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M. CRAMER
L. RÜSCHENDORF

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Convergence of a branching type recursion

by

M. CRAMER and L. RÜSCHENDORF

University of Freiburg, Institut für Mathematische Stochastik, Hebelstr. 27, Freiburg 79104, Germany.

ABSTRACT. – The asymptotic distribution of a branching type recursion (L_n) is investigated. The recursion is given by $L_n \stackrel{d}{=} \sum_{i=1}^K X_i L_{n-1}^{(i)} + Y$, where (X_i) is a random sequence, $(L_{n-1}^{(i)})$ are iid copies of L_{n-1}, K is a random number and $K, (L_{n-1}^{(i)}), \{(X_i), Y\}$ are independent. This recursion has been studied intensively in the literature in the case that $X_i \geq 0$, K is nonrandom and $Y = 0$. Included in the more general recursion are branching processes, a model of Mandelbrot (1974) for studying turbulence, which was also investigated in connection with infinite particle systems and the expansion of total mass in the construction of random multifractal measures. The stability in the recursion arises from the fact that it includes a smoothing part (addition) and on the other hand a part increasing the fluctuation (random multiplier, random number and random immigration). Our treatment of this recursion is based on a contraction technique which applies under some restrictions on the first two moments of the involved random variables. We obtain a quantitative approximation result. The exponential rate of convergence can be observed empirically. Typically after 8 iterations the limiting distribution of the recursion is well approximated.

Key words: Branching type recursion, contraction method.

RÉSUMÉ. – On étudie la distribution asymptotique d'une relation de récurrence de type branchement donnée par $L_n \stackrel{d}{=} \sum_{i=1}^K X_i L_{n-1}^{(i)} + Y$. Ici (X_i) est une suite de variables aléatoires réelles, $L_{n-1}^{(i)}$ sont des v.a. indépendantes de la même distribution que L_{n-1}, K un nombre

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aléatoire et $K, (L_{n-1}^{(i)}), \{(X_i), Y\}$ sont indépendantes. Dans la littérature on a étudié cette récurrence dans le cas où $X_i \geq 0$, K non-aléatoire et $Y = 0$. Entre autres la relation de récurrence plus générale décrit des processus de branchement, un model de Mandelbrot (1974) pour l'étude de turbulence, et l'expansion de masse totale dans la construction des mesures aléatoires multifractales. La stabilité dans la récurrence vient du fait qu'elle contient une partie de lissage (addition) et conversement une partie augmentant la fluctuation (multiplicateur aléatoire, nombre aléatoire et immigration aléatoire). L'analyse dans cet article est fondée sur une technique de contraction applicable sous quelques restrictions aux deux premiers moments des v.a. données. Nous obtenons des résultats d'approximation quantitatifs. La rapidité de convergence exponentielle est observable empiriquement. Après environ 8 itérations la distribution limite de la relation de récurrence est bien approchée.

1. INTRODUCTION

Consider the following recursive sequence (L_n) defined by

$$L_0 \equiv 1, \quad L_n \stackrel{d}{=} \sum_{i=1}^{K} X_i L_{n-1}^{(i)} + Y \tag{1}$$

where $L_{n-1}^{(i)}$ are iid copies of $L_{n-1}, (X_i)$ is a real random sequence, K a random number in $\mathbb{N}_0$ and Y a random immigration such that $K, \{(X_i), Y\}, (L_{n-1}^{(i)})$ are independent. $\stackrel{d}{=}$ denotes equality in distribution. (1) induces a transformation T on M^1, the set of probability distributions on $(\mathbb{R}^1, \mathcal{B}^1)$, by letting $T(\mu)$ be the distribution of $\sum_{i=1}^{K} X_i Z_i + Y$, where (Z_i) are iid μ-distributed, $(Z_i), \{(X_i), Y\}, K$ independent.

Some special cases of this transformation resp. recursion have been studied intensively in the literature. If $X_i \equiv 1$ then (1) describes a Galton-Watson process with immigration Y and the number of descendants of an individuum described by K. (1) can be considered from this point of view as a branching process with random multiplicative weights. The special case where K is constant, $Y = 0, (X_i)$ iid and non negative has been introduced by Mandelbrot (1974) to analyse a model of turbulence of Yaglom and Kolmogorov. This case has been studied by Kahane and Peyrière (1976) and Guivarch (1990) who considered the question of nontrivial fixed points of T, existence of moments of the fixed points and convergence of (L_n). For $X_i \equiv K^{-1/\alpha}$ the solutions of the fixed point equation $Z \stackrel{d}{=} \sum_{i=1}^{K} K^{-1/\alpha} Z_i$ are Paretian stable distributions (if $Z_i \geq 0$). For this reason the solutions are called semi-stable in Guivarch (1990). In this paper we will be mainly interested in the case of multipliers X_i and solutions Z_i with moments of some order ≥ 2. While the analysis of Kahane and Peyrière (1976) is based on an associated martingale, Guivarch (1990) uses a more elementary martingale property together with a conjugation relation and moment type estimates for the L_p-distance, $0 < p < 1$.

Motivated by some problems in infinite particle systems Holley and Liggett (1981) and Durrett and Liggett (1983) considered this kind of smoothing transformation with (X_i) not necessarily independent and $X_i \geq 0$, K constant, $Y = 0$. In the last mentioned paper a complete analysis of this case could be given. In particular a necessary and sufficient condition for the existence and the characterization of (all) fixed points and a general sufficient condition for convergence was derived as well as a generalization of the result of Kahane and Peyrière on the existence of moments. The method of the paper of Durrett and Liggett is based on an associated branching random walk.

The use of contraction properties of minimal L_p-metrics in this paper allows to obtain quantitative approximation results for the recursion (1). Under the moment assumptions used in this paper the recursion converges exponentially fast to the limiting distribution. This is demonstrated by simulations for several examples. Also it is possible to dismiss with the assumption of nonnegativity, to deal with a random number K and with immigration Y. This allows to include branching process applications as well as the consideration of the development of total mass in the construction of multifractal measures (of e.g. Arbeiter (1991)). This example was motivating the work on this paper. After finishing essentially the investigation on this paper we were informed by G. Letac about the history of the problem. We are grateful to him for his remarks and indications.

In section 2 we consider the case $Y \equiv 0$, discuss a robustness property of the problem, relations to previous work and give some numerical examples. In section 3 we consider an extension to the case with immigration under assumptions on the X_i, Y and L_0 assuring the stationarity of the first two moments. The method in this paper is based on a contraction method w.r.t. suitable metrics as developed in a sequence of further examples in Rachev and Rüschendorf (1991).

2. BRANCHING TYPE RECURSION WITH MULTIPLICATIVE WEIGHTS

In this section we consider the recursion (1) with possibly dependent multipliers X_i and immigration $Y \equiv 0$, *i.e.*

$$L_0 \equiv 1, \quad L_n \stackrel{d}{=} \sum_{i=1}^{K} X_i L_{n-1}^{(i)}, \tag{2}$$

where $\left(L_{n-1}^{(i)}\right)$ are iid copies of $L_{n-1}, (X_i)$ is a square integrable real random sequence, K a random number in $\mathbb{N}_0$ and $K, (X_i), \left(L_{n-1}^{(i)}\right)$ are independent.

To determine the correct normalization of (L_n) we at first consider the first moments of (L_n). Denote $l_n := EL_n$, $c := E\left(\sum_{i=1}^{K} X_i\right)$, $v_n := Var(L_n), a := E\left(\sum_{i=1}^{K} X_i^2\right)$ and $b := Var\left(\sum_{i=1}^{K} X_i\right)$.

PROPOSITION 2.1. – $l_0 = 1$, $l_n = c^n$.
If $a \neq c^2 \neq 0$, *then* $v_0 = 0$,

$$v_n = bc^{2n} \frac{1 - (\frac{a}{c^2})^n}{c^2 - a}, \qquad n \geq 1. \tag{3}$$

If $a = c^2$, *then* $v_n = nba^{n-1}$.

Proof. – Using the independence assumption in (2) and conditional expectations we obtain

$$\begin{aligned} l_n &= E\left(E\left(\sum_{i=1}^{K} X_i L_{n-1}^{(i)} \middle| K\right)\right) \\ &= E\left(\sum_{i=1}^{K} EX_i L_{n-1}^{(i)}\right) = E\left(\sum_{i=1}^{K} EX_i\right) l_{n-1} \\ &= c\; l_{n-1}; \ i.e. \ \ l_n = c^n. \end{aligned}$$

Similarly,

$$\begin{aligned} v_n &= EL_n^2 - (EL_n)^2 \\ &= E\left[E\left(\left(\sum_{i=1}^{K} X_i L_{n-1}^{(i)}\right)^2 \middle| K\right)\right] - c^2\; l_{n-1}^2 \end{aligned}$$

$$= E\left[\sum_{i=1}^{K} E\Big(X_i L_{n-1}^{(i)}\Big)^2 + \sum_{i\neq j} E\Big(X_i X_j L_{n-1}^{(i)} L_{n-1}^{(j)}\Big)\right] - c^2\, l_{n-1}^2$$
$$= E\left[E L_{n-1}^2 \sum_{i=1}^{K} E X_i^2 + l_{n-1}^2 \sum_{i\neq j} E(X_i X_j)\right] - c^2\, l_{n-1}^2$$
$$= E\left[\sum_{i=1}^{K} E X_i^2 (Var L_{n-1} + l_{n-1}^2) + l_{n-1}^2 \sum_{i\neq j} E(X_i X_j)\right] - c^2\, l_{n-1}^2$$
$$= E\left(\sum_{i=1}^{K} X_i^2\right) v_{n-1} + Var\left(\sum_{i=1}^{K} X_i\right) l_{n-1}^2$$
$$= a\ v_{n-1} + b\ c^{2(n-1)}$$
$$= b \sum_{k=0}^{n-1} a^k c^{2(n-1-k)}$$
$$= \begin{cases} bc^{2n-2}\frac{1-(\frac{a}{c^2})^n}{1-\frac{a}{c^2}} = bc^{2n}\frac{1-(\frac{a}{c^2})^n}{c^2-a}, & \text{if } a \neq c^2 \neq 0 \\ nba^{n-1}, & \text{if } a = c^2. \quad \square \end{cases}$$

In the case $b = 0$, $v_n = 0$ for all n. Therefore we consider only the case $b > 0$.

From (3) we obtain that for $a < c^2$, $\sqrt{v_n}$ is of the same order as l_n. This makes it possible to use the simple normalization by l_n. Define for $c \neq 0$

$$\widetilde{L}_n := L_n / c^n \tag{4}$$

then $E\widetilde{L}_n = 1$ and $Var(\widetilde{L}_n) \longrightarrow \frac{b}{c^2-a}$.

$\widetilde{L}_n$ satisfies the modified recursion

$$\widetilde{L}_n \stackrel{d}{=} \frac{1}{c}\sum_{i=1}^{K} X_i \widetilde{L}_{n-1}^{(i)}, \tag{5}$$

where $\widetilde{L}_{n-1}^{(i)} := \frac{L_{n-1}^{(i)}}{c^{n-1}}$.

Define D_2 to be the set of distributions on $(\mathbb{R}^1, \mathcal{B}^1)$ with finite second moments and first moment equal to one and define $T : D_2 \longrightarrow D_2$ by

$$T(G) = \mathcal{L}\left(\frac{1}{c}\sum_{i=1}^{K} X_i Z_i\right), \tag{6}$$

where (Z_i) are i.i.d. random variables with distribution $G, (X_i), (Z_i), K$ independent. Let l_2 denote the minimal L_2-metric on D_2 defined by

$$l_2(\mu, \nu) = \inf\left\{ \left(E(V-W)^2\right)^{1/2}; V \stackrel{d}{=} \mu, W \stackrel{d}{=} \nu \right\}$$
$$= \left(\int_0^1 \left(F^{-1}(u) - G^{-1}(u)\right)^2 du \right)^{1/2}, \tag{7}$$

where F, G are the distribution functions of μ, ν respectively. If $a < c^2$, then T is a contraction w.r.t. l_2.

PROPOSITION 2.2. – *Assume that $a < c^2$, then for $F, G \in D_2$*

$$l_2(T(F), T(G)) \leq \sqrt{\frac{a}{c^2}}\, l_2(F, G) \tag{8}$$

Proof. – Let $U^{(i)} \stackrel{d}{=} F, V^{(i)} \stackrel{d}{=} G, i \in \mathbb{N}$, be choosen on $(\Omega, \mathcal{A}, P)$ such that $\|U^{(i)} - V^{(i)}\|_2 = l_2(F, G). \forall i$ and $K, (X_i), (U^{(1)}, V^{(1)}), (U^{(2)}, V^{(2)}), \ldots$ independent. Then

$$\begin{aligned}
l_2^2(T(F), T(G)) &\leq \left\| \frac{1}{c} \sum_{i=1}^K X_i U^{(i)} - \frac{1}{c} \sum_{i=1}^K X_i V^{(i)} \right\|_2^2 \\
&= \frac{1}{c^2} E\left(E\left[\left(\sum_{i=1}^K X_i U^{(i)} - \sum_{i=1}^K X_i V^{(i)} \right)^2 \middle| K \right] \right) \\
&= \frac{1}{c^2} E\left[\sum_{i=1}^K E\left(X_i^2 \left(U^{(i)} - V^{(i)}\right)^2 | K \right) \right. \\
&\quad \left. + \sum_{i \neq j} E\left[X_i \left(U^{(i)} - V^{(i)}\right) X_j \left(U^{(j)} - V^{(j)}\right) | K \right] \right] \\
&= \frac{1}{c^2} E\left(\sum_{i=1}^K E X_i^2 E\left(U^{(i)} - V^{(i)}\right)^2 \right) \\
&= \frac{a}{c^2}\, l_2^2(F, G) \quad \square
\end{aligned}$$

As consequence of Proposition 2.2 T has exactly one fixed point in D_2 with variance equal to $b/(c^2 - a)$. The fixed point equation is given in terms of random variables $Z, Z_i \in D_2, Z_i \stackrel{d}{=} Z, (Z_i)$ independent, by

$$Z \stackrel{d}{=} \frac{1}{c} \sum_{i=1}^{K} X_i Z_i \tag{9}$$

and we obtain as corollary:

THEOREM 2.3. – *If* $a = E\left(\sum_{i=1}^{K} X_i^2\right) < c^2$, *then*

$$l_2(\widetilde{L}_n, Z) \leq \left(\frac{a}{c^2}\right)^{n/2} \frac{\sqrt{b}}{\sqrt{c^2 - a}} \tag{10}$$

In particular $\widetilde{L}_n$ *converges in distribution to* Z.

PROPOSITION 2.4. – *If* K *is constant and* $E\left(\sum_{i=1}^{K} |X_j|^k\right) < c^k \quad \forall\, 2 \leq k \leq h$ *then* $E|Z|^h < \infty$.

Proof. – $\widetilde{L}_n$ can be equivalently represented by Y_n of the following form

$$Y_0 = 1, \quad Y_n = \frac{1}{c^n} \sum_{(j_1,\ldots,j_n)\in\{1,\ldots K\}^n} \prod_{k=1}^{n} X_{j_1,\ldots,j_k},$$

where $(X_{j_1,\ldots,j_{k-1},1}, \ldots, X_{j_1,\ldots,j_{k-1},K}) \stackrel{d}{=} (X_1, \ldots, X_K)$ (*cf.* Guivarch, 1990). (Y_n) is a martingale and therefore $|Y_n|^k$ is a submartingale. Representing the Y_n in the recursive way $Y_n = \frac{1}{c}\sum_{j=1}^{K} X_j Y_{n-1}^{(j)}$, where $Y_{n-1}^{(j)}$ are independent copies of Y_{n-1}, one obtains

$$\begin{aligned} c^k\; E|Y_n|^k \leq & \left(E\sum_{j=1}^{K} |X_j|^k\right) E|Y_{n-1}|^k \\ & + \sum_{\substack{k_1+\ldots+k_K=k \\ k_i\leq k-1}} \binom{k}{k_1, \ldots, k_K} E\prod_{j=1}^{K} |X_j|^{k_j} \prod_{j=1}^{K} E|Y_{n-1}|^{k_j} \end{aligned}$$

One can deduce from Theorem 2.3 that $E|Y_n|^k$ is uniformly bounded for $k \leq 2$. By induction over $k \leq h$ one sees that the lower order terms in the above equation are uniformly bounded, say by C. Since $E|Y_n|^k \geq E|Y_{n-1}|^k$ one obtains

$$E|Y_n|^k \left[c^k - E\left(\sum_{i=1}^{K} |X_j|^k\right)\right] \leq C.$$

Therefore, the assumptions of this proposition ensure that $E|Y_n|^k$ is uniformly bounded for all $k \leq h$. The submartingale convergence theorem now yields the existence of an integrable almost sure limit of $|Y_n|^h$. Since $\widetilde{L}_n \stackrel{d}{=} Y_n$ the weak limit Z of $\widetilde{L}_n$ is absolutely h-integrable. $\square$

For solutions of the stationary equation (9) it is possible to obtain a stability result in terms of l_p metrics defined as in (7) with 2 replaced by p. Suppose we want to approximate the solution S of the equation

$$S \stackrel{d}{=} \sum_{i=1}^{K} X_i S_i \tag{11}$$

by the solution of the "approximate" equation

$$S^* \stackrel{d}{=} \sum_{i=1}^{K} X_i^* S_i^*,$$

where we assume w.l.g. $c = 1$ and consider the case of independent sequences $(X_i), (X_i^*)$, such that $(X_i), (S_i)$ and $(X_i^*), (S_i^*)$ are independent and K is constant.

Proposition 2.5. – *If K is constant, $\sum_{i=1}^{K} l_p(X_i, X_i^*) < \varepsilon$ and $\sum_{i=1}^{K} ||X_i||_p < 1$, then*

$$l_p(S, S^*) \leq \frac{\varepsilon ||S^*||_p}{1 - \sum_{i=1}^{K} ||X_i||_p} \tag{12}$$

Proof. – From the definition of S, S^*

$$\begin{aligned} l_p(S, S^*) &= l_p\left(\sum_{i=1}^{K} X_i S_i, \sum_{i=1}^{K} X_i^* S_i^*\right) \\ &\leq \sum_{i=1}^{K} l_p(X_i S_i, X_i^* S_i^*) \\ &\leq \sum_{i=1}^{K} (l_p(X_i S_i, X_i S_i^*) + l_p(X_i S_i^*, X_i^* S_i^*) \\ &\leq \left(\sum_{i=1}^{K} ||X_i||_p\right) l_p(S, S^*) + ||S^*||_p \cdot \varepsilon \end{aligned}$$

This implies that

$$l_p(S, S^*) \leq \frac{\varepsilon ||S^*||_p}{1 - \sum_{i=1}^K ||X_i||_p} \quad \square$$

A similar idea for establishing robustness of equations can be found in Rachev (1991), Chapter 19.3.

For random K we replace Proposition 2.5 by

PROPOSITION 2.6. – *If* $E\left(\sum_{i=1}^K (X_i - X_i^*)^2\right) \leq \varepsilon^2, EX_i = EX_i^*$ *and* $a = E\left(\sum_{i=1}^K X_i^2\right) < 1$, *then*

$$l_2(S, S^*) \leq \frac{\varepsilon}{1 - \sqrt{a}} ||S^*||_2 \tag{13}$$

Proof. – By the triangle inequality and the independence assumption and the assumption $EX_i = EX_i^*$

$$\begin{aligned}
l_2(S, S^*) &= l_2\left(\sum_{i=1}^K X_i S_i, \sum_{i=1}^K X_i^* S_i^*\right) \\
&\leq l_2\left(\sum_{i=1}^K X_i^* S_i^*, \sum_{i=1}^K X_i S_i^*\right) + l_2\left(\sum_{i=1}^K X_i S_i^*, \sum_{i=1}^K X_i S_i\right) \\
&\leq \left(E\left(\sum_{i=1}^K X_i^2\right)\right)^{1/2} l_2(S, S^*) + ||S^*||_2 \left(E \sum_{i=1}^K (X_i - X_i^*)^2\right)^{1/2} \\
&= \sqrt{a} l_2(S, S^*) + \varepsilon ||S^*||_2
\end{aligned}$$

Therefore, $l_2(S, S^*) \leq \frac{\varepsilon ||S^*||_2}{1 - \sqrt{a}}$. $\square$

Remark. – (*a*) In the case of constant K and nonnegative X_i Durrett and Liggett (1981) proved that the stationary solution Z of (9) has moments of order β if and only if

$$v(\beta) = \log\left(\frac{1}{c^\beta} \sum_{i=1}^K EX_i^\beta\right) < 0 \tag{14}$$

For $\beta = 2$, (14) is equivalent to the condition $a < c^2$ used in Theorem 1. In this sense this condition is sharp when using l_2-distances. Guivarch (1990) has shown how to dismiss with the second moment assumption.

(*b*) For the normalized recursion (5) with (X_i) i.i.d., K constant (where we assume w.l.g. $c = 1$), we can use the explicit form (cf. the proof of Proposition 2.4)

$$\widetilde{L}_0 = 1, \quad \widetilde{L}_n = \sum_{(j_1,\ldots,j_n)\in\{1,\ldots,K\}^n} \prod_{k=1}^{n} X_{j_1,\ldots,j_k}, \tag{15}$$

where $(X_{j_1,\ldots,j_k})$ are independent and distributed as X_1, *i.e.* $\widetilde{L}_n$ is the sum over product weights in the complete K-ary tree. For nonnegative multipliers X_i one can consider further functionals as e.g.

$$M_n = \max_{P_n} \prod_{k=1}^{n} X_{j_1,\ldots,j_k}, \tag{16}$$

the max product over all paths of length n. Taking logarithms

$$\begin{aligned} -\ln M_n &= -\max_{P_n} \sum_{k=1}^{n} \ln(X_{j_1,\ldots,j_k}) \\ &= \min_{P_n} \sum_{k=1}^{n} (-\ln(X_{j_1,\ldots,j_k})) \end{aligned}$$

and application of Kingman's subadditive ergodic theorem yields for some constant β

$$\frac{1}{n}\log M_n \longrightarrow \beta \quad a.s. \tag{17}$$

This shows that in some sense the max product weight is not larger in the order than the average product weight. For some cases β is known, e.g. for $X_i \stackrel{d}{=} U[0,1], \beta \approx -0.23196$ (*cf.* Mahmoud (1992), p. 165).

(*c*) For some cases explicit solutions of (9) are known.

1) If K is constant, $\frac{1}{c}X_i \stackrel{d}{=} \beta(\frac{a}{K}, a - \frac{a}{K})$ is Beta-distributed, then $Z \stackrel{d}{=} \Gamma(a,\beta)$ is Gamma-distributed (*cf.* Guivarch, 1990).

2) If $Z_1 \stackrel{d}{=} \frac{1}{K}\sum_{i=1}^{K} X_i Z_i$, then with (Y_i) i.i.d., $\overline{X} \stackrel{d}{=} X_1, Y_1 \stackrel{d}{=} X_1 Z_1$ holds $Y_1 \stackrel{d}{=} \frac{1}{K}\sum_{i=1}^{K} Y_i \overline{X}$. Conversely, if $Y_1 \stackrel{d}{=} \frac{1}{K}\sum_{i=1}^{K} Y_i X_1$, then with (X_i) i.i.d., $Z_i \stackrel{d}{=} \frac{1}{K}\sum_{j=1}^{K} Y_j$, the (Z_i) solve $Z_1 \stackrel{d}{=} \frac{1}{K}\sum_{i=1}^{K} X_i Z_i$ (*cf* Durrett and Liggett (1981)).

3) If (Z_i) solve $Z_1 \stackrel{d}{=} \sum_{i=1}^{K} X_i Z_i, X_i \geq 0$, then $Y_i = Z_i^{1/\vartheta} W_i, 0 < \vartheta \leq 2, W_i$ stable rv's of index ϑ, solve

$$\sum_{i=1}^{K} X_i^{1/\vartheta} Y_i \stackrel{d}{=} Y_1. \tag{18}$$

To prove (18), $\sum_{i=1}^K X_i^{1/\vartheta} Z_i^{1/\vartheta} W_i \stackrel{d}{=} \left(\sum_{i=1}^K X_i Z_i\right)^{1/\vartheta} W_1 \stackrel{d}{=} Z_1^{1/\vartheta} W_1 = Y_1$. This interesting transformation property is used in Guivarch (1990) to reduce the case of moments of X_i of low order to the case with moments of higher order.

4) If $\sum_{i=1}^K X_i^2 \equiv c^2 \neq 0$, then $Z \stackrel{d}{=} Z_i \stackrel{d}{=} \mathcal{N}(0, \sigma^2)$ normally distributed solve (9).

5) If Z solves (9) and $\overline{Z}$ is an independent copy of Z, then $Z^* := Z - \overline{Z}$ solves

$$Z^* \stackrel{d}{=} \frac{1}{c} \sum_{i=1}^K X_i^* Z_i^*,$$

where$X_i^* = \tau_i X_i$ and the τ_i are arbitrary random signs. In particular the case $K = 2$, $X_i^* \stackrel{d}{=} U[-1, 1]$ independent is solved by $Z^* := Z - \overline{Z}$ where $Z \stackrel{d}{=} \Gamma(2, \frac{1}{2}, 0)$.

(*d*) The following simulations (Figures 1 and 2) of $\widetilde{L}_n, K = 2$, X_1, X_2 independent, $X_1 \stackrel{d}{=} X_2 \stackrel{d}{=} U[0,1]$, respectively $X_i \stackrel{d}{=} \beta(2,2)$ show good coincidence with the theoretical Gamma-distribution.

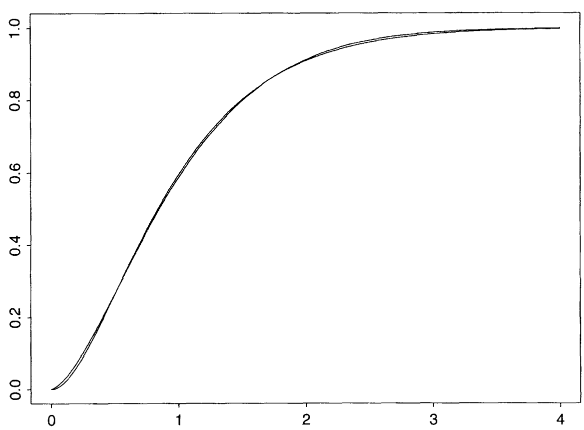

FIG. 1. – Empirical df, $X_1 \stackrel{d}{=} U[0, 1]$, $n = 10$, theoretical Gamma $\Gamma(2, \frac{1}{2}, 0)$

(*e*) In the case $K = 2$, X_1, X_2 independent, $X_1 \stackrel{d}{=} X_2 \stackrel{d}{=} U\left[-\frac{1}{8}, \frac{9}{8}\right]$ no explicit solution of (9) is known. Nevertheless the following simulations (Figure 3) show that $\widetilde{L}_n$ converges very fast to the fixed point of (9). The empirical distribution functions of $\widetilde{L}_{10}$ and $\widetilde{L}_{12}$ can hardly be distinguished. Therefore they may be regarded as the limit distribution function. The empirical distribution function of $\widetilde{L}_6$ is already very close to it.

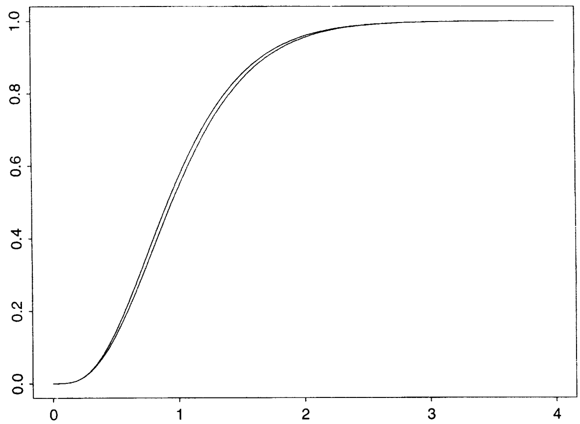

FIG. 2. – Empirical df, $X_1 \stackrel{d}{=} \beta(2, 2)$, $n = 8$, theoretical Gamma $\Gamma(4, \frac{1}{4}, 0)$

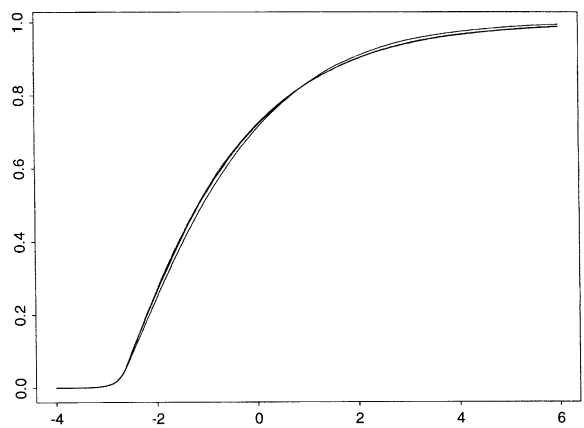

FIG. 3. – Empirical df of $\widetilde{L}_n$ for $n = 6, 10$ and 12, $X_1 \stackrel{d}{=} U[-\frac{1}{8}, \frac{9}{8}]$

(*f*) *Branching processes.* – Equation (2) includes the Galton-Watson process as special case. A Galton Watson process is defined by the recursion

$$Z_0 = 1, \quad Z_{n+1} = \sum_{k=1}^{Z_n} X_k^n, \tag{19}$$

where $X_k^n \stackrel{d}{=} X$ are i.i.d. with reproduction distribution in $\mathbb{N}_0$. Define $K \stackrel{d}{=} X$ and $X_i \equiv 1$, then

$$L_n \stackrel{d}{=} Z_n \tag{20}$$

for all n.

The equality can be seen by induction in n. First $Z_0 = L_0 = 1$. If $Z_k \stackrel{d}{=} L_k$ for $k \leq n$, then

$$Z_{n+1} \stackrel{d}{=} \sum_{k=l}^{L_n} X_k^n \stackrel{d}{=} \sum_{i=1}^{K} \sum_{k=1+\sum_{j=1}^{i-1} L_{n-1}^{(j)}}^{\sum_{j=1}^{i} L_{n-1}^{(j)}} X_k^n$$

$$\stackrel{d}{=} \sum_{i=1}^{K} \left(\sum_{k=1}^{L_{n-1}} X_k^n \right)^{(i)} \stackrel{d}{=} \sum_{i=1}^{K} \left(\sum_{k=1}^{Z_{n-1}^{(i)}} X_k^n \right)^{(i)} = \sum_{i=1}^{K} Z_n^{(i)}$$

$$\stackrel{d}{=} \sum_{i=1}^{K} L_n^{(i)} \stackrel{d}{=} L_{n+1}.$$

The assumption $a < c^2$ is equivalent to the condition $EX > 1$. $\square$

From equality (20) one can derive explicit stationary distributions and extinction probabilities in some cases. If e.g. X is geometrically distributed, $P(X = k) = p(1-p)^k, k \in \mathbb{N}_0$, then $c = EX = \frac{1-p}{p} > 1$ if $p < \frac{1}{2}$ and $Var(X) = \frac{1-p}{p^2}$. The normalized Galton-Watson process $\frac{Z_n}{\sqrt{Var(Z_n)}}$ converges to a (unique) solution of the fixed point equation

$$Z \stackrel{d}{=} \frac{1}{EX} \sum_{i=1}^{X} Z_i, \quad EZ = \sqrt{\frac{EX(EX-1)}{Var(X)}} \tag{21}$$

The extinction probability is easily seen to be $\frac{p}{1-p}$. For the normalized continuous part an equation identical to (21) (but with different variances) holds. It is well known that this equation is solved by the geometric stable distribution of order 1, *i.e.* the exponential distribution. This implies finally

$$Z \stackrel{d}{=} \frac{p}{1-p}\delta_0 + \frac{1-2p}{1-p} \exp\left(\frac{\sqrt{1-2p}}{1-p} \right) \tag{22}$$

since $EZ = \sqrt{1-2p}, EZ^2 = 2(1-p)$.

3. A RANDOM IMMIGRATION TERM

In this section we admit an additional immigration term.

$$L_n \stackrel{d}{=} \sum_{i=1}^{K} X_i L_{n-1}^{(i)} + Y \tag{23}$$

where $\{(X_i), Y\}, K, L^{(1)}_{n-1}, L^{(2)}_{n-2}, \ldots$ are independent, X_i and Y have finite second moments. The analysis of (23) is essentially simplified if we assume for $l_0 := EL_0, v_0 := Var(L_0)$,

$$l_0 = \frac{EY}{1-c} \quad \text{if } c \neq 1, \quad v_0 = \frac{Var(l_0 \sum_{i=1}^K X_i + Y)}{1-a}, \quad a < 1 \tag{24}$$

If $c = 1$, then $EY = 0$ and l_0 arbitrary.

LEMMA 3.1. – *Under assumption* (24) *holds*

$$l_n = EL_n = l_0, \quad v_n = Var(L_n) = v_0, \qquad \forall n \in \mathbb{N}. \tag{25}$$

Proof. – From (24) $l_n = c\, l_{n-1} + EY = \frac{EY}{1-c} = l_{n-1}$.

$$\begin{aligned}
v_n &= Var(L_n) = EL_n^2 - l_n^2 \\
&= E\left[\sum_{i=1}^K E(X_i^2 \left(L^{(i)}_{n-1}\right)^2 | K) + \sum_{i \neq j} E(X_i X_j L^{(i)}_{n-1} L^{(j)}_{n-1} | K) \right. \\
&\quad \left. + E(Y^2|K) + 2E\left(\sum_{i=1}^K Y X_i L^{(i)}_{n-1} | K\right)\right] - l_0^2 \\
&= a(v_{n-1} + l_0^2) + \left(\sum_{i \neq j} EX_i X_j\right) l_0^2 \\
&\quad + EY^2 + l_0 2E\left(\sum_{i=1}^K Y X_i\right) - l_0^2 \\
&= av_{n-1} + Var\left(l_0 \sum_{i=1}^K X_i + Y\right) = v_{n-1} \quad \square
\end{aligned}$$

Condition (24) can be achieved with a two point distribution for L_0. It allows to use the technique of proof of section 2. A change of the initial condition leads to the necessity to change the method of proof and leads to a great variety of different cases to be considered. We, therefore, restrict to (24) in this paper.

As in section 2 we introduce the operator

$$T : M(l_0, v_0) \to M(l_0, v_0), T(G) = \mathcal{L}\left(\sum_{i=1}^K X_i V_i + Y\right) \tag{26}$$

where $M(l_0, v_0)$ is the set of distributions with mean l_0 and variance v_0 and (V_i) are i.i.d., $V_1 \stackrel{d}{=} G, (V_i), \{(X_i), Y\}, K$ independent.

Similarly as in Proposition 2.2 we can show the contraction

$$l_2(T(F), T(G)) \leq \sqrt{a}\, l_2(F, G) \tag{27}$$

which implies the convergence of L_n to the unique fixed point of T in $M(l_0, v_0)$ w.r.t. the l_2-metric, the contraction factor being $\sqrt{a}$.

We obtain a sharper result (*i.e.* a smaller contraction factor) by the use of the Zolotarev-metric ζ_r instead of l_2.

It is defined by

$$\zeta_r(F, G) = \sup \{|E(f(X) - f(Y))|; |f^{(m)}(x) - f^{(m)}(y)| \leq |x - y|^\alpha\} \tag{28}$$

for $r = m + \alpha, m \in \mathbb{N}_0, 0 < \alpha \leq 1$.

PROPOSITION 3.2.

$$\zeta_r(T(F), T(G)) \leq E\left(\sum_{i=1}^{K} |X_i|^r\right) \zeta_r(F, G) \tag{29}$$

Proof. – Note that ζ_r is ideal of order r w.r.t. summation, *i.e.*

$$\zeta_r(X + Z, Y + Z) \leq \zeta_r(X, Y)$$

for Z independent of X, Y and

$$\zeta_r(cX, cY) = |c|^r \zeta_r(X, Y).$$

For $(Z_i), (W_i)$ i.i.d. distributed according to F, G we have with $X = (X_i)$

$$\begin{aligned}
\zeta_r(TF, TG) &= \sup \left\{ \left| Ef\left(\sum_{i=1}^{K} X_i Z_i + Y\right) - Ef\left(\sum_{i=1}^{K} X_i W_i + Y\right) \right|; \right. \\
&\qquad\qquad \left. |f^{(m)}(X) - f^{(m)}(Y)| \leq |x - y|^2 \right\} \\
&\leq \int \zeta_r\left(\sum_{i=1}^{K} x_i Z_i + y, \sum_{i=1}^{K} x_i W_i + y\right) dP^{(X,Y,K)} \quad (x, y, k) \\
&\leq \int \sum_{i=1}^{k} |x_i|^r \zeta_r(Z_i, W_i) dP^{(X,Y,K)}\ (x, y, k) \\
&= E\left(\sum_{i=1}^{K} |X_i|^r\right) \zeta_r(F, G). \quad \square
\end{aligned}$$

Note that for our recursion defined by T the first two moments are conserved. Therefore, we can apply (29) with $r \leq 3$ and obtain as corollary:

THEOREM 3.3. – *If $c \neq 1$ and $l_0 = \frac{EY}{1-c}$ or $c = 1$ and $EY = 0$ and if $v_0 = \frac{Var(l_0 \sum_{i=1}^K X_i + Y)}{1-a}$ for $a < 1$, then for $0 < r \leq 3$*

$$a_r := E \sum_{i=1}^{K} |X_i|^r < 1$$

implies

$$\zeta_r(L_n, Z) \leq \frac{a_r^n}{1 - a_r} \zeta_r(L_0, L_1) < \infty,$$

where Z is a fixed point of T in $M(l_0, v_0)$.

In particular L_n converges in distribution to Z.

So also in the case with immigration one obtains an exponential rate of convergence. As a consequence after a few iterations the limiting distribution is well approximated.

Consider the following example: $L_0 \stackrel{d}{=} \frac{1}{10} \delta_{-5} + \frac{2}{5} \delta_0 + \frac{1}{2} \delta_2$, $K = 2$, X_1, X_2 independent, $X_1 \stackrel{d}{=} X_2 \stackrel{d}{=} U\left[-\frac{1}{2}, 1\right]$, $Y \stackrel{d}{=} \frac{17}{32} \delta_{-1} + \frac{5}{64} \delta_0 + \frac{25}{64} \delta_2$.

In this situation (24) is fulfilled. The fast convergence is confirmed by the closeness of the empirical distribution functions of L_6 and L_8 in the following simulation.

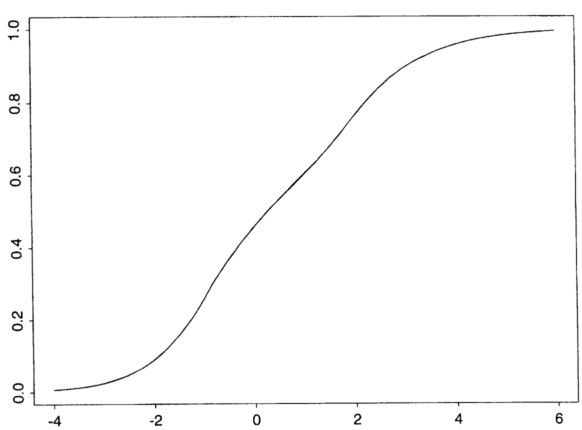

FIG. 4. – Empirical distribution functions for L_6 and L_8.

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