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by

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RÉSUMÉ. — Nous allons nous occuper d'une sous-classe spéciale de la classe des Z_2-extensions d'automorphismes ayant un spectre rationnel discret.

Quelques connections avec des points strictement transitifs dans l'espace $\{0,1\}^Z$ (particulièrement avec la théorie d'automates, la théorie de suites de Morse et de Toeplitz) sont établies.

On construit quelques exemples d'extension de cette sorte ayant $2k$-spectres lebesguiens ($k \geqq 1$) (sur l'orthocomplémentaire de l'espace des fonctions propres).

Une nouvelle classe de points strictement transitifs dans $\{0,1\}^Z$, nommée k-Morse-suites ($k \geqq 1$) est introduite. Nous développons leur théorie métrique en démontrant qu'il y a continuum non isomorphique 2-Morse suites qui possède un 2-spectre lebesguien (sur l'orthocomplémentaire de l'espace des fonctions propres).

ABSTRACT. — We shall concern ourselves with some special subclass of ergodic Z_2-extensions of automorphisms with rational discrete spectra. Some connections with stricly transitive points in the space $\{0,1\}^Z$ (in particular with automata theory, Morse sequences theory, Toeplitz sequences theory) are exhibited. Examples of such extensions with $2k$-fold, $k \geqq 1$, Lebesgue spectrum (in the orthocomplement of the space of eigenfunctions) are constructed. A new class of strictly transitive points in $\{0,1\}^N$ called k-Morse sequences, $k \geqq 1$, is introduced. We develop their metric theory proving there are continuum of 2-Morse sequences pairwise nonisomorphic having 2-fold Lebesgue spectrum (in the orthocomplement of the space of eigenfunctions).

Key words : Lebesgue spectrum, Toeplitz sequence, ergodic theory, automata theory.

INTRODUCTION

The theory presented in the paper combines methods and results from a few papers [1], [5], [8], [10], [11], [12], [14], [16] and it is mainly connected with the possibility of the construction of ergodic automorphisms with nonsingular spectra and finite spectral multiplicity.

So far, there are two extremely interesting examples of such automorphisms. The first arises from automata theory. In [6] there is a proof that Rudin-Shapiro automat has a Lebesgue component in its spectrum. Queffelec in her dissertation developed spectral theory of dynamical systems generated by automata. She calculated that Rudin-Shapiro automat has twofold Lebesgue spectrum (in the orthocomplement of the eigenfunctions space). The second example (in fact continuum examples) was constructed in [14]. The authors considered some Z_2-extensions over an automorphism with discrete spectrum and proved that such examples had twofold Lebesgue spectrum in the orthocomplement of the eigenfunctions space.

Let us consider another two examples. In 1968 Keane [8] introduced to ergodic theory a class of strictly transitive points called generalized Morse sequences. However, in 1980 in [1] a new class of 0-1- and strictly transitive sequences was provided. The authors called them also generalized Morse sequences.

Are there some connections between those four examples? First of all, we see that the dynamical systems arising from the strictly transitive points above are always Z_2-extensions of automorphisms with rational pure point spectra. But their main feature in common is that for such an $x \in \{0, 1\}^{\mathrm{N}}$ the sequence $\hat{x} \in \{0, 1\}^{\mathrm{N}}$ is a regular Toeplitz sequence ([5], [18]) ($\hat{x}[n] = x[n] + x[n+1] \bmod 2$). On the other hand Mathew-Nadkarni's construction also leads to a class of regular Toeplitz sequences. In particular the results of this paper show that the theory of Rudin-Shapiro automat strictly includes in the Mathew-Nadkarni's theory.

Below, we intend to clarify in some details our approach to the problems arising.

Let $(\mathrm{X}, \mathscr{B}, \mu)$ be a Lebesgue space and $\mathrm{T} : (\mathrm{X}, \mathscr{B}, \mu) \circlearrowleft$ be its automorphism. Suppose T to be ergodic and to have a rational discrete spectrum, i. e. there is a sequence $\{n_t\}$, $n_t \mid n_{t+1}$, $t \geqq 0$ such that

$$\mathrm{Sp}(\mathrm{T}) = \mathrm{G}\{n_t : t \geqq 0\},$$

the group generated by $\{\exp 2\pi i/n_t : i = 0, \ldots, n_t - 1\}$, $t \geqq 0$ [here and in what follows Sp(T) denotes always the point spectrum of an automorphism

T]. Consider now a measurable map $\Theta : X \to Z_2 = \{0, 1\}$. We can define T_Θ, a Z_2-extension of T by

$$\left.\begin{aligned} T_\Theta : \quad (X \times Z_2, \tilde{\mu}) \circlearrowleft \\ T_\Theta(x, i) = (Tx, \Theta(x) + i), \end{aligned}\right\} \tag{1}$$

where $\tilde{\mu}$ is the product measure $\mu \times \nu (\nu(i) = 1/2, i = 1, 0)$ and + is always meant in Z_2 (unless it states otherwise). We will require Θ to satisfy some additional properties. Our concept lies in the following. It is well-known that if T is ergodic then T has a rational pure point spectrum with $\mathrm{Sp}(T) = G\{n_t : t \geqq 0\}$ iff there is a sequence $\{D^{n_t}\}$ of n_t-T-stacks such that $D^{n_t} \nearrow \varepsilon$ (the point partition), where $D^{n_t} = \{D_0^{n_t}, \ldots, D_{n_t - 1}^{n_t}\}$, $T^i D_0^{n_t} = D_i^{n_t} \pmod{n_t}$ and D^{n_t} is a partition of X. Hence, to define a mesurable $\Theta : X \to Z_2$ [i. e. two measurable sets $\Theta^{-1}(0)$, $\Theta^{-1}(1)$] it is sufficient to assume

$$\Theta^{-1}(i) \leqq \{D^{n_t}\}, \qquad i = 0,1. \tag{2}$$

This condition simply means that $\Theta^{-1}(i)$ consists of some levels of D^{n_t}, $t \geqq 0$. In other words there exists a partition of the set $\{0, 1, \ldots, n_t - 1\}$ into A^t, B^t, C^t such that

$$\begin{aligned} i \in A^t \quad & \text{iff } \Theta \big|_{D_i^{n_t}} = 0, \\ i \in B^t \quad & \text{iff } \Theta \big| D_i^{n_t} = 1, \\ i \in C^t \quad & \text{iff } \Theta \big| D_i^{n_t} \text{ is not constant.} \end{aligned} \tag{3}$$

Let us pass to the stack $D^{n_{t+1}}$. There is again a partition A^{t+1}, B^{t+1}, C^{t+1} of $\{0, 1, \ldots, n_{t+1} - 1\}$ satisfying (3). Denote

$$\lambda_{t+1} = n_{t+1}/n_t,$$

Then

$$\begin{aligned} & A^t, A^t + n_t, \ldots, A^t(\lambda_{t+1} - 1)n_t \subset A^{t+1}, \\ & B^t, B^t + n_t, \ldots, B^t + (\lambda_{t+1} - 1)n_t \subset B^{t+1}. \end{aligned} \tag{4}$$

Further, given A^t, B^t, C^t we can define a finite sequence $\Theta^{(t)}$ of 0's and 1's and some "holes", $\Theta^{(t)} = (\Theta_0^{(t)}, \ldots, \Theta_{n_t - 1}^{(t)})$, where

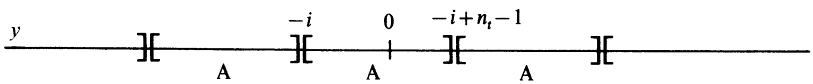
$$\Theta_i^{(t)} = \begin{matrix} 0 & \text{if } i \in A^t \\ 1 & \text{if } i \in B^t \\ \infty & \text{if } i \in C^t \end{matrix}$$

Due to (4) $\Theta^{(t+1)}$ arises from $\Theta^{(t)}$ by repeating $\Theta^{(t)}$ λ_{t+1} times and by filling up some "holes". This is just a method of constructing of Toeplitz sequences ([5], [18]). It is obvious that the method presented of constructing Θ leads always to a measurable function defined a. e. However, in order to assure T_Θ to have some "good" properties we claim Θ to be a regular Toeplitz sequence [18]. Now, we will consider some special examples of such Z_2-extensions.

Let $\Theta \in \{0, 1\}^N$ be a regular Toeplitz sequence. Put $X = X(\Theta)$ (where $X(u) = \overline{\{\tau^i u : i \in Z\}}$, $u \in \{0, 1\}^Z$, τ is the shift). Thus (X, τ) is a strictly ergodic dynamical system ([5], [18]), so there is a unique (ergodic) τ-invariant measure μ on X. Let $\{n_t\}$, $n_t \mid n_{t+1}\, t \geqq 0$, define a period structure of Θ (*see* [18]). Then there is a sequence $D^{n_t} = \{D_0^{n_t}, \ldots, D_{n_t-1}^{n_t}\}$, $t \geqq 0$, and $D_i^{n_t}$ are simultaneously open and closed, $\tau^i D_0^{n_t} = D_i^{n_t} \pmod{n_t}$ and $D^{n_t} \nearrow \varepsilon$ [18]. Moreover,

$$D_i^{n_t} = \{y \in X(\Theta) : y \text{ has the same } n_t\text{-skeleton as } \tau^i \Theta\} \quad [18] \qquad (5)$$

(here Θ is meant as two-sided sequence). In other words if n_t-skeleton of Θ is equal to, say, A, then y is of the form (*Picture* 1)



Picture 1

where the "holes" of A are filled up and y need not be a Toeplitz sequence at all.

It is well-known that $(X(\Theta), \tau, \mu)$ has a rational pure point spectrum with $\mathrm{Sp}(\tau) = G\{n_t : t \geqq 0\}$ where $\{n_t\}$ defines a period structure of Θ. Consider, now, the function $\Theta : X(\Theta) \to Z_2$ given by

$$\Theta(y) = y[0].$$

This function is continuous and taking into considerations D^{n_t} and n_t-skeleton of Θ we observe that $\Theta \mid_{D_i^{n_t}}$ is constant and equal to $\Theta[i]$ whenever $\Theta[i]$ is not the "hole" in the n_t-skeleton of Θ. Such extensions we will call *Toeplitz Z_2-extensions.*

So, we will consider regular Toeplitz sequences as maps $\Theta : X \to Z_2$ and we will examine properties of τ_Θ. This is our first point of view.

To present our second approach (combinatorial) we have to introduce some operations on 0-1-sequences and blocks.

Let $B=(b_0, \ldots, b_{r-1})$ be a block, $b_i \in \{0, 1\}$, $|B|=r$ is called the length of B. Then, by $\hat{B}$ we mean the block $\hat{B}=(\hat{b}_0, \ldots, \hat{b}_{r-2})$, $\hat{b}_i=b_i+b_{i+1}$, $i=0, \ldots, r-2$. The inverse operation $\check{B}$ yields two blocks C and $\tilde{C}$, where $\tilde{C}$ is got from C by interchanging all of the 0's and 1's ($\tilde{C}$ is the mirror image of C). These notations are easily adapted to the cases one-sided or two-sided sequences. In the sequel we treat $\check{}$ rather as a function. To avoid some difficulties, from now on, we will assume that if Θ is a regular Toeplitz sequence then it has the following property: *a block* B *appears on* $\check{\Theta}$ *iff the block* $\tilde{B}$ *appears on it.* In the case of strict ergodicity of $\check{\Theta}$ this condition is equivalent to the one that $X(\check{\Theta})$ is mirror-invariant. It will turn out (Section I) that there are some connections between τ on $X(\check{\Theta})$ and τ_{Θ}. Moreover, from metric theory point of view these two automorphisms are indistinguishable.

Now, we briefly describe further results of the paper.

In Section II we present spectral theory of Z_2-extensions. As a corollary we will get that in order to obtain a Z_2-extension with Lebesgue maximal spectral type (in the orthocomplement of the eigenfunctions space) it is sufficient to construct a regular Toeplitz sequence $\Theta \in \{0,1\}^{N}$ such that $\sum_{i \geq 1} |a_i^{\Theta}|^2 < \infty$, where

$$a_i^{\Theta} = \mu\Big(\bigcup_{\substack{|B|=i \\ \text{B contains} \\ \text{an even} \\ \text{number} \\ \text{of 1's}}} B\Big) - \mu\Big(\bigcup_{\substack{|B|=i \\ \text{B contains} \\ \text{an odd} \\ \text{number} \\ \text{of 1's}}} B\Big)$$

and μ is the measure defined by the average frequencies of blocks on Θ.

We also reach an estimation of spectral multiplicity of τ_{Θ}. This allows to provide for every $k \geq 1$ a class of Z_2-extensions with 2^k-fold Lebesgue spectrum (in the orthocomplement of the space of eigenfunctions) (*see* III.2).

In [1] the authors proposed to study properties of generalized Morse sequences. We do so for the rest of the paper. First of all we introduce a new class of strictly transitive points called k-Morse sequences, $k \geq 1$. Generalized Morse sequences introduced by Keane [8] are included in this theory for $k=1$, those introduced in [1] for $k=2^s$, $s \geq 1$. We develop metric theory of dynamical systems arising from such sequences. The tools used resemble those presented by the author in [10], [11], [12] and by M. K.

Mentzen and the author in [13]. In particular, we prove that Mathew-Nadkarni's construction contains continuum pairwise nonisomorphic dynamical systems.

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I. ERGODIC TOEPLITZ Z_2-EXTENSIONS AND SHIFT DYNAMICAL SYSTEMS

Let $\Theta \in \{0, 1\}^N$ be a regular Toeplitz sequence. We recall hare that τ_Θ is ergodic with respect to $\tilde{\mu}$ iff τ_Θ is uniquely ergodic [17]. Then for any partition $Q=(Q_0, Q_1)$ of $X(\Theta) \times Z_2$ and any $(x, i) \in X(\Theta) \times Z_2$ there exist the average frequencies of all blocks on $Q-\infty$-name of (x, i) i. e. let

$$b_j = \begin{cases} 0 & \text{if} \quad \tau_\Theta^j(x, i) \in Q_0 \\ 1 & \text{if} \quad \tau_\Theta^j(x, i) \in Q_1 \end{cases}$$

[the one-sided sequence b is then called $Q-\infty$-name of (x, i)], then

$$\lim_n \frac{1}{n} \operatorname{fr}(B, b[0, n-1]),$$

exists, where

$$\operatorname{fr}(B, b[0, n-1]) = \operatorname{card}\{i : b[i, i+|B|-1] = B\},$$
$$B \in \{0, 1\}^k$$

$\left(\text{in fact} \lim_n \frac{1}{n} \operatorname{fr}(B, b[i, i+n-1]) \text{ exists uniformly with respect to } i\right).$

THEOREM 1. — *Let τ_Θ be an ergodic Toeplitz Z_2-extension. Then Θ is strictly transitive.*

Proof. — Let us take $Q_i = X(\Theta) \times \{i\}$, $i=0, 1$. Then Q is a partition of $X(\Theta)$ into two open sets. Now, we will examine $Q-n$-names of points from Q_0. We have $D_0^{n_t} \times \{0\} \subset Q_0$, $t \geqq 0$ and fix $(x, 0) \in D_0^{n_t} \times \{0\}$. Let $\Theta^{(t)}$ be n_t-skeleton of Θ and suppose m_t to be the first place of appearing of a "hole", i. e.

$$\Theta^{(t)} = \Theta[0], \ldots, \Theta[m_t - 1], \infty, \ldots$$

and $\Theta[i] \in \{0, 1\}$, $0 \leqq i \leqq m_t - 1$.

Then

$$m_t \leqq m_{t+1} \quad \text{and} \quad m_t \underset{t}{\to} \infty \tag{7}$$

Let us observe that

$$\left.\begin{array}{c}\tau_\Theta^k(x, 0) = (\tau^k x, \Theta(\tau^{k-1} x) + \ldots + \Theta(\tau x) + \Theta(x)),\\ k = 0, 1, \ldots\end{array}\right\} \tag{8}$$

From (8) it follows that

$$Q - n\text{-name of } (x, 0) = (0, \Theta(x), \Theta(\tau x) + \Theta(x), \ldots, \Theta(\tau^{n-1} x) + \ldots + \Theta(\tau x) + \Theta(x))$$

This implies immediately that

$$(Q - m_t\text{-name of } (x, 0))^{\wedge} = \Theta[0, m_t - 1]. \tag{9}$$

Now, take into considerations the sets

$$X_k = \{(x, 0) \in Q_0 : Q - m_k\text{-name of } (x, 0) = \check{\Theta}[0, m_k - 1], \check{\Theta}[0] = 0\}$$

Then X_k is closed since τ_Θ is continuous (Θ is continuous) and from (9) $X_k \neq \varnothing$. Further

$$X_k \supset X_{k+1} \supset \ldots$$

in view of (7). Since $X(\Theta) \times Z_2$ is compact, $\bigcap_{t \geqq 1} X_t \neq \varnothing$ and therefore there is $(x, 0) \in Q_0$ such that $Q - \infty$-name of $(x, 0) = \check{\Theta}$. Hence $\check{\Theta}$ has the average frequencies of all blocks.

Remark. — If $\tilde{\mu}$ is the only ergodic measure then τ_Θ is in addition minimal since $\tilde{\mu}$ has positive measure on all open nonempty sets. Therefore in this case $\check{\Theta}$ is almost periodic (Q_i are open, $i = 0, 1$).

LEMMA 1. — *Let Θ be a Toeplitz sequence and $\check{\Theta}$ be almost periodic. Then* $X(\check{\Theta}) = X(\Theta)^{\vee} = \{\check{y} : y \in X(\Theta)\}$.

Proof. — If $y \in X(\Theta)$, then there must exist $\{i_s\}$ such that $\tau^{i_s}\Theta \to y$, this implies $\tau^{i_s}\check{\Theta} \to \check{y}$. Therefore $X(\Theta)^{\vee} \subset X(\check{\Theta})$. To see the inverse inclusion it is sufficient to prove $X(\Theta)^{\vee}$ is closed. So, let us assume $\check{y}^n \to z$,

$z \in X(\check{\Theta})$. We have, $\{\check{y}^n\}$ is a Cauchy sequence, so the sequence $\{y^n\}$ is also. Hence $y^n \to y \in X(\Theta)$ and $\check{y}^n \to \check{y}$.

Now, we are able to reverse Theorem 1.

THEOREM 2. — *Let Θ be a regular Toeplitz sequence and $\check{\Theta}$ be almost periodic. If $\check{\Theta}$ is also strictly transitive, then τ_Θ is (uniquely) ergodic.*

Proof. — Suppose, on the contrary, τ_Θ not to be ergodic. Then there is a measurable, nonconstant function $f : X(\Theta) \to S^1$ such that

$$f(\tau y) = (-1)^{\Theta(y)} f(y) \text{ a. e. } y \in X(\Theta) \quad [17]. \tag{10}$$

Let us take $g : X(\check{\Theta}) \to S^1$, $g(z) = (-1)^{z[0]} f(\hat{z})$. Then g is well defined from Lemma 1, measurable and it is not constant. We wish to show g is τ-invariant. To this aim we calculate

$$g(\tau z) = (-1)^{\tau z[0]} f(\tau z) = (-1)^{z[1]} f(\tau \hat{z}) \overset{(10)}{=} (-1)^{z[1]+\Theta(z)} f(\hat{z}).$$

To prove g is τ-invariant it remains to show $z[0] = z[1] + \Theta(\hat{z})$. However, from (6) $\Theta(\hat{z}) = \hat{z}[0]$ and the proof is complete.

As we have seen a regular Toeplitz sequence induces an ergodic Z_2-extension iff it induces some strictly ergodic dynamical system. This does not specifically exclude the possibility that these two dynamical systems are distinct. The following theorem clarifies this situation.

THEOREM 3. — *Let Θ be a regular Toeplitz sequence and τ_Θ be ergodic. Then τ_Θ and τ on $X(\check{\Theta})$ are metrically isomorphic.*

Proof. — From the proof of Theorem 1 it follows that $(X(\check{\Theta}), \tau)$ is isomorphic to the factor of τ_Θ generated by the partition Q. All we have to show is that Q is a generator of τ_Θ.

Let $\mathscr{B}_Q$ be a σ-algebra generated by $\bigvee_{-\infty}^{\infty} \tau_\Theta^i Q$ and η_Q be the corresponding measurable partition. Denote by η the partition of $X(\Theta) \times Z_2$ with the atoms of the form

$$\{(z, 0), (z, 1)\}, \qquad z \in X(\Theta).$$

Then $\eta_Q \geqq \eta$ since $\tau : X(\check{\Theta}) \circlearrowleft$ has $\tau : X(\Theta) \circlearrowleft$ as a factor and η is a canonical partition [15]. Moreover $\eta_Q \neq \eta$ since $Q_i \in \mathscr{B}_Q$ and $Q_i \notin \mathscr{B}(\eta)$ (the σ-algebra of η-measurable sets). This forces η_Q to be equal to ε because η_Q must have the same number of points in a. e. atom.

II. SOME SPECTRAL INVESTIGATIONS

II.1. Maximal spectral type

Let $T : (X, \mathscr{B}, \mu) \circlearrowleft$ be an ergodic automorphism with discrete spectrum and $Sp(T) = G\{n_t : t \geqq 0\}$. Let $\varphi : X \to Z_2$ be measurable. Then

$$L^2(X \times Z_2, \tilde{\mu}) = \mathscr{D} \oplus \mathscr{C}$$

where

$$\mathscr{D} = \{ f \in L^2(X \times Z_2, \tilde{\mu}) : f\sigma = f \}, \qquad \mathscr{C} = \{ f \in L^2(X \times Z_2, \tilde{\mu}) : f\sigma = -f \},$$

where

$$\sigma : (X \times Z_2, \tilde{\mu}) \circlearrowleft, \qquad \sigma(x, i) = (x, i+1).$$

Then $U_{T_\varphi}|_{\mathscr{D}}$ is spectrally isomorphic to $U_T : L^2(X, \mu)$.

Let us assume T_φ is ergodic. It is known [12] that if there is any point spectrum on $\mathscr{C}$ then T_φ has again pure point spectrum.

Define also $V^\varphi : L^2(X, \mu) \circlearrowleft$, by

$$V^\varphi(f)(x) = (-1)^{\varphi(x)} f(Tx).$$

Then V^φ is unitary and moreover

$$\left.\begin{array}{c} U_{T_\varphi}|_{\mathscr{C}} \text{ is spectrally isomorphic to } V^\varphi \\ ([3], [14]). \end{array}\right\} \qquad (11)$$

It turns out that when dealing with spectral properties of T_φ, the base role is played by the sequence

$$((V^\varphi)^n 1, 1) = \int_X (V^\varphi)^n (1)\, d\mu.$$

It is clear that

$$\int_X (V^\varphi)^n 1\, d\mu = \mu\{x : \varphi(x) + \ldots + \varphi(T^{n-1}x) = 0\} - \mu\{x : \varphi(x) + \ldots + \varphi(T^{n-1}x) = 1\}.$$

Now, let Θ be a regular Toeplitz sequence. Denote by

$$a_n^{\Theta}=\mu\Big(\bigcup_{\substack{|B|=n\\ \text{B has an}\\ \text{even number}\\ \text{of 1's}}} B\Big)-\mu\Big(\bigcup_{\substack{|B|=n\\ \text{B has an}\\ \text{odd number}\\ \text{of 1's}}} B\Big) \tag{12}$$

$n=1, 2, \ldots$

PROPOSITION 1. – *Let Θ be a regular Toeplitz sequence. Then*

$$((V^{\Theta})^k 1, 1)=a_k^{\Theta}, \qquad k=1, 2, \ldots \tag{13}$$

Proof. – The assertion follows from the following simple facts:

$$\frac{1}{n_t}(\text{number of the "holes" in } n_t\text{-skeleton of } \Theta) \underset{t}{\rightarrow} 0, \tag{14}$$

$$\mu(D_i^{n_t})=1/n_t, \tag{15}$$

there are the average frequencies of all blocks on Θ. (16)

PROPOSITION 2. – *Let Θ be a regular Toeplitz sequence such that Θ is strictly transitive. Then*

$$a_n^{\Theta}=\mu_{\check\Theta}(00^{(n)}\cup 11^{(n)})-\mu_{\check\Theta}(01^{(n)}\cup 10^{(n)}),$$

where $\mu_{\check\Theta}$ is the measure given by the average frequencies on $\check\Theta$ and

$$\mu(ij^{(n)})=\mu\{y\in X(\check\Theta) : y[0]=i,\ y[n-1]=j\}$$
$$=\lim_{N}\frac{1}{N}\operatorname{fr}(i\underbrace{\infty\infty\ldots\infty}_{n-2}j, \check\Theta[0, N-1]).$$

Proof. – Let us observe that if $\Theta[i]=1$, then

$$\check\Theta[i]+\check\Theta[i+1]=1$$

(i. e. the pair $(\check\Theta[i], \check\Theta[i+1])$ is disagreed). So, if $(\Theta_i, \Theta_{i+1}, \ldots, \Theta_j)$ contains an even (odd) number of 1's, then $(\check\Theta_i, \check\Theta_{i+1}, \ldots, \check\Theta_{j+1})$ contains an even (odd) number of pairs disagreed.

The following result is well-known (for instance [16]).

PROPOSITION 3. – *The maximal spectral type of V^{φ} is equal to*

$$\sigma_m=\sum_{k=1}^{\infty} 2^{-k}(\sigma * \delta_{\alpha_k})$$

where σ *is the spectral measure of* 1, *i. e.* $\sigma=\sigma_1$ *and* $\{\alpha_k\}$ *is the sequence of all eigenvalues of* T.

II.2. Ergodicity, mixing and Lebesgue spectrum

Suppose T_φ to be ergodic and take $f(x, i)=(-1)^i$, $x\in X$, $i\in Z_2$. Then using Birkhoff theorem we see that

$$\frac{1}{N}\sum_{k=0}^{N-1}[f(T_\varphi^k(x, i))] \underset{N}{\rightarrow} 0.$$

By Lebesgue integral theorem

$$\frac{1}{N}\sum_{k=0}^{N-1}\int_X [(-1)^{\varphi(T^{k-1}x)+\ldots+\varphi(x)}]\,d\mu(x) \underset{N}{\rightarrow} 0.$$

Thus, we obtain

$$\frac{1}{N}\sum_{k=0}^{N-1}((V^\varphi)^k 1, 1) \underset{N}{\rightarrow} 0 \tag{17}$$

as a necessary condition T_φ to have the ergodicity property.

Denote

$$((V^\varphi)^n 1, 1)=a_n^\varphi, \qquad n=0, 1, \ldots$$

COROLLARY 1. — T_φ *is ergodic iff*

$$\frac{1}{N}\sum_{i=0}^{N-1}\alpha_k^i a_i^\varphi \underset{N}{\rightarrow} 0, \quad \textit{for all } \alpha_k\in \mathrm{Sp}(T). \tag{18}$$

Proof. — The condition (18) gives

$$\frac{1}{N}\sum_{s=0}^{N-1}(U_{T_\varphi}^s F_k, F_k) \underset{N}{\rightarrow} 0,$$

where $F_k(x, s)=(-1)^s g_k(x)$ and g_k is the eigenfunction corresponding to $\alpha_k\in \mathrm{Sp}(T)$.

But $\{F_k\}$ is an orthonormal base on $\mathscr{C}$, so

$$\frac{1}{N}\sum_{s=0}^{N-1}(U^s_{T_\varphi}f,f)\underset{N}{\rightarrow}0$$

holds on the whole space $\mathscr{C}$. On $\mathscr{D}$ (19) is valid since T was ergodic.

COROLLARY 2. — *T_φ is weakly mixing on $\mathscr{C}$ iff*

$$\frac{1}{N}\sum_{s=0}^{N-1}|a^\varphi_i|\underset{N}{\rightarrow}0, \tag{20}$$

iff

$$a^\varphi_n \underset{n\notin M}{\rightarrow} 0,$$

where the density of M *is equal to* 0.

Proof. — Let us notice that σ_1 is continuous iff (20) holds and σ_1 is continuous iff σ_m is continuous.

COROLLARY 3. — *T_φ is mixing on $\mathscr{C}$ iff $a^\varphi_n \underset{n}{\rightarrow} 0$.*

Proof. — Let g_k be as in the proof of Corollary 1. Hence

$$((V^\varphi)^n g_k, g_k)=\alpha^n_k a^\varphi_n, \quad \text{so}\ ((V^\varphi)^n g_k, g_k)\underset{n}{\rightarrow}0.$$

As a consequence of Proposition 3 we get then

$$\int_{S^1} z^n\, d\sigma_m \underset{n}{\rightarrow} 0.$$

So, from the generalized Lebesgue-Riemann lemma $\int_{S^1} z^n\, d\nu \underset{n}{\rightarrow} 0$ whenever $\nu \ll \sigma_m$.

COROLLARY 4. — (*a*) *$\sigma_m \equiv \lambda$ iff $\sigma_1 \ll \lambda$ (λ is the Lebesgue measure)*;
(*b*) *if $\sigma_m \ll \lambda$, then $\sigma_m \equiv \lambda$;*
(*c*) *if $\sum_{n\geq 1}|\sigma_1(n)|^2<\infty$, then $\sigma_m \equiv \lambda$;*
(*d*) *σ_m is singular iff σ_1 is singular.*

Proof. — For the proof of (*a*) we notice that the set of eigenvalues $\{\alpha_k\}$ of T gives a dense set on S^1. The remain statements are clear.

II. AN ESTIMATION OF SPECTRAL MULTIPLICITY

THEOREM 4. — *Let Θ be a regular Toeplitz sequence with the following property: any n_t-skeleton $\Theta^{(t)}$ has at most* K *of the "holes" and $\Theta^{(t)}_{n_t-1}=\infty$. Then the maximal spectral multiplicity of V^Θ is also at most* K.

Proof. — We have $\Theta^{(t)}=(\Theta^{(t)}_0, \ldots, \Theta^{(t)}_{n_t-1})$. Let $i_1, \ldots, i_K$ be the all indices such that $\Theta^{(t)}_{i_s}=\infty$, $s=1, \ldots, K$. Next, we define functions $f^t_1, \ldots, f^t_K \in L^2(X(\Theta), \mu)$ putting

$$f^t_s=\chi_{D^{n_t}_{i_s+1}}, \quad s=2, \ldots, K, \qquad f^t_1=\chi_{D^{n_t}_0}. \tag{21}$$

Now, let $Z(f^t_1, \ldots, f^t_K)$ be the smallest V^Θ-invariant subspace of $L^2(X(\Theta), \mu)$.

Since

$$(V^\Theta)^r f^t_s=(-1)^{\varphi+\varphi\tau+\ldots+\varphi\tau^{r-1}} f^t_s(\tau^r),$$

if Θ is constant on $D^{n_t}_{i_s+1}, \ldots, D^{n_t}_{i_s+r}$ i.e. $\Theta^{(t)}_{i_s+1}, \ldots, \Theta^{(t)}_{i_s+r}$ are not the "holes"), then

$$(V^\Theta)^r f^t_s= \pm f^t_s(\tau^r)= \pm \chi_{D^{n_t}_{i_s+r}} (\operatorname{mod} n_t). \tag{22}$$

This leads to the following

$$L^2(D^{n_t}) \subset Z(f^t_1, \ldots, f^t_K), \tag{23}$$

where $L^2(D^{n_t})$ is the space of all measurable functions constant on the elements of D^{n_t}. Further

$$Z(f^t_1, \ldots, f^t_K) \subset Z(f^{t+1}_1, \ldots, f^{t+1}_K). \tag{24}$$

Indeed, it is sufficient to show $f^t_s \in Z(f^{t+1}_1, \ldots, f^{t+1}_K)$ and even, due to (23), that $f^t_s \in L^2(D^{n_{t+1}})$. But the latter is obvious, since $D^{n_t} \leqq D^{n_{t+1}}$. In virtue of $D^{n_t} \nearrow \varepsilon$ we get

$$\overline{\bigcup_t Z(f^t_1, \ldots, f^t_K)}=L^2(X, \mu)$$

and (24) completes the proof.

III. EXAMPLES

III.1. Generalized Morse sequences [8]

We refer the reader to the Morse shifts theory to [8] and [11]. Our notations are as in [11].

It is well-known that the dynamical systems arising from Morse sequences are Z_2-extensions (*see* for instance [9]). We intend to determine the form of the function defining the Z_2-extension.

To this end, let us introduce the so called $*$-product (*see* [2]). Let $B \in \{0, 1\}^n$, $C \in \{0, 1\}^m$

$$B * C = \begin{cases} B\, c_0\, B\, c_1 \ldots B\, c_{m-1}\, B \text{ if B contains an even number of 1's} \\ B\, \tilde{c}_0\, B\, \tilde{c}_1 \ldots B\, \tilde{c}_{m-1}\, B \text{ otherwise} \end{cases}$$

Then if d^0, d^1, ... is a sequence of blocks with lengths at least one and starting with 1 then the formula

$$z = d^0 * d^1 * \ldots \tag{25}$$

defines a Toeplitz sequence. The classic Toeplitz sequence is then equal to $1 * 1 * \ldots$ [4].

Let us denote $|d^i| = \lambda_i - 1$, $\lambda_i \geqq 2$. Then $n_t = \lambda_0 \ldots \lambda_t$-skeleton of z is equal to $z^{(t)} = d^0 * \ldots * d^t \infty$, $|z^{(t)}| = n_t$.

In other words the formula (25) leads always to the so called 1-spaced Toeplitz sequences (n_t-skeleton must contain only one "hole" at the last place). The reader can check that any 1-spaced Toeplitz sequence is of the form (25).

Now, we determine $\check{z}$. To this end let us recall that

$$B \times C = B^{c_0} B^{c_1} \ldots B^{c_{m-1}}, \qquad B^0 = B, \qquad B^1 = \tilde{B}.$$

The following formula holds

$$(B \times C)\hat{} = \hat{B} * \hat{C}. \tag{26}$$

Therefore $\check{z} = \check{d}^0 \times \check{d}^1 \times \ldots$

So, if $x = b^0 \times b^1 \times \ldots$ is a Morse sequence, then $\hat{x} = \hat{b}^0 * \hat{b}^1 * \ldots$ In addition n_t-skeleton of $\hat{x}$ is equal to

$$\hat{b}^0 * \hat{b}^1 * \ldots * \hat{b}^t \infty = \hat{c}_t \infty \; (c_t = b^0 \times \ldots \times b^t).$$

Morse sequences play a peculiar role in general ergodic theory (*see* [12]) and especially in the theory of Z_2-extensions over automorphisms with rational discrete spectra. In [12] there is a proof that Morse sequences compose a "typical" set of automorphisms (in the strong topology). In fact, Katok-Stepin theory of an odd approximation of Z_2-extensions is included in the theory of Morse sequences (for the speed of approximation at least $o(1/n^2)$ [12]).

As an immediate consequence of Theorem 4 we get all Morse sequences have simple spectra.

III.2. A class of Toeplitz Z_2-extensions with nonsingular spectra

Let $m=2^k$, $k \geqq 1$. We will construct a class of Toeplitz Z_2-extensions such that V^Θ will have uniform Lebesgue spectrum with maximal spectral multiplicity equal to m.

We will consider the Toeplitz sequence Θ as a limit of some periodic sequences $\mathrm{Per}\,\Theta^{(t)}$ (*see* [5]), where

$$\mathrm{Per}\,\Theta^{(n)} = |\, a_1^{(n)} a_2^{(n)} \ldots a_{k_n}^{(n)} \,|\, a_1^{(n)} a_2^{(n)} \ldots a_{k_n}^{(n)} \,| \ldots \in \{0, 1, \infty\}^{\mathrm{N}}$$

and Θ is constructed from $\{\mathrm{Per}\,\Theta^{(n)}\}$ in the following way. We first write down $\mathrm{Per}\,\Theta^{(1)}$, next we write down $\mathrm{Per}\,\Theta^{(2)}$ but only in the "holes", $\mathrm{Per}\,\Theta^{(3)}$ is written down in the remains "holes" and so on. The periodic part of $\mathrm{Per}\,\Theta^{(n)}$ will be said to be an n-base-skeleton of Θ and denoted by $\mathrm{per}\,\Theta^{(n)}$.

Now, set

$$\mathrm{per}\,\Theta^{(r)} = a_1^{(r)} \infty\, a_2^{(r)} \infty \ldots \infty\, a_{2^k-1}^{(r)} \infty\, \tilde{a}_1^{(r)} \infty\, \tilde{a}_2^{(r)} \infty \ldots \infty\, \tilde{a}_{2^k-1}^{(r)} \infty,$$
$$a_i^{(r)} \in \{0, 1\}, \qquad r = 1, 2, \ldots$$

So, the corresponding Toeplitz sequence arises in the following way:

$$\begin{array}{l}
a_1^{(1)} \infty\, a_2^{(1)} \infty\, a_3^{(1)} \infty\, a_4^{(1)} \infty \ldots \quad \infty\, \tilde{a}_{2^k-1}^{(1)} \infty \,|\; a_1^{(1)} \infty\, a_2^{(1)} \infty\, a_3^{(1)} \infty\, a_4^{(1)} \infty \ldots \infty\, \tilde{a}_{2^k+1}^{(1)} \infty \,| \ldots \\
\quad a_1^{(2)} \quad \infty \quad a_2^{(2)} \quad \infty \ldots \quad a_{2^k\ 1}^{(2)} \quad \infty \qquad \tilde{a}_1^{(2)} \quad \infty \quad \tilde{a}_2^{(2)} \quad \infty \ldots \tilde{a}_{2^k-1}^{(2)} \quad \infty \ldots \\
\qquad\quad a_1^{(3)} \qquad\qquad \infty \ldots \\
\qquad\quad \ldots
\end{array}$$

Any sequence Θ arising as the above has the following properties:

$$a_1^\Theta = 0. \tag{27}$$

since on each base-skeleton the frequencies of 0's and 1's are the same,

$$2^r\text{-skeleton of } \Theta = \Theta^{(r)} = A_1^{(r)} \infty A_2^{(r)} \infty \ldots \infty A_m^{(r)} \infty, \tag{28}$$

where

$$|A_i^{(r)}| = 2^r - 1, \qquad i = 1, \ldots, m.$$

We wish to get

$$a_n^\Theta = 0 \qquad \text{for} \quad n \geqq 2. \tag{29}$$

Let us fix $n \geqq 2$. Then, find $r \geqq 1$ such that

$$2^r \leqq n \leqq 2^{r+1} \tag{30}$$

and consider 2^r-skeleton of Θ (*Picture* 2).

$$\underbrace{A_1^{(r)} \infty \ldots \infty A_m^{(r)} \infty \,|}_{\mathscr{C}_1} \; \underbrace{A_1^{(r)} \infty A_2^{(r)} \infty \ldots \infty A_m^{(r)} \infty \,|}_{\mathscr{C}_2} \; \underbrace{A_1^{(r)} \infty A_2^{(r)} \infty \ldots \infty A_m^{(r)} \infty \,| \ldots}_{\mathscr{C}_3}$$

Picture 2

In each group $\mathscr{C}_i$ we number the "holes", in the same way (*Picture* 3)

$$\underbrace{A_1^{(r)} \overset{1}{\infty} A_2^{(r)} \overset{2}{\infty} \ldots \overset{2^k-1}{\infty} A_{2^k}^{(r)} \overset{2^k}{\infty}}_{\mathscr{C}_i}$$

Picture 3

Now, we fix our attention on $r+1, r+2, \ldots, r+k$-base skeletons of Θ. We get

1^0 $(r+1)$-base-skeleton fills up in each $\mathscr{C}_i$ 2^{k-1} of the "holes" at the places of the form $2^0 + 2^1 s$ (in each group $\mathscr{C}_i$ the same places!)

2^0 $(r+2)$-base-skeleton fills up in each $\mathscr{C}_i$ 2^{k-2} of the "holes" at the places of the form $2^1 + 2^2 s$ (in each group $\mathscr{C}_i$ the same places!)

. . .

j^0 $(r+j)$-base-skeleton fills up in each $\mathscr{C}_i$ 2^{k-j} of the "holes" at the places of the form $2^{j-1}+2^j s$ (in each group $\mathscr{C}_i$ the same places!)

. . .

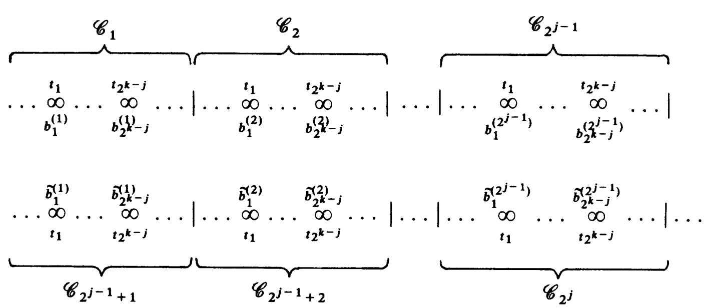
k^0 $(r+k)$-base-skeleton fills up in each $\mathscr{C}_i$ 1 of the "holes" at the place 2^{k-1} (in each $\mathscr{C}_i$ the same place!).

In addition, we notice that for any j, $1 \leqq j \leqq k$, $r+j$-base-skeleton can be written down as follows:

$$(b_1^{(1)} \infty \dots \infty b_{2^{k-j}}^{(1)}) \infty (b_1^{(2)} \infty \dots \infty b_{2^{k-j}}^{(2)}) \infty \dots \infty (b_1^{(2^{j-1})} \infty \dots \infty b_{2^{k-j}}^{(2^{j-1})}) \infty (\tilde{b}_1^{(1)} \infty \dots \infty \tilde{b}_{2^{k-j}}^{(1)}) \infty (\tilde{b}_1^{(2)} \infty \dots \infty \tilde{b}_{2^{k-j}}^{(2)}) \infty \dots \infty (\tilde{b}_1^{(2^{j-1})} \infty \dots \infty \tilde{b}_{2^{k-j}}^{(2^{j-1})}) \infty = \text{per}\, \Theta^{(r+j)}$$

Now, let $|\mathrm{B}|=n$. Due to (28) and (30) if we consider blocks of such length on 2^r-skeleton of Θ, then B must contain at least one "holes" and at most two "holes".

We take into consideration the case B is a subblock of $\mathrm{A}_i^{(r)} \infty \mathrm{A}_{i+1}^{(r)}$, $i \neq 2^k$. Then, there is a unique j, $1 \leqq j \leqq k-1$ that the i-th "hole" is filled up in $r+j$-base-skeleton. Now, the easy proof follows from Picture 4 below [$t_1, \dots, t_{2^{k-j}}$ are the numbers of "holes" that are filled up by $(r+j)$-base-skeleton]

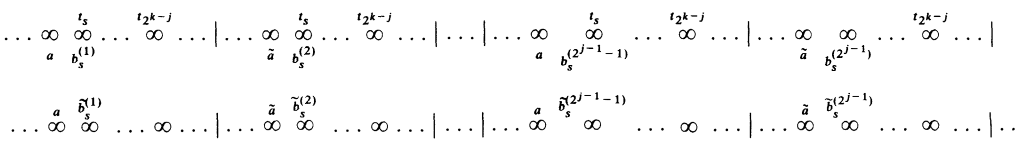


Picture 4

The frequencies of blocks of length n, which are subblocks of $\mathrm{A}_i^{(r)} \infty \mathrm{A}_{i+1}^{(r)}$, $i \neq 2^k$ simply cancel out.

If B is a subblock of $A_{2^k}^{(r)} \infty A_1^{(r)}$ then it is sufficient to notice that the Toeplitz sequence arising as the limit of Per $\Theta^{(r+k+1)}$, Per $\Theta^{(r+k+2)}, \ldots$ has the same frequencies of 0's and 1's.

Now, take into consideration more complicated case B contains two "holes". Then one of them *must* be filled up by $(r+1)$-base-skeleton and there is a unique j, $2 \leqq j \leqq k$ that the second is filled up by $(r+j)$-base skeleton (we exclude for a moment the case one of the "holes" has the number 2^k). The "corresponding" picture (*Picture* 5) looks like the below



Picture 5

To complete the proof of (29) it remains to consider the case B is a subblock of $A_{2^k}^{(r)} \infty A_1^{(r)} \infty A_2^{(r)}$ or $A_{2^k-1}^{(r)} \infty A_{2^k}^{(r)} \infty A_1^{(r)}$ (*Picture* 6)

$$A_1^{(r)} \infty \ldots \infty \underset{a}{A_{2^k-1}^{(r)} \infty} A_{2^k}^{(r)} \infty \,|\, A_1^{(r)} \infty \ldots \infty \underset{\tilde a}{A_{2^k-1}^{(r)} \infty} A_2^{(r)} \infty \,|\, A_1^{(r)} \infty \ldots \infty \underset{a}{A_{2^k-1}^{(r)} \infty} A_{2^k}^{(r)} \infty \,|\, \ldots$$

Picture 6

The $(r+k+1)$-base-skeleton fills up precisely those places near by a. The Toeplitz sequence arising from Per $\Theta^{(r+k+2)}$, Per $\Theta^{(r+k+3)}, \ldots$ fills up the remain "holes" (that are near by a). These remarks finish the proof of (29).

All we have shown, so far, is that

$$((V^{\Theta})^n 1, 1) = 0 \quad \text{for all } n \geqq 1 \tag{31}$$

and therefore for all $n \leqq -1$.

Thus V^{Θ} has Lebesgue spectrum with multiplicity at most 2^k (by Theorem 4). It remains to prove that it is at least 2^k.

To this end, let us take any i, $0 \leqq i \leqq 2^k - 1$ and consider all blocks (of a fixed length) that start at places of the form

$$i + s\,2^k, \qquad s = 0, 1, \ldots \tag{32}$$

We assert that for any n the frequencies of blocks of length n starting at places of the form (32) cancel out. Indeed, for $n \geqq 2$ this is the reasoning we have dealt with before a moment. It remains the case $n=1$. For i even the result is obvious. For i odd the reasoning as in case B contains one "hole" is deciding.

Now, notice that

$$((V^{\Theta})^n i, \chi_{D_i^{2^k}}) = \lim_N \frac{1}{N} \operatorname{fr}(B : |B| = n,$$
$$\text{B starts at a place of the form (32)}$$
$$\text{and B contains an even number of 1's, } \Theta\,[0, N-1])$$
$$-\lim_N \frac{1}{N} \operatorname{fr}(B : |B| = n,$$
$$\text{B starts at a place of the form (32)}$$
$$\text{and B contains an odd number of 1's, } \Theta\,[0, N-1]). \quad (33)$$

Combining the considerations above and (33) we get

$$((V^{\Theta})^n 1, \chi_{D_i^{2^k}}) = 0 \quad \text{for } n \geqq 1 \qquad (i=0, \ldots, 2^k-1). \tag{34}$$

The following formulae are also obvious

$$((V^{\Theta})^n \chi_{D_i^{2^k}}, \chi_{D_i^{2^k}}) = \left\{ \begin{array}{ll} ((V^{\Theta})^n 1, \chi_{D_i^{2^k}}) & \text{if} \quad n = i \bmod 2^k \\ 0 & \text{otherwise} \end{array} \right\} \tag{35}$$

$$((V^{\Theta})^n \chi_{D_i^{2^k}}, \chi_{D_j^{2^k}}) = \left\{ \begin{array}{ll} ((V^{\Theta})^n 1, \chi_{D_j^{2^k}}) & \text{if} \quad n+i-j = j \bmod 2^k \\ 0 & \text{otherwise} \end{array} \right\} \tag{36}$$

Combining (34), (35) and (36) we obtain that the spectral multiplicity is at least 2^k and we have proved the desired result.

III.3. Generalized Morse sequences [1]

In [1] the authors proposed to examine properties of the following sequences.

Let $u = u_{l-1} \ldots u_0 \in \{0, 1, \infty\}^l$, $u_{l-1} \neq \infty$.

Now, define an infinite sequence $x_u \in \{0, 1\}^{\mathbb{N}}$ putting

$$x_u[n] = \operatorname{card}\{ i \geqq 0 : (\forall j = 0, \ldots, l-1) u_j \neq \infty \rightarrow u_j = e_{j+1}(n) \} \bmod 2,$$

where

$$n=\sum_{i=0}^{\infty} e_i(n)\,2^n$$

$$[e_i(n)=0 \text{ for } i>\log n/\log 2],$$

i. e. $x_u[n]$ is the number of appearences of u on n written in binary expansion.

In [1] there is a proof that if u is not trivial (i. e. $u_i=1$ for some i, $0\leqq i\leqq l-1$), then u is recognizable by a 2-substitution. Moreover, there is more precise description in terms of $(k, 2k)$-substitutions. Namely, let u be nontrivial and let $\mathrm{E}(u)$ be the set of all $v=(v_0, \ldots, v_{l-1})\in\{0,1\}^l$ obtaining from u by filling up all the "holes". Put

$$\mathrm{Sp}(u)=\left\{\sum_{i=0}^{l-1} v_i 2^i : v\in\mathrm{E}(u)\right\} \quad \text{and} \quad k=2^{|u|-1}.$$

Then x_u is a fixed point of the $(k, 2k)$-substitution

$$\sigma(y_0, \ldots, y_{k-1})=\Theta^{(\alpha_0)}y_0\,\Theta^{(\alpha_1)}y_1\ldots\Theta^{(\alpha_{k-1})}y_{k-1}. \tag{37}$$

$y_i\in\{0,1\}$ and

$$\begin{array}{llll}\alpha_j=0 & \text{if} \quad 2j\notin\mathrm{Sp}(u) & \text{and} & 2j+1\notin\mathrm{Sp}(u)\\ \alpha_j=1 & \text{if} \quad 2j\in\mathrm{Sp}(u) & \text{and} & 2j+1\notin\mathrm{Sp}(u)\\ \alpha_j=2 & \text{if} \quad 2j\notin\mathrm{Sp}(u) & \text{and} & 2j+1\in\mathrm{Sp}(u)\\ \alpha_j=3 & \text{if} \quad 2j\in\mathrm{Sp}(u) & \text{and} & 2j+1\in\mathrm{Sp}(u)\end{array}$$

and $\Theta^{(0)}$, $\Theta^{(1)}$, $\Theta^{(2)}$, $\Theta^{(3)}$ are all of the continuous substitutions on $\{0,1\}$ of length two, i. e. $\Theta^{(i)}0=\Theta^{(i)}(1)$ [1].

Let us observe that the column number [16] of the substitution (37) is equal to 2. It suggests that from ergodic theory point of view they are rational Z_2-extensions. For a precise proof of this fact it is sufficient to show that $\hat{x}_u$ is always a regular Toeplitz sequence. However, we do not prove this at the stage since we will introduce a more general class of strictly transitive sequences including both generalized Morse sequences introduced by Keane and those from [1].

III.4. k-Morse sequences

Let k be a natural number, $k \geqq 1$. We will consider blocks the lengths of which are some multiple of k (we call them k-blocks).

Let $B \in \{0, 1\}^{rk}$, $C \in \{0, 1\}^{sk}$. Then

$$B = B_1 \ldots B_k, \qquad |B_i| = r,$$
$$C = (c_0, \ldots, c_{k-1})(c_k, \ldots, c_{2k-1}) \ldots (c_{(s-1)k}, \ldots, c_{sk-1}).$$

We define the k-product of k-blocks,

$$B \times_k C = B_1^{c_0} B_2^{c_1} \ldots B_k^{c_{k-1}} B_1^{c_k} B_2^{c_{k+1}} \ldots B_k^{c_{2k-1}} \ldots B_1^{c_{(s-1)k}} B_2^{c_{(s-1)k+1}} \ldots B_k^{c_{sk-1}} \tag{38}$$

where

$$B_i^0 = B_i, \qquad B_i^1 = \tilde{B}_i, \qquad i = 1, \ldots, k.$$

We have

$$|B \times_k C| = krs. \tag{39}$$

It is also worthwhile to note that k-product is an associative operation.

Now, let $b^0, b^1, b^2, \ldots$ be k-blocks and

$$|b^i| = k\lambda_i, \qquad \lambda_i \geqq 2,$$
$$b^i = (b_0^{(i)}, \ldots, b_{k-1}^{(i)})(b_k^{(i)}, \ldots, b_{2k-1}^{(i)}) \ldots (b_{(\lambda_i-1)k}^{(i)}, \ldots, b_{\lambda_i k-1}^{(i)})$$
$$(b_0^{(i)}, \ldots, b_{k-1}^{(i)}) = (0, \ldots, 0).$$

Then, we can define a one-sided sequence

$$x = b^0 \times_k b^1 \times_k b^2 \times_k \ldots \tag{40}$$

The sequences of the form (40) are said to be *k-Morse sequences* provided that x is almost periodic and strictly transitive. Of course, 1-Morse sequences are precisely those introduced in [8].

Now, we fix our attention on $(k, 2k)$-substitutions of the form (37). Assume in addition

$$\Theta^{(\alpha_0)} 0 \ldots \Theta^{(\alpha_{k-1})} 0 = \underbrace{0 \ldots 0}_{k} b_{k+1} \ldots b_{2k-1} = B_1 \ldots B_k \tag{41}$$

and put $B=\Theta^{(\alpha_0)}0\ldots\Theta^{(\alpha_{k-1})}0$. Then $|B|=2k$ and the k-Morse sequence

$$B\times_k B\times_k\ldots$$

is a fixed point of σ. In particular

$x_1=01\times_1 01\times_1\ldots$ – classic Thue-Morse sequence.

$x_{11}=0001\times_2 0001\times_2\ldots$ – Rudin-Shapiro sequence (a 2-Morse sequence).

Now, let $x=b^0\times_k b^1\times_k\ldots$ be a k-Morse sequence. By c_t we mean the product $b^0\times_k b^1\times_k\ldots\times_k b^t$,

$$|c_t|=kn_t,\qquad n_t=\lambda_0\ldots\lambda_t,\qquad x_t=b^t\times_k b^{t+1}\times_k\ldots$$

THEOREM 5. – *Let $x=b^0\times_k b^1\times_k\ldots$ be a k-Morse sequence. Then $\hat{x}$ is a regular Toeplitz sequence and moreover the sequence $\{kn_t\}$ is its period structure. The kn_t-skeleton of x has exactly k "holes" for any $t\geqq 0$.*

Proof. – Due to the associativity property we can write down

$$x=b^0\times_k(b^1\times_k b^2\times_k\ldots)=b^0\times_k x_1.$$

The sequence x_1 we divide into sequence of blocks of length k, i. e.

$$x_1=(e_0^{(1)},\ldots,e_{k-1}^{(1)})(e_k^{(1)},\ldots,e_{2k-1}^{(1)})\ldots$$

Since $b^0=b_1^{(0)}b_2^{(0)}\ldots b_k^{(0)}$, directly from the definition of k-product it follows that x is a juxtaposition of the form

$$b_1^{(0)\,\square}\,b_2^{(0)\,\square}\ldots b_k^{(0)\,\square}\,b_1^{(0)\,\square}\,b_2^{(0)\,\square}\ldots b_k^{(0)\,\square}\ldots$$

where $\square$ is equal to 0 or 1 (i. e. we take the block or take its mirror image). However, $b_j^{(0)}=\hat{\tilde{b}}_j^{(0)}$, so kn_0-skeleton is of the form

$$\hat{b}_1^{(0)}\infty\,\hat{b}_2^{(0)}\infty\ldots\infty\,\hat{b}_k^{(0)}\infty\,|\,\hat{b}_1^{(0)}\infty\,\hat{b}_2^{(0)}\infty\ldots\infty\,\hat{b}_k^{(0)}\infty\,|\ldots$$

Now, $x=c_t\times_k x_{t+1}$ and deducing as before we get that kn_t-skeleton of x is of the form

$$\hat{b}_1^{(01\ldots t)}\infty\ldots\infty\,\hat{b}_k^{(01\ldots t)}\infty\,|\,\hat{b}_1^{(01\ldots t)}\infty\ldots\infty\,\hat{b}_k^{(01\ldots t)}\infty\,|\ldots$$

which completes the proof.

At present, we turn back again to $(k,2k)$-substitutions of the form (41) to give more detailed description of the structure of the base-skeletons. We will assume that k is an even number.

To compute s-base-skeleton of σ we take into considerations

$$\underbrace{B\times_k\ldots\times_k B}_{s-1}\times_k B=C_1^{b_0}\ldots C_k^{b_{k-1}}C_1^{b_k}\ldots C_k^{b_{2k-1}},$$

where

$$\underbrace{B\times_k\ldots\times_k B}_{s-1}=C_1\ldots C_k,\qquad |C_i|=|C_j|,\quad i,j=1,\ldots,k.$$

But k is an even number so

$$B\times_k\ldots\times_k B\times_k B=(C_1^{b_0}C_2^{b_1})\times\ldots\times(C_{k-1}^{b_{k-2}}C_k^{b_{k-1}})(C_1^{b_k}C_2^{b_{k+1}})\ldots(C_{k-1}^{b_{2k-2}}C_k^{b_{2k-1}}).$$

Thus, from the proof of Theorem 5 we have obtained that s-base skeleton is equal to

$$\begin{aligned}(C_1[|C_1|-1]+C_2[0]+b_0+b_1)\\ \infty(C_3[|C_3|-1]+C_4[0]+b_2+b_3)\infty\ldots\\ \infty(C_{k-1}[|C_{k-1}|-1]+C_k[0]+b_{k-2}+b_{k-1})\\ \infty(C_1[|C_1|-1]+C_2[0]+b_k+b_{k+1})\\ \infty(C_3[|C_3|-1]+C_4[0]+b_{k+2}+b_{k+3})\infty\ldots\\ \infty(C_{k-1}[|C_{k-1}|-1]+C_k[0]+b_{2k-2}+b_{2k-1})\end{aligned}\tag{42}$$

Combining (41) and (42) we obtain that s-base-skeleton is of the form

$$u_1^{(s)}\infty u_2^{(s)}\infty\ldots\infty u_{k/2}^{(s)}\infty u_1^{(s)}+b_k+b_{k+1}\infty u_2^{(s)}+b_{k+2}+b_{k+3}\infty\ldots\infty u_{k/2}^{(s)}+b_{2k-2}+b_{2k-1}\infty$$

Due to III.2 we have proved the following

THEOREM 6. — *For any substitution* (41) *of the form*

$$\sigma:\ \underbrace{0\ldots0}_{2^s}\mapsto 0000\ldots00(i_1,1+i_1)(i_2,1+i_2)\ldots(i_{2^s-1},1+i_{2^s-1})$$

the dynamical system arising from σ has Lebesgue spectrum in the orthocomplement of the eigenfunctions space with spectral multiplicity equal to 2^s. In particular x_{11}, x_{10} have Lebesgue spectrum with multiplicity 2, *$x_{1\infty1}$, $x_{1\infty0}$ have Lebesgue spectrum with multiplicity* 4 *and in general $x_{1\infty\ldots\infty1}$, $x_{1\infty\ldots\infty}0$ have Lebesgue spectrum with multiplicity $2^{|u|-1}$.*

So far, we have not mentioned whether any sequence defined by (40) is a k-Morse sequence or not (i. e. whether it is almost periodic and strictly transitive). We will deal with this problem to get a new class of strictly ergodic dynamical systems as well as to get a class of ergodic Toeplitz Z_2-extensions.

THEOREM 7. – *Let* $x=b^0\times_k b^1\times_k\ldots$ *be a sequence such that*

$$x_t=b^t\times_k b^{t+1}\times_k\ldots=B_1^{(t)}B_2^{(t)}\ldots,$$

$|B_i^{(t)}|=k$, *has the property*

$$\textit{for any } C\in\{0,1\}^k \textit{ there is } j \textit{ such that } C=B_j^{(t)}. \tag{43}$$

Then x *is almost periodic.*

Proof. – Let B be any block. Then, there is t such that $B\subset c_t$ (i. e. B is a subblock of c_t). Hence, it is sufficient to prove that c_t appears on x with a bounded frequency.

We have the operations

$$c_t^{i_1\ldots i_k},\qquad i_s\in\{0,1\},\quad s=1,\ldots,k,$$

where $c_t=B_1^{(t)}\ldots B_k^{(t)}$ and

$$c_t^{i_1\ldots i_k}=(B_1^{(t)})^{i_1}\ldots(B_k^{(t)})^{i_k}. \tag{44}$$

Given $i_1,\ldots,i_k$ we seek such $s_{i_1,\ldots,i_k}$ that

$$c_t^{i_1\ldots i_k}\subset c_{s_{i_1,\ldots,i_k}}$$

[the block $c_t^{i_1,\ldots,i_k}$ appears on x in view of (43)]. Next, we select a common s to have

$$c_t^{i_1\ldots i_k}\subset c_s\quad\text{for any } i_1,\ldots,i_k.$$

It is possible by the simple fact $c_T\subset c_{T'}$ for $T\leqq T'$. Further, we seek T with the property

$$c_T=B_1^{(T)}\ldots B_k^{(T)},\qquad |B_i^{(T)}|>|c_s|.$$

This simply means that on $B_1^{(T)}$ all the blocks (44) appear. Moreover $x=c_T\times_k x_{T+1}$, so x is a juxtaposition

$$B_1^{(T)\square}\ldots B_k^{(T)\square}B_1^{(T)\square}\ldots B_k^{(T)\square}\ldots$$

So, to get our claim it is enough to notice that

$$c_t^{i_1 \dots i_k} \subset B_1^{(T)} \quad \text{iff } c_t^{i_1 \dots i_k} \subset \tilde{B}_1^{(T)}$$

[which is an easy consequence of the fact that on $B_1^{(T)}$ all the blocks (44) appear].

Now, let $x = b^0 \times_k b^1 \times_k \dots$ be a sequence and $B = (b_0, \dots, b_{sk-1})$ be any k-block.

For $j = 0, 1$ we put

$$\begin{gathered} \mathrm{fr}_i(j, B) = \mathrm{fr}(j : (b_i, b_{k+i}, \dots, b_{(s-1)k+i})) \\ i = 0, 1, \dots, k-1. \end{gathered} \tag{45}$$

We define some condition (★)

$$(\forall t \geqq 0)\, \mathrm{fr}_i(0, (b^t \times_k \dots \times_k b^{t+s})_j) - \mathrm{fr}_i(1, (b^t \times_k \dots \times_k b^{t+s})_j) \underset{s}{\to} 0, \quad (★)$$

$$i = 0, \dots, k-1$$

and

$$\begin{gathered} b^t \times_k \dots \times_k b^{t+s} = (b^t \times_k \dots \times_k b^{t+s})_1 \dots (b^t \times_k \dots \times_k b^{t+s})_{k'}, \\ j = 1, \dots, k. \end{gathered}$$

LEMMA 2. – *If* (★) *holds and* $k^2 \,|\, |b^t|$, $t \geqq 0$, *then* x *is strictly transitive.*

Proof. – For a purpose of this proof we employ the symbol fr(B, C) to indicate the average frequency $\frac{1}{|C|}\mathrm{fr}(B, C)$ and also we employ $\mathrm{fr}_i(j, B)$ to indicate $\frac{1}{|B|}\mathrm{fr}_i(j, B)$.

A. Suppose, first, that we consider $\mathrm{fr}(d, b \times_k c)$ but we assume $|d| \ll |b|$. If $b = B_1 \dots B_k$, then we can write

$$\begin{aligned} \mathrm{fr}(d, b \times_k c) \approx \mathrm{fr}(d, B_1)\,\mathrm{fr}_0(0, c) &+ \mathrm{fr}(\tilde{d}, B_1)\,\mathrm{fr}_0(1, c) \\ &+ \mathrm{fr}(d, B_2)\,\mathrm{fr}_1(0, c) + \mathrm{fr}(\tilde{d}, B_2)\,\mathrm{fr}_1(1, c) \\ &+ \dots \\ &+ \mathrm{fr}(d, B_k)\,\mathrm{fr}_{k-1}(0, c) + \mathrm{fr}(\tilde{d}, B_k)\,\mathrm{fr}_{k-1}(1, c). \end{aligned}$$

since the average frequencies obtained by appearences of d on blocks of the form $B_i^{\square} B_{i+1}^{\square}$ or $B_k^{\square} B_1^{\square}$ are not essential.

Similarly

$$\mathrm{fr}(\bar{d}, b\times_k c)\approx \mathrm{fr}(d, B_1)\,\mathrm{fr}_0(1, c)+\mathrm{fr}(\bar{d}, B_1)\,\mathrm{fr}_0(0, c) + \ldots + \mathrm{fr}(d, B_k)\,\mathrm{fr}_{k-1}(1, c)+\mathrm{fr}(\bar{d}, B_k)\,\mathrm{fr}_{k-1}(0, c).$$

Thies implies

$$\begin{aligned}|\mathrm{fr}(d, b\times_k c)-\mathrm{fr}(\bar{d}, b\times_k c)| &\approx |\mathrm{fr}(d, B_1)-\mathrm{fr}(\bar{d}, B_1)|\,.\,|\mathrm{fr}_0(0, c)-\mathrm{fr}_0(1, c)|\\ &+|\mathrm{fr}(d, B_2)-\mathrm{fr}(\bar{d}, B_2)|\,|\mathrm{fr}_1(0, c)-\mathrm{fr}_1(1, c)|\\ &+\ldots\\ &+|\mathrm{fr}(d, B_k)-\mathrm{fr}(\bar{d}, B_k)|\,|\mathrm{fr}_{k-1}(0, c)-\mathrm{fr}_{k-1}(1, c)|\\ &=\sum_{i=0}^{k-1}|\mathrm{fr}(d, B_{i+1})-\mathrm{fr}(\bar{d}, B_{i+1})|\,|\mathrm{fr}_i(0, c)-\mathrm{fr}_i(1, c)|. \qquad (46)\end{aligned}$$

Now, suppose that

$$b\times_k c=D_1\ldots D_k, \qquad |c|\gg|b|, \qquad k^2\,|\,|c|,$$
$$b=B_1\ldots B_k,$$
$$c=(c_0, \ldots, c_{k-1})(c_k, \ldots, c_{2k-1})\ldots(c_{(r-1)k}, \ldots, c_{rk-1}).$$

Thus

$$b\times_k c=\underbrace{B_1^{c_0}\ldots B_k^{c_{k-1}}B_1^{c_k}\ldots B_k^{c_{2k-1}}\ldots B_1^{c_{r-1}}}_{D_1}\ldots\underbrace{B_1^{c_{(r-1)k}}\ldots B_k^{c_{rk-1}}}_{D_k}$$

We have

$$\begin{aligned}\mathrm{fr}(d, D_1)&\approx \mathrm{fr}(d, B_1)\,\mathrm{fr}_0(0, c_0\ldots c_{r-1})+\mathrm{fr}(\bar{d}, B_1)\,\mathrm{fr}_0(1, c_0\ldots c_{r-1})\\ &+\mathrm{fr}(d, B_2)\,\mathrm{fr}_1(0, c_0\ldots c_{r-1})+\mathrm{fr}(\bar{d}, B_2)\,\mathrm{fr}_1(1, c_0\ldots c_{r-1})\\ &+\ldots\\ &+\mathrm{fr}(d, B_k)\,\mathrm{fr}_{k-1}(0, c_0\ldots c_{r-1})+\mathrm{fr}(\bar{d}, B_k)\,\mathrm{fr}_{k-1}(1, c_0\ldots c_{r-1})\end{aligned}$$

and generally

$$\begin{aligned}\mathrm{fr}(d, D_i)&\approx \mathrm{fr}(d, B_1)\,\mathrm{fr}_0(0, c_{(i-1)r}\ldots c_{ir-1})\\ &+\mathrm{fr}(\bar{d}, B_1)\,\mathrm{fr}_0(1, c_{(i-1)r}\ldots c_{ir-1})\\ &+\ldots\end{aligned}$$

$$+\mathrm{fr}(d, \mathrm{B}_k)\,\mathrm{fr}_{k-1}(0, c_{(i-1)r} \dots c_{ir-1})$$
$$+\mathrm{fr}(\tilde{d}, \mathrm{B}_k)\,\mathrm{fr}_{k-1}(1, c_{(i-1)r} \dots c_{ir-1}) \quad (47)$$

Establishing the analogous formula for fr $(\tilde{d}, \mathrm{D}_i)$ we get

$$|\mathrm{fr}(d, \mathrm{D}_i)-\mathrm{fr}(\tilde{d}, \mathrm{D}_i)| \approx \sum_{j=1}^{k} |\mathrm{fr}(d, \mathrm{B}_j)-\mathrm{fr}(\tilde{d}, \mathrm{B}_j)|$$
$$\times |\mathrm{fr}_{j-1}(0, c_{(i-1)r} \dots c_{ir-1})-\mathrm{fr}_{j-1}(1, c_{(i-1)r} \dots c_{ir-1})| \quad (48)$$

B. $\lim_{\mathrm{T}} (\mathrm{fr}(d, c_{\mathrm{T}}) - \mathrm{fr}(\tilde{d}, c_{\mathrm{T}})) = 0$ and also

$$\lim_{\mathrm{T}} (\mathrm{fr}(d, \mathrm{B}_i^{(\mathrm{T})})-\mathrm{fr}(\tilde{d}, \mathrm{B}_i^{(\mathrm{T})}))=0, \qquad c_{\mathrm{T}}=\mathrm{B}_1^{(\mathrm{T})} \dots \mathrm{B}_k^{(\mathrm{T})},$$
$$i=1, \dots, k.$$

It is enough to prove the latter statement.

From (47) we have the following

$$\Big\| \mathrm{fr}(d, \mathrm{B}_i^{(\mathrm{T})})-\mathrm{fr}(\tilde{d}, \mathrm{B}_i^{(\mathrm{T})})| - \sum_{j=1}^{k} |\mathrm{fr}(d, \mathrm{B}_j^{(t)})-\mathrm{fr}(\tilde{d}, \mathrm{B}_j^{(t)})|$$
$$|\mathrm{fr}_{j-1}(0, (b^t \times_k \dots \times_k b^{\mathrm{T}})_i-\mathrm{fr}_{j-1}(1, (b^t \times_k \dots \times_k b^{\mathrm{T}})_i \Big\| \underset{\mathrm{T}}{\rightarrow} 0$$

From (★) we obtain

$$\sum_{j=1}^{k} |\mathrm{fr}(d, \mathrm{B}_j^{(t)})-\mathrm{fr}(\tilde{d}, \mathrm{B}_j^{(t)}|$$
$$\times |\mathrm{fr}_{j-1}(0, (b^t \times_k \dots \times_k b^{\mathrm{T}})_i)-\mathrm{fr}_{j-1}(1, (b^t \times_k \dots \times_k b^{\mathrm{T}})_i)| \underset{\mathrm{T}}{\rightarrow} 0$$

which implies

$$|\mathrm{fr}(d, \mathrm{B}_i^{(\mathrm{T})})-\mathrm{fr}(\tilde{d}, \mathrm{B}_i^{(\mathrm{T})})| \underset{\mathrm{T}}{\rightarrow} 0.$$

C. $\lim_{\mathrm{T}} \mathrm{fr}(d, c_{\mathrm{T}})$ exists and also $\lim_{\mathrm{T}} \mathrm{fr}(d, \mathrm{B}_i^{(\mathrm{T})})$ exists for $i=1, \dots, k$.

First, take t_0 such that

$$|\mathrm{fr}(d, \mathrm{B}_j^{(t_0)})-\mathrm{fr}(\tilde{d}, \mathrm{B}_j^{(t_0)})| < \varepsilon, \qquad j=1, \dots, k.$$

Then, applying (47)

$$\Big|\operatorname{fr}(d, B_i^{(T)}) - \sum_{j=1}^{k} [\operatorname{fr}(d, B_j^{(t_0)}) \operatorname{fr}_{j-1}(0, (b^{t_0} \times_k \ldots \times_k b^T)_i) + \operatorname{fr}(\tilde{d}, B_j^{(t_0)} \operatorname{fr}_{j-1}(1, (b^{t_0} \times_k \ldots \times_k b^T)_i)]\Big| < \varepsilon$$

for $T \geqq T_0$.

But

$$\sum_{j=1}^{k} [\operatorname{fr}(d, B_j^{(t_0)}) \operatorname{fr}_{j-1}(\ldots) + \operatorname{fr}(\tilde{d}, B_j^{(t_0)} \operatorname{fr}_{j-1}(\ldots)]$$

$$= \sum_{j=1}^{k} [\operatorname{fr}(d, B_j^{(t_0)} \operatorname{fr}_{j-1}(\ldots) + \operatorname{fr}(\tilde{d}, B_j^{(t_0)})(1 - \operatorname{fr}_{j-1}(0, \ldots)]$$

$$= \sum_{j=1}^{k} \operatorname{fr}_{j-1}(0, (b^{t_0} \times_k \ldots \times_k b^T)_i) \,|\operatorname{fr}(d, B_j^{(t_0)}) - \operatorname{fr}(\tilde{d}, B_j^{(t_0)}| + \sum_{j=1}^{k} \operatorname{fr}(\tilde{d}, B_j^{(t_0)}).$$

The first sum is estimated by $k\,\varepsilon$, so

$$\Big|\operatorname{fr}(d, B_i^{(T)}) - \sum_{j=1}^{k} \operatorname{fr}(d, B_j^{(t_0)})\Big| < 100\, k^2 \varepsilon$$

$$\wr\wr$$

$$\sum_{j=1}^{k} \operatorname{fr}(\tilde{d}, B_j^{(t_0)})$$

so

$$|\operatorname{fr}(d, B_i^{(T)}) - \operatorname{fr}(d, B_i^{(T')})| < 200\, k^2 \varepsilon$$

whenever $T, T' > T_0$. In other words $\{\operatorname{fr}(d, B_i^{(T)})\}$ is a Cauchy sequence. $D.x$ is strictly transitive.

If we take a long sector $x[k, n]$, then it is a juxtaposition $c_T^{i_1, \ldots, i_k}$ (without some ends) and also it is a juxtaposition

$$B_1^{(T)\square}\, B_2^{(T)\square} \ldots B_k^{(T)\square} \ldots$$

On any $B_j^{(T)} (\tilde{B}_j^{(T)})$ there are frequencies of all blocks.

Remark. – The proof of Lemma 2 stems from the proof of Lemma 3 in [8].

There is a condition likewise the Keane's one [8] which implies the condition (★) holds. Namely

LEMMA 3. — *If* $x=b^0\times_k b^1\times_k\dots$ *and* $k^2\,||\,b^i|$, $i\geqq 0$ *and if*

$$\sum_{t\geqq 0}\left(\sum_{i=0}^{k-1}\min\left(\frac{1}{\lambda_t}\mathrm{fr}_i(0,\, b^{t\,(i+1)}),\frac{1}{\lambda_t}\mathrm{fr}_i(1,\, b^{t\,(i+1)})\right)\right)=\infty \tag{49}$$

where $b^t=b^{t\,(1)}\,b^{t\,(2)}\dots b^{t\,(k)}$, *then* (★) *holds.*

Proof. — First, we derive a formula for

$$\frac{1}{tr}\mathrm{fr}_j(0,(b\times_k c)_p),\qquad p=1,\,\dots,\,k,\quad j=0,\,\dots,\,k-1.$$

Let

$$b\times_k c=(b\times_k c)_1\,(b\times_k c)_2\dots(b\times_k c)_k,$$
$$|b|=tk,\qquad |c|=rk,\qquad |b\times_k c|=trk,$$
$$b=\mathbf{B}_1\dots\mathbf{B}_k,$$
$$c=(c_0,\,\dots,\,c_{k-1})\,(c_k\dots c_{2k-1})\times\dots(c_{(r-1)\,k}\dots,\,c_{rk-1})$$

Due to the assumption $k\,|\,r$ we have

$$\begin{aligned}\frac{1}{tr}\mathrm{fr}_j(0,\,(b\times_k c)_p)=&\frac{1}{t}\mathrm{fr}_j(0,\,\mathbf{B}_1)\frac{1}{r}\mathrm{fr}_0(0,\,c_{(p-1)\,r}\dots c_{pr-1})\\&+\frac{1}{t}\mathrm{fr}_j(1,\,\mathbf{B}_1)\frac{1}{r}\mathrm{fr}_0(1,\,c_{(p-1)\,r}\dots c_{pr-1})\\&+\dots\\&+\frac{1}{t}\mathrm{fr}_j(0,\,\mathbf{B}_k)\frac{1}{r}\mathrm{fr}_{k-1}(0,\,c_{(p-1)\,r}\dots c_{pr-1})\\&+\frac{1}{t}\mathrm{fr}_j(1,\,\mathbf{B}_k)\frac{1}{r}\mathrm{fr}_{k-1}(1,\,c_{(p-1)\,r}\dots c_{pr-1}).\end{aligned}$$

This implies

$$\frac{1}{tr}\mathrm{fr}_j(0,\,(b\times_k c)_p)=\sum_{s=0}^{k-1}\left[\frac{1}{t}\mathrm{fr}_j(0,\,\mathbf{B}_{s+1})\frac{1}{r}\mathrm{fr}_s(0,\,c^{(p)})+\frac{1}{t}\mathrm{fr}_j(1,\,\mathbf{B}_{s+1})\frac{1}{r}\mathrm{fr}_s(1,\,c^{(p)})\right] \tag{50}$$

where $c^{(p)} = c_{(p-1)r} \ldots c_{pr-1}$.

Similarly

$$\frac{1}{tr}\mathrm{fr}_j(1,(b\times_k c)_p) = \sum_{s=0}^{k-1}\left[\frac{1}{t}\mathrm{fr}_j(0,\,\mathrm{B}_{s+1})\frac{1}{r}\mathrm{fr}_s(1,\,c^{(p)}) + \frac{1}{t}\mathrm{fr}_j(1,\,\mathrm{B}_{s+1})\frac{1}{r}\mathrm{fr}_s(0,\,c^{(p)})\right] \quad (51)$$

Combining (50) and (51) we get

$$\left|\frac{1}{tr}\mathrm{fr}_j(0,(b\times_k c)_p) - \frac{1}{tr}\mathrm{fr}_j(1,(b\times_k c)_p)\right| = \sum_{s=0}^{k-1}\left|\frac{1}{t}\mathrm{fr}_j(0,\,\mathrm{B}_{s+1}) - \frac{1}{t}\mathrm{fr}_j(1,\,\mathrm{B}_{s+1})\right| \times \left|\frac{1}{r}\mathrm{fr}_s(0,\,c^{(p)}) - \frac{1}{r}\mathrm{fr}_s(1,\,c^{(p)})\right| \leqq \sum_{s=0}^{k-1}\left|\frac{1}{t}\mathrm{fr}_j(0,\,\mathrm{B}_{s+1}) - \frac{1}{t}\mathrm{fr}_j(1,\,\mathrm{B}_{s+1})\right| \times \left|\frac{1}{r}\mathrm{fr}_s(0,\,c^{(p)}) - \frac{1}{r}\mathrm{fr}_s(1,\,c^{(p)})\right|$$

By an easy induction

$$\left|\frac{1}{n_{t-1}}\mathrm{fr}_j(0,(b^0\times_k b^1\times_k \ldots \times_k b^{t-1})_p) - \frac{1}{n_{t-1}}\mathrm{fr}_j(1,\,(b^0\times_k b^1\times_k \ldots \times_k b^{t-1})_p)\right|$$
$$\leqq \sum_{s_1=0}^{k-1}\sum_{s_2=0}^{k-1}\ldots\sum_{s_{t-1}=0}^{k-1}\left|\frac{1}{\lambda_{t-1}}\mathrm{fr}_{s_{t-1}}(0,\,b^{t-1\,(p)}) - \frac{1}{\lambda_{t-1}}\mathrm{fr}_{s_{t-1}}(1,\,b^{t-1\,(p)})\right|$$
$$\times\left|\frac{1}{\lambda_{t-2}}\mathrm{fr}_{s_{t-2}}(0,\,b^{t-2\,(p)}) - \frac{1}{\lambda_{t-2}}\mathrm{fr}_{s_{t-2}}(1,\,b^{t-2\,(p)})\right|$$
$$\times \ldots$$
$$\times\left|\frac{1}{\lambda_1}\mathrm{fr}_{s_1}(0,\,b^{1\,(p)}) - \frac{1}{\lambda_1}\mathrm{fr}_{s_1}(1,\,b^{1\,(p)})\right|$$
$$\times\left|\frac{1}{\lambda_0}\mathrm{fr}_j(0,\,b^{0\,(s_1+1)}) - \frac{1}{\lambda_0}\mathrm{fr}_j(1,\,b^{0\,(s_1+1)})\right|$$

$$\leq \sum_{s_{t-1}=0}^{k-1} \left| \frac{1}{\lambda_{t-1}} \mathrm{fr}_{s_{t-1}}(0, b^{t-1\,(p)}) - \frac{1}{\lambda_{t-1}} \mathrm{fr}_{s_{t-1}}(1, b^{t-1\,(p)}) \right|$$

$$\times \sum_{s_{t-2}=0}^{k-1} \left| \frac{1}{\lambda_{t-2}} \mathrm{fr}_{s_{t-2}}(0, b^{t-2\,(p)}) - \frac{1}{\lambda_{t-2}} \mathrm{fr}_{s_{t-2}}(1, b^{t-2\,(p)}) \right| \dots$$

$$\times \sum_{s_1=0}^{k-1} \left| \frac{1}{\lambda_0} \mathrm{fr}_j(0, b^{0\,(s_1+1)}) - \frac{1}{\lambda_0} \mathrm{fr}_j(1, b^{0\,(s_1+1)}) \right|$$

$$\left| \frac{1}{\lambda_1} \mathrm{fr}_{s_1}(0, b^{1\,(p)}) - \frac{1}{\lambda_1} \mathrm{fr}_{s_1}(1, b^{1\,(p)}) \right|$$

It is well-known [8] that if the series (49) is divergent then the infinite product obtained converges to zero. Therefore (★) is satisfied (this reasoning does not require us to start with b^0).

We can put Lemma 2 and 3 together to obtain

THEOREM 8. — *Let* $x = b^0 \times_k b^1 \times_k \dots$ *and* $|b^i| = k\lambda_i$, $k | \lambda_i$, $i \geqq 0$ *and*

$$\sum_{t=0}^{\infty} \left(\sum_{i=0}^{k-1} \min\left(\frac{1}{\lambda_t} \mathrm{fr}_i(0, b^{t\,(i+1)}), \frac{1}{\lambda_t} \mathrm{fr}_i(1, b^{t\,(i+1)}) \right) \right) = \infty.$$

then x *is strictly transitive.*

IV. ON METRIC THEORY OF k-MORSE SEQUENCES

In this section we develop metric theory of k-Morse sequences. Our main aim is to show how to distinguish between some k-Morse sequences.

Let $x = b^0 \times_k b^1 \times_k \dots$ be a k-Morse sequence,

$$c_t = b^0 \times_k \dots \times_k b^t = B_1^{(t)} \dots B_k^{(t)},$$
$$|b^i| = k\lambda_i, \qquad |c_t| = kn_t = k\lambda_0 \dots \lambda_t.$$

In addition, we assume that for every $t \geqq 0$

$$x_t = b^t \times_k b^{t+1} \times_k \dots = (a_{11}^{(t)} \dots a_{1k}^{(t)})(a_{21}^{(t)} \dots a_{2k}^{(t)}) \dots$$

has the property:

$$\text{for every } B \in \{0,1\}^k \text{ there is } j \text{ such that } B=(a_{j1}^{(t)} \dots a_{jk}^{(t)}). \tag{52}$$

Thus, x is a juxtaposition of the blocks

$$c_t^{i_1 \dots i_k}=(B_1^{(t)})^{i_1}(B_2^{(t)})^{i_2} \dots (B_k^{(t)})^{i_k}, \quad i_s \in \{0,1\}. \tag{53}$$

Any block $c_t^{i_1 \dots i_k}$ will be said to be a t-symbol. Any $B_j^{(t)}$ (and its mirror image) we will call a sub-t-symbol, $j=1, \dots, k$.

We treat x as two-sided sequence (*see* [8]) and by $X(x)$ we mean the closure of trajectory of x via the shift $\tau:\{0,1\}^Z \circlearrowleft$, i.e.

$$X(x)=\overline{\{\tau^i x : i \in Z\}}.$$

We have got then a strictly ergodic dynamical system $(X(x), \tau, \mu_x)$ called a k-Morse dynamical system.

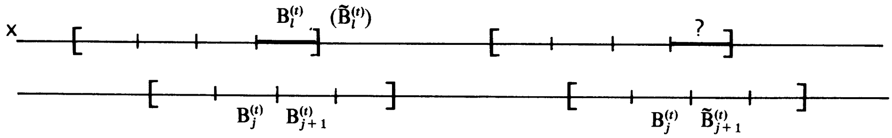
LEMMA 4 [the structure of $X(x)$]. — *Let $y \in X(x)$. Then for any $t \geqq 0$ there is a unique j_t, $0 \leqq j_t \leqq n_t - 1$ such that*

$$y[-j_t+rn_t, -j_t+(r+1)n_t-1]=(B_{k_r}^{(t)})^{i_r}, \quad r=0, \pm 1, \dots \tag{54}$$

Moreover, for a. e. $y \in X(x)$

$$j_t \underset{t}{\nearrow} \infty, \qquad k_t=n_t-j_t \underset{t}{\nearrow} \infty. \tag{5)}$$

Proof. — It is sufficient to prove (54) for x (see *Picture* 7).



Picture 7

The proof of (54) is now obvious from Picture 7 because we have assumed (52).

On the other hand any $y \in \{0, 1\}^Z$ satisfying (54) and (55) must belong to $X(x)$. So, by denoting

$$C^{n_t}_{j,0}(r) = \{ y \in X(x) : y[rn_t, (r+1)n_t - 1] = (B^{(t)}_{k_r})^{i_r},\ y[0, n_t - 1] = (B^{(t)}_r)^j \}$$
$$j = 0, 1, \qquad r = 1, \ldots, k$$

we see that $C^{n_t}_{j,0}(r)$ may be considered to be the base of an n_t-τ-stack. In this way we obtain $2k$ of n_t-τ-stacks (mutually disjoint) and moreover the corresponding sequence of partitions will tend to the point partition as soon as we prove (55).

To prove (55) we put

$$D^{n_t}_0 = C^{n_t}_{0,0}(1) \cup C^{n_t}_{1,0}(1) \cup \ldots \cup C^{n_t}_{0,0}(k) \cup C^{n_t}_{1,0}(k).$$

Then

$$D^{n_t}_0,\ \tau D^{n_t}_0,\ \ldots,\ \tau^{n_t - 1} D^{n_t}_0 \text{ are pairwise disjoint and } \bigcup_{i=0}^{n_t - 1} D^{n_t}_i = X(x). \quad (57)$$

Since there is a homomorphism h from $(X(x), \tau)$ to $(X(\hat{x}), \tau)$ (*see* Section I) and any n_t-τ-stack satisfying (57) is unique (due to the ergodicity property) $h^{-1} D^{n_t} = \bar{D}^{n_t}$, where $\bar{D}^{n_t}$ is the n_t-stack in $X(\hat{x})$ described in (5). The claim (55) is true for regular Toeplitz sequences ([5], [18]), so also for x.

Remark. — Another proof of Lemma 4 can be done on the basis of Lemma 1.

Remark. — In fact we have proved that the rank of a k-Morse sequence is bounded by $2k$. It is well-known that in that case spectral multiplicity is bounded by $2k$, too. But Theorem 4 strengthens this estimation. The spectral multiplicity of any k-Morse sequence is bounded by k (Theorem 5).

Lemma 4 says that a k-Morse sequence $x = b^0 \times_k b^2 \times_k \ldots$ has a unique decomposition into a juxtaposition of sub-t-symbols. However, to develop metric theory of k-Morse sequences we will require x to have a unique decomposition into a juxtaposition of t-symbols, $t \geqq 0$.

We start with a very simple observation.

LEMMA 5. — *Let* $x = b^0 \times_k b^1 \times_k \ldots$ *be a k-Morse sequence. Then*

$$d(c_t^{i_1 \ldots i_k}, c_t^{j_1 \ldots j_k}) = \begin{cases} 0 & \text{if } (i_1 \ldots i_k) = (j_1 \ldots j_k) \\ \geqq 1/k & \text{otherwise} \end{cases} \quad (58)$$

Let $x=b^0\times_k b^1\times_k\ldots$ be a k-Morse sequence such that for each t, if

$$x[j+skn_t,\ j+(s+1)kn_t-1]=c_t^{i_1^{(s)}\ldots i_k^{(s)}}\qquad\text{for}\quad s-0,1,\ldots,$$

then $kn_t\,|\,j$.

Hence, it is clear that for every $y\in X(x)$ and every t there is a unique j_t, $0\leqq j_t\leqq kn_t-1$ such that

$$y[-j_t+skn_t,\ -j_t+(s+1)kn_t-1]=c_t^{i_1^{(s)}\ldots i_k^{(s)}},$$
$$s=0,\ \pm 1,\ \ldots$$

In this case one can define 2^k kn_t-stacks with the bases

$$E_0^{i_1\ldots i_k,t}=\{y\in X(x):y[skn_t,(s+1)kn_t-1]=c_t^{i_1^{(s)}\ldots i_k^{(s)}}\text{ and } y[0,\ kn_t-1]=c_t^{i_1\ldots i_k}\}.\quad(59)$$

$(i_1,\ \ldots\ i_k)\in\{0,1\}^k$ and the corresponding sequence of partitions tends to the unit partition.

In others words we will assume that

for every t there is L_t such that if a string $\eta_1^{(t)}\ldots\eta_{L_t}^{(t)}$ appears on x at a place j and $\eta_i^{(t)}$ are t-symbols, then $kn_t\,|\,j$. (60)

An extremely interesting case is when the sequence $\{L_t\}$ is bounded, i. e.

there is $L=L(x)$ such that for every t if a string $\eta_1^{(t)}\ldots\eta_L^{(t)}$ appears on x at a place j and $\eta_i^{(t)}$ are t-symbols, then $kn_t\,|\,j$. (61)

The next lemma is an essential strengthener of the property (61) for the number $L(x)$.

LEMMA 6. — *If* $x=b^0\times_k b^1\times_k\ldots$ *is a* k*-Morse sequence such that* $k\,|\,\lambda_t$, $\sup k\lambda_t=k\lambda<\infty$ *and* $L=L(x)$ *satisfies* (61), *then*

$$(\exists\,\delta>0)\quad(\forall\, t\geqq 0)\quad[\textit{if}\ \eta=\eta_1^{(t)}\ldots\eta_{L(x)}^{(t)},$$
$$\eta_i^{(t)}\ \textit{is a t-symbol and}\ d(\eta,\ x[j,\ j+|\eta|-1])<\delta$$
$$\textit{then}\ d(\eta,\ x[j,\ j+|\eta|-1])=0\ \textit{and}\ kn_t\,|\,j].$$

Proof. – The proof goes by induction.

We put $\delta = 1/k\lambda L$.

1^0 $t=0$. If $\eta = \eta_1^{(0)} \dots \eta_L^{(0)}$ appears on x at a place j then $|\eta| = L k \lambda_0 \leqq k \lambda L$, so if $d(\eta, x[j, j + L k \lambda_0 - 1]) < \delta$, then it is equal to 0 and from (61) $k\lambda_0 | j$.

2^0 Let

$$\eta^{(t+1)} = \eta_1^{(t+1)} \dots \eta_L^{(t+1)} \quad \text{and} \quad d(\eta^{(t+1)}, x[j, j + L k n_{t+1} - 1]) < \delta.$$

Since $\eta_i^{(t+1)}$ is a $(t+1)$-symbol,

$$\eta_i^{(t+1)} = \eta_{i,1}^{(t)} \dots \eta_{i,\lambda_{t+1}}^{(t)} \quad \text{and} \quad \eta_{i,k}^{(t)} \text{ are } t\text{-symbols.} \tag{62}$$

Indeed, due to the condition $k | \lambda_{t+1}$ we have

$$\begin{aligned} c_{t+1} = c_t \times_k b^{t+1} &= (B_1^{(t)})^{b^{t+1}[0]} (B_2^{(t)})^{b^{t+1}[1]} \dots \\ &\qquad (B_k^{(t)})^{b^{t+1}[k-1]} (B_1^{(t)})^{b^{t+1}[k]} = \dots \\ &= c_t^{b^{t+1}[0], \dots, b^{t+1}[k-1]} c_t^{b^{t+1}[k], \dots, b^{t+1}[2k-1]} = \dots \\ &= \underbrace{c_t^{b^{t+1}[0], \dots, b^{t+1}[k-1]}}_{B_1^{(t+1)}} \dots c_t^{b^{t+1}[s_1 k], \dots, b^{t+1}[(s_1+1)k-1]} \dots \\ & \hspace{20em} B_k^{(t+1)} \end{aligned}$$

Therefore $c_{t+1}^{i_1 \dots i_k}$ is a juxtaposition of t-symbols since any $B_i^{(t+1)}$ was a juxtaposition of t-symbols.

Since $|B_i^{(t+1)}| = n_{t+1}$ and $|c_t^{j_1 \dots j_k}| = k n_t$, the number of t-symbols in $c_{t+1}^{i_1 \dots i_k}$ is equal to $(n_{t+1}/k n_t) k = \lambda_{t+1}$.

We write down $\eta^{(t+1)}$ as follows

$$\eta^{(t+1)} = \underbrace{\eta_{1,1}^{(t)} \dots \eta_{1,\lambda_{t+1}}^{(t)} \eta_{2,1}^{(t)}}_{L} \dots \eta_{2,\lambda_{t+1}}^{(t)} \dots \underbrace{\eta_{L,1}^{(t)} \dots \eta_{L,\lambda_{t+1}}^{(t)}}_{L},$$

i.e. as λ_{t+1} groups of L consecutive t-symbols. Then, there must exist such a group that it appears on x within δ. Applying the induction hypothesis we come that

$$k n_t | j. \tag{63}$$

In view of this fact, in the representation

$$d(\eta^{(t+1)}, x[j, j + L k n_{t+1} - 1]) < \delta$$

in opposite of t-symbols from $\eta^{(t+1)}$ there are t-symbols from x. So, if the distance (64) is positive, then by virtue of Lemma 5

$$d(\eta^{(t+1)}, x[j, j+\mathrm{L}\,kn_{t+1}-1]) \geqq \frac{1}{k}|c_t|/\mathrm{L}\,|c_{t+1}| = 1/\mathrm{L}\,k\,\lambda_{t+1} \geqq \delta$$

and we get a contradiction. In order to obtain $kn_{t+1}\,|\,j$ it is enough to apply (61).

Remark. – The transparent example of a 2-Morse sequence satisfying (61) is Rudin-Shapiro sequence $0001 \times_2 0001 \times_2 \ldots$ Therefore Lemma 6 is true for it. The significance of the lemma was presented (in case of substitutions) in [13]. In particular by the methods developed in [13] we are able to prove the following property of Rudin-Shapiro dynamical system:

Let (X, T, ν) *be any ergodic automorphism of a Lebesgue space. Suppose Rudin-Shapiro dynamical system to be a factor of* T *and let* U *be another factor of* T. *If* T *is of local rank one and* $\tau \times \mathrm{U}$ *is ergodic, then* U *is a rotation on a finite group.*

Now, we will examine some class $\mathscr{R}$ of 2-Morse sequences including Rudin-Shapiro sequence.

We put

$$x = b^0 \times_2 b^1 \times_2 \ldots \in \mathscr{R} \quad \text{iff } b^i = 0001 \text{ or } 0010. \tag{65}$$

Arguing as in II.4 (proof of Theorem 6) we see that any such x determines a 2-Morse shift with twofold Lebesgue spectrum (in the orthocomplement of the eigenfunctions space).

We intend to prove

THEOREM 9. – *If* $x = b^0 \times_2 b^1 \times_2 \ldots$, $y = \beta^0 \times_2 \beta^1 \times_2 \ldots \in \mathscr{R}$ *and* x *and* y *are measure-theoretically isomorphic, then*

$$b^t = \beta^t \tag{66}$$

for t large enough.

As a simple consequence of this theorem we get a continuum pairwise nonisomorphic dynamical systems with twofold nonsingular spectra.

Before the proof of Theorem 9 we will settle down some auxiliary informations.

Let us suppose $\varphi : (\mathrm{X}(x), \tau) \to (\mathrm{X}(y), \tau)$ to establish an isomorphism between

$$x = b^0 \times_2 b^1 \times_2 \ldots \in \mathscr{R} \qquad \text{and} \qquad y = \beta^0 \times_2 \beta^1 \times_2 \ldots \in \mathscr{R}.$$

Given $\varepsilon > 0$ and using ergodic theorem we can find a finite code $\varphi_\varepsilon : X(x) \to \{0, 1\}^Z$ (i. e. a measurable map, $\varphi_\varepsilon \tau = \tau \varphi_\varepsilon$), such that

$$d(\varphi_\varepsilon(z), \varphi(z)) = \lim_m d(\varphi_\varepsilon(z)[-m, m], \varphi(z)[-m, m]) < \varepsilon$$

$$\text{a. e. } z \in X(x) \qquad (67)$$

and the length of φ_ε is equal to n ($|\varphi_\varepsilon| = n$), i. e. $\varphi_\varepsilon(z)$ [0] depends only on $z[-n, n]$ and n is the smallest number with this property (for more details *see* [13]).

Now, we will establish some properties of finite codes.

$$\liminf_m d(\varphi_\varepsilon(z)[-m, m], \varphi_\varepsilon(\tilde{z})[-m, m]) > 1 - 2\varepsilon$$

$$\text{for a. e. } z \in X(x) \qquad (68)$$

Indeed, it follows from (67) and from the simple fact that

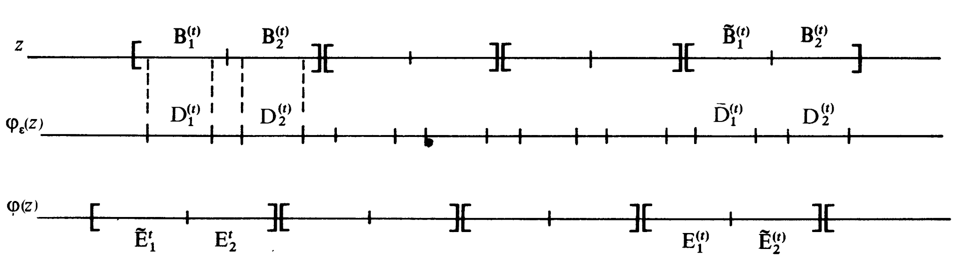
$$\varphi(\tilde{z}) = \varphi(z) \quad \text{a. e. } z \in X(x).$$

(for every Z_2-extension T_θ the map $(x, i) \mapsto (x, i+1)$ belongs to the second centralizer of T_θ [13]).

Denote by $B_1^{(t)}$, $\tilde{B}_1^{(t)}$, $B_2^{(t)}$, $\tilde{B}_2^{(t)}$ ($E_1^{(t)}$, $\tilde{E}_1^{(t)}$, $E_2^{(t)}$, $\tilde{E}_2^{(t)}$) all sub-t-symbols of $x(y)$ and by

$$D_1^{(t)}, \bar{D}_1^{(t)}, D_2^{(t)}, \bar{D}_2^{(t)} \qquad (|D_i^{(t)}| = |B_i^{(t)}| - 2n)$$

their codes via φ_ε (see *Picture* 8)



Picture 8

we will consider $t \in N$ such that

$$n/n_t < \varepsilon. \qquad (69)$$

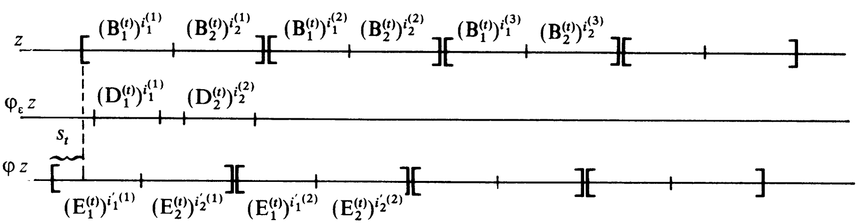
In view of (68) we may assume

$$\left.\begin{array}{r} d(\mathrm{D}_i^{(t)}, \bar{\mathrm{D}}_i^{(t)}) > 1-3\varepsilon \\ i=1,\ 2,\ \text{for } t \text{ large enough.} \end{array}\right\} \tag{70}$$

Let us fix t satisfying (69), (70) and denote by s_t the shift of sub-t-symbols of x with respect to sub-t-symbols of y (*Picture* 9).

We assert that

$$s_t/n_t < 1/8 \quad \text{or} \quad (n_t - s_t)/n_t < 1/8. \tag{71}$$



Picture 9

If (71) were false then from (67), (69) and (70) we would have that there are two functions $f_i: \{0,\ 1\} \to \{0,\ 1\}^2$ such that

$$f_i(1) = \widetilde{f_i(0)}, \qquad i=1,\ 2, \tag{72}$$

$$d(\mathrm{D}_1^{(t)}, (\mathrm{E}_1^{(t)})^{f_1(0)[0]}(\mathrm{E}_2^{(t)})^{f_1(0)[1]}) < 100\,\varepsilon, \quad f_i(0) = (f_i(0)[0], f_i(0)[1]), \tag{73}$$

$$d(\mathrm{D}_2^{(t)}, (\mathrm{E}_2^{(t)})^{f_2(0)[0]}(\mathrm{E}_1^{(t)})^{f_2(0)[1]}) < 100\,\varepsilon, \tag{74}$$

and

$$\left.\begin{array}{r} d(\mathrm{D}_1^{(t)}, (\mathrm{E}_1^{(t)})^{k_1}(\mathrm{E}_2^{(t)})^{k_2}) \geqq 1-100\,\varepsilon, \\ d(\mathrm{D}_2^{(t)}, (\mathrm{E}_2^{(t)})^{l_1}(\mathrm{E}_2^{(t)})^{l_2}) \geqq 1-100\,\varepsilon \end{array}\right\} \tag{75}$$

for any pairs

$$(k_1, k_2) \neq (f_1(0)[0], f_1(0)[1]), \qquad (l_1, l_2) \neq (f_2(0)[0], f_2(0)[1])$$

(for a suitable choice of ε).

For ε as small as we need one can assume there are $i_1^{(k)}$, $i_2^{(k)}$, $k=1, \ldots, 8$ such that (*Picture* 9)

$$f_1(i_1^{(1)}) = (i_1'^{(1)}, i_2'^{(1)}), \qquad f_2(i_2^{(1)}) = i_2'^{(1)}, i_1'^{(2)}, \ldots, f_2(i_2^{(8)}) = (i_2'^{(8)}, i_1'^{(9)}) \tag{76}$$

Put $f_1(0)=(0, 0)$, $f_2(0)=(0, 1)$ and $i_1'^{(0)}=0$. Then applying (76) we compute that

$$i_1'^{(1)} \ldots i_1'^{(9)} = 001100110. \tag{77}$$

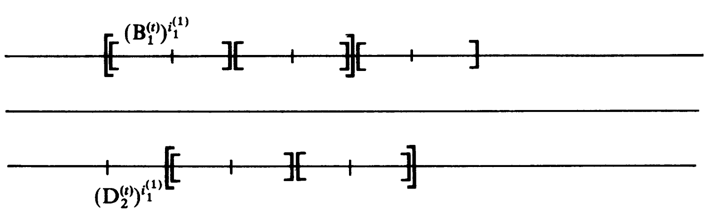
But $i_1'^{(1)} \ldots i_1'^{(9)}$ appears on $x_{t+1} \in \mathscr{R}$ and

$$x_{t+1} = \mathrm{BC} \times {}_2 x_{t+2} = \mathrm{B}^{\square} \mathrm{C}^{\square} \mathrm{B}^{\square} \mathrm{C}^{\square} \ldots$$

and $\mathrm{B} \neq \mathrm{C}, \tilde{\mathrm{C}}$, a contradiction.

The remaining cases are left to the reader. The proof of (71) is now complete.

Remark. – The proof of (71) does not exclude the following situation (*Picture* 10).



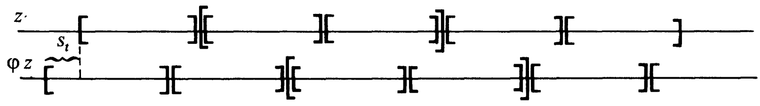
Picture 10

This case reduces to the preceding by passing from t into $t+1$.

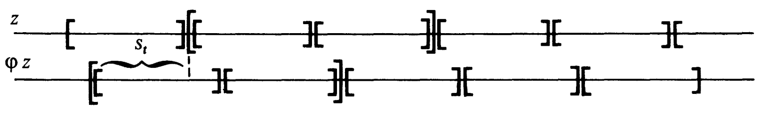
Now, we assert that

$$s_t \text{ is constant for } t \text{ large enough.} \tag{78}$$

An easy proof of it follows from Picture 11 and Picture 12 and the proof of the remark above.



Picture 11



Picture 12

So, we can assume $s=s_t$, $t \geqq 0$ and by considering

$$\varphi\tau^{-s} \text{ instead of } \varphi$$

we may assume that the numbers j_t, k_t, $t \geqq 0$ defined in (55) are *the same for* z *and* $\varphi(z)$, i. e.

$$j_t(z)=j_t(\varphi(z)), \qquad k_t(z)=k_t(\varphi(z)). \tag{79}$$

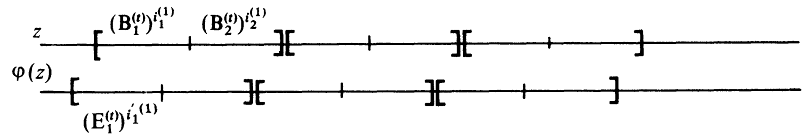
Proof of Theorem 9. — Let t be large enough (i. e. determined by the condition $s/n_t < \varepsilon$).

Due to (71) we can define two functions $g_i: \{0, 1\} \to \{0, 1\}$, $i=1, 2$ such that

$$g_i(1)=g_i(0), \qquad i=1, 2 \tag{80}$$

$$\left.\begin{aligned} d(D_1^{(t)}, (E_1^{(t)})^{g_1(0)} E_2^{(t)}) &< 100\,\varepsilon, \\ d(D_2^{(t)}, (E_2^{(t)})^{g_2(0)} E_1^{(t)}) &< 100\,\varepsilon \quad (\textit{Picture } 13) \end{aligned}\right\} \tag{81}$$

$$\left.\begin{array}{l} d(D_1^{(t)}, (E_1^{(t)})^{g_1(1)} E_2^{(t)}) \geqq 1-100\,\varepsilon, \\ d(D_2^{(t)}, (E_2^{(t)})^{g_2(1)} E_1^{(t)}) \geqq 1-100\,\varepsilon. \end{array}\right\} \tag{82}$$



Picture 13

Like in (76) we may assume

$$i_1'^{(1)} = g_1(i_1^{(1)}), \ldots, i_8'^{(2)} = g_2(i_8^{(2)}), \tag{83}$$

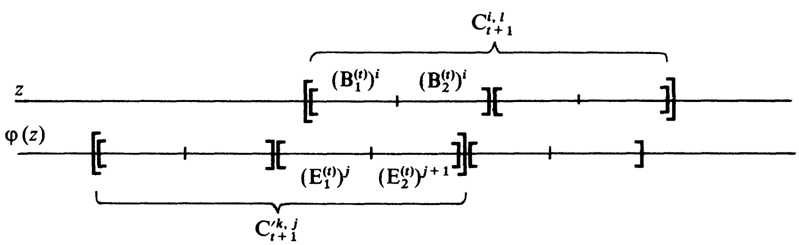
and moreover

$$\{i_1^{(1)}, i_2^{(1)}, \ldots, i_8^{(1)}\} = \{0, 1\}. \qquad \{i_1^{(2)}, i_2^{(2)}, \ldots, i_8^{(2)}\} = \{0, 1\}. \tag{84}$$

we assert that

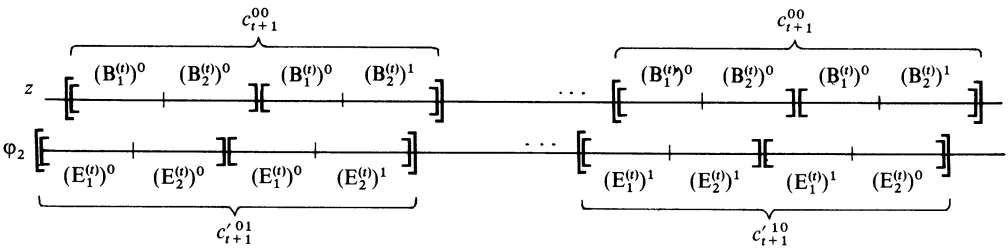
$$g_1(i) = g_2(i), \qquad i = 1, 2. \tag{85}$$

This follows from the following simple observation (*Picture* 14)



Picture 14

i. e. any pair (i, i) implies that $(t+1)$-symbol arises by joining the t-symbol considered $[(B_1^{(t)})^i (B_2^{(t)})^i$ on Picture 14] with the another from the right. Moreover, any pair $(j, j+1)$ implies that $(t+1)$-symbol arises by joining the t-symbol considered $[(E_1^{(t)})^j (E_2^{(t)})^{j+1}$ on Picture 14] with the another from the left (this rule is valid for *any* $x \in \mathscr{R}$). Hence (85) is true because of (79).



Picture 15

Now, suppose that $b^{t+1}=0001$ and $\beta^{t+1}=0010$.

Then one of the following situations has to appear (*Picture* 15). This leads to an obvious contradiction since from the rule above $(t+2)$-symbols on z are obtained by joining those from the right and on $\varphi(z)$ those from the left,

$$j_{t+2}(z) \neq j_{t+2}(\tau^{-s}\varphi z).$$

If we assume $y=x$, then any $\varphi:(X(x), \tau, \mu_x))$, $\varphi\tau=\tau\varphi$ has the property $\tau^{-s}\varphi$ acts inside of the atoms of the partition η [it follows from (79)], where $A\in\eta$ iff $A=\{z, \tilde{z}\}$, $z\in X(x)$. Then, immediately from the ergodicity of τ on $X(x)$ we obtain.

THEOREM 10. — *Let $x\in\mathscr{R}$ then the centralizer* $C(\tau)$ *of* τ *on* $X(x)$. *i. e.*

$$C(\tau)=\{\varphi:\varphi:(X(x), \mu_{x_x})\circlearrowleft, \varphi\tau=\tau\varphi\}$$

is of the form

$$C(\tau)=\{\tau^i\sigma^j: i\in Z, j=0, 1\}$$

where σ is the mirror map.

Two problems. First, observe that all Toeplitz Z_2-extensions with 2^k-fold Lebesgue spectrum we have considered in III.2 possess the point spectrum equal to $G\{2^t: t\geqq 0\}$. Therefore for any odd number s, τ_Θ^s is ergodic and has $2^k s$-fold Lebesgue spectrum (in the orthocomplement of proper functions), $k\geqq 1$. Hence for *any even number r* we get a class of Toeplitz Z_2-extensions with Lebesgue spectral multiplicity equal to r. The problem is, when are odd numbers the values of Lebesgue spectral multiplicity of Toeplitz Z_2-extensions (the most interesting case, $r=1$).

As we have seen in III.4 (Theorem 5) the spectral multiplicity of any k-Morse sequence is at most k. In what cases is it equal to k?

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