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Monotonicity properties of some skew tent maps

by

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ABSTRACT. — We investigate properties of monotonicity of kneading sequences and topological entropy for some unimodal maps of the interval, called skew tent maps given by the formula

$$F_{\lambda,\mu}(x)=\begin{cases} 1+\lambda x & \text{if} \quad x\leqq 0 \\ 1-\mu x & \text{if} \quad x\geqq 0 \end{cases}$$

when the parameters belong to the region

$$D_1=\{(\lambda,\mu)\,|\,\lambda\leqq 1,\ \mu>1\}$$

The case $\lambda\geqq 1$ has been studied in [M-V] and the authors proved that kneading sequences and entropy are stricly increasing functions of λ and μ. Here we prove, when $\lambda<1$, that kneading sequences and entropy are increasing and we identify some regions where these functions of λ and μ are constant. We prove also that the set of (λ, μ) where the kneading sequences is a primary itinerary $\underline{M}$ is the graph of a decreasing function $\lambda=\beta_{\underline{M}}(\mu)$ as in [M-V].

Key words : Unimodal maps, tent maps, kneading sequences, topological entropy.

RÉSUMÉ. — On étudie les propriétés de croissance des itinéraires et de l'entropie topologique de certaines applications unimodales de l'intervalle, appelées applications tentes et données par les formules

$$F_{\lambda,\mu}(x)=\begin{cases} 1+\lambda x & \text{si} \quad x\leqq 0 \\ 1-\mu x & \text{si} \quad x\geqq 0 \end{cases}$$

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les paramètres appartenant au domaine

$$D_1 = \{ (\lambda, \mu) \mid \lambda \leqq 1, \mu > 1 \}$$

Le cas $\lambda > 1$ a été étudié dans [M-V] et les auteurs ont montré que les itinéraires et l'entropie sont des fonctions strictement croissantes de λ et μ. Ici nous montrons que si $\lambda < 1$ les itinéraires et l'entropie sont des fonctions croissantes et nous identifions les régions ou ces fonctions sont constantes. Nous montrons également que l'ensemble des (λ, μ) ou l'itinéraire $\underline{M}$ est probablement primaire est le graphe d'une fonction décroissante $\lambda = \beta_{\underline{M}}(\mu)$ comme dans [M-V].

INTRODUCTION

The problem of the monotonicity of the entropy for various classes of unimodal maps of the interval depending on real parameters is very natural. Few answers are given today to this question, only the following cases, to our knowledge, have a positive answer.

For the tent maps $F_{\mu, \mu}$, the answer is simple, the entropy is $\log \mu$ [M-S]. For the expensive skew tent maps $(\lambda \geqq 1, \mu > 1)$ the answer is also given in [M-V] and uses the theory of kneading sequences. The entropy is a strictly increasing function of λ and μ, and recently in [B-M-T] the authors gave elegant proof for the case above by studying trapezoidal maps, but they only proved monotonicity (not strict monotonicity). If $\mu < 1$, there exists an attractive fixed point which attracts everything except perhaps one point and the kneading sequence is $\mathbf{R}^\infty$. If $\mu = 1$ then there exists a whole interval of periodic points of period 2 (except a fixed point) and the kneading sequence is RC. In both cases the entropy is 0.

For the quadratic maps $g_\mu(x) = 1 - \mu x^2$, there exists also a positive answer as a corollary of the study of the complex maps $g_c(z) = z^2 + c$, [D]. It will be interesting to have a proof using only real technics for this case.

Here we study the properties of monotonicity of the skew tent maps

$$F_{\lambda, \mu}(x) = \begin{cases} 1 + \lambda x & \text{if } x \leqq 0 \\ 1 - \mu x & \text{if } x \geqq 0 \end{cases}$$

when the parameters belong to the region

$$D_1 = \{ (\lambda, \mu) \mid \lambda \leqq 1, \mu > 1 \}$$

We consider the relation $(\lambda', \mu') > (\lambda, \mu)$ if $\lambda' \geqq \lambda$, $\mu' \geqq \mu$ and at least one of these inegalities is strict. Our results are in the same spirit as those of [M-V] and we use some results that are proved there.

This paper is organised as follows. In section 1, we state the results. In section 2 we prove theorem 1. Section 3 is very technical, we investigate the partial derivative associated to kneading sequences. In section 4 we prove the main theorems 2 and 3.

1. STATEMENT OF RESULTS

We consider a partition of the region $D_1 = \{(\lambda, \mu) \mid \lambda \leqq 1, \mu > 1\}$ by the sets

$$\mathcal{R}_m = \{(\lambda, \mu) \in D_1 \mid \lambda^{m-1}\mu > \lambda^{m-1} + \lambda^{m-2} + \ldots + 1, \; \lambda^m \mu \leqq \lambda^m + \lambda^{m-1} + \ldots + 1\}$$
$$m = 1, 2 \ldots$$

Then each set $\mathcal{R}_m$ is decomposed into three disjoint sets

$$A_m = \{(\lambda, \mu) \in \mathcal{R}_m \mid \lambda^m \mu \leqq 1\},$$
$$B_m = \{(\lambda, \mu) \in \mathcal{R}_m \mid \lambda^m \mu \geqq 1, \; \lambda^m \mu^2 \leqq \lambda + \mu\},$$
$$C_m = \{(\lambda, \mu) \in \mathcal{R}_m \mid \lambda^m \mu^2 > \lambda + \mu\}.$$

For $m = 1, 2, 3$ these sets are pictured in Figure 1.

For $m \geqq 1$ the graphs of the functions

$$\begin{cases} \lambda^{m-1}\mu = \lambda^{m-1} + \lambda^{m-2} + \ldots + 1 \\ \lambda^m \mu = 1 \end{cases}$$

have one common point with coordinates $(\lambda_{m,}, \mu_m)$ such that

$$\lambda_m^m + \lambda_m^{m-1} + \ldots + \lambda_m = 1 \qquad \text{and} \qquad \mu_m = \frac{1}{\lambda_m^m}.$$

So the values λ_m are decreasing to $\frac{1}{2}$, and the values μ_m are increasing to infinity.

For $m \geqq 2$ the graphs of functions

$$\begin{cases} \lambda^{m-1}\mu = \lambda^{m-1} + \lambda^{m-2} + \ldots + 1 \\ \lambda^m \mu^2 = \lambda + \mu \end{cases}$$

have one common point with coordinates (λ'_m, μ'_m) such that

$$(\lambda'_m)^{2m} + (\lambda'_m)^{2m-1} + \ldots + (\lambda'_m)^{m+2} + 1 = (\lambda'_m)^m + (\lambda'_m)^{m-1} + \ldots + \lambda'_m$$

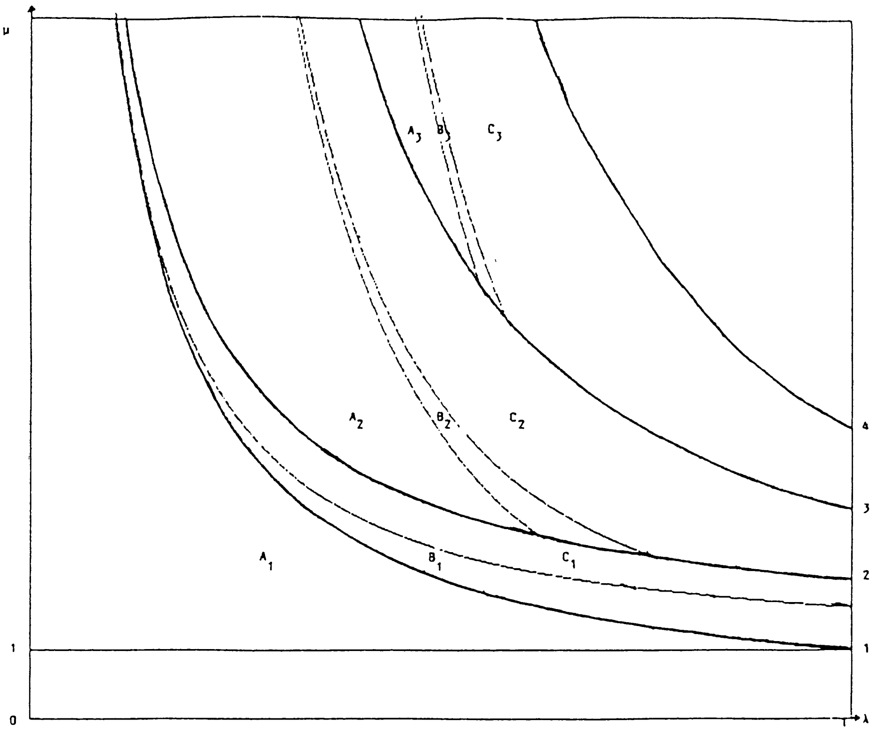

FIG. 1.

and

$$\mu'_m = \frac{(\lambda'_m)^m - 1}{(\lambda'_m)^m - (\lambda'_m)^{m-1}}$$

So the values λ'_m are also decreasing to $\frac{1}{2}$.

As in [M-V] we shall denote by $K(\lambda, \mu)$, $h(\lambda, \mu)$ the kneading sequence and the entropy of the skew tent map $F_{\lambda, \mu}$.

We shall also use for kneading sequences the notations of [C-E].

A maximal admissible sequence $\underline{M}$ is called **strictly primary** if

(1) $\underline{M} > RLR^{\infty}$.

(2) If $\underline{M} = \underline{A} * \underline{B}$ with $\underline{A} \neq 0$, then $\underline{B} = C$.

We shall denote by $\mathscr{P}$ the class of strictly primary sequences.

A maximal admissible sequence $\underline{M}$ is **primary** if

(1) $\underline{M} > R^{*\infty}$.

(2) If $\underline{M} = \underline{A} * \underline{B}$ with $\underline{A} \neq 0$, $\underline{B} \neq C$ then $\underline{A} = R^{*m}$ for some $m > 0$.

We shall denote by $\mathscr{M}$ the class of primary sequences.

Now we state the main results.

THEOREM 1. — $(\lambda, \mu) \in \mathcal{R}_m$ *if and only if* $RL^{m-1}C < K(\lambda, \mu) \leqq RL^m C$.
More precisely:

$(\lambda, \mu) \in \mathcal{A}_m$ *if and only if* $K(\lambda, \mu) = RL^{m-1} * R^\infty = (RL^m)^\infty$.

$(\lambda, \mu) \in \mathcal{B}_m$ *if and only if* $K(\lambda, \mu) = RL^{m-1} * \underline{B}$ *with* $\underline{B} \in \mathcal{M}$.

$(\lambda, \mu) \in \mathcal{C}_m$ *if and only if* $K(\lambda, \mu) = RL^m R \ldots$ *and* $K(\lambda, \mu) \in \mathcal{P}$.

THEOREM 2. — *Consider* $(\lambda, \mu), (\lambda', \mu') \in D_1$ *such that* $(\lambda', \mu') > (\lambda, \mu)$.

(1) *If* (λ, μ) *and* (λ', μ') *belong to different sets* $\mathcal{R}_m$ *then* $K(\lambda', \mu') > K(\lambda, \mu)$.

(2) *If* $(\lambda, \mu), (\lambda', \mu') \in A_m$ *then* $K(\lambda', \mu') = K(\lambda, \mu) = (RL^m)^\infty$.

(3) *If* $(\lambda, \mu), (\lambda', \mu') \in B_m \cup C_m$ *then* $K(\lambda', \mu') > K(\lambda, \mu)$.

COROLLARY. — *Let* $(\lambda', \mu'), (\lambda, \mu) \in D_1$ *such that* $(\lambda', \mu') > (\lambda, \mu)$.

(1) *If* (λ, μ) *and* (λ', μ') *belong to different sets* $\mathcal{R}_m$ *then* $h(\lambda', \mu') > h(\lambda, \mu)$.

(2) *If* $(\lambda, \mu), (\lambda', \mu') \in A_m \cup B_m$ *then* $h(\lambda', \mu') = h(\lambda, \mu)$.

(3) *If* $(\lambda, \mu), (\lambda', \mu') \in C_m$ *then* $h(\lambda', \mu') > h(\lambda, \mu)$.

This corollary is an immediate consequence of theorem 2 because we know the relations between kneading sequences and entropy ([J-R], [M-S]). In particular the entropy is constant on the intervals $[RL^{m-1} * R^\infty, RL^{m-1} * RL^\infty]$.

THEOREM 3. — *For each* $\underline{M} \in \mathcal{M}$ *(except* RL^∞*) there exists a number* $\gamma(\underline{M})$ *and a continuous decreasing function* $\beta_{\underline{M}} : [\gamma(\underline{M}), \infty[\to]0,1]$ *such that* $K(\lambda, \mu) = \underline{M}$ *if and only if* $\lambda = \beta_{\underline{M}}(\mu)$.

The graphs of the functions $\beta_{\underline{M}}$ *fill up all the sets* $\bigcup_{m=1}^{\infty} C_m$ *and* B_1. *And we have* $\lim_{\mu \nearrow \infty} \beta_{\underline{M}}(\mu) = 0$.

In fact for each $\underline{M}$ these functions are the continuous extensions for $\lambda < 1$ of the functions $\beta_{\underline{M}}$ given in the theorem C of [M-V] for $\lambda \geqq 1$.

2. KNEADING SEQUENCES OF SKEW TENT MAPS

We set $x_n = F^n_{\lambda, \mu}(1)$. The kneading sequence of $F_{\lambda, \mu}$ is denoted by

$$K(\lambda, \mu) = A_0 \ldots A_{n-1} C \quad \text{if} \quad x_i \neq 0 \text{ for } i = 0, \ldots, n-1 \text{ and } x_n = 0;$$
$$K(\lambda, \mu) = A_0 A_1 \ldots \quad \text{if} \quad x_i \neq 0 \text{ for } i \geqq 0,$$

where $A_i = L$ or R when $x_i < 0$ or $x_i > 0$ respectively.

If $\underline{A} = A_0 \ldots A_{n-1}$ is a finite word with alphabet (L, R) we define $|\underline{A}|$ as the number of letters in A and $\theta_1(\underline{A})$, $\theta_2(\underline{A})$ as the numbers of L, R respectively in $\underline{A}$. We set also $\varepsilon(\underline{A}) = (-1)^{\theta_2(\underline{A})}$. We will say that a unimodal map f has a n-renormalisation if there exists an interval I_0 containing 0, invariant by f^n, such that $f^n_{|I_0}$ is topologically conjugate to an unimodal map g. If a skew tent $F_{\lambda,\mu}$ has a n-renormalisation, it is clear that the map g is also a skew tent map with slopes given by the following lemma.

LEMMA 2.1. – *$K(\lambda, \mu)=\underline{A}*\underline{B}$ if and only if $F_{\lambda,\mu}$ have a $|A|+1$-renormalisation. Moreover if $K(\lambda, \mu)=\underline{A}*\underline{B}$ the map $F^{|A|+1}_{\lambda,\mu\,|I_0}$ is linearly conjugate to the skew tent map $F_{\varphi_{\underline{A}}(\lambda,\mu)}$ where*

$$\varphi_{\underline{A}}(\lambda, \mu)=\begin{cases}(\lambda^{\theta_1(\underline{A})+1}\mu^{\theta_2(\underline{A})}, \lambda^{\theta_1(\underline{A})}\mu^{\theta_2(\underline{A})+1}) & \text{if } \varepsilon(\underline{A})=1\\ (\lambda^{\theta_1(\underline{A})}\mu^{\theta_2(\underline{A})+1}, \lambda^{\theta_1(\underline{A})+1}\mu^{\theta_2(\underline{A})}) & \text{if } \varepsilon(\underline{A})=-1\end{cases} \tag{2.1}$$

and $K(\varphi_{\underline{A}}(\lambda, \mu))=\underline{B}$.

Proof. – The equivalence is proved in [C-E], p. 147.

Now if $K(\lambda, \mu)=\underline{A}*\underline{B}$ the map $F^{|A|+1}_{\lambda,\mu}$ has an invariant segment I_0 containing 0 such that $F^{|A|+1}_{\lambda,\mu\,|I_0}$ is unimodal and piecewise linear. By the linear conjugacy $\psi(x)=F^{|A|+1}_{\lambda,\mu}(0)\,x$, we have again a kew tent map given by

$$F_{\varphi_{\underline{A}}(\lambda,\mu)}(x)=\frac{1}{F^{|A|+1}_{\lambda,\mu}(0)}F^{|A|+1}_{\lambda,\mu}(x\,.\,F^{|A|+1}_{\lambda,\mu}(0)).$$

By the definition of the $*$ product, if $\varepsilon(\underline{A})=1$ then $F^{|A|+1}_{\lambda,\mu}(0)>0$ and by conjugacy we have

$$\varphi_{\underline{A}}(\lambda, \mu)=(\lambda^{\theta_1(\underline{A})+1}\mu^{\theta_2(\underline{A})}, \lambda^{\theta_1(\underline{A})}\mu^{\theta_2(\underline{A})+1}).$$

If $\varepsilon(\underline{A})=-1$ then $F^{|A|+1}_{\lambda,\mu}(0)<0$ and the conjugacy reverses the slopes. This proves formulas (2.1).

Moreover $K(\varphi_{\underline{A}}(\lambda, \mu))=K(F^{|A|+1}_{\lambda,\mu}|I_0)=\underline{B}$.

LEMMA 2.2. – *If $K(\lambda, \mu)=\underline{A}*\underline{B}$ then we have*

$$\lambda^{\theta_1(\underline{A})+1}\mu^{\theta_2(\underline{A})+1}\leqq\lambda+\mu. \tag{2.2}$$

Proof. – By [M-V], lemma 3.1, if $K(\lambda, \mu)=\underline{A}*\underline{B}$, the slopes $\varphi_{\underline{A}}(\lambda, \mu)$ belong to $\mathcal{R}=\{x, y\,|\,x>0, y>0, xy\leqq x+y\}$. Using formula (2.1) we obtain (2.2).

Now if $\underline{A}=A_0A_1\ldots A_{|\underline{A}|}$ is a finite word with alphabet (L, R) we denote by $\underline{A}_j$ the word

$$\underline{A}_j=A_0A_1\ldots A_{j-1} \quad \text{for } j=1, 2\ldots, |\underline{A}|$$

where $\underline{A}_0=\varnothing$ and therefore $\varepsilon(\underline{A}_0)=1$, $\theta_1(\underline{A}_0)=\theta_2(\underline{A}_0)=0$.

To each finite word we associate the following characteristic polynomial

$$x_{\underline{A}}(\lambda, \mu)=\varepsilon(\underline{A})\sum_{j=0}^{|A|}\varepsilon(\underline{A}_j)\lambda^{\theta_1(\underline{A})-\theta_1(\underline{A}_j)}\mu^{\theta_2(\underline{A})-\theta_2(\underline{A}_j)}$$

If $K(\lambda, \mu)=\underline{A}\ldots$ is a kneading sequence where $\underline{A}=A_0 A_1 \ldots A_n$ we have

$$F^{|\underline{A}|}_{(\lambda,\mu)}(1)=F^{n}_{(\lambda,\mu)}(1)=\begin{cases} 1-\mu(F^{n-1}_{(\lambda,\mu)}(1)) & \text{if } A_n=R \\ 1+\lambda(F^{n-1}_{(\lambda,\mu)}(1)) & \text{if } A_n=L \end{cases}$$

So by iteration we obtain that

$$F^{|A|}_{(\lambda,\mu)}(1)=x_{\underline{A}}(\lambda, \mu).$$

For the finite kneading sequence $K(\lambda, \mu)=\underline{A}C$ we have

$$x_{\underline{A}}(\lambda, \mu)=F^{|A|}_{(\lambda,\mu)}(1)=0.$$

Remarks 2.1. — In fact the two-variable polynomials $x_A(\lambda, \mu)$ generalise those introduced in [D-G-P] for the study of tent maps. So they satisfy the functional equation:

$$x_{\underline{A}*\underline{B}}(\lambda, \mu)=x_{\underline{A}}(\lambda, \mu).x_{\underline{B}}(\varphi_{\underline{A}(\lambda,\mu)}).$$

LEMMA 2.3. — *Assume* $(\lambda, \mu)\in D_1$ *and* $m\in N^*$.

(1) $K(\lambda, \mu)=RL^{m-1}C$ *if and only if*

$$\lambda^{m-1}\mu=\lambda^{m-1}+\lambda^{m-2}+\ldots+1.$$

(2) $K(\lambda, \mu)=RL^{m-1}*RC$ *if and only if*

$$\begin{cases} \lambda^{m-1}\mu>\lambda^{m-1}+\lambda^{m-2}+\ldots+1 \\ \lambda^{m}\mu=1 \end{cases}$$

(3) $K(\lambda, \mu)=RL^{m-1}*RL^{\infty}$ *if and only if*

$$\begin{cases} \lambda^{m-1}\mu>\lambda^{m-1}+\ldots+1 \\ \lambda^{m}\mu^{2}=\lambda+\mu \end{cases}$$

(4) $K(\lambda, \mu)=RL^{m-1}*R^{\infty}$ *if and only if*

$$\begin{cases} \lambda^{m-1}\mu>\lambda^{m-1}+\lambda^{m-2}+\ldots+1 \\ \lambda^{m}\mu<1. \end{cases}$$

Proof. — (1) Suppose $K(\lambda, \mu)=RL^{m-1}C$, then we have

$$x_{RL^{m-1}}(\lambda, \mu)=1+\lambda+\ldots+\lambda^{m-1}-\lambda^{m-1}\mu=0.$$

Now suppose $K(\lambda, \mu)\neq RL^{m-1}C$, then either $K(\lambda, \mu)>RL^{m-1}C$ or $K(\lambda, \mu)<RL^{m-1}C$.

For the first case $K(\lambda, \mu)=RL^{m-1}L\ldots$ then $x_{RL^{m-1}}(\lambda, \mu)<0$.

For the other case $K(\lambda, \mu)=RL^{p}R\ldots$ with $1\leqq p\leqq m-1$ ($p\geqq 1$ because $\mu>1$).

So we have

$$x_{RL^{p}}(\lambda, \mu)=1+\lambda+\ldots-\lambda^{p}\mu>0.$$

Since

$$x_{RL^{m-1}}(\lambda, \mu) = 1 + \lambda + \ldots + \lambda^{m-p+1} x_{RL^p}(\lambda, \mu)$$

we have $x_{RL^{m-1}}(\lambda, \mu) > 0$. This proves (1).

(2) By lemma (2.1) the equalities

$$K(\lambda, \mu) = RL^{m-1} * RC = RL^m RL^{m-1} C$$

are equivalent to $x_{RL^{m-1}}(\lambda, \mu) < 0$ and there exists a renormalisation with kneading sequence $K(\varphi_{RL^{m-1}}(\lambda, \mu)) = RC$.

Applying formula (2.1) we obtain

$$\varphi_{RL^{m-1}}(\lambda, \mu) = (\lambda^{m-1} \mu^2, \lambda^m \mu) \tag{2.3}$$

As $x_{RC}(\lambda, \mu) = \lambda - 1$ we obtain the equivalence of $K(\lambda, \mu) = RL^{m-1} * RC$ with

$$x_{RL^{m-1}}(\lambda, \mu) < 0 \quad \text{and} \quad \lambda^m \mu - 1 = 0.$$

This proves (2).

(3) By lemma (2.1) the equalities

$$K(\lambda, \mu) = RL^{m-1} * RL^\infty = RL^m (RL^{m-1} R)^\infty$$

are equivalent to $x_{RL^{m-1}}(\lambda, \mu) < 0$ and there exists a renormalisation with kneading sequence $K(\varphi_{RL^{m-1}}(\lambda, \mu)) = RL^\infty$. We obtain the equivalence of $K(\lambda, \mu) = RL^{m-1} * RL^\infty$ with

$$x_{RL^{m-1}}(\lambda, \mu) < 0 \quad \text{and} \quad \lambda^m \mu^2 = \lambda + \mu.$$

This proves (3).

(4) By lemma (2.1) the equalities $K(\lambda, \mu) = RL^{m-1} * R^\infty = (RL^m)^\infty$ are equivalent to $x_{RL^{m-1}}(\lambda, \mu) < 0$ and there exists a renormalisation with kneading sequence $K(\varphi_{RL^{m-1}}(\lambda, \mu)) = R^\infty$.

As the kneading sequence R^∞ is associated to $\mu < 1$, by lemma (2.1) and formula (2.3) we obtain the equivalence of $K(\lambda, \mu) = RL^{m-1} * R^\infty$ with

$$x_{RL^{m-1}}(\lambda, \mu) < 0 \quad \text{and} \quad \lambda^m \mu < 1.$$

This proves (4).

Proof of Theorem 1

(1) Assume that $RL^{m-1} C < K(\lambda, \mu) \leqq RL^m C$, then $K(\lambda, \mu) = RL^m R \ldots$ or $K(\lambda, \mu) = RL^m C$. This is equivalent to

$$\begin{cases} x_{RL^k}(\lambda, \mu) = 1 + \ldots + \lambda^k - \lambda^k \mu < 0 & \text{for } k = 1 \ldots m-1, \\ x_{RL^m}(\lambda, \mu) = 1 + \ldots + \lambda^m \mu \geqq 0. \end{cases}$$

Taking into account that the first inequalities are equivalent to

$$x_{RL^{m-1}}(\lambda, \mu)=1+\lambda+\ldots+\lambda^{m-1}-\lambda^{m-1}\mu<0,$$

this proves the first assertion of theorem 1.

(2) The second assertion follows immediatly from part (4) of lemma (2.3).

(3) Assume that $K(\lambda, \mu)=RL^{m-1} * \underline{B}$ with $\underline{B}\in\mathcal{M}$. Necessarly we get $x_{RL^{m-1}}(\lambda, \mu)<0$ and there exists a renormalisation with kneading sequence $K(\varphi_{RL^{m-1}}(\lambda, \mu))=\underline{B}$ such that $R^{\infty}<\underline{B}\leqq RL^{\infty}$. From formulas (2.2) and (2.3) we obtain

$$\lambda^{m-1}\mu>\lambda^{m-1}+\ldots+1, \qquad \lambda^{m}\mu>1, \qquad \lambda^{m}\mu^{2}\leqq\lambda+\mu,$$

so $(\lambda, \mu)\in B_m$.

Conversely, if $(\lambda, \mu)\in B_m$, as $B_m\subset R_m$ we have $K(\lambda, \mu)=RL^m\ldots$ Therefore $\lambda\mu^2\leqq\lambda+\mu$, so the maps $F_{\lambda,\mu}^{m+1}$ with slopes $(\lambda^{m-1}\mu^2, \lambda^m\mu)$ near the turning point have an invariant segment I_0. From lemma 2.1 we get $K(\lambda, \mu)=RL^{m-1} * \underline{B}$ with $\underline{B}=K(F_{\lambda,\mu}^{m+1}|_{I_0})$.

Notice that $\mu>1>\lambda$, so $\lambda^{m-1}\mu^2>\lambda^m\mu>1$ then from ([M-V], theorem B) we have $\underline{B}\in\mathcal{M}$. This proves the third assertion.

(4) Suppose that for $(\lambda, \mu)\in C_m$ the kneading sequence is not strictly primary. Then there exist $\underline{A}$ and $\underline{B}$ such that $K(\lambda, \mu)=\underline{A} * \underline{B}$ with $\underline{A}=RL^m RL^k R\ldots$ and $k\leqq m-1$.

By maximality of the kneading sequence there cannot be more than m consecutives symbols L after the third R. In C_m we have $\lambda^l\mu<1$, for $l=1, \ldots, m$, so we get

$$\lambda^{\theta_1(\underline{A})}\mu^{\theta_2(\underline{A})}>\lambda^{m-1}\mu,$$

and

$$\lambda^{\theta_1(\underline{A})+1}\cdot\mu^{\theta_2(\underline{A})+1}>\lambda^{m}\mu^{2}>\lambda+\mu.$$

Lemma (2.2) shows that this is impossible and this completes the proof of theorem 1.

3. ESTIMATES OF SOME PARTIAL DERIVATIVES

We set $F=F_{\lambda,\mu}$ and $x_n=F^n(1)$ and $a_n=\dfrac{\partial x_n}{\partial\lambda}$, $b_n=\dfrac{\partial x_n}{\partial\mu}$ if $x_i\neq 0$ for all $i<n$.

We have the recursive formulas

$$a_0=b_0=0$$

$$a_{n+1}=\left\{\begin{array}{lll} x_n+\lambda a_n & \text{if} & x_n<0, \\ -\mu a_n & \text{if} & x_n>0, \end{array}\right\} \tag{3.1}$$

$$b_{n+1}=\left\{\begin{array}{ll} b\lambda_n & \text{if } x_n<0, \\ -(x_n+\mu b_n) & \text{if } x_n>0. \end{array}\right\} \tag{3.2}$$

If $\underline{A}_n=A_0 A_1 \ldots A_{n-1}$, to simplify the notations we shall denote $\theta_i(n)=\theta_i(\underline{A}_n)$ ($i=1$ or 2) and $\varepsilon_n=(-1)^{\theta_2(n)}$. If $\varepsilon_n=+1$ respectively (-1), we shall write A_1^+ (respectively A_n^-) for the following symbol.

Moreover for $(\lambda, \mu)\in C_m$ we shall set $X=\lambda^m\mu$. As $\lambda^m\mu^2<\lambda+\mu$, we get $X<1+\dfrac{\lambda}{\mu}$ and $X^2-1=\dfrac{\lambda}{\mu}(X+1)>0$.

In the whole section $(\lambda, \mu)\in C_m$ with $m\geqq 1$ fixed

The aim of this section is to prove the following proposition.

PROPOSITION 3.1. – *If $(\lambda, \mu)\in C_m$ and $K(\lambda, \mu)=A_0 A_1 \ldots$ is an infinite sequence belonging to $\tilde{C}_m$, then there exists n_0 such that $A_{n_0}=R^-$ and for this n_0 we have:*

$$b_{n_0}<-\frac{b_{m+1}X}{X^2-1}, \qquad a_{n_0}<-\frac{a_{m+1}X}{X^2-1}.$$

The proof is very technical and requires an analysis of kneading sequences for $(\lambda, \mu)\in C_m$.

Taking into account the ordering between kneading sequences and using theorem 1 we have $(\lambda, \mu)\in C_m$ if and only if $K(\lambda, \mu)\in\tilde{C}_m=[RL^{m-1}*RL^\infty, RL^m C]$ and $K(\lambda, \mu)$ is a strictly primary admissible sequence.

The interval $\tilde{C}_m$ can be split into disjoint intervals

$$\tilde{C}_m=\bigcup_{p=1}^{\infty} S_{p,m}$$

where

$$S_{p,m}=[RL^m(RL^{m-1}R)^p C, RL^m(RL^{m-1}R)^{p-1}C].$$

To simplify the notations we set $\underline{A}_p=RL^m(RL^{m-1}R)^p$.

Again, each interval $S_{p,m}$ can be split into

$$S_{p,m}=\bigcup_{i=0}^{m} T^i_{p,m},$$

where

$$T^i_{p,m}=]\underline{A}_p L^i C, \underline{A}_p L^{i+1}C] \qquad \text{for } i=0, 1, \ldots, m-2,$$
$$T^{m-1}_{p,m}=]\underline{A}_p L^{m-1}C, \underline{A}_{p-1}RL^{m-2}C],$$
$$T^m_{p,m}=]\underline{A}_{p-1}RL^{m-2}C, \underline{A}_{p-1}C].$$

Now each interval $T^{i}_{p,m}$ for $i=0, 1, \ldots, m-1$ can be split into

$$U^{i,n}_{p,m}=]\underline{A}_p L^i (LRL^{m-1})^{2n-2} C, \underline{A}_p L^i (LRL^{m-1})^{2n} C] \quad \text{for} \quad n=1, 2\ldots$$
$$V^{i}_{p,m}=]\underline{A}_p L^i (LRL^{m-1})^{\infty}, \underline{A}_p L^{i+1} C] \quad \text{for} \quad i \leq m-2$$
$$V^{m-1}_{p,m}=[\underline{A}_p L^{m-1} (LRL^{m-1})^{\infty}, \underline{A}_{p-1} RL^{m-2} C] \quad \text{for} \quad i=m-1$$

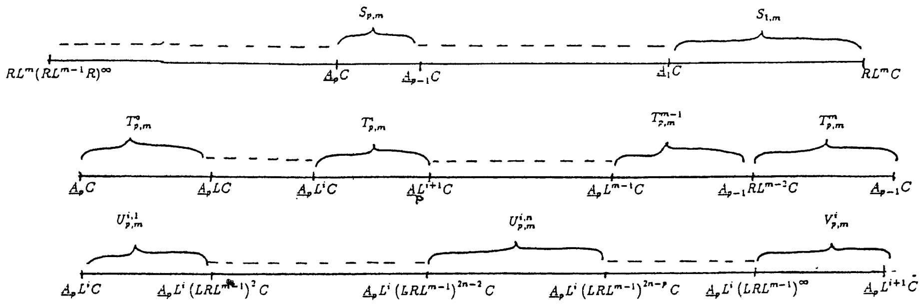

FIG. 2.

These splittings are pictured in Figure 2.

The next lemma gives the beginning of infinite kneading sequences which belong to these intervals and its proof is an obvious consequence of the ordering and the maximality properties of kneading sequences. Let $\mathring{T}$, $\mathring{V}$, $\mathring{U}$ will denote the interior of the corresponding intervals.

LEMMA 3.1. – *Let* K *infinite kneading sequence in* $\tilde{C}_m$.
If $K \in \mathring{T}^m_{p,m}$ *then* $K=\underline{A}_{p-1} RL^i R^+ \ldots$ *with* $i=0\ldots m-2$,
If $K \in \mathring{T}^i_{p,m} (i<m)$, *then* $K=\underline{A}_p L^{i+1} R^- \ldots$
If $K \in \mathring{U}^{i,n}_{p,m}$ *then*
(1) *For* $p>1$ *we have* $K=\underline{A}_p L^i (LRL^{m-1})^{2n} RR^+ \ldots$
(2) *For* $p=1$ *we have two possibilities*

$$K=\underline{A}_1 L^i (LRL^{m-1})^{2n} RL^k R^+ \ldots$$

with $k=0, \ldots, i$

$$K=\underline{A}_1 L^i (LRL^{m-1})^{2n} RL^i (LRL^{m-1})^{2k} R^+ \ldots$$

with $k=0, \ldots, n$.

To prove the proposition 3.1 we shall need estimates of partial derivatives at the ends of the intervals of kneading sequences definited above. As $K=RL^m\ldots$ we always have $x_m=1+\lambda+\ldots+\lambda^{m-1}-\lambda^{m-1}\mu<0$ and $x_m=1+\lambda x_{m-1}$.

We shall denote by β_0 and $\beta_1=1-\mu\beta_0$ the periodic points of F with period $(m+1)$ and with itineraries respectively $(RRL^{m-1})^\infty$ and $(RL^{m-1}R)^\infty$ $(0<\beta_0<\beta_1<1)$. We have

$$F^{m+1}(\beta_0)=1+\lambda+\ldots+\lambda^{m-1}-\lambda^{m-1}\mu+\lambda^{m-1}\mu^2\beta_0=x_m+\lambda^{m-1}\mu^2\beta_0.=\beta_0$$

Hence

$$\beta_0 = \frac{-x_m}{\lambda^{m-1}\mu^2 - 1}. \tag{3.3}$$

Moreover as

$K = \underline{A}_p \ldots = RL^m (RL^{m-1} R)^p > \ldots > RL^{m-1} * RL^\infty = RL^m (RL^{m-1} R)^\infty$

we get from the definition of the order of kneading sequences that

$$x_{p(m+1)} < \beta_1 \qquad (p \geqq 1). \tag{3.4}$$

Let α_p be the positive functions defined by the following recursive relations

$$\alpha_0 = 1, \qquad \alpha_1 = X - 1,$$
$$\alpha_p = \lambda^{m-1}\mu^2 \alpha_{p-1} - 1 \quad \text{for } p \geqq 2 \qquad (X = \lambda^m \mu).$$

It is easy to verify by induction that these (α_p) are such that

$$x_{\underline{A}_p} = x_{(p+1)(m+1)} = 1 + \mu \alpha_p x_m.$$

We set

$$B = -b_{m+1} = \lambda^m$$

and

$$b^*_{p(m+1)} = b_{p(m+1)} + x_{pm+p-1} \qquad (p \geqq 2).$$

LEMMA 3.2. – (1) *Assume that* $K = \underline{A}_p \ldots$, *then we have*

$$b_{2m+2} < b^*_{2m+2} < -B\mu(-x_m) < 0, \tag{3.5}$$

$$b^*_{(p+1)(m+1)} < b_{2m+2} \frac{\alpha_p}{X-1} < 0 \qquad \textit{for} \quad p \geqq 2. \tag{3.6}$$

(2) *Assume that* $K = \underline{A}_{p-1} RL^i \ldots$ *for* $p \geqq 1$, $0 \leqq i \leqq m-2$, *then we have*

$$b_{m+1+i+1} > B(-x_m) > 0, \tag{3.7}$$

$$b_{p(m+1)+i+1} > -\frac{b_{2m+2}}{\lambda^{m-i-1}\mu} \frac{\alpha_p}{X-1} > 0 \qquad \textit{for} \quad p \geqq 2. \tag{3.8}$$

Proof:

(1) *Estimate of* b_{2m+2} *for* $K = \underline{A}_1 \ldots$ – By iterations of (3.2) we get

$$\left.\begin{aligned} b_{2m+2} &= -\lambda^{m-1}\mu^2 B + \lambda^{m-1}\mu x_{m+1} - x_{2m+1}, \\ b^*_{2m+2} &= -\lambda^{m-1}\mu^2 B + \lambda^{m-1}\mu x_{m+1}. \end{aligned}\right\} \tag{$*$}$$

Using (3.4) and $1 = \beta_1 + \mu\beta_0$ we obtain

$$b^*_{2m+2} < -\lambda^{m-1}\mu^2 B(\beta_1 + \mu\beta_0) + \lambda^{m-1}\mu\beta_1.$$

Since $B = \lambda^m$ we find

$$b^*_{2m+2} < -\lambda^{m-1}\mu^3\beta_0 B - \lambda^{m-1}\mu\beta_1 (X-1),$$

From this it follows that

$$b^*_{2m+2} < -\lambda^{m-1}\mu^3\beta_0 B = -\frac{\lambda^{m-1}\mu^3 B(-x_m)}{\lambda^{m-1}\mu^2 - 1} < -B\mu(-x_m),$$

and that

$$\lambda^{m-1}\mu\beta_1 < -\frac{b^*_{2m+2}}{X-1}. \tag{3.5)'}$$

(2) *Estimate of* $b^*_{(p+1)(m+1)}$ *for* $\underline{A}_p\ldots(p \geqq 2)$. – By iterations of (3.2) we get

$$\left.\begin{aligned} b_{(p+1)(m+1)} &= \lambda^{m-1}\mu^2 b_{p(m+1)} + \lambda^{m-1}\mu x_{p(m+1)} - x_{(p+1)m+p}, \\ b^*_{(p+1)(m+1)} &= \lambda^{m-1}\mu^2 b_{p(m+1)} + \lambda^{m-1}\mu x_{p(m+1)}. \end{aligned}\right\} (*, *)$$

By (3.4) we obtain

$$b^*_{(p+1)(m+1)} < \lambda^{m-1}\mu^2 b_{p(m+1)} + \lambda^{m-1}\mu\beta_1.$$

Now using (3.5)′ and the following recursive property

$$b_{2m+2} = b_{2m+2}\frac{\alpha_1}{X-1} \qquad \text{for} \quad p=2$$

$$b_{p(m+1)} < b^*_{p(m+1)} < b_{2m+2}\frac{\alpha_{p-1}}{X-1} \qquad \text{for} \quad p>2$$

we obtain for $p \geqq 2$,

$$b^*_{(p+1)(m+1)} < b_{2m+2}\frac{\lambda^{m-1}\mu^2\alpha_{p-1}-1}{X-1} = b_{2m+2}\frac{\alpha_p}{X-1}.$$

(3) *Estimate of* $b_{m+1+i+1}$ *for* $\underline{A}_0\,\mathrm{RL}^i$. – By iteration of (3.2) we get

$$b_{m+1+i+1} = \lambda^i\mu B - \lambda^i x_{m+1} = \frac{\lambda^{m-1}\mu^2 - \lambda^{m-1}\mu x_{m+1}}{\lambda^{m-i-1}\mu}.$$

The numerator is the expression of $(-b^*_{2m+2})$ obtained in the first part $(*)$. So by the same steps we get (recall that $\lambda \leqq 1$)

$$b_{m+1+i+1} > \frac{B\lambda^{m-1}\mu^3\beta_0}{\lambda^{m-i-1}\mu} > \frac{B(-x_m)}{\lambda^{m-1-i}} > B(-x_m).$$

(4) *Estimate of* $b_{p(m+1)+i+1}$ *for* $\underline{A}_{p-1}\,\mathrm{RL}^i\ldots(p \geqq 2)$. – We have

$$b_{p(m+1)+i+1} = \lambda^i\mu b_{p(m+1)} - \lambda^i x_{p(m+1)} = \frac{\lambda^{m-1}\mu^2 b_{p(m+1)} - \lambda^{m-1}\mu x_{p(m+1)}}{\lambda^{m-i-1}\mu}.$$

The numerator is the expression of $(-b^*_{(m+1)(p+1)})$ in the second part $(*, *)$. So by the same steps we get

$$b_{p(m+1)+i+1} > -\frac{b_{2m+2}}{\lambda^{m-i-1}\mu}\,\frac{\alpha_p}{X-1},$$

and this proves the lemma 3.2.

Now we shall prove a similar lemma for the partial derivatives a_n with $(\lambda, \mu) \in C_m$. We shall denote

$$A = -a_{m+1} = -(\lambda^{m-1} x_1 + \lambda^{m-2} x_2 + \ldots + x_m)$$

where $x_1, x_2, \ldots, x_m$ are all negatives.

LEMMA 3.3. – (1) *Assume that* $K = \underline{A}_p \ldots$, *then we have*

$$a_{2m+2} < -A\mu(-x_m) < 0 \tag{3.9}$$

$$a_{(p+1)(m+1)} < a_{2m+2} \frac{\alpha_p}{X-1} < 0, \quad \textit{for} \quad p \geqq 2, \tag{3.10}$$

(*with equality if* $p = 1$).

(2) *Assume that* $K = \underline{A}_{p-1} RL^i \ldots$ *for* $p \geqq 1$ *and* $0 \leqq i \leqq m-2$ *then we have*

$$a_{m+1+i+1} > A(-x_m) > 0, \tag{3.11}$$

$$a_{p(m+1)+i+1} > -\frac{a_{2m+2}}{\lambda^{m-i-1}\mu} \frac{\alpha_p}{X-1} > 0 \quad \textit{for} \quad p \geqq 2. \tag{3.12}$$

Proof:

(1) *Estimate of* a_{2m+2} *for* $K = \underline{A}_1 \ldots$ – By iterations of (3.1) we get

$$a_{2m+2} = -\lambda^{m-1}\mu^2 A - \mu(\lambda^{m-2} x_{m+2} + \lambda^{m-3} x_{m+3} + \ldots + x_{2m}). \tag{$*$}$$

By (3.4) we have $x_{m+1} < \beta_1$, hence $x_{m+2} = 1 - \mu x_{m+1} > 1 - \mu\beta_1 = F(\beta_1)$ and $x_{m+3} = 1 + \lambda x_{m+2} > 1 + \lambda F(\beta_1)$ and so on. Therefore

$$x_{m+i} > F^{i-1}(\beta_1) \quad \text{for} \quad 2 \leqq i \leqq m.$$

It follows that

$$a_{2m+2} < -\lambda^{m-1}\mu^2 A - \mu[\lambda^{m-2} F(\beta_1) + \ldots + F^{m-1}(\beta_1)].$$

Using $1 = \beta_1 + \mu\beta_0$ we get

$$a_{2m+2} < -\lambda^{m-1}\mu^3\beta_0 A - \lambda^{m-1}\mu^2\beta_1 A - \mu[\lambda^{m-2} F(\beta_1) + \ldots + F^{m-1}(\beta_1)].$$

As $\beta_1 < 1$, it is easy to check that

$$x_i\beta_1 < F^i(\beta_1) < 0 \quad \text{for} \quad 1 \leqq i \leqq m-1.$$

So that

$$-A\beta_1 = \lambda^{m-1} x_1\beta_1 + \ldots + x_m\beta_1 < \lambda^{m-1} x_1\beta_1 + \ldots + \lambda x_{m-1}\beta_1 < \lambda^{m-1} F(\beta_1) + \ldots + \lambda F^{m-1}(\beta_1)$$

and

$$-\lambda^{m-1}\mu^2\beta_1 A < \mu X[\lambda^{m-2} F(\beta_1) + \ldots + F^{m-1}(\beta_1)]$$

Therefore

$$a_{2m+2} < -\lambda^{m-1}\mu^3\beta_0 A + \mu(X-1)[\lambda^{m-2} F(\beta_1) + \ldots + F^{m-1}(\beta_1)].$$

It follows that

$$-\mu[\lambda^{m-2}F(\beta_1)+\ldots+F^{m-1}(\beta_1)]<-\frac{a_{2m+2}}{X-1}. \qquad (3.9)'$$

and using (3.3) we get

$$a_{2m+2}<-\lambda^{m-1}\mu^3\beta_0 A<-A\mu(-x_m).$$

(2) *Estimates of* $a_{(p+1)(m+1)}$ *for* $\underline{A}_p\ldots(p\geqq 2)$. – By iterations of (3.1) we get

$$a_{(p+1)(m+1)}=\lambda^{m-1}\mu^2 a_{p(m+1)} -\mu[\lambda^{m-2}x_{p(m+1)+1}+\ldots+x_{p(m+1)+m-1}]. \qquad (**)$$

By (3.4) we easily check that $x_{p(m+1)+i}>F^i(\beta_i)$, so that

$$a_{(p+1)(m+1)}<\lambda^{m-1}\mu^2 a_{p(m+1)}-\mu[\lambda^{m-2}F(\beta_1)+\ldots+F^{m-1}(\beta_1)].$$

Using (3.9)′ and the recursive property we get

$$a_{(p+1)(m+1)}<a_{2m+2}\frac{\lambda^{m-1}\mu^2\alpha_{p-1}-1}{X-1}=a_{2m+2}\frac{\alpha_p}{X-1}.$$

(3) *Estimate of* $a_{m+1+i+1}$ *for* $K=\underline{A}_0 RL^i (0\leqq i\leqq m-2)$. – By iterations of (3.1) we get

$$a_{m+1+i+1}=\lambda^i a_{m+2}+\lambda^{i-1}x_{m+2}+\ldots+x_{m+i+1}.$$

As $a_{m+2}=-\mu a_{m+1}=\mu A$ we can write

$$a_{m+1+i+1}=\frac{1}{\lambda^{m-i-1}\mu}[\lambda^{m-1}\mu^2 A+\mu(\lambda^{m-2}x_{m+2}+\ldots+\lambda^{m-i-1}x_{m+i+1})].$$

Using $1=\beta_1+\mu\beta_0$ and $x_{m+i}>F^{i-1}(\beta_1)$ we get

$$a_{m+1+i+1}>\frac{1}{\lambda^{m-i-1}\mu}[\lambda^{m-1}\mu^3\beta_0 A+\lambda^{m-1}\mu^2\beta_1 A +\mu[\lambda^{m-2}F(\beta_1)+\ldots+\lambda^{m-i-1}F^i(\beta_1)]].$$

From the inequality (*) we obtain

$$A\beta_1>\lambda^{m-1}F(\beta_1)+\ldots+\lambda^{m-i-1}F^i(\beta_1).$$

Hence

$$a_{m+1+i+1}>\frac{1}{\lambda^{m-i-1}\mu}[\lambda^{m-1}\mu^3\beta_0 A-\mu(X-1) \times[-\lambda^{m-2}F(\beta_1)\ldots\lambda^{m-i-1}(F^i(\beta_1)]].$$

As all the $F^j(\beta_1)$ for $j=1\ldots i$ are negative we get

$$a_{m+1+i+1}>\frac{\lambda^{m-1}\mu^3\beta_0 A}{\lambda^{m-i-1}\mu}>\frac{A(-x_m)}{\lambda^{m-i-1}}>A(-x_m).$$

(4) *Estimate of* $a_{p(m+1)+i+1}$ *for* $K=\underline{A}_{(p-1)}RL^i\ldots$ $(p\geqq 2)$. – By iterations of (3.1) we get

$$a_{p(m+1)+i+1}=\lambda^i a_{p(m+1)+1}+\lambda^{i-1}x_{p(m+1)+2}+\ldots+x_{p(m+1)+i}.$$

As $a_{p(m+1)+1}=-\mu\, a_{p(m+1)}$ we can write

$$a_{p(m+1)+i+1}=\frac{1}{\lambda^{m-i-1}\mu}[-\lambda^{m-1}\mu^2 a_{p(m+1)} + \mu(\lambda^{m-2}x_{p(m+1)+1}+\ldots+\lambda^{m-i-1}x_{p(m+1)+i})].$$

Using $x_{p(m+1)+j}>F^j(\beta_1)$ for $j=1\ldots i$ we get

$$a_{p(m+1)+i+1}>\frac{1}{\lambda^{m-i-1}\mu}[-\lambda^{m-1}\mu^2 a_{p(m+1)} + \mu[\lambda^{m-2}F(\beta_1)+\ldots+\lambda^{m-i-1}F^i(\beta_1)]].$$

From (3.9)′ and as $F^j(\beta_1)<0$ we have also

$$\lambda^{m-2}F(\beta_1)+\ldots+\lambda^{m-i-1}F^i(\beta_1)>\frac{a_{2m+2}}{X-1}.$$

so by the same steps as in (2) we prove the last inequality.

Remarks 3.1. – From the equalities

$$x_{\underline{A}_p}=1+\mu\alpha_p x_m=1-\mu(1+\lambda\, x_{\underline{A}_{p-1}RL^{m-2}}).$$

we obtain

$$x_{\underline{A}_{p-1}RL^{m-2}}=\frac{1}{\lambda}(\alpha_p(-x_m)-1).$$

By lemma 3.2 and 3.3, this polynomial in μ and λ is strictly increasing. If $K\in T^m_{p,m}$ we have $K>\underline{A}_{p-1}RL^{m-2}$ and $x_{\underline{A}_{p-1}RL^{m-2}}>0$ hence $\alpha_p(-x_m)>1$.

On the other hand if $K\in\bigcup_{i=0}^{m-1}T^i_{m,p}$, we have $\alpha_p(-x_m)\leqq 1$.

Now the next lemma gives estimates of partial derivatives for kneading sequences $K\in U^{i,n}_{p,m}$ and such that $K=\underline{A}_p L^i(LRL^{m-1})^{2n}\ldots$ To simplify we shall denote by $B_n(A_n)$ the partial derivatives associated to $\underline{A}_p L^i(LRL^{m-1})^{2n}$.

LEMMA 3.4. – *Suppose that* $K=\underline{A}_p L^i(LRL^{m-1})^{2n}\ldots$ $(n\geqq 0)$, *then we have the following inequalities*

$$\mu B_n<b^*_{(p+1)(m+1)}-X^{2n}\lambda^i\mu\, x_{(p+1)m+p}<0, \qquad (3.13)$$

$$\mu A_n<a_{(p+1)(m+1)} + X^{2n}\mu[\lambda^{i-1}x_{(p+1)(m+1)}+\ldots+x_{(p+1)(m+1)+i-1}]<0. \qquad (3.14)$$

Proof. – We shall prove the lemma by induction on n.

(1) $n=0$. We have

$$B_0=\lambda^i b_{(p+1)(m+1)}=\lambda^i b^*_{(p+1)(m+1)}-\lambda^i x_{(p+1)(m+1)},$$
$$\mu B_0=\lambda^i \mu b^*_{(p+1)(m+1)}-\lambda^i \mu x_{(p+1)(m+1)}.$$

As $\lambda^i \mu>\lambda^m \mu>1$ for $0\leqq i\leqq m-1$ and $b^*_{(p+1)m+p}<0$ we get (3.13).

For the other derivative we have

$$A_0=\lambda^i a_{(p+1)(m+1)}+[\lambda^{i-1} x_{(p+1)(m+1)}+\ldots+x_{(p+1)(m+1)+i-1}],$$
$$\mu A_0=\lambda^i \mu a_{(p+1)(m+1)}+\mu[\lambda^{i-1} x_{(p+1)(m+1)}+\ldots+x_{(p+1)(m+1)+i-1}].$$

As $\lambda^i\mu>1$ and $a_{(p+1)(m+1)}<0$ we get (3.14).

(2) Now let us assume that (3.13) and (3.14) hold for $n-1$ and prove them for n.

We shall first prove the following inequality

$$(X^{2n}\mu\lambda^i-1)\frac{\alpha_p}{X-1}-(1+X+\ldots+X^{2n-2})>0. \qquad (3.15)$$

As $X-1>\frac{\lambda}{\mu}$, we get $\frac{\alpha_p}{X-1}\geqq 1$. The inequality (3.15) is obvious if $\lambda^i\mu\geqq n+1$. Now we consider the case $\lambda^i\mu<n+1$ $(0\leqq i\leqq m-1)$. We denote by Q_n the characteristic polynomial of $\underline{A}_p L^i(LRL^{m-1})^{2n}$. By induction we see that Q_{n-1} is strictly decreasing and negative.

$$Q_{n-1}=(X-1)(-x_m)(1+X^2+\ldots+X^{2n-4})+X^{2n-2}Q_0,$$

and

$$Q_0=1+\lambda+\ldots+\lambda^i+\lambda^i\mu\alpha_p x_m<1+\lambda^i\mu\alpha_p x_m.$$

Using $\frac{X^2 Q_{n-1}}{(X-1)(-x_n)}<0$ we find

$$X^{2n}\mu\lambda^i\frac{\alpha_p}{X-1}-(X^2+\ldots+X^{2n-2})>\frac{X^{2n}}{(X-1)(-x_m)}.$$

and

$$(X^{2n}\mu\lambda^i-1)\frac{\alpha_p}{X-1}-(1+X^2+\ldots+X^{2n+2})>\frac{X^{2n}}{(X-1)(-x_m)}-\frac{\alpha_p}{X-1}-1.$$

According to remark (3.1) we have $\alpha_p(-x_m)<1$ so that

$$\frac{X^{2n}}{(X-1)(-x_m)}-\frac{\alpha_p}{X-1}-1=\frac{X^{2n}-\alpha_p(-x_m)}{(X-1)(-x_m)}-1>\frac{2n}{(-x_m)}-1.$$

But we have $(-x_m)=\lambda^{m-1}\mu-\lambda^{m-1}\ldots 1<\lambda^i\mu-1<n$, hence

$$\frac{2n}{(-x_m)}-1>1>0.$$

and this proves (3.15).

To prove (3.13) we shall compare B_{k+1} and B_k. We have

$$\underline{A}_p L^i (LRL^{m-1})^{2k+2} = \underline{A}_p L^i (LRL^{m-1})^{2k} (LR^- L^m R^+ L^{m-1}).$$

Therefore

$$B_{k+1} = \lambda^{2m} \mu^2 B_k + \lambda^{2m-1} \mu x_{j-m-1} - \lambda^{m-1} x_j.$$

with x_j corresponding to R^+. This x_j satisfies the induction formulas

$$x_j = 1 + \lambda + \ldots + \lambda^m - \lambda^m \mu - \lambda^{m-1} \mu x_{j-m-2} = x_{m+1} - \lambda^{m+1} \mu x_{j-m-2}.$$

As $x_{j-m-2} < 0$ we have $-x_j < -x_{m+1}$ and as $0 < x_{j-m-1} < 1$ we get

$$B_{k+1} < X^2 B_k + \lambda^{2m-1} \mu - \lambda^{m-1} x_{m+1}.$$

Using the relation (*) of the proof of lemma 3.2 we obtain

$$\mu B_{k+1} < X^2 \mu B_k - b^*_{2m+2}.$$

The partial derivative of $\underline{A}_p L^i$ is $\lambda^i b_{p(m+1)}$, hence by iterations we get

$$\left.\begin{aligned}
&\mu B_n < X^{2n} \mu \lambda^i b_{(p+1)(m+1)} - b^*_{2m+2} (1 + X^2 + \ldots + X^{2n-2}),\\
&\mu B_n < X^{2n} \mu \lambda^i b^*_{(p+1)(m+1)} - X^{2n} \mu \lambda^i x_{(p+1)m+p}\\
&\qquad\qquad - b^*_{2m+2} (1 + X^2 + \ldots + X^{2n-2}).\\
&\mu B_n < b^*_{(p+1)(m+1)} - X^{2n} \mu \lambda^i x_{(p+1)m+p}\\
&\quad + (X^{2n} \mu \lambda^i - 1) b^*_{(p+1)(m+1)} - b^*_{2m+2} (1 + X^2 + \ldots + X^{2n-2}).
\end{aligned}\right\} \quad (3.16)$$

Recalling formula (3.6) in lemma 3.2 we obtain

For $p \geqq 2$

$$b^*_{(p+1)(m+1)} < b_{2m+2} \frac{\alpha_p}{X-1} < b^*_{2m+2} \frac{\alpha_p}{X-1} < 0,$$

For $p = 1$

$$b^*_{2m+2} = b^*_{2m+2} \frac{\alpha_p}{X-1} \qquad (\alpha_1 = X - 1).$$

Hence for each $p \geqq 1$ we have

$$\begin{aligned}
&(X^{2n} \mu \lambda^i - 1) b^*_{(p+1)(m+1)} - b^*_{2m+2} (1 + X^2 + \ldots + X^{2n-2})\\
&\qquad \leqq \left[(X^{2n} \mu \lambda^i - 1) \frac{\alpha_p}{X-1} - (1 + X^2 + \ldots + X^{2n-2}) \right] b^*_{2m+2}.
\end{aligned}$$

As $b^*_{2m+2} < 0$, from (3.15) and (3.16) we deduce (3.13).

To prove (3.14) we proceed analogously. Comparing A_{k+1} and A_k we obtain

$$A_{k+1} < X^2 A_k + \lambda^{m-1} \mu A + (\lambda^{m-2} x_{m+2} + \ldots + x_{2m}).$$

Using the relation (*) in the proof of lemma 3.3 we obtain

$$\mu A_{k+1} < X^2 \mu A_k - a_{2m+2}.$$

By iteration

$$\mu A_n < X^{2n} \mu [\lambda^i a_{(p+1)(m+1)} + \lambda^{i-1} x_{(p+1)(m+1)} + \ldots + x_{(p+1)(m+1)+i-1}] - a_{2m+2}(1+X^2+\ldots+X^{2n-2})$$

$$\mu A_n < a_{(p+1)(m+1)} + X^{2n} \mu [\lambda^{i-1} x_{(p+1)(m+1)} + \ldots + x_{(p+1)(m+1)+i-1}] + (X^{2n}\mu\lambda^i - 1) a_{(p+1)(m+1)} - a_{2m+2}(1+\ldots+X^{2n-2})).$$

Using (3.10), (3.15) and $a_{2m+2}<0$ we get

$$(X^2 \mu\lambda^i - 1) a_{(p+1)(m+1)} - a_{2m+2}(1+X^2+\ldots+X^{2n-2}) < 0,$$

and this proves (3.14).

Proof of the proposition (3.1)

Remarks 3.2. — Recall that $m \geqq 1$ is fixed and we have set $B=-b_{m+1}$, $A=-a_{m+1}$.

Let $K=A_0 A_1 \ldots$ be a infinite kneading sequence in $\tilde{C}_m$. Assume that for n_1 we have $A_{n_1}=R^+$ and $b_{n_1} > \frac{B}{X^2-1}$, $a_{n_1} > \frac{A}{X^2-1}$, then it is easy to infer that for the first $n_0 > n_1$ such that $A_{n_0}=R^-$ we have $b_{n_0} < -\frac{BX}{X^2-1}$, $a_{n_0} < -\frac{AX}{X^2-1}$.

We shall now study all cases given by lemma 3.1.

(1) Assume that $K \in \mathring{T}^m_{p,m}$, then $K=\underline{A}_{p-1} L^i R^+ \ldots$ and set

$$n_1 = p(m+1)+i+1 \qquad (0 \leqq i \leqq m-2).$$

Using (3.8) and (3.5) we get for $p \geqq 2 \, (\lambda \leqq 1)$,

$$b_{n_1} > -\frac{b_{2m+2}}{\lambda^{m-i-1}\mu} \frac{\alpha_p}{X-1} > -\frac{b^*_{2m+2}}{\lambda^{m-i-1}\mu} \frac{\alpha_p}{X-1} > \frac{B\mu(-x_m)\alpha_p}{\lambda^{m-i-1}\mu(X-1)},$$

so

$$b_{n_1} > \frac{B(-x_m)\alpha_p}{X-1}.$$

As $\alpha_1 = X-1$, by (3.7) this inequality is also true for $p=1$.

In the same way by (3.12), (3.9) and (3.11) we get for $p \geqq 1$

$$a_{n_1} > \frac{A(-x_m)\alpha_p}{X-1}.$$

Recall that in $\mathring{T}^m_{p,m}$ we have $\alpha_p(-x_m)>1$ (Remark 3.1). Therefore as $(X-1)<(X^2-1)$ we get

$$b_{n_1}>\frac{B}{X^2-1} \quad \text{and} \quad a_{n_1}>\frac{A}{X^2-1}.$$

The remark 3.2 gives the n_0 of the proposition 3.1.

(2) Assume that $K\in\mathring{V}^i_{p,m}(0\leqq i\leqq m-1)$ then $K=\underline{A}_p L^{i+1}R^+\dots$ and set $n_0=(p+1)(m+1)+i+1$. We denote by γ_0 the periodic point of F with period $(m+1)$ and itinerary $(LRL^{m-1})^\infty$. It is easy to prove that $\gamma_0=\dfrac{x_m}{X+1}<0$.

Now as $K>\underline{A}_p L^i(LRL^{m-1})^\infty$, by the ordering property of itineraries we get $x_{(p+1)(m+1)+i}<\gamma_0$, *i.e.*,

$$x_{(p+1)(m+1)+i}=1+\lambda+\dots+\lambda^i+\lambda^i\mu\alpha_p x_m<\frac{x_m}{X+1},$$

and

$$\lambda^i\mu\alpha_p(-x_m)(X+1)>X+1+(-x_m).$$

As $(\lambda,\mu)\in\tilde{C}_m$ we have $X+1+(-x_m)>\lambda^{m-1}\mu=\dfrac{X}{\lambda}$.

Hence if $K\in\mathring{V}^i_{p,m}$ we get the condition

$$\lambda^{i+1}\mu\alpha_p(-x_m)(X+1)>X. \tag{3.17}$$

Now we have by (3.6)-(3.10)

$$b_{n_0}=\lambda^{i+1}b_{(p+1)(m+1)}<\lambda^{i+1}b^*_{(p+1)(m+1)}$$

$$<\lambda^{i+1}b_{2m+2}\frac{\alpha_p}{X-1} \quad \text{for} \quad p\geqq 2,$$

$$b_{n_0}=\lambda^{i+1}b_{2m+2}\frac{\alpha_p}{X-1} \quad \text{for} \quad p=1,$$

$$a_{n_0}<\lambda^{i+1}a_{(p+1)(m+1)}\leqq\lambda^{i+1}a_{2m+2}\frac{\alpha_p}{X-1} \quad \text{for} \quad p\geqq 1.$$

And by (3.5)-(3.9) we get

$$b_{n_0}<-B\lambda^{i+1}\mu\alpha_p(-x_m)\frac{(X+1)}{X^2-1},$$

$$a_{n_0}<-A\lambda^{i+1}\mu\alpha_p(-x_m)\frac{X+1}{X^2-1}.$$

Using (3.17) we prove that

$$b_{n_0}<-\frac{BX}{X^2-1} \quad \text{and} \quad a_{n_0}<-\frac{AX}{X^2-1}.$$

(3) Assume that $K \in \mathring{U}^{i,n}_{p,m}$, then $K = \underline{A}_p L^i (LRL^{m-1})^{2n} R^- \dots$
As we have $K > \underline{A}_p L^i (LRL^{m-1})^\infty$ the condition (3.17) becomes

$$\lambda^i \mu \alpha_p (-x_m)(X+1) > X. \tag{3.18}$$

(*a*) Suppose that $p > 1$, then $K = \underline{A}_p L^i (LRL^{m-1})^{2n} RR^+ \dots$ and let

$$n_1 = (p+1)(m+1) + i + 2n(m+1) + 1.$$

With the notations of lemma 3.4 we have

$$b_{n_1} > -\mu B_n - 1.$$

So using (3.13) we get

$$b_{n_1} > b^*_{(p+1)(m+1)} + X^{2n} \lambda^i \mu x_{(p+1)m+p} - 1.$$

As $x_{(p+1)m+p} > 0$ and $F^j(x_{(p+1)m+p}) < 0$ for $1 \leqq j \leqq i+1$ we have

$$F^j(x_{(p+1)m+p}) = 1 + \lambda + \dots + \lambda^i - \lambda^i \mu x_{(p+1)m+p} < 0.$$

Using $X^{2n} > 1$, we find therefore $X^{2n} \lambda^i \mu x_{(p+1)m+p} - 1 > 0$, and

$$b_{n_1} > -b^*_{(p+1)(m+1)}. \tag{3.19}$$

As before using lemma 3.2 we get

$$b_{n_1} > -b_{2m+2} \frac{\alpha_p}{X-1} > B \mu \frac{(-x_m)\alpha_p}{X-1}.$$

By (3.18) we obtain

$$b_{n_1} > \frac{BX}{\lambda^i (X^2-1)} > \frac{B}{X^2-1}.$$

In the same way we have

$$a_{n_1} = -\mu A_n > -a_{(p+1)(m+1)}.$$

Using lemma 3.3 we get

$$a_{n_1} > -a_{2m+2} \frac{\alpha_p}{X-1} > \frac{A \mu (-x_m) \alpha_p}{X-1}.$$

By (3.18) we obtain

$$a_{n_1} > \frac{A}{X^2-1}.$$

The remarks 3.2 gives the n_0 of the proposition 3.1.

(*b*) Suppose that $p = 1$, lemma 3.1 gives two possibilities for the kneading sequences. First assume that

$$K = \underline{A}_1 L^i (LRL^{m-1})^{2n} RL^i (LRL^{m-1})^{2k} R^+ \dots \quad \text{for} \quad 0 \leqq k \leqq n.$$

We shall denote by $B_{n,k}$ ($A_{n,k}$) the partial derivatives associated to

$$\underline{A}_1 L^i (LRL^{m-1})^{2n} RL^i (LRL^{m-1})^{2k} \quad \text{for} \quad 0 \leqq k \leqq n.$$

We compare $B_{n,k+1}$ and $B_{n,k}$. We have

$$\underline{A}_1 L^i (LRL^{m-1})^{2n} RL^i (LRL^{m-1})^{2k+2} = \underline{A}_1 L^i (LRL^{m-1})^{2n} RL^i (LRL^{m-1})^{2k} LRL^m R^- L^{m-1}.$$

We denote by j the index corresponding to R^- in the preceding formula, by iteration of (3.2) we get

$$B_{n,k+1} = \lambda^{2m} \mu^2 B_{n,k} + \lambda^{2m-1} \mu x_{j-m-1} - \lambda^{m-1} x_j.$$

Let $\gamma_1 = F(\gamma_0) > 0$ be the periodic point of itinerary $(RL^m)^\infty$.

From the inequality $K > \underline{A}_1 L^i (LRL^{m-1})^{2n} RL^i (LRL^{m-1})^\infty$ we deduce that $x_j < \gamma_1$, $x_{j-m+1} > \gamma_1$, and that

$$B_{n,k+1} > \lambda^{2n} \mu^2 B_{n,k} + \lambda^{2m-1} \mu \gamma_1 - \lambda^{m-1} \gamma_1,$$
$$B_{n,k+1} > X^2 B_{n,k} + \lambda^{m-1} \gamma_1 (X-1) > B_{n,k}.$$

And by iterations we get

$$B_{n,k+1} > B_{n,0} \quad \text{for} \quad 0 \leqq k \leqq n-1.$$

By (3.19) with $p=1$ we obtain

$$B_{n,k} \geqq B_{n,0} > -\lambda^i b^*_{2m+2} \quad \text{for} \quad 0 \leqq k \leqq n.$$

Let $n_1 = 2m+2+i+2n(m+1)+i+1+2k(m+1)$, with our notations we have $b_{n_1} = B_{n,k}$ and by the same proof as in (*a*) we obtain

$$b_{n_1} > \frac{B}{X^2-1}.$$

Now we compare $A_{n,k+1}$ and $A_{n,k}$. Taking into account that

$$K > \underline{A}_1 L^i (LRL^{m-1})^{2n} RL_i (LRL^{m-1})^\infty$$

by iteration of (3.2) we get

$$A_{n,k+1} < \lambda^{2m} \mu^2 A_{n,k} + \lambda^{2m-1} \mu^2 \gamma_0 - \lambda^{m-1} \mu [\lambda^{m-1} F(\gamma_0) + \ldots + \lambda F^{m-1}(\gamma_0)] - \lambda^{m-1} \mu \gamma_0 + [\lambda^{m-2} F(\gamma_0) + \ldots + F^{m-1}(\gamma_0)]$$

As $X = \lambda^m \mu > 1$ we have

$$-\lambda^{m-1} \mu [\lambda^{m-1} F(\gamma_0) + \ldots + \lambda F^{m-1}(\gamma_0)] + [\lambda^{m-2} F(\gamma_0) + \ldots + F^{m-1}(\gamma_0)] > 0.$$

So that

$$A_{n,k+1} > X^2 A_{n,k} + \lambda^{m-1} \mu (X-1) \gamma_0. \tag{3.20}$$

Now we shall prove that

$$A_{n,k} > -\lambda^i a_{2m+2}, \quad \text{for} \quad 0 \leqq k \leqq n. \tag{3.21}$$

So with our notations we have $a_{n_1} = A_{n,k}$ and by the same proof as in (*a*) we shall obtain

$$a_{n_1} > \frac{A}{X^2 - 1}.$$

the remark 3.2 will give the n_0 for the proposition 3.2.

We prove (3.21) by induction on k.

(i) For $k=0$ we have $K = \underline{A}_1 L^i (LRL^{m-1})^{2^n} R^- L^i$.

We denote by j the index corresponding to R^-. By iterations of (3.1) we get with the notations of lemma 3.4

$$A_{n,0} = -\lambda^i \mu A_n + \lambda^{i-1} x_{j+1} + \ldots + x_{j+i}.$$

But we have $x_{j+1} > x_1, \ldots, x_{j+i} > x_i$, so that

$$A_{n,0} > -\lambda^i \mu A_n + \lambda^{i-1} x_1 + \ldots + x_i.$$

Therefore, by (3.14) we get

$$A_{n,0} > -\lambda^i a_{2m+2} - \lambda^i X^{2^n} \mu \times [\lambda^{i-1} x_{2m+2} + \ldots + x_{2m+2+i-1}] + \lambda^{i-1} x_1 + \ldots + x_i.$$

We observe that

$$x_{2m+2+i} = 1 + \lambda + \ldots + \lambda^{i-(k-1)} + \lambda^{i-(k-2)} x_{2m+k} < 0 \qquad \text{for} \quad 2 \leqq k \leqq i+1.$$

Therefore

$$-\lambda^{i-(k-2)} x_{2m+k} > 1,$$
$$-\lambda^i X^{2^n} \mu x_{2m+k} > \lambda^{k-2} \mu - \lambda^{k-2} \ldots 1 = x_{k-1},$$

hence

$$A_{n,0} > -\lambda^i a_{2m+2}.$$

(ii) Assume that $A_{n,k} > -\lambda^i a_{2m+2}$, by (3.20) we get

$$A_{n,k+1} > -\lambda^i a_{2m+2} X^2 + \lambda^{m-1} \mu (X-1) \gamma_0 = -\lambda^i a_{2m+2} - (X^2-1) \lambda^i a_{2m+2} + \lambda^{m-1} \mu (X-1) \gamma_0.$$

Taking into account that $\gamma_0 = \dfrac{x_m}{X+1}$, $X+1>0$, and (3.9) we get

$$A_{n,k+1} > -\lambda^i a_{2m+2} + \lambda^i \mu (X-1)(-x_m) \left[A - \frac{\lambda^{m-i-1}}{X+1} \right].$$

But recall that

$$A = -a_{m-1} = -x_m - \ldots - \lambda^{m-1} x_1 > -\lambda^{m-1} x_1 = \lambda^{m-1} (\mu - 1)$$

If $(\lambda, \mu) \in C_m$ we have $\lambda^{m-1}\mu > \lambda^{m-1} + \lambda^{m-2} + \ldots + 1$, hence $A > 1 > \frac{\lambda^{m-i-1}}{X+1}$ so

$$A_{n,k+1} > -\lambda^i a_{2m+2},$$

and this prove (3.21)

(4) The last possibility to be considered is

$$K = \underline{A}_1 L^i (LRL^{m-1})^{2n} RL^k R^+ \ldots \quad \text{with } k = 0, \ldots, i.$$

We set $n_1 = (2m+2) + i + 2n(n+1) + k + 1$ then

$$b_{n_1} > \lambda^k(-\mu B_n - 1)$$
$$a_{n_1} > -\lambda^k \mu A_n$$

Using the same arguments as in the proof of (*a*) we get

$$b_{n_1} > \frac{\lambda^k BX}{\lambda^i(X^2-1)} > \frac{B}{X^2-1}$$
$$a_{n_1} > \frac{\lambda^k AX}{\lambda^i(X^2-1)} > \frac{A}{X^2-1}$$

and then we use remark 3.2 as before

This completes the proof of proposition 3.1.

4. PROOFS OF THEOREMS 2 AND 3

We keep the same notations as before. If $(\lambda, \mu) \in C_m$ recall that $X = \lambda^m \mu > 1 + \frac{\lambda}{\mu}$ and $B = -b_{m+1}$, $A = -a_{m+1}$.

LEMMA 4.1. — *Let $(\lambda, \mu) \in C_m$ and let $K(\lambda, \mu) = A_0 A_1 \ldots$ be an infinite kneading sequence. We know that there exists n_0 such that*

$$A_{n_0} = R^-, \quad \text{and} \quad b_{n_0} < -\frac{BX}{X^2-1}, \quad a_{n_0} < -\frac{AX}{X^2-1}.$$

Set $c = -\frac{BX}{X^2-1} - b_{n_0}$ and $d = -\frac{AX}{X^2-1} - a_{n_0}$.

Then c, d are strictly positive and we have for $n \geq 0$:

If $\varepsilon_{n+n_0} = -1$ then

$$b_{n+n_0} < -\frac{BX}{X^2-1} - c\lambda^{\theta'_1(n)}\mu^{\theta'_2(n)},$$
$$a_{n+n_0} < -\frac{AX}{X^2-1} - d\lambda^{\theta'_1(n)}\mu^{\theta'_2(n)}.$$

If $\varepsilon_{n+n_0} = +1$ then

$$b_{n+n_0} > +\frac{B}{X^2-1} + c\lambda^{\theta_1'(n)}\mu^{\theta_2'(n)},$$

$$a_{n+n_0} > +\frac{A}{X^2-1} + d\lambda^{\theta_1'(n)}\mu^{\theta_2'(n)}.$$

where $\theta_i'(n) = \theta_i(n+n_0) - \theta_i(n_0)$ $(i=1, 2)$.

Proof. – c and d are positive by hypothesis, and we prove the other assertions by induction.

For $n=0$ it is obvious.

Assume that the inequalities hold for n prove them for $n+1$.

(1) Assume that $A_{n+n_0} = L^-$, then $\varepsilon_{n+n_0+1} = -1$.

Let l be the first index such that $A_{n_0+n-l} = R$. By the maximality property of kneading sequences we have $0 \leqq l \leqq m$. By iterations of (3.1) and (3.2) we get

$$b_{n_0+n+1} < -\lambda^l \mu b_{n_0+n-l} < -\frac{\lambda^l \mu B}{X^2-1} - c\lambda^{\theta_1'(n+1)}\mu^{\theta_2'(n+1)}$$

$$a_{n_0+n+1} < -\lambda^l \mu a_{n_0+n-l} < -\frac{\lambda^l \mu A}{X^2-1} - d\lambda^{\theta_1'(n+1)}\mu^{\theta_2'(n+1)}.$$

Therefore as $-\lambda^l \mu \leqq -\lambda^m \mu = -X$ $(\lambda \leqq 1)$ we obtain

$$b_{n_0+n+1} < -\frac{BX}{X^2-1} - c\lambda^{\theta_1'(n+1)}\mu^{\theta_2'(n+1)},$$

$$a_{n_0+n+1} < -\frac{AX}{X^2-1} - d\lambda^{\theta_1'(n+1)}\mu^{\theta_2'(n+1)}.$$

(2) Assume that $A_{n+n_0} = L^+$, then $\varepsilon_{n+n_0+1} = +1$.

Let l be the first index such that $A_{n_0+n-l} = R^-$ $(0 \leqq l \leqq m)$.

By iterations of (3.1) and (3.2) we get

$$b_{n_0+n+1} = -\lambda^l \mu b_{n_0+n-l} - \lambda^l x_{n_0+n-l},$$

$$a_{n_0+n+1} = -\lambda^l \mu a_{n_0+n-l} + \lambda^{l-1} x_{n_0+n-l+1} + \ldots + x_{n+n_0}.$$

First, we have $0 \leqq x_{n_0+n-l} \leqq 1$ so that

$$b_{n_0+n+1} \geqq -\lambda^l \mu b_{n_0+n-l} - \lambda^l = \frac{1}{\lambda^{m-l}}[-X b_{n_0+n-l} - \lambda^m].$$

Therefore as $B = \lambda^m$ we find

$$b_{n_0+n+1} > \frac{1}{\lambda^{m-l}}\left[\frac{BX^2}{X^2-1} - B\right] + c\lambda^{\theta_1'(n+1)}\mu^{\theta_2'(n+1)}.$$

As $\lambda \leqq 1$ we have $\dfrac{1}{\lambda^{m-l}} > 1$, then

$$b_{n_0+n+1} > \frac{B}{X^2-1} + c\lambda^{\theta'_1(n+1)}\mu^{\theta'_2(n+1)}.$$

For the other derivative we use

$$x_{n_0+n-l+1} > x_1, \ldots, x_{n_0+n} > x_l.$$

Hence

$$a_{n_0+n+1} > \frac{\lambda^l \mu X A}{X^2-1} + \lambda^{l-1} x_1 + \ldots + x_l + d\lambda^{\theta'_1(n+1)}\mu^{\theta'_2(n+1)}.$$

We know that $x_1, \ldots, x_m < 0$, so

$$\frac{\lambda^l \mu X A}{X^2-1} + \lambda^{l-1} x_1 + \ldots + x_l$$
$$\geqq \frac{1}{\lambda^{m-l}}\left(\frac{\lambda^m \mu X A}{X^2-1} + \lambda^{m-1} x_1 + \ldots + x_m\right)$$
$$= \frac{1}{\lambda^{m-l}}\left(\frac{X^2 A}{X^2-1} - A\right) = \frac{1}{\lambda^{m-l}}\frac{A}{X^2-1} \geqq \frac{A}{X^2-1}.$$

Then

$$a_{n_0+n+1} > \frac{A}{X^2-1} + d\lambda^{\theta'_1(n+1)}\mu^{\theta'_2(n+1)}.$$

(3) The two cases $A_{n+n_0} = R^+$ and $A_{n+n_0} = R^-$ are the same as in (1) and (2) with $l=0$.

This completes the proof of lemma 4.1.

Now recall that $\varepsilon_n = (-1)^{\theta_2(n)}$ where $\theta_2(n)$ is the number of symbols R *among* $A_0 A_1 \ldots A_{n-1}$.

If the kneading sequence $K(\lambda, \mu) = \underline{A}C$ is finite we set $\varepsilon_n = 0$ for $n > |\underline{A}|$.

PROPOSITION 4.1. – *Assume that* $(\lambda, \mu) \in C_m$. *Then for* $n \geqq 2$, *we have either* $\varepsilon_n = 0$ *or* $\varepsilon_n a_n > 0$ *and* $\varepsilon_n b_n > 0$.

Moreover for infinite kneading sequences we have

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |b_n| = \infty.$$

Proof. – In the proof of proposition (3.1) we have studied all the beginnings of kneading sequences in $\tilde{C}_m$ up to the index n_0 and we control the sign of the derivatives a_n and b_n.

It is easy to verify that if $\varepsilon_n \neq 0$ we have $\varepsilon_n a_n > 0$ and $\varepsilon_n b_n > 0$ for $n < n_0$.

Now for $n \geqq n_0$ by lemma 4.1 we have also $\varepsilon_n a_n < 0$ and $\varepsilon_n b_n < 0$ if $\varepsilon_n \neq 0$.

If $K(\lambda, \mu)$ is an infinite kneading sequence in $\tilde{C}_m$, by the maximality property, there are at most m successive symbols L in $K(\lambda, \mu)$. Hence $\theta'_1(n) \leqq m\theta'_2(n)$ and as $\lambda \leqq 1$ we find $\lambda^{\theta'_1(n)}\mu^{\theta'_2(n)} \geqq (\lambda^m\mu)^{\theta'_2(n)}$. Recall that in C_m we have $\lambda^m\mu > 1$ hence by lemma 4.1 we find

$$\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} |b_n| = \infty.$$

Proof of Theorem 2

(1) The regions $\mathscr{R}_m$ are bounded by the graphs of the decreasing functions

$$\lambda^{m-1}\mu = \lambda^{m-1} + \ldots + 1$$
$$\lambda^m\mu = \lambda^m + \ldots + 1.$$

Hence if (λ, μ), (λ', μ') belong to different regions $\mathscr{R}_m$ and if $(\lambda', \mu') > (\lambda, \mu)$ we have $m' > m$. By theorem 1 we deduce that

$$K(\lambda, \mu) = RL^m R\ldots, \qquad K(\lambda', \mu') = RL^{m'} R\ldots$$

Therefore we have $K(\lambda', \mu') > K(\lambda, \mu)$.

(2) Assume that (λ, μ), $(\lambda', \mu') \in A_m$. Then by theorem 1 we have

$$K(\lambda, \mu) = K(\lambda', \mu') = (RL^m)^\infty.$$

(3) Assume that (λ, μ), $(\lambda', \mu') \in B_m \cup C_m$ and $(\lambda', \mu') > (\lambda, \mu)$.

As the graphs bounding the regions B_m and C_m are decreasing, there are three cases:

(i) (λ, μ), $(\lambda', \mu') \in B_m$;

(ii) $(\lambda, \mu) \in B_m$ and $(\lambda', \mu') \in C_m$;

(iii) (λ, μ), $(\lambda', \mu') \in C_m$.

(i) Assume that

$$(\lambda, \mu), (\lambda', \mu') \in B_m = \{(\lambda, \mu) \mid \lambda^m\mu \geqq 1,\ \lambda^m\mu^2 \leqq \lambda + \mu,\ \lambda^{m-1}\mu > \lambda^{m-1} + \ldots + 1\}.$$

By theorem 1 we have

$$K(\lambda, \mu) = RL^{m-1} * \underline{B}, \qquad K(\lambda', \mu') = RL^{m-1} * \underline{B}'$$

with $\underline{B}$, $\underline{B}'$ primary.

By lemma 2.1, the map F admits a $m+1$-renormalisation $F_{\varphi_{RL^{m-1}}}$ such that

$$K(\varphi_{RL^{m-1}}(\lambda, \mu)) = \underline{B}, \qquad K(\varphi_{RL^{m-1}}(\lambda', \mu')) = \underline{B}'$$

with $\varphi_{RL^{m-1}}(\lambda, \mu) = (\lambda^{m-1}\mu^2, \lambda^m\mu)$.

Notice that $\mu > 1 \geqq \lambda$ so $\lambda^{m-1}\mu^2 > \lambda^m \mu > 1$. Hence φ_{RL^m} maps the region B_m into

$$D = \left\{ (\lambda, \mu) \mid \lambda \geqq 1, \mu > 1, \frac{1}{\lambda} + \frac{1}{\mu} \geqq 1 \right\}.$$

As the map $\varphi_{RL^{m-1}}$ is strictly increasing by theorem 1 of [M-V] we have $\underline{B}' > \underline{B}$. The application $RL^{m-1} *$ is also strictly increasing so we get

$$K(\lambda', \mu') > K(\lambda, \mu).$$

(ii) Assume that $(\lambda, \mu) \in B_m$ and $(\lambda', \mu') \in C_m$.
Using theorem 1 we find

$$K(\lambda', \mu') > RL^{m-1} * RL^{\infty} > K(\lambda, \mu).$$

(iii) Assume that (λ, μ), $(\lambda', \mu') \in C_m$.
If $(\lambda, \mu) \in C_m$ by proposition 4.1 we have for $n \geqq 2$.
$\varepsilon_n a_n > 0$ and $\varepsilon_n b_n > 0$ if $\varepsilon_n \neq 0$ and for infinite kneading sequences

$$\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} |b_n| = \infty.$$

We have the same results as in [M-V] (proposition 3.6), so in the same way we prove that

$$K(\lambda', \mu') > K(\lambda, \mu) \quad \text{as claimed.}$$

Proof of theorem 3

Let $\underline{M} \in \mathcal{M}$, the function $\gamma(\underline{M})$ is given in [M-V] (lemma 6.1).
(1) Assume that $\underline{M} \geqq RLR^{\infty}$.

Recall that $\beta_{RLR^{\infty}}(\mu) = \dfrac{\mu}{\mu^2 - 1}$. If $\mu > \gamma(\underline{M})$ we have by theorem 2

$$K\left(\frac{\mu}{\mu^2 - 1}, \mu\right) \leqq \underline{M} < K(1, \mu).$$

Hence by the intermediate value theorem [M] there exists $\beta_{\underline{M}}(\mu)$ such that $\underline{M} = K(\beta_{\underline{M}}(\mu), \mu)$.

The unicity and the decreasing property of the function $\lambda = \beta_{\underline{M}}(\mu)$ follow from theorem 2.

If $\underline{M} > RLR^{\infty}$ there exists $m > 0$ such that $RL^m C \leqq \underline{M} < RL^{m-1} C$ and then $\beta_{RL^m C} \leqq \beta_{\underline{M}} \leqq \beta_{RL^{m-1} C}$. It follows that $\lim_{\mu \to \infty} \beta_{\underline{M}}(\mu) = 0$. The functions $\beta_{\underline{M}}$ map $[\gamma_{\underline{M}}, \infty[$ into $]0, 1]$. By the same proof as in ([M-V], Theorem C), we prove that the functions $\beta_{\underline{M}}$ are continuous, and by theorem 1, theirs graphs fill up the set $\bigcup_{m=1}^{\infty} C_m$.

(2) Assume that $\underline{M} < RLR^{\infty}$, then $\underline{M} = R * \underline{B}$.

By the renormalisation construction (lemma 2.1) we have

$$(K(\lambda, \mu) = \underline{M} \quad \text{if and only if } K(\mu^2, \lambda\mu) = \underline{B}.$$

As $\mu^2 > \lambda\mu > 1$, by theorem C of [M-V], this is equivalent to $\mu^2 = \beta_{\underline{B}}(\lambda\mu)$ or $\lambda = \frac{1}{\mu} \beta_{\underline{B}}^{-1}(\mu^2)$.

These functions are continuous and decreasing from $[\gamma_{\underline{M}}, \infty[$ into $]0, 1]$ and fill up the whole set B_1.

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