_n^0\geq ny]\leq P\left[\sum_{i=1}^{\lfloor ny\rfloor}Y_i\leq n\right].$$

But

$$P\left[\sum_{i=1}^{\lfloor ny\rfloor} Y_i \le n\right] \le P\left[\frac{1}{\lfloor ny\rfloor} \sum_{i=1}^{\lfloor ny\rfloor} Y_i \le \frac{1}{y}\right]$$

so we ask about large deviations of the arithmetic mean of a sequence of i.i.d. random variables. Cramér's theorem (see [2], Theorem 2.2.3) implies that the distributions of $\frac{1}{\lfloor ny \rfloor} \sum_{i=1}^{\lfloor ny \rfloor} Y_i$ (or $\frac{1}{\lceil ny \rceil} \sum_{i=1}^{\lceil ny \rceil} Y_i$) satisfy a LDP with

speed $\lfloor ny \rfloor$ (or $\lceil ny \rceil$) and rate function Λ^* . Note that $Y_1 \geq 0$, $E[Y_1] = \infty$ hence $\Lambda^*(y) \to 0$ for $y \to \infty$. Since we have

$$\frac{1}{n} \log P \left[\frac{1}{\lfloor ny \rfloor} \sum_{i=1}^{\lfloor ny \rfloor} Y_i \le \frac{1}{y} \right] = \frac{\lfloor ny \rfloor}{n} \frac{1}{\lfloor ny \rfloor} \log P \left[\frac{1}{\lfloor ny \rfloor} \sum_{i=1}^{\lfloor ny \rfloor} Y_i \le \frac{1}{y} \right]$$

and
$$\frac{\lfloor ny \rfloor}{n} \xrightarrow[n \to \infty]{} y$$
, the claim follows.

In order to prove Corollary 1, we need the following preliminary proposition.

Proposition 2. — Let $\psi(n) \to 0$, $\psi(n)g(n) \to \infty$. Then, for each x > 0, $\frac{1}{\psi(n)}P[\frac{L_n^0}{g(n)\psi(n)} \le x] \underset{n \to \infty}{\longrightarrow} x$.

Proof of Proposition 2

1. We have

$$\begin{split} P\big[L_n^0 \leq g(n)\psi(n)x\big] &\leq P\left[\sum_{j=1}^{\lceil g(n)\psi(n)x\rceil} Y_j \geq n\right] \\ &\leq P\bigg[\max_{1\leq j\leq \lceil g(n)\psi(n)x\rceil} Y_j \geq \frac{n}{\lceil g(n)\psi(n)x\rceil}\bigg] \\ &= 1 - \left(1 - P\bigg[Y_1 \geq \frac{n}{\lceil g(n)\psi(n)x\rceil}\bigg]\right)^{\lceil g(n)\psi(n)x\rceil} \\ &\leq 1 - \left(1 - \frac{1}{g(\frac{n}{\lceil g(n)\psi(n)x\rceil})}\right)^{\lceil g(n)\psi(n)x\rceil} \end{split}$$

where we used Proposition 1 in the last inequality. Since $1-z \le -\log z$, the last term is

$$\leq -\lceil g(n)\psi(n)x\rceil\log\left(1-\frac{1}{g(\frac{n}{\lceil g(n)\psi(n)x\rceil})}\right).$$

Hence

$$\frac{1}{\psi(n)} P\left[\frac{L_n^0}{g(n)\psi(n)} \le x\right]$$

$$\le -\frac{\left[g(n)\psi(n)x\right]}{g(n)\psi(n)} \log\left(1 - \frac{1}{g(\frac{n}{\left[g(n)\psi(n)x\right]})}\right)^{g(n)}.$$
(16)

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Provided that

$$\frac{g(n)}{g(\frac{n}{g(n)\psi(n)})} \xrightarrow{n \to \infty} 1, \qquad (17)$$

(16) implies that

$$\limsup_{n \to \infty} \frac{1}{\psi(n)} P\left[\frac{L_n^0}{g(n)\psi(n)} \le x\right] \le x. \tag{18}$$

But (17) holds true since

$$g(n) \ge g(\frac{n}{g(n)\psi(n)}) \ge g(\frac{n}{g(n)})$$

and $\frac{g(n)}{g(n/g(n))} \xrightarrow[n \to \infty]{} 1$ due to Lemma 1.

2.

$$P[L_n^0 \le g(n)\psi(n)x] \ge P\left[\sum_{j=1}^{\lfloor g(n)\psi(n)x\rfloor} Y_j \ge n\right]$$

$$\ge P\left[\max_{1 \le j \le \lfloor g(n)\psi(n)x\rfloor} Y_j \ge n\right]$$

$$= 1 - (1 - P[Y_1 \ge n])^{\lfloor g(n)\psi(n)x\rfloor}.$$

Now we use the inequality $1-z \ge -z\log z \pmod z \pmod z = (1-P[Y_1 \ge n])^{\lfloor g(n)\psi(n)x\rfloor}$ to get

$$P[L_n^0 \le g(n)\psi(n)x] \ge -\frac{\lfloor g(n)\psi(n)x\rfloor}{g(n)x} \log(1 - P[Y_1 \ge n])^{g(n)x} \cdot (1 - P[Y_1 \ge n])^{\lfloor g(n)\psi(n)x\rfloor}. \tag{19}$$

Proposition 1 implies that

$$(1 - P[Y_1 \ge n])^{g(n)x} \xrightarrow[n \to \infty]{} e^{-x}$$

and therefore

$$(1 - P[Y_1 \ge n])^{\lfloor g(n)\psi(n)x\rfloor} \underset{n \to \infty}{\longrightarrow} 1.$$

We conclude from (19) that

$$\liminf_{n \to \infty} \frac{1}{\psi(n)} P \left[\frac{L_n^0}{g(n)\psi(n)} \ge x \right] \ge x.$$

Proof of Corollary 1

1. Let $d \in \mathbb{R}$, $J(d) > \frac{1}{c}$, choose $\delta > 0$ such that $J(d) - \delta > \frac{1}{c}$, and fix any d' > d. We show that

$$P[\sup_{j=0,1,\dots,n-\lfloor c\log g(n)\rfloor} \eta_{n,j} \ge d' \text{ for infinitely many } n] = 0. \quad (20)$$

Let $\psi(n)=(\log g(n))^{\gamma}$ where $\gamma>1$. Since we can take the sup in $\sup_{j=0,1,\cdots,n-\lfloor c\log g(n)\rfloor}\eta_{n,j}$ over those j with $X_j=(0,0)$ only, without changing the value, and since $\eta_{n,j}$ has the same distribution as $\eta_{n,0}$ for those j, we have

$$P[\sup_{j=0,1,\dots,n-\lfloor c\log g(n)\rfloor} \eta_{n,j} \ge d]$$

$$\le P[L_n^0 \ge g(n)\psi(n)] + \psi(n)g(n)P[\eta_{n,0} \ge d].$$
 (21)

Now we have to estimate the terms on the r.h.s. of (21):

$$P[L_n^0 \ge g(n)\psi(n)] \le e^{-\psi(n)(1-\delta)}$$
 (22)

for n big enough, due to Theorem 1 and

$$P[\eta_{n,0} \ge d] \le e^{-c\log g(n)(J(d)-\delta)} \tag{23}$$

for n big enough, due to Theorem 2.

Let $\lambda > 1$, $n_0 = 0$ and $n_k = \lceil g^{-1}(\lambda^k) \rceil$, $k = 1, 2, \cdots$. Then we see from (22) and (23), applying the Borel-Cantelli lemma, that

$$P[\sup_{j=0,1,\cdots,n_k-\lfloor c\log g(n_k)\rfloor}\eta_{n_k,j}\geq d \text{ for infinitely many } k]=0\,.$$

In other words, we have proved (20) along the subsequence (n_k) with d replacing d'. Let $n_k \leq n \leq n_{k+1}$ and observe that, for $j=0,1,\cdots,n-\lfloor c\log g(n) \rfloor$,

$$\eta_{n,j} \le \eta_{n_{k+1,j}} \frac{\log g(n_{k+1})}{\log g(n)} \le \eta_{n_{k+1,j}} \frac{\log g(n_{k+1})}{\log g(n_k)}$$

$$\le \eta_{n_{k+1,j}} \frac{k+1}{k}$$

For k big enough, $\eta_{n_{k+1,j}} < d$ implies $\eta_{n,j} < d'$. This completes the proof of (20).

2. Let $d \in \mathbb{R}$, $J(d) < \frac{1}{c}$. Choose $\delta > 0$ and $\lambda > 1$ such that $\lambda(J(d) + \delta) < \frac{1}{c}$. We will construct a subsequence n_k such that

$$P[\sup_{0 \le j \le n_k - \lfloor c \log g(n_k) \rfloor} \eta_{n_k,j} < d \text{ for infinitely many } \mathbf{k}] = 0. \quad (24)$$

Fixing n, let $j_0^n:=0$, $j_m^n:=\inf\{j:j>j_{m-1}^n+\lfloor c\log g(n)\rfloor, X_j=\{0,0\}\}$, $M^n:=M^n(\omega)=\max\{m:j_m^n\leq n\}$ and $J^n:=\{j_0^n,\ldots,j_{M^n-1}^n\}$. Then $(\eta_{n,j})_{j\in J^n}$ are i.i.d. with the same distribution as $\eta_{n,0}$. Let $\psi(n)$, to be determined below, satisfy the assumptions of Proposition 2. We have

$$P\left[\sup_{0 \le j \le n - \lfloor c \log g(n) \rfloor} \eta_{n,j} < d\right]$$

$$\le P\left[M^n < \frac{\lfloor g(n)\psi(n) \rfloor}{\lfloor c \log g(n) \rfloor}\right] + P\left[\eta_{n,0} < d\right]^{\frac{\lfloor g(n)\psi(n) \rfloor}{\lfloor c \log g(n) \rfloor}}. (25)$$

But, for each $\delta > 0$, and all n large enough,

$$P\left[M^{n} < \frac{\lfloor g(n)\psi(n)\rfloor}{\vert c\log g(n)\vert}\right] \le P\left[L_{n}^{0} < \lfloor g(n)\psi(n)\rfloor\right] \le (1+\tilde{\delta})\psi(n) \quad (26)$$

for n large enough, where we used Proposition 2 in the last inequality. Turning now to the second term in (25), we first note that, by Theorem 2, for all n large enough,

$$P[\eta_{n,0} \ge d] \ge e^{-c \log g(n)(J(d) + \delta)} \ge e^{-\beta \log g(n)}$$

for n large enough, where $\beta:=c(J(d)+\delta)<1.$ Hence

$$P[\eta_{n,0} < d]^{\frac{\lfloor g(n)\psi(n)\rfloor}{\lfloor c\log g(n)\rfloor}} \le (1 - e^{-\beta\log g(n)})^{\frac{\lfloor g(n)\psi(n)\rfloor}{\lfloor c\log g(n)\rfloor}}$$

$$\le e^{-\frac{(1-\delta)\psi(n)g(n)^{1-\beta}}{c\log g(n)}}$$
(27)

for n large enough. Considering (26) and (27), it remains to specify a subsequence (n_k) and a positive function $\psi(\cdot)$ such that $\psi(n) \underset{n \to \infty}{\longrightarrow} 0$, $\psi(n)g(n) \underset{n \to \infty}{\longrightarrow} \infty$ and

$$\sum_{k} \psi(n_k) < \infty \tag{28}$$

$$\sum_{k} e^{-\frac{(1-\delta)\psi(n_k)g(n_k)^{1-\beta}}{c\log g(n_k)}} < \infty \tag{29}$$

Then, (24) follows from (25), (26) and (27) together with the Borel-Cantelli lemma. We finish the proof by observing that (28) and (29)

are satisfied for
$$n_k = g^{-1}(2^k)$$
 and $\psi(n) = \log g(n)/g(n)^{\gamma}$ where $0 < \gamma < 1 - \beta$.

Proof of Theorem 3. – We begin by proving a finite distribution result, from which the required LDP will follow by standard projective limits arguments. Note first that for $0=x_0< x_1< x_2< \cdots < x_k \leq 1$, and $0=a_0\leq a_1\leq a_2\leq \cdots \leq a_k<\infty$, and with Y_i as in the proof of Theorem 1,

$$P[\overline{L}_{n}(x_{1}) \geq a_{1}, \overline{L}_{n}(x_{2}) \geq a_{2}, \cdots, \overline{L}_{n}(x_{k}) \geq a_{k}]$$

$$\leq P\left[\sum_{i=1}^{\lfloor g(n)\psi(n)a_{1}\rfloor} Y_{i} \leq t(n, x_{1}), \cdots, \sum_{i=1}^{\lfloor g(n)\psi(n)a_{k}\rfloor} Y_{i} \leq t(n, x_{k})\right]$$

$$\leq P\left[\sum_{i=1}^{\lfloor g(n)\psi(n)a_{1}\rfloor} Y_{i} \leq t(n, x_{1}), \sum_{i=\lfloor g(n)\psi(n)a_{2}\rfloor}^{\lfloor g(n)\psi(n)a_{2}\rfloor} Y_{i} \leq t(n, x_{2}), \cdots, \sum_{i=\lfloor g(n)\psi(n)a_{k}\rfloor}^{\lfloor g(n)\psi(n)a_{k}\rfloor} Y_{i} \leq t(n, x_{k})\right]$$

$$= \prod_{j=1}^{k} P\left[\sum_{i=\lfloor g(n)\psi(n)a_{j-1}\rfloor+1}^{\lfloor g(n)\psi(n)a_{j}\rfloor} Y_{i} \leq t(n, x_{j})\right].$$

Write $g(n)\psi(n)=g\Big(t(n,x_j)\Big)\overline{\psi}_j\Big(t(n,x_j)\Big)$, then for any $\delta>0$ and n large enough,

$$P\left[\overline{L}_{n}(x_{1}) \geq a_{1}, \cdots, \overline{L}_{n}(x_{k}) \geq a_{k}\right]$$

$$\leq \prod_{j=1}^{k} P\left[\sum_{i=\lfloor g(t(n,x_{j}))\overline{\psi}_{j}(t(n,x_{j}))a_{j}\rfloor}^{\lfloor g(t(n,x_{j}))\overline{\psi}_{j}(t(n,x_{j}))a_{j}\rfloor} Y_{i} \leq t(n,x_{j})\right]$$

$$\leq \prod_{j=1}^{k} P\left[\sum_{i=1}^{\lfloor g(t(n,x_{j}))\overline{\psi}_{j}(t(n,x_{j}))(a_{j}-a_{j-1})\rfloor-1} Y_{i} \leq t(n,x_{j})\right]$$

$$\leq \prod_{j=1}^{k} \exp\left(-(a_{j}-a_{j-1})\frac{\overline{\psi}_{j}(t(n,x_{j}))g(t(n,x_{j}))}{g(\frac{t(n,x_{j})}{\overline{\psi}_{j}(t(n,x_{j}))g(t(n,x_{j}))})}(1-\delta)\right)$$

$$= \prod_{j=1}^{k} \exp\left(-(a_{j}-a_{j-1})\frac{\psi(n)g(n)}{g(\frac{t(n,x_{j})}{\overline{\psi}(n)g(n)})}(1-\delta)\right)$$

where the last inequality holds for n large enough and follows from the proof of the upper bound in Theorem 1. Therefore, using the assumption (5),

$$\limsup_{n \to \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P[\overline{L}_n(x_1) \ge a_1, \dots, \overline{L}_n(x_k) \ge a_k]$$

$$\le -\sum_{j=1}^k (a_j - a_{j-1}) \frac{(1-\delta)}{x_j}.$$

Taking now $\delta \rightarrow 0$ yields

$$\limsup_{n \to \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P[\overline{L}_n(x_1) \ge a_1, \cdots, \overline{L}_n(x_k) \ge a_k]$$

$$\le -\sum_{j=1}^k \frac{(a_j - a_{j-1})}{x_j},$$
(30)

proving a finite dimensional upper bound.

We next turn to a complementary lower bound. We first show that

$$\liminf_{n \to \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P[\overline{L}_n(x_1) \ge a_1, \cdots, \overline{L}_n(x_k) \ge a_k]$$

$$\ge -\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}.$$
(31)

Indeed, assume w.l.o.g. $a_{j-1} < a_j, j=1,2,\cdots,k$. We have, setting $\varphi_{n,j}:=\lceil g(n)\psi(n)a_j \rceil$,

$$P\left[\frac{L_{t(n,x_{j})}^{o}}{g(n)\psi(n)} \geq a_{j}, j = 1, 2, \cdots, k\right]$$

$$\geq P\left[\sum_{i=1}^{\varphi_{n,1}} Y_{i} \leq t(n, x_{1}), \sum_{i=\varphi_{n,1}+1}^{\varphi_{n,2}} Y_{i} \leq t(n, x_{2}) - t(n, x_{1}), \cdots\right]$$

$$\sum_{i=\varphi_{n,k-1}+1}^{\varphi_{n,k}} Y_{i} \leq t(n, x_{k}) - t(n, x_{k-1})\right]$$

$$\geq \prod_{j=1}^{k} P\left[\sum_{\varphi_{n,j-1}+1}^{\varphi_{n,j}} Y_{i} \leq t(n, x_{j}) - t(n, x_{j-1})\right]. \tag{32}$$

Observe that for $j = 1, 2, \dots, n$

$$P\left[\sum_{i=\varphi_{n,j-1}+1}^{\varphi_{n,j}} Y_{i} \leq t(n,x_{j}) - t(n,x_{j-1})\right]$$

$$\geq P\left[\max_{\varphi_{n,j-1}+1 \leq i \leq \varphi_{n,j}} Y_{i} \leq \frac{t(n,x_{j}) - t(n,x_{j-1})}{\varphi_{n,j} - \varphi_{n,j-1} - 1}\right]$$

$$\geq P\left[\max_{\varphi_{n,j-1}+1 \leq i \leq \varphi_{n,j}} Y_{i} \leq \frac{t(n,x_{j}) - t(n,x_{j-1})}{\varphi_{n,j}}\right]$$

$$\geq \left(1 - \frac{1}{g(\frac{t(n,x_{j}) - t(n,x_{j-1})}{\varphi_{n,j}})}\right)^{\varphi_{n,j} - \varphi_{n,j-1} - 1}$$

$$(33)$$

where the last inequality is due to Proposition 1. Note that due to (5) and (6),

$$\frac{g(\gamma_n)}{g(\frac{t(n,x_j)-t(n,x_{j-1})}{\lceil g(n)\psi(n)a_i \rceil})} \xrightarrow[n \to \infty]{} \frac{1}{x_j}$$
(34)

(31) now follows from (32), (33) and (34).

In the second step, we prove that, for $0 < \delta < \min\{a_j - a_{j-1}, j = 1, 2, \dots k\}$ we have

$$\liminf_{n \to \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \times \log P[\overline{L}_n(x_1) \in (a_1 - \delta, a_1 + \delta), \dots, \overline{L}_n(x_k) \in (a_k - \delta, a_k + \delta)]$$

$$\geq -\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}.$$
(35)

To prove (35), observe that

$$P\left[\frac{L_{t(n,x_j)}^o}{\psi(n)g(n)} \in (a_j - \delta, a_j + \delta), j = 1, 2, \cdots, k\right]$$

$$\geq P\left[\frac{L_{t(n,x_j)}^o}{g(n)\psi(n)} \geq a_j - \delta, j = 1, 2, \cdots, k\right]$$

$$-\sum_{\ell=1}^k P\left[\frac{L_{t(n,x_j)}^o}{g(n)\psi(n)} \geq a_j - \delta, j \neq \ell, \frac{L_{t(n,x_\ell)}^o}{g(n)\psi(n)} \geq a_\ell + \delta\right].$$

Since

$$\liminf_{n \to \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P\left[\frac{L_{t(n,x_j)}^o}{g(n)\psi(n)} \ge a_j - \delta, j = 1, 2, \dots, k\right] \\
\ge -\frac{a_1 - \delta}{x_1} - \sum_{j=2}^k \frac{a_j - a_{j-1}}{x_j} \ge -\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}$$

due to the first step, it is enough to show that for $\ell = 1, 2, \dots, k$ we have

$$\limsup_{n \to \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P\left[\frac{L_{t(n,x_j)}^o}{g(n)\psi(n)} \ge a_j - \delta, j \ne \ell, \frac{L_{t(n,x_\ell)}^o}{g(n)\psi(n)} \ge a_\ell + \delta\right]$$

$$< -\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}.$$

But, using the upper bound (30), we have

$$\limsup_{n \to \infty} \frac{g(\gamma_n)}{g(n)\psi(n)} \log P \left[\frac{L_{t(n,x_j)}^o}{g(n)\psi(n)} \ge a_j - \delta, j \ne \ell, \frac{L_{t(n,x_\ell)}^o}{g(n)\psi(n)} \ge a_\ell + \delta \right] \\
\le -\sum_{j=1}^{\ell-1} \frac{a_j - a_{j-1}}{x_j} - \frac{a_\ell + 2\delta - a_{\ell-1}}{x_\ell} - \frac{a_{\ell+1} - a_\ell - 2\delta}{x_{\ell+1}} - \sum_{j=\ell+2}^k \frac{a_j - a_{j-1}}{x_j} \\
< -\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}$$

where we used $\frac{2\delta}{x_\ell} - \frac{2\delta}{x_{\ell+1}} > 0$ in the last inequality. This completes the proof of the lower bound.

It now follows from (30) and (35) that for $0 < x_1 < \cdots < x_k < 1$, the random vector $\{\overline{L}_n(x_j)\}_{j=1}^k$ satisfies in \mathbb{R}^k the LDP with good rate function

$$I_k(y_1, \dots, y_k) = \sum_{j=1}^k \frac{(y_j - y_{j-1})}{x_j},$$

where $y_0 := 0$. By [2], Thm 4.6.1 (see Section 5.1 in [2] for a similar argument), we have that the random monotone function $\{\overline{L}_n(x)\}_{x\in[0,1]}$ satisfies the LDP in $M_+^\omega([0,1])$ (with $M_+^\omega([0,1])$ denoting $M_+([0,1])$ equipped with the topology of pointwise convergence) with good rate function

$$I_{\chi}(m) = \sup_{0=x_0 < x_1 < \dots < x_k < 1} \sum_{i=1}^k \frac{m(x_i) - m(x_{i-1})}{x_i}$$
.

It then follows by monotone convergence that

$$I_{\chi}(m) = I(m) = \int_0^1 \frac{m(dx)}{x}.$$

Finally, note that the topology in $M_+^{\omega}([0,1])$ is stronger than the topology in $M_+([0,1])$, which concludes the proof of the theorem by an application of [2], Corollary 4.2.6.

Proof of Theorem 4. – Let $0 = a_0 < a_1 < \cdots < a_k \le 1$ as before. Recall that with $\psi(n) \equiv 1$, (30) and (31) imply that

$$P\left(\frac{L^0_{t(n,x_j)}}{g(n)} \ge a_j, j = 1, 2, \cdots, k\right) \underset{n \to \infty}{\longrightarrow} \exp\left(-\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}\right).$$

But sets of the form $A=\{f: f(x_j)\geq a_j, j=1,2,\cdots,k\}$ generate the Borel σ -field on M_+ , hence in order to prove convergence of the finite-dimensional marginals of $\frac{L^0_{t(n,\cdot)}}{g(n)}$ to those of Z_x , we only have to check that

$$P[Z_{x_j} \ge a_j, j = 1, 2, \dots, k] = \exp\left(-\sum_{j=1}^k \frac{a_j - a_{j-1}}{x_j}\right),$$

which follows from an explicit computation using (7). Tightness of the distributions of $\frac{L_{t(n,\cdot)}^0}{q(n)}$ is immediate from Prohorov's theorem.

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(Manuscript received July 11, 1997; Revised April 29, 1998.)