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Approximating a helix in finitely many dimensions

by

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ABSTRACT. — Consider $\alpha \in]0, 1[$. We prove that there exists a constant $K(\alpha)$, depending on α only, such that for $p \geq 1$, there exists a map F from $\mathbb{R}$ to $\mathbb{R}^p$ such that for $s, t \in \mathbb{R}$, we have

$$\|F(s) - F(t)\| |s - t|^{\alpha - 1} \leq K(\alpha)/p^\alpha.$$

RÉSUMÉ. — Pour $\alpha \in]0, 1[$, il existe une constante $K(\alpha)$, dépendant de α seulement, telle que pour $p \geq 1$, il existe une application F de $\mathbb{R}$ dans $\mathbb{R}^p$ telle que, pour tous réels s, t on ait

$$\|F(s) - F(t)\| |s - t|^{\alpha - 1} \leq K(\alpha)/p^\alpha.$$

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1. INTRODUCTION

A helix is a map h from $\mathbb{R}$ to a Hilbert space H such that $\|h(s) - h(t)\| = \|h(s-t)\|$ for $s, t \in \mathbb{R}$. Within isometries, a helix is determined by the function

$$(1) \quad \psi(t) = \|h(t)\|^2.$$

It is a theorem of I. J. Shoenberg that the functions $\psi(t)$ given by (1) are exactly the functions of negative type. In this note, we are interested in the case $\|h(t)\| = |t|^\alpha$, for a certain $\alpha \in]0, 1[$. The case $\alpha = 1/2$ corresponds to Wilson's helix, that is realized by Brownian motion.

P. Assouad and L. A. Shepp raised the question whether the helix corresponding to $\|h(t)\| = |t|^{1/2}$ (Wilson's helix) can be approximated in the p -dimensional euclidean space. This was settled by J. P. Kahane [2] who obtained the following result. (Throughout the paper, $\|\cdot\|$ denotes the euclidean norm.)

THEOREM 1 (J. P. Kahane). — There exists a universal constant K such that for $p \geq 1$, there exists a map F from $\mathbb{R}$ to $\mathbb{R}^p$ such that

$$\forall s, t \in \mathbb{R}, \quad 1 - \frac{K}{p} \leq \frac{\|F(s) - F(t)\|}{|s-t|^{1/2}} \leq 1 + \frac{K}{p}.$$

On the other hand, P. Assouad [1] proved that for all $\alpha \in]0, 1[$, $p \geq p_0$, there exists a map F from $\mathbb{R}$ to $\mathbb{R}^p$ such that

$$(2) \quad \forall s, t \in \mathbb{R}, \quad \frac{1}{K} \leq \frac{\|F(s) - F(t)\|}{|s-t|^\alpha} \leq K$$

where K depends on α only. The estimate of (2) does not improve when $p \rightarrow \infty$. The purpose of the present note is to improve upon (2).

THEOREM 2. — *Given $\alpha \in]0, 1[$, there exists a constant $K(\alpha)$, depending on α only, such that for $p \geq 1$, there exists a map F from $\mathbb{R}$ to $\mathbb{R}^p$ that satisfies*

$$(3) \quad \forall s, t \in \mathbb{R}, \quad 1 - \frac{K(\alpha)}{p^\alpha} \leq \frac{\|F(s) - F(t)\|}{|s-t|^\alpha} \leq 1 + \frac{K(\alpha)}{p^\alpha}.$$

In the case $\alpha = 1/2$, this gives an error in $K/\sqrt{p}$, and unfortunately does not recover the error K/p of Kahane's Theorem 1. It is not difficult to see that this error K/p is of optimal order in Kahane's theorem; but when $\alpha \neq 1/2$, we do not have a nontrivial lower bound for the error in (3).

2. THE APPROACH

We fix $\alpha \in]0, 1[$, and $p \geq 1$. For convenience, we assume that p is a multiple of 4 (so that $p \geq 4$). We set, for $n \geq 0$,

$$D_n = \left\{ \frac{i}{p 2^n}; 0 \leq i \leq p 2^n \right\}.$$

For $0 \leq q \leq 2^{n+1} - 2$, we set

$$I_{n,q} = \left[\frac{q}{2^{n+1}}, \frac{q+2}{2^{n+1}} \right].$$

Thus $I_{n,q} \subset [0, 1] = I_{0,0}$. For $n \geq 1$, $0 \leq q \leq 2^{n+1} - 2$, we find $l(q) (= l_n(q))$ such that $I_{n,q} \subset I_{n-1,l(q)}$. When $0 < q < 2^{n+1} - 2$, and when q is even, there are two possible choices. We make an arbitrary choice; the construction will actually not depend on that choice.

Consider a map $t \rightarrow x(t)$ from $\mathbb{R}$ to a Hilbert space H that satisfies $\|x(t) - x(s)\| = |t - s|^\alpha$. We first construct affine maps $\theta_{n,q}$ from H to $\mathbb{R}^p$ that satisfy

$$(4) \quad \forall s, t \in D_n \cap I_{n,q}, \quad \|\theta_{n,q}(x(t)) - \theta_{n,q}(x(s))\| = |t - s|^\alpha$$

$$(5) \quad \forall t \in D_{n-1} \cap I_{n,q}, \quad \theta_{n,q}(x(t)) = \theta_{n-1,l(q)}(x(t)).$$

We proceed to this easy construction, by induction over n . A basic observation is that $D_n \cap I_{n,q}$ has $p+1$ points. The affine span of these points is isometric to $\mathbb{R}^p$; thus for each n, q , one can find an affine map $\xi_{n,q}$ from H to $\mathbb{R}^p$ that satisfies

$$\forall s, t \in D_n \cap I_{n,q}, \quad \|\xi_{n,q}(x(t)) - \xi_{n,q}(x(s))\| = |t - s|^\alpha.$$

We take $\theta_{0,0} = \xi_{0,0}$. If all the maps $\theta_{n,q}$ have been constructed, for a certain n and for all $q \leq 2^{n+1} - 2$, we take $\theta_{n+1,q} = U \circ \xi_{n+1,q}$, where U is an isometry of $\mathbb{R}^p$ such that $U(\xi_{n+1,q}(x(t))) = \theta_{n,l(q)}(x(t))$ for $t \in D_{n-1} \cap I_{n,q}$. By isometry we mean that $\|U(x) - U(y)\| = \|x - y\|$ for $x, y \in \mathbb{R}^p$. The existence of U follows from the following elementary fact, that will be used repeatedly: if S is a map from a subset A of $\mathbb{R}^p$ to $\mathbb{R}^p$ such that $\|S(x) - S(y)\| = \|x - y\|$ for $x, y \in A$, then we can find an isometry U of $\mathbb{R}^p$ such that $U(x) = S(x)$ for $x \in A$.

For the simplicity of notation, we will write $x_{n,q,t} = \theta_{n,q}(x(t))$. The idea of the preceding construction is that the points $x_{n,q,t}$, $t \in D_n \cap I_{n,q}$ have the correct position with respect to each other. Also, a certain degree of consistency is obtained through (5). One would like to have $F(t) = x_{n,q,t}$ for $t \in D_n \cap I_{n,q}$. The problem is that it is not possible to insure that $x_{n,q,t} = x_{n,q+1,t}$ for $t \in D_n \cap I_{n,q} \cap I_{n,q+1}$. To solve that difficulty, for $t \in I_{n,q}$, we will construct an isometry $R_{n,q,t}$ of $\mathbb{R}^p$. We require the following

properties.

(6) For $t \in D_n \cap I_{n,q} \cap I_{n,q+1}$, we have

$$R_{n,q,t}(x_{n,q,t}) = R_{n,q+1,t}(x_{n,q+1,t}).$$

(7) For $t \in D_{n-1} \cap I_{n,q}$, $y = x_{n,q,t} = x_{n-1,l(q),t}$, we have

$$R_{n,q,t}(y) = R_{n-1,l(q),t}(y).$$

(8) For $s, t \in I_{n,q}$, $x, y \in \mathbb{R}^p$, we have

$$\|R_{n,q,s}(x) - R_{n,q,s}(y) - (R_{n,q,t}(x) - R_{n,q,t}(y))\| \leq K \|x - y\| |t - s|.$$

(There, as in the sequel, K is a constant depending on α only, that is not necessarily the same at each occurrence; on the other hand, $K_1, K_2, \dots$ denote specific constants depending on α only).

(9) If $x = x_{n,q,u}$ for $u \in I_{n,q} \cap D_n$, then for $s, t \in I_{n,q} \cap D_n$, we have

$$\|R_{n,q,s}(x) - R_{n,q,t}(x)\| \leq K \frac{2^{n(1-\alpha)}}{p^\alpha} |s - t|.$$

(10) For t in $[(q+1)2^{-n-1}, (q+2)2^{-n-1}]$, the isometry $R_{n,q,t}^{-1} \circ R_{n,q+1,t}$ does not depend on t .

The construction of these isometries will be done in section 3; but, before, we provide motivation by proving Theorem 2.

For $t \in D_n \cap I_{n,q}$, we set

$$(11) \quad F(t) = R_{n,q,t}(x_{n,q,t}).$$

Given n , there are two consecutive values of q for which $t \in I_{n,q}$; it follows from (6) that the value of $F(t)$ does not depend on which value of q we use. Also, it follows from (7) that the value of $F(t)$ does not depend on which value of n we consider. Thus, (11) actually defines $F(t)$ for $t \in D = \bigcup_{n \geq 0} D_n$.

Consider now $u, v \in D_n$ such that $|u - v| \leq 2^{-n-1}$. Thus $u, v \in I_{n,q}$ for some q . Let $\tau = (q+1)2^{-n-1}$. It follows from (4), since $R_{n,q,\tau}$ is an isometry, that

$$\|R_{n,q,\tau}(x_{n,q,u}) - R_{n,q,\tau}(x_{n,q,v})\| = |u - v|^\alpha.$$

Thus, by (9), used for $s = u$, $t = \tau$, and for $s = v$, $t = \tau$, we have

$$(12) \quad \begin{aligned} \|F(u) - F(v)\| - |u - v|^\alpha &\leq \|R_{n,q,u}(x_{n,q,u}) - R_{n,q,\tau}(x_{n,q,u})\| \\ &\quad + \|R_{n,q,v}(x_{n,q,v}) - R_{n,q,\tau}(x_{n,q,v})\| \leq K \frac{2^{-n\alpha}}{p^\alpha}. \end{aligned}$$

It follows in particular that

$$(13) \quad \|F(u) - F(v)\| \leq K 2^{-n\alpha}.$$

LEMMA. — For $s, t \in D$, we have $\|F(s) - F(t)\| \leq K|s - t|^\alpha$.

Proof. — Consider the largest n such that $|s - t| \leq 2^{-n}$, so that $2^{-n} \leq 2|s - t|$. We observe that, given $s \in [0, 1]$, we can find $u \in D_n$ such that $|s - u| \leq 2^{-n}/p \leq 2^{-n-2}$. We thus construct sequences $(u_k), (v_k)$ $k \geq n$, such that $u_k, v_k \in D_{k-2}$, $|u_k - s| \leq 2^{-k}$, $|v_k - u| \leq 2^{-k}$. Thus $|u_n - v_n| \leq 2^{-n+2}$, $|u_k - u_{k+1}|, |v_k - v_{k+1}| \leq 2^{-k+1}$. We can and do assume that $u_k = s$, $v_k = t$ for k large enough. Then

$$\|F(u) - F(v)\| \leq \|F(u_n) - F(v_n)\| + \sum_{k \geq n} (\|F(u_k) - F(u_{k+1})\| + \|F(v_k) - F(v_{k+1})\|).$$

By (13), this implies that

$$\|F(u) - F(v)\| \leq K 2^{-\alpha n} \leq K|s - t|^\alpha. \quad \square$$

The lemma implies in particular that F can be extended by continuity to the closure of D , i. e. to $[0, 1]$, and that

$$(14) \quad \|F(s) - F(t)\| \leq K|s - t|^\alpha$$

for $s, t \in [0, 1]$.

Consider now $s, t \in [0, 1]$ and the largest n such that $|s - t| \leq 2^{-n-1}$, so that $2^{-n} \leq 4|s - t|$. Consider q such that $s, t \in I_{n,q}$. Thus we can find $u, v \in I_{n,q} \cap D_n$ such that $|s - u| \leq 2^{-n}/p$, $|t - v| \leq 2^{-n}/p$. By (14), we have

$$\|F(s) - F(u)\| \leq \frac{K 2^{-n\alpha}}{p^\alpha}; \quad \|F(t) - F(v)\| \leq \frac{K 2^{-n\alpha}}{p^\alpha}.$$

Thus

$$|\|F(s) - F(t)\| - \|F(u) - F(v)\|| \leq \frac{K 2^{-n\alpha}}{p^\alpha}.$$

From (12), we have

$$|\|F(u) - F(v)\| - |u - v|^\alpha| \leq \frac{K 2^{-n\alpha}}{p^\alpha}.$$

Thus, since $|s - t| \geq 2^{-n-2}$, we have

$$(15) \quad \left| \frac{\|F(s) - F(t)\|}{|s - t|^\alpha} - 1 \right| \leq \frac{K}{p^\alpha} + \left| \frac{|u - v|^\alpha}{|s - t|^\alpha} - 1 \right|.$$

We have $\|u - v\| - |s - t| \leq 2^{-n+1}/p$. Using that $|(1+x)^\alpha - 1| \leq K|x|$ for $|x| \leq 4$, we get that

$$\left| \frac{|u - v|^\alpha}{|s - t|^\alpha} - 1 \right| \leq \frac{K}{p} \leq \frac{K}{p^\alpha}.$$

Thus, we have constructed a map F from $[0, 1]$ to $\mathbb{R}^p$ such that

$$(16) \quad \forall s, t \in [0, 1], \quad \left| \frac{\|F(s) - F(t)\|}{|s - t|^\alpha} - 1 \right| \leq \frac{K}{p^\alpha}.$$

There is no loss of generality to assume $F(1/2)=0$. Consider an ultra-filter $\mathcal{U}$ on $\mathbb{N}$, and define

$$G(t) = \lim_{n \rightarrow \mathcal{U}} n^\alpha F\left(\frac{1}{2} + \frac{t}{n}\right).$$

The limit exists since, from (14) and $F\left(\frac{1}{2}\right)=0$, we have

$$n^\alpha \left\| F\left(\frac{1}{2} + \frac{t}{n}\right) \right\| \leq K |t|^\alpha.$$

Moreover it is immediate to check that, for $s, t \in \mathbb{R}$, we have $\|G(s) - G(t)\| / |s - t|^\alpha - 1 \leq K/p^\alpha$. This completes the proof of Theorem 2.

The reader has observed that conditions (8) and (10) have not been used. Condition (8) is used during the construction as a preliminary step for conditions (9). Condition (10) helps to keep control of the situation as the induction continues.

3. CONSTRUCTION

The construction proceeds by induction on n . For $t \in [0, 1]$, we set $R_{0,0,t} = \text{Identity}$. We now perform the induction step from $n-1$ to n . Consider q , $-1 \leq q \leq 2^{n+2} - 2$, and set

$$\tau = (q+1)2^{-n-1}, \quad \tau' = (q+2)2^{-n-1}, \quad I = [\tau, \tau'].$$

For $t \in I$, we construct isometries $T_{n,q,t}, S_{n,q,t}$ of $\mathbb{R}^p$, such that the following holds (where we set $l(-1)=0$)

$$(17) \quad T_{n,q,\tau} = R_{n-1,l(q),\tau}, \quad S_{n,q,\tau'} = R_{n-1,l(q+1),\tau'}.$$

$$(18) \quad \forall t \in I \cap D_n, \quad T_{n,q,t}(x_{n,q,t}) = S_{n,q,t}(x_{n,q+1,t})$$

$$(19) \quad \text{For } t \in D_{n-1} \cap I, \text{ we have}$$

$$\begin{aligned} T_{n,q,t}(x_{n,q,t}) &= R_{n-1,l(q),t}(x_{n,q,t}) \\ S_{n,q,t}(x_{n,q+1,t}) &= R_{n-1,l(q+1),t}(x_{n,q+1,t}). \end{aligned}$$

$$(20) \quad \text{For } s, t \in I, x, y \in \mathbb{R}^p, \text{ we have}$$

$$\begin{aligned} \|T_{n,q,s}(x) - T_{n,q,s}(y) - (T_{n,q,t}(x) - T_{n,q,t}(y))\| &\leq K_1 2^n |s - t| \|x - y\| \\ \|S_{n,q,s}(x) - S_{n,q,s}(y) - (S_{n,q,t}(x) - S_{n,q,t}(y))\| &\leq K_1 2^n |s - t| \|x - y\|. \end{aligned}$$

$$(21) \quad \text{For } u, s, t \in I \cap D_n, x = x_{n,q,u}, y = x_{n,q+1,u}, \text{ we have}$$

$$\begin{aligned} \|T_{n,q,s}(x) - T_{n,q,t}(x)\| &\leq K_2 \frac{2^{n(1-\alpha)}}{p^\alpha} |s - t| \\ \|S_{n,q,s}(y) - S_{n,q,t}(y)\| &\leq K_2 \frac{2^{n(1-\alpha)}}{p^\alpha} |s - t|. \end{aligned}$$

$$(22) \quad \text{For } t \in I, \text{ the isometry } T_{n,q,t}^{-1} \circ S_{n,q+1,t} \text{ does not depend on } t.$$

Before we proceed to the construction of the isometries $T_{n,q,t}$, $S_{n,q,t}$, we show how to construct the isometries $R_{n,q,t}$ for $0 \leq q \leq 2^{n+1} - 2$. For $t \in [q 2^{-n-1}, (q+1) 2^{-n-1}]$ we set $R_{n,q,t} = S_{n,q-1,t}$; for $t \in [(q+1) 2^{-n-1}, (q+2) 2^{-n-1}]$, we set $R_{n,q,t} = T_{n,q,t}$. Condition (17) ensures that $S_{n,q-1,\tau} = T_{n,q,\tau}$ so that $R_{n,q,t}$ is well defined. It is simple to see that conditions (6) to (10) follow from conditions (18) to (22) respectively.

We now construct the isometries $T_{n,q,t}$, $S_{n,q,t}$. Set $l = l(q)$, $l' = l(q+1)$. Thus, we either have $l' = l$ or $l' = l+1$. For $t \in [(l+1) 2^{-n}, (l+2) 2^{-n}]$, we have by induction hypothesis and (10) that, if $l' = l+1$,

$$(23) \quad R_{n-1,l,t}^{-1} \circ R_{n-1,l',t} = \text{Constant isometry} = V.$$

If $l' = l$, the above also holds, for $V = \text{identity}$. We set for simplicity $A = R_{n-1,l,\tau}$; $B = R_{n-1,l',\tau}$. It is simple to see that $\tau \in [(l+1) 2^{-n}, (l+2) 2^{-n}]$; thus, by (23), we have $A^{-1} \circ B = V$.

Given $t \in I \cap D_{n-1}$, we have

$$R_{n-1,l,t}(x_{n-1,l,t}) = R_{n-1,l',t}(x_{n-1,l',t}).$$

This is obvious if $l' = l$; if $l' = l+1$, this follows from (6). Remembering that $R_{n-1,l,t}^{-1} \circ R_{n-1,l',t} = V = A^{-1} \circ B$, we get

$$\forall t \in I \cap D_{n-1}, \quad A(x_{n-1,l,t}) = B(x_{n-1,l',t}).$$

It then follows from (5) that

$$(24) \quad \forall t \in I \cap D_{n-1}, \quad A(x_{n,q,t}) = B(x_{n,q+1,t}).$$

Since A, B are isometries, it follows from (4) that

$$\forall s, t \in I \cap D_n, \quad \|A(x_{n,q,s}) - A(x_{n,q,t})\| = \|B(x_{n,q+1,s}) - B(x_{n,q+1,t})\|.$$

Thus, there exists an isometry U of $\mathbb{R}^p$ such that

$$(25) \quad \forall t \in I \cap D_n, \quad U \circ A(x_{n,q,t}) = B(x_{n,q+1,t}).$$

Since $\text{card } I \cap D_n = p/2 + 1 < p$, we can assume that $\det U = 1$ (by composing if necessary U by a reflection through a hyperplane containing the points $A(x_{n,q,t})$, $t \in I \cap D_n$). It is then clear that we can find a semi-group $U(t)$ of isometries of $\mathbb{R}^p$, with $U(1) = U$, such that

$$(26) \quad \left\{ \begin{array}{l} \forall a, b \in \mathbb{R}, \quad \forall x, y \in \mathbb{R}^q, \\ \left\| U(a)(x) - U(a)(y) - U(b)(x) + U(b)(y) \right\| \leq K_3 |b - a| \|x - y\| \end{array} \right.$$

(actually one can take $K_3 = 2\pi$).

For $t \in I$, we set

$$\begin{aligned} T_{n,q,t} &= R_{n-1,l,t} \circ A^{-1} \circ U(\varphi(t)) \circ A \\ S_{n,q,t} &= R_{n-1,l',t} \circ B^{-1} \circ U(\varphi(t) - 1) \circ B \end{aligned}$$

where $\varphi(t) = 2^{n+1}(t - \tau)$. Thus $\varphi(\tau) = 0$, $\varphi(\tau') = 1$. Thus (17) holds.

It remains to prove (18) to (22).

Proof of (18). — It follows from (25) that, for $t \in D_n \cap I$, we have

$$A(x_{n,q,t}) = U^{-1} \circ B(x_{n,q+1,t})$$

so that

$$(27) \quad U(\varphi(t)) \circ A(x_{n,q,t}) = U(\varphi(t)-1) \circ B(x_{n,q+1,t}).$$

Since $R_{n-1,l,t}^{-1} \circ R_{n-1,l',t} = A^{-1} \circ B$, we have

$$R_{n-1,l',t} \circ B^{-1} = R_{n-1,l,t} \circ A^{-1},$$

and, combined with (27) and the definition of $T_{n,q,t}$, $S_{n,q,t}$, this implies (18).

Proof of (19). — We consider only the case of $T_{n,q,t}$, and leave the other case to the reader. By (24), (25), we have

$$t \in I \cap D_{n-1} \Rightarrow U \circ A(x_{n,q,t}) = A(x_{n,q,t})$$

so have

$$U(s) \circ A(x_{n,q,t}) = A(x_{n,q,t})$$

for all $s \in \mathbb{R}$. Thus

$$A^{-1} \circ U(\varphi(t)) \circ A(x_{n,q,t}) = x_{n,q,t},$$

which implies the result.

Proof of (20). — We prove this inequality for the constant $K_1 = 4K_3$, where K_3 occurs in (26) and we again consider only the case of $T_{n,q,t}$. We have

$$\|T_{n,q,s}(x) - T_{n,q,s}(y) - (T_{n,q,t}(x) - T_{n,q,t}(y))\| \leq (I) + (II)$$

where

$$(I) = \|R_{n-1,l,t} \circ A^{-1} \circ U(\varphi(t)) \circ A(x) - R_{n-1,l,t} \circ A^{-1} \circ U(\varphi(t)) \circ A(y) \\ - R_{n-1,l,t} \circ A^{-1} \circ U(\varphi(s)) \circ A(x) + R_{n-1,l,t} \circ A^{-1} \circ U(\varphi(s)) \circ A(y)\| \\ (II) = \|R_{n-1,l,t}(x') - R_{n-1,l,t}(y') - R_{n-1,l,s}(x') + R_{n-1,l,s}(y')\|$$

for $x' = A^{-1} \circ U(\varphi(s)) \circ A(x)$, $y' = A^{-1} \circ U(\varphi(s)) \circ A(y)$. Since A and $U(\varphi(s))$ are isometries, we have $\|x' - y'\| = \|x - y\|$. We observe that (8) holds with the same value $K = K_1$ of the constant K than (20); thus, by induction hypothesis, we have

$$(II) \leq K_1 2^{n-1} |s - t| \|x - y\|.$$

Since $R_{n-1,l,t} \circ A^{-1}$ is an isometry, we have

$$(I) = \|U(\varphi(t)) \circ A(x) - U(\varphi(t)) \circ A(y) - U(\varphi(s)) \circ A(x) + U(\varphi(s)) \circ A(y)\|.$$

Since $\|A(x) - A(y)\| = \|x - y\|$, $|\varphi(t)| \leq 2^{n+1} |s - t|$, by (26) we have

$$(I) \leq K_3 2^{n+1} |s - t| \|x - y\|.$$

Thus

$$(I) + (II) \leq (K_1 2^{n-1} + K_3 2^{n+1}) |s - t| \|x - y\| \leq K_1 2^n |s - t| \|x - y\|$$

since $K_1 = 4K_3$.

Proof of (21). — We prove (21) for $K_2 = 2^\alpha K_1 (1 - 2^{-(1-\alpha)})^{-1}$, and again we consider only the case of T. We proceed by induction, observing that (9) holds with the same constant K_2 . We find v in $I \cap D_{n-1}$ with $|u - v| \leq 2^{-n+1}/p$. We set $z = x_{n,q,v}$. We have

$$\|T_{n,q,s}(x) - T_{n,q,t}(x)\| \leq (I) + (II)$$

where

$$(I) = \|T_{n,q,s}(z) - T_{n,q,t}(z)\|$$

$$(II) = \|T_{n,q,t}(x) - T_{n,q,t}(z) - (T_{n,q,s}(x) - T_{n,q,s}(z))\|.$$

We recall that by (24), (25)

$$U \circ A(z) = B(x_{n,q+1,v}) = A(z),$$

so that, by definition of $T_{n,q,t}$

$$(I) = \|R_{n-1,l(q),s}(z) - R_{n-1,l(q),t}(z)\|$$

and, by induction hypothesis,

$$(I) \leq K_2 \frac{2^{(n-1)(1-\alpha)}}{p^\alpha} \cdot |s - t|.$$

If we recall that (20) holds for the constant K_1 we have

$$(II) \leq K_1 2^n |s - t| \|x - z\|.$$

Since, by (4),

$$\|x - z\| = |u - v|^\alpha \leq 2^{(-n+1)\alpha}/p^\alpha,$$

we have $(II) \leq K_1 2^\alpha 2^{n(1-\alpha)} |s - t| p^{-\alpha}$. Thus

$$(I) + (II) \leq \frac{2^{n(1-\alpha)}}{p^\alpha} [2^\alpha K_1 + K_2 2^{-(1-\alpha)}] |s - t| = \frac{2^{n(1-\alpha)}}{p^\alpha} K_2 |s - t|.$$

Proof of (22). — We have, for $t \in J$,

$$T_{n,q,t}^{-1} \circ S_{n,q,t} = A^{-1} \circ U(-\varphi(t)) \circ A \circ R_{n-1,l,t}^{-1} \circ R_{n-1,l',t} \circ B^{-1} \circ U(\varphi(t) - 1) \circ B.$$

Since, by (23), we have $R_{n-1,l,t}^{-1} \circ R_{n-1,l',t} = A^{-1} \circ B$, we have

$$T_{n,q,t}^{-1} \circ S_{n,q,t} = A^{-1} \circ U(-\varphi(t)) \circ U(\varphi(t) - 1) \circ B = A^{-1} \circ U^{-1} \circ B,$$

and this does not depend on t .

The proof is complete.

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