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Some results on the continuity of stable processes and the domain of attraction of continuous stable processes

by

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ABSTRACT. — We study the continuity of p-stable stochastic processes ($1 \leqq p < 2$) and their domain of attraction on the Banach space C. We can improve several recent results by weakening the assumptions of the metric entropy conditions. We also give some necessary conditions for the continuity of p-stable random Fourier integrals which extend results of Nisio and Salem-Zygmund for the case $p = 2$.

RÉSUMÉ. — Nous étudions la continuité des processus p-stables ($1 \leqq p < 2$) et leur domaine d'attraction sur l'espace de Banach C. Notre étude permet d'affaiblir les hypothèses sur l' « entropie métrique » dans plusieurs travaux récents. Nous donnons aussi des conditions nécessaires pour la continuité des « intégrales de Fourier aléatoires » qui étendent au cas p-stable des résultats de Nisio et Salem-Zygmund.

In [*8*] necessary and sufficient conditions are obtained for the continuity of strongly stationary p-stable random Fourier series. Methods used in [*8*] enable us to improve upon some of the results in [*1*] [*3*] and [*4*] relating to the continuity of p-stable stochastic processes and the characterization of their domains of attraction by weakening the requirements on the size of the metric entropy used in these results. We do this in Section 1. In Section 2 we give some necessary conditions for the continuity of p-stable random Fourier series and integrals, $1 \leq p < 2$, that are analogous to the classical results of Salem and Zygmund for series in the case $p = 2$.

1. SUFFICIENT CONDITIONS FOR CONTINUITY AND THE CENTRAL LIMIT THEOREM

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing convex function with $\Phi(0) = 0$. For any probability space $(\Omega, \mathcal{F}, \mathrm{P})$ we denote by $\mathrm{L}^{\Phi}(d\mathrm{P})$ the so called « Orlicz space » formed by all measurable functions $f : \Omega \to \mathrm{C}$ for which there is a $c > 0$ such that

$$\int \Phi\left(\frac{|f|}{c}\right) d\mathrm{P} < \infty .$$

We equip this space with the norm

$$\| f \|_{\Phi} = \inf \left\{ c > 0 : \mathrm{E}\Phi\left(\frac{|f|}{c}\right) \leq 1 \right\}.$$

We define

$$\psi_q(x) = \exp |x|^q - 1, \qquad 2 \leq q < \infty$$

and

$$\psi_{\infty}(x) = \exp(\exp |x|) - e .$$

We will consider the Orlicz spaces $\mathrm{L}^{\psi_q}(d\mathrm{P})$, $2 \leq q \leq \infty$. We will also be concerned with the weak $\mathrm{L}_{p,\infty}$ spaces defined as follows: These are the spaces of all real valued random variables for which $\mathrm{P}(|\mathrm{X}| > \lambda) = 0(1/\lambda^p)$, $1 \leq p \leq 2$. For these spaces we consider the function

$$\Lambda_p(\mathrm{X}) = \sup_{\lambda > 0} (\lambda^p \mathrm{P}(|\mathrm{X}| > \lambda))^{1/p},$$

which is equivalent to a norm for $p > 1$.

Let (T, ρ) be a compact metric or pseudo-metric space. We define by $\mathrm{N}(\mathrm{T}, \rho ; \varepsilon)$ the minimum number of open balls of radius $\varepsilon > 0$ in the ρ metric or pseudo-metric, with centers in T, that is necessary to cover T. We define

$$\mathrm{J}_q(\rho ; d) = \int_0^d (\log \mathrm{N}(\mathrm{T}, \rho ; \varepsilon))^{1/q} d\varepsilon, \qquad 2 \leq q < \infty ,$$

$$\mathrm{J}_{\infty}(\rho ; d) = \int_0^d \log^+ \log \mathrm{N}(\mathrm{T}, \rho, \varepsilon) d\varepsilon ,$$

and $J_q(\rho\,;\infty) = J_q(\rho)$, $2 \leq q \leq \infty$. The first lemma is a variant of Dudley's Theorem for Gaussian processes. It has been observed in various forms by many authors.

LEMMA 1.1. — Let $\{ X(t), t \in T \}$ be in $L^{\psi_q}(dP)$, $2 \leq q \leq \infty$ and satisfy

$$(1.1) \qquad \forall s, \quad t \in T, \qquad \| X(s) - X(t) \|_{\psi_q} \leq \rho(s, t)$$

If $J_q(\rho) < \infty$, $2 \leq q \leq \infty$ then $\{ X(t), t \in T \}$ has a version with continuous sample paths and

$$(1.2) \qquad E\Big(\sup_{\substack{\rho(s,t) \leq d \\ s,t \in T}} | X(s) - X(t) | \Big) \leq D_q [J_q(\rho\,; d) + \hat{\rho}\phi_q(d/4\hat{\rho})]$$

where $\phi_q(u) = u\,(\log^+ \log 1/u)^{1/q}$, $2 \leq q < \infty$, $\phi_\infty = u\,(\log^+ \log^+ \log 1/u)$, $\hat{\rho} = \sup_{s,t \in T} \rho(s, t)$ and D_q is a constant depending only on q. Furthermore, for any $t_0 \in T$

$$(1.2\,a) \qquad E(\sup_{t \in T} | X(t) |) \leq D'_q [J_q(\rho) + \hat{\rho} + E\,| X(t_0) |]$$

where D'_q is a constant depending only on q.

Proof. — This is proved in Chapter II, Theorem 3.1 [*7*], in the case $q=2$. As we comment in Lemma 3.2 [*8*], the proof for $2 < q < \infty$ is completely similar since (1.1) implies

$$P\left[\frac{| X(s) - X(t) |}{\rho(s, t)} > u \right] \leq 2e^{-|u|^q}, \qquad u > 0\,.$$

Likewise, the proof when $q = \infty$ follows because in this case (1.1) implies

$$P\left[\frac{| X(s) - X(t) |}{\rho(s, t)} > u \right] \leq 2e^{-e^{|u|}}, \qquad \forall u > 0\,.$$

The next Theorem gives conditions which imply tightness for normed sums of i. i. d. C(T) valued random variables, where C(T) denotes the Banach space of continuous functions on T with the sup norm.

THEOREM 1.2. — Let $\{ X(t), t \in T \}$ be a real valued stochastic process with continuous sample paths. If $p > 1$, we assume that $E\,| X(t) | < \infty$ and $EX(t) = 0$ for all $t \in T$. If $p = 1$, let $\{ X(t), t \in T \}$ be symmetric. Let $\{ X_k(t) \}$ be i. i. d. copies of $\{ X(t), t \in T \}$. Let τ be a continuous metric or pseudo-metric on T and define

$$(1.3) \qquad \| X \|_\tau = \sup_{s,t \in T} \frac{| X(s) - X(t) |}{\tau(s, t)}\,.$$

Then, in the notation of Lemma 1.1,

$$(1.4)\quad \Lambda_p\left(\sup_{\substack{\tau(s,t)\leq\delta\\ s,t\in T}} n^{-1/p}\left|\sum_{k=1}^{n}(X_k(s)-X_k(t))\right|\right) \leq C_p\Lambda_p(\|X\|_\tau)[J_q(\tau;\delta)+\hat{\tau}\phi_q(\delta/4\hat{\tau})]$$

where $\frac{1}{p}+\frac{1}{q}=1, 2<q\leq\infty$, and $\hat{\tau}=\sup_{s,t\in T}\tau(s,t)$. ($\phi_q$ is defined in Lemma 1.1).

Proof. — Let $\{X_k\}$ be defined on the probability space $(\Omega', \mathcal{F}', P)$ with expectation $E_{\Omega'}$. Let $\{\varepsilon_k\}$ be a Rademacher sequence independent of $\{X_k\}$ defined on the probability space $(\Omega'', \mathcal{F}'', P'')$ with expectation operator E_ε. We consider $n^{-1/p}\sum_{k=1}^{n}\varepsilon_k(X_k(t)-X_k(s))$ defined on the probability space $(\Omega'\times\Omega''\ \mathcal{F}'\times\mathcal{F}'', P'\times P'')$ which is equivalent to (i. e. has the same finite joint distribution as) $n^{-1/p}\sum_{k=1}^{n}(X_k(t)-X_k(s))$.

Note that by the contraction principle, see e. g. (4.8) Chapter II [7], and the fact that ψ_q is convex, we have for $s, t\in T$ and $\omega\in\Omega'$ fixed,

$$(1.5)\quad \left\|n^{-1/p}\sum_{k=1}^{n}\varepsilon_k(X_k(s,\omega)-X_k(t,\omega))\right\|_{\psi_q} \leq \left\|n^{-1/p}\sum_{k=1}^{n}\varepsilon_k\|X_k(\omega)\|_\tau\right\|_{\psi_q}\tau(s,t).$$

By Lemma 3.1 [8] (and its obvious extension for $q=\infty$) this is

$$(1.6)\qquad \leq C'_p\left(\sup_{1\leq k\leq n} k^{1/p}\frac{\|X_k(\omega)\|_\tau^*}{n^{1/p}}\right)\tau(s,t)$$

where $\{\|X_k(\omega)\|_\tau^*\}$ is a non-decreasing rearrangement of $\{\|X_k(\omega)\|_\tau\}$ and C'_p is a constant depending only on p. Let

$$(1.7)\qquad R_n(\omega)=\left(\sup_{1\leq k\leq n} k^{1/p}\frac{\|X_k(\omega)\|_\tau^*}{n^{1/p}}\right)$$

and

$$\mathrm{H}_q(\tau\,;\delta) = \mathrm{J}_q(\tau\,;\delta) + \hat{\tau}\phi_q(\delta/4\hat{\tau}), \qquad 2 < q < \infty\,. \tag{1.8}$$

By Theorem 3.3 [8], for $c > 0$

$$\mathrm{P}[\mathrm{H}_q(\tau\,;\delta)\mathrm{R}_n(\omega) > c] \leq \lambda_p^p c^{-p}\mathrm{H}_q^p(\tau,\delta)\Lambda_p^p(\|\mathrm{X}\|_\tau) \tag{1.9}$$

where λ_p is a constant depending only on p.

In order to obtain (1.4) we note that

$$\begin{aligned}
&\mathrm{P}\Big(\sup_{\substack{\tau(s,t)\leq\delta\\ s,t\in\mathrm{T}}} n^{-1/p}\Big|\sum_{k=1}^{n}\varepsilon_k(\mathrm{X}_k(s)-\mathrm{X}_k(t))\Big| > c\Big) \qquad (1.10)\\
&\leq \mathrm{P}\Big(\Big\{\sup_{\substack{\tau(s,t)\leq\delta\\ s,t\in\mathrm{T}}} n^{-1/p}\Big|\sum_{k=1}^{n}\varepsilon_k(\mathrm{X}_k(s)-\mathrm{X}_k(t))\Big| > c,\ \mathrm{H}_q(\tau\,;\delta)\mathrm{R}_n(\omega)\leq c\Big\}\Big)\\
&\quad + \mathrm{P}(\mathrm{H}_q(\tau\,;\delta)\mathrm{R}_n(\omega) > c)\\
&\leq c^{-2}\mathrm{E}_\Omega\mathrm{E}_\varepsilon\Big(\sup_{\substack{\tau(s,t)\leq\delta\\ s,t\in\mathrm{T}}} n^{-1/p}\Big|\sum_{k=1}^{n}\varepsilon_k(\mathrm{X}_k(s)-\mathrm{X}_k(t))\Big|\,\mathrm{I}_{[\mathrm{H}_q(\tau,\delta)\mathrm{R}_n(\omega)\leq c]}\Big)^2\\
&\quad + \lambda_p^p c^{-p}\mathrm{H}_q^p(\tau\,;\delta)\Lambda_p^p(\|\mathrm{X}\|_\tau)
\end{aligned}$$

where we use (1.9) at the last step.

We now consider

$$\mathrm{E}_\varepsilon\Big(\sup_{\substack{\tau(s,t)\leq\delta\\ s,t\in\mathrm{T}}} n^{-1/p}\Big|\sum_{k=1}^{n}\varepsilon_k(\mathrm{X}_k(s,\omega)-\mathrm{X}_k(t,\omega))\Big|\Big)^2.$$

Let

$$\rho(s,t) = \mathrm{C}'_p\mathrm{R}_n(\omega)\tau(s,t) \tag{1.11}$$

and

$$d = \delta\mathrm{C}'_p\mathrm{R}_n(\omega) \tag{1.12}$$

and note that for $b > 0$, $\mathrm{N}_{b\tau}(\varepsilon) = \mathrm{N}_\tau(\varepsilon/b)$. Using (1.11) and (1.12) and a change of variables of integration we see that

$$\mathrm{H}_q(\rho\,;d) = \mathrm{C}'_p\mathrm{R}_n(\omega)\mathrm{H}_q(\tau\,;\delta)\,. \tag{1.13}$$

Therefore by Lemma 1.1 (1.5) (1.6) and (1.7), with ρ and d as given in (1.11) and (1.12) and the fact that the second and first moments of Rademacher series are equivalent we get

$$E_\varepsilon\Big(\sup_{\substack{\tau(s,t)\le\delta\\ s,t\in T}} n^{-1/p}\Big|\sum_{k=1}^n \varepsilon_k(X_k(s)-X_k(t))\Big|\Big)^2$$

$$= E_\varepsilon\Big(\sup_{\substack{\rho(s,t)\le d\\ s,t\in T}} n^{-1/p}\Big|\sum_{k=1}^n \varepsilon_k(X_k(s)-X_k(t))\Big|\Big)^2$$

$$< (D_q H_q(\rho\,;d))^2 = (D_q C'_p R_n(\omega) H_q(\tau\,;\delta))^2$$

where at the last step we use (1.13). Therefore

$$(1.14)\quad E_{\Omega'}E_\varepsilon\Big(\sup_{\substack{\tau(s,t)\le\delta\\ s,t\in T}} n^{-1/p}\Big|\sum_{k=1}^n \varepsilon_k(X_k(s)-X_k(t))\Big|\Big)^2 I_{[H_q(\tau;\delta)R_n(\omega)\le c]}$$

$$\le (D_q C'_p)^2 E_{\Omega'}(H_q(\tau;\delta)R_n(\omega))^2 I_{[H_q(\tau;\delta)R_n(\omega)\le c]}$$

$$\le (D_q C'_p)^2 \frac{2}{2-p}\,[\lambda_p H_q(\tau\,;\delta)\Lambda_p(\|X\|_\tau)]^p,$$

where at the last step we use (1.9). We now get (1.4) (in the symmetric case) by combining (1.10) and (1.14). We can remove the condition of symmetry when $p > 1$, since in this case, it is easy to show that for Z, Z′ i. i. d. there exists a constant C''_p depending only on p such that

$$(C''_p)^{-1}\Lambda_p(\|Z\|) \leqslant \Lambda_p(\|Z-Z'\|) \le C''_p\Lambda_p(\|Z\|).$$

This completes the proof of the theorem.

We note that the proof of this theorem was suggested by Theorem 2.1 [*3*].

REMARK 1.3. — When $p=2$ (1.4) is valid if $\Lambda_p(\ \ )$ is replaced by $(E|\ \ |^2)^{1/2}$ in the two places where it appears. The proof is essentially the same as the one given here but somewhat easier.

We now give a central limit theorem for stochastic processes in the domain of normal attraction of p-stable processes. Continuing the notation of Theorem 1.2 we say that $\{X(t), t\in T\}$ is in the domain of normal attraction of a p-stable process on a Banach space B if the measures induced by $n^{-1/p}\sum_{k=1}^n X_k(t)$ converge weakly to a stable measure on B, $1 \le p < 2$.

COROLLARY 1.4. — Let $\{ X(t), t \in T \}$ be as in Theorem 1.2. Let $1 \leq p < 2$ and assume that

i) The n-dimensional finite joint distributions of $\{ X(t), t \in T \}$ are in the domain of normal attraction of a p-stable measure on R^n, for all $1 \leq n < \infty$;

ii) $\Lambda_p(\| X \|_\tau) < \infty$, and

iii) $J_q(\tau) < \infty$.

Then $\{ X(t), t \in T \}$ is in the domain of normal attraction of a p-stable measure on C(T). (The measure is determined by the limits of the finite joint distributions in (*i*).)

Proof. — There is nothing to prove. Theorem 1.2 gives tightness and this and (*i*) are all that is needed for this Corollary.

REMARK 1.5. — When $p=2$ we can replace (*ii*) in Corollary 1.4 by $E \| X \|_\tau^2 < \infty$ and $J_q(\tau)$ by $J_2(\tau)$. Then we get the central limit theorem of [*6*]. In this case (*i*) can be simplified to $EX(t_0)^2 < \infty$ for some $t_0 \in T$, since this condition and $E \| X \|_\tau^2 < \infty$ imply $\sup_{t \in T} EX^2(t) < \infty$ and this implies (*i*) as given in the statement of Corollary 1.4. Note that by Remark 1.3 we also have a proof of the central limit theorem of [*6*]. When $1 \leq p < 2$ Corollary 1.4 seems to be the correct generalization of this central limit theorem. Corollary 1.4 improves upon Theorem 3.1 [*3*], in that it weakens the metric entropy condition. Theorem 2.1 [*3*], which deals with triangular arrays can be similarly improved.

We now obtain a sufficient condition for a stable process to have continuous sample paths. Let $(U, \mathscr{U}, m)$ be a measure space where m is a positive σ-finite measure. We say that M is an independently scattered random p-stable measure on $(U, \mathscr{U})$ with control measure m if the following two conditions are satisfied:

(1.15) If $U_k \in \mathscr{U}$, $k = 1, \ldots,$ are disjoint then $M(U_k)$, $k=1, \ldots$ are independent and

(1.16) There exists a p, $1 \leq p \leq 2$ such that for all $U \in \mathscr{U}$, $M(U) \overset{\mathscr{D}}{=} m^{1/p}(U)\theta$ where θ is a canonical p-stable random variable, i. e.

$$E \exp (it\theta) = \exp(- | t |^p), \qquad 1 \leq p \leq 2,$$

and "$\overset{\mathscr{D}}{=}$" means equal in distribution.

Let m be a finite positive symmetric measure on the boundary of the unit ball B of C(T) centered at 0. Let $\mathscr{C}$ denote the Borel sets of C(T) (with respect to the sup norm) and let M be an independently scattered random p-stable measure on $(C, \mathscr{C})$ with control measure m. Let $x \in C(T)$ and for fixed $t \in T$ let x_t denote the value of x at t. The stochastic integral $\int_B x_t M(dx)$ is well defined and satisfies

$$(1.17) \qquad \Lambda_p\left(\int_B x_t M(dx)\right) \leq C_p\left(\int |x_t|^p m(dx)\right)^{1/p}$$

for some constant C_p depending only on p. We will be concerned with the stochastic process

$$(1.18) \qquad Z(t) = \int_B x_t M(dx), \qquad t \in T$$

which we will also denote, simply, as

$$(1.19) \qquad Z = \int_B x M(dx).$$

Note that by using (1.17) on finite linear combinations of $Z(t)$ we see that $\{Z(t), t \in T\}$ has stable finite dimensional distributions. We will now give a sufficient condition for Z to have a version in C(T).

THEOREM 1.6. — Let Z be given by (1.19) and let (T, τ) be a compact metric or pseudo-metric space. For $x \in C(T)$ define

$$(1.20) \qquad \|x\|_\tau = \sup_{s,t \in T} \frac{|x_t - x_s|}{\tau(s,t)}.$$

Let $1 \leq p < 2, \frac{1}{p} + \frac{1}{q} = 1$. Then, if $J_q(\tau;\delta) < \infty$ for some $\delta > 0$, $\{Z(t), t \in T\}$ has a version $\{\overline{Z}(t), t \in T\}$ with continuous sample paths satisfying

$$(1.21) \quad \Lambda_p\Big(\sup_{\substack{\tau(s,t) \leq \delta \\ s,t \in T}} |\overline{Z}(t) - \overline{Z}(s)| \leq C'_p\left(\int_B \|x\|_\tau^p m(dx)\right)^{1/p} [J_q(\tau;\delta) + \hat{\tau}\phi_q(\delta/4\hat{\tau})]$$

for some constant C'_p depending only on p.

Proof. — The proof follows that of Corollary 3.2 [3] except that we use our Theorem 1.2. Without loss of generality we assume that $m(B) = 1$. Let X be a C(T) valued random variable with distribution m and let θ be a real valued random variable, independent of X, with characteristic function $\mathrm{E} \exp iu\,\theta = \exp - |u|^p$. Let $C_p = \sup u^p P(|\theta| > u)$. Using the notation of (1.3) we note that

$$(1.22)\quad \Lambda_p^p(\|\theta X\|_\tau) = \sup_{u>0} u^p E_X P_\theta(|\theta|\,\|X\|_\tau > u) \le C_p E_X(\|X\|_\tau^p) = \left(\int_B \|x\|_\tau^p m(dx)\right).$$

Let $\{\theta_k X_k\}$ denote independent copies of θ and X and let $\{\theta_k\}$ and $\{X_k\}$ be independent of each other. By Theorem 1.2 and (1.22) we have

$$(1.23)\quad \Lambda_p\left(\sup_{\substack{\tau(s,t)\le\delta\\ s,t\in T}} n^{-1/p}\left|\sum_{k=1}^{n} \theta_k(X_k(t) - X_k(s))\right|\right) \le c_p^{1/p} C_p\left(\left(\int_B \|X\|_\tau^p m(dx)\right)^{1/p} [J_q(\tau;\delta) + \hat{\tau}\phi_q(\delta/4\hat{\tau})]\right).$$

One can check that the finite joint distributions of $n^{-1/p}\sum_{k=1}^{n}\theta_k X_k$ converge to those of Z. This fact and (1.23) shows that Z has a version with continuous sample paths and that θX is in the domain of attraction of Z. We get (1.21) from simple facts on the weak convergence of measures.

REMARK 1.7. — Theorem 1.6 is also valid when $p=2$ with $\Lambda_p(\ )$ replaced by $(E|\ \ |^2)^{1/2}$, although in this case the result is a simple consequence of Dudley's sufficient condition for the continuity of Gaussian processes. Theorem 1.6 improves upon [*1*] by relaxing the condition on the metric entropy and extending the result to the case $p = 1$. Theorem 1.6 can be quite sharp. This was pointed out in [*1*] in the case $p = 2$ and the same argument can be used for $1 \le p \le 2$. Consider the random Fourier series

$$(1.24)\qquad \sum_{k=1}^{\infty} a_k\theta_k e^{ikt}, \qquad t \in [0, 2\pi]$$

where θ is p-stable, $1 \le p \le 2$. Choose a function $\tau(s,t) = \tilde{\tau}(|s-t|)$ such that $u/\tilde{\tau}(u) = 0(1)$ as $u \downarrow 0$ and $\tilde{\tau}(u)$ is non-decreasing for $u \ge 0$. Under these conditions

$$J_q(\tau) \approx \sum_{k=2}^{\infty} \frac{\tau(1/k)}{k(\log k)^{1/p}}$$

where, as usual, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p \le 2$. (A $\approx$ B means there exist $0 < C_1$,

$C_2 < \infty$ such that $C_1B < A < C_2B$.) In order to represent (1.24) as a stochastic integral of the type (1.8) we take $f_t(u) = e^{itu}$ and $m(\{e^{itk}\}) = |a_k|^p$, $k = 1, \ldots$. Thus

$$\int_U \| e^{i(\cdot)u} \|_\tau^p m(du) = \sum_{k=1}^{\infty} \left| \frac{a_k}{\tau(1/k)} \right|^p. \tag{1.25}$$

Note that $J_\infty(\tau) < \infty$ if $\tau(1/k) = (\log^+ \log k)^{-1} (\log^+ \log^+ \log k)^{-(1+\varepsilon)}$, $\varepsilon > 0$ and $J_q(\tau) < \infty$ for $\tau(1/k) = (\log k)^{-1/q} (\log^+ \log k)^{-(1+\varepsilon)}$, $2 \le q < \infty$. Using this in (1.25) along with Theorem 1.6 we get, for $p = 1$, that

$$\sum_{k=1}^{\infty} |a_k| (\log^+ \log k)(\log^+ \log^+ \log k)^{1+\varepsilon} < \infty$$

is a sufficient condition for (1.24) to converge uniformly a. s. and for $1 < p \le 2$ we get that

$$\sum_{k=1}^{\infty} |a_k|^p (\log k)^{p/q} (\log^+ \log k)^{p(1+\varepsilon)} < \infty$$

is a sufficient condition for (1.24) to converge uniformly a. s. (The best we can do, by more subtle arguments, is that $\sum_{k=2}^{\infty} |a_k| (\log^+ \log k) < \infty$ is sufficient when $p = 1$ and $\sum_{k=2}^{\infty} \left(\sum_{n=k}^{\infty} |a_n|^p \right)^{1/p} / k (\log k)^{1/p} < \infty$ is sufficient when $1 < p \le 2$. In fact these conditions are necessary and sufficient when $|a_k|$ is non-increasing.)

In the last topic of this section we exhibit certain p-stable processes along with stochastic processes in their domain of normal attraction. The results that we give here are the principle results of [*4*], however, as in Theorems 1.2 and 1.6 we can weaken the condition on the metric entropy that is used in [*4*]. Following [*4*] we define a class of random measures that extends (1.15) and (1.16).

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(U, \mathscr{U})$ be a measurable space. Let $\tilde{M} = \tilde{M}(., \omega)$ be a random measure on $(U, \mathscr{U})$. That is, if $U_1, \ldots, U_n \in U$ are disjoint sets then $\tilde{M}\left(\bigcup_{i=1}^{n} U_i, \omega \right) = \sum_{i=1}^{n} \tilde{M}(\overline{U}_i, \omega)$ a. s. and if furthermore

$\bigcup_{i=1}^{n} U_n \uparrow \overline{U}$ then $\sum_{i=1}^{n} \tilde{M}(U_i, \omega)$ converges to $\tilde{M}(\overline{U}, \omega)$ in probability. We require that $\tilde{M}$ satisfy the following properties :

(1.26) Let $U_k \in \mathcal{U}$, $k = 1, \ldots,$ be disjoint and let $\{ \varepsilon_k \}$ be a Rademacher sequence independent of $\tilde{M}$. Then $\{ \tilde{M}(U_k) \}$ and $\{ \varepsilon_k \tilde{M}(U_k) \}$ have the same probability law;

(1.27) There exists a $1 \leq p < 2$ and a real positive finite measure m on $(U, \mathcal{U})$ such that for each finite collection of disjoint sets $U_1, \ldots, U_n \in \mathcal{U}$, the random vector $(\tilde{M}(U_1), \ldots, \tilde{M}(U_n))$ is in the domain of normal attraction of $(m^{1/p}(U_1)\theta_1, \ldots, m^{1/p}(U_n)\theta_n)$ where the random variables $\theta_1, \ldots, \theta_n$ are i. i. d. with

$$E \exp (iu\theta_1) = \exp (- | u |^p), \qquad - \infty < u < \infty ;$$

(1.28)
$$\overline{\lim_{t \to \infty}} \sup_{U \in \mathcal{U}} t^p P \left\{ \left| \frac{\tilde{M}(U)}{(m(U))^{1/p}} \right| > t \right\} \leq c_p$$

for some constant c_p.

Note that an independently scattered random p-stable measure (see (1.15) and (1.16)) satisfies (1.26)-(1.28). In fact, for a given control measure m we will be interested in the corresponding independently scattered random p-stable measure M and the corresponding class of random measures $\tilde{M}$ that satisfy (1.26)-(1.28) for this measure m.

The definition of the stochastic integral with respect to M, given in (1.17) and (1.18) can be extended to stochastic integrals with respect to $\tilde{M}$. Thus for $t \in T$ we can extend (1.17) to

(1.29)
$$\Lambda_p\left(\left| \int_B x_t \tilde{M}(dx) \right| \right) \leq C_p \left(\int_B | x_t |^p m(dx) \right)^{1/p}$$

and define

(1.30)
$$\tilde{Z}(t) = \int_B x_t \tilde{M}(dx), \qquad t \in T,$$

or simply

(1.31)
$$\tilde{Z} = \int_B x \tilde{M}(dx)$$

as we did in (1.18) and (1.19). Note that Z defined in (1.19) is now a special case of (1.31).

In the next theorem we take $T = [- 1/2, 1/2]^n$, for some integer n and have the control measure m supported on the functions $X = e^{i\langle \cdot, u\rangle}$ where $u \in \mathbb{R}^n$ so that $x_t = e^{i\langle t,u\rangle}$, $t \in [-1/2, 1/2]^n$. We define, for $1 < p \leq 2$,

$$(1.32)\quad \kappa(s,t) = \mathrm{E}\left|\int (x_t - x_s)d\tilde{\mathrm{M}}\right|$$
$$\leq \mathrm{C}_p\left(\int |x_t - x_s|^p dm\right)^{1/p} = \mathrm{C}_p\left(\int |e^{i\langle t,u\rangle} - e^{i\langle s,u\rangle}|^p dm\right)^{1/p},$$

in which the inequality follows from (1.29). We have the following theorem:

THEOREM 1.8. — Fix p, $1 < p < 2$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Let m be a finite positive symmetric measure on $C([-1/2, 1/2]^n)$ supported on the functions $e^{i\langle \cdot, u\rangle}$, $u \in \mathbb{R}^n$. Let $\tilde{\mathrm{M}}$ be a random measure satisfying (1.26)-(1.28) for this p and control measure m. Assume that

$$(1.33)\qquad \mathrm{J}_{q'}(\kappa) < \infty$$

where $2 < q' < q$ and κ is given in (1.32). Then the stochastic integral

$$\tilde{\mathrm{Y}}(t) = \int e^{i\langle t,u\rangle} d\tilde{\mathrm{M}}, \qquad t \in [-1/2, 1/2]^n,$$

has a version with continuous sample paths and is in the domain of normal attraction of the p-stable process

$$\mathrm{Y}(t) = \int e^{i\langle t,u\rangle} d\mathrm{M}, \qquad t \in [-1/2, 1/2]^n,$$

where M is an independently scattered random p-stable measure with control measure m (i. e. M satisfies (1.15) and (1.16).

Proof. — This theorem is the same as Theorem 4.1 [*4*] except that $\mathrm{J}_2(\kappa) < \infty$ is replaced by (1.33). We can use the proof of Theorem 4.1 [*4*] if we only improve one point. Let $\{\mathrm{A}_{r,j}\}_{j=1}^{\infty}$ be disjoint sets covering the support of m and let $e^{i\langle t,\lambda_{r,j}\rangle} \in \mathrm{A}_{r,j}$. Define

$$\tilde{\mathrm{Y}}^r(t) = \sum_{j=1}^{\infty} e^{i\langle t,\lambda_{r,j}\rangle}\tilde{\mathrm{M}}(\mathrm{A}_{r,j}), \qquad t \in [-1/2, 1/2]^n$$

and

$$\kappa_r(s,t) = 2\mathrm{C}_p\left(\sum_{j=1}^{\infty} |e^{i\langle t,\lambda_{r,j}\rangle} - e^{i\langle s,\lambda_{r,j}\rangle}|^p m(\mathrm{A}_{r,j})\right)^{1/p}.$$

We can adopt the proof of Theorem 4.1 [4], to prove this theorem if we show

$$(1.34)\qquad \mathrm{E}\sup_{t\in\mathrm{T}} |\tilde{\mathrm{Y}}^r(t)| \le \mathrm{D}_q\left[\left(\sum_{j=1}^{\infty} m(\mathrm{A}_{r,j})\right)^{1/p} + \mathrm{J}_{q'}(\kappa_r)\right]$$

for some constant D_q depending only on q.

By symmetry $\tilde{\mathrm{Y}}^r(t)$ is equivalent to

$$(1.35)\qquad \tilde{\mathrm{Y}}^r(t) = \sum_{j=1}^{\infty} \varepsilon_j e^{i\langle t,\lambda_{rj}\rangle}\tilde{\mathrm{M}}(\mathrm{A}_{r,j})$$

where $\{\varepsilon_j\}_{j=1}^{\infty}$ is a Rademacher sequence independent of $\{\tilde{\mathrm{M}}(\mathrm{A}_{r,j})\}_{j=1}^{\infty}$. Fix $\{\tilde{\mathrm{M}}(\mathrm{A}_{r,j})\}$ in (1.35) and consider $\tilde{\mathrm{Y}}^r(t)$ only as a function of $\{\varepsilon_j\}_{j=1}^{\infty}$. We have for $\frac{1}{p'}+\frac{1}{q'}=1$,

$$(1.36)\qquad \left\|\sum_{j=1}^{\infty}\varepsilon_j\left(e^{i\langle t,\lambda_{r,j}\rangle}-e^{i\langle s,\lambda_{r,j}\rangle}\right)\tilde{\mathrm{M}}(\mathrm{A}_{r,j})\right\|_{\psi_{q'}}$$
$$\le \mathrm{C}_p \sup_j j^{1/p'}\left(m^{1/p}(\mathrm{A}_{r,j})\,|e^{i\langle t-s,\lambda_{r,j}\rangle}-1|\,\xi_j\right)^*$$
$$\le \mathrm{C}_p\left(\sum_{j=1}^{\infty}|\xi_j|^{p'}m^{p'/p}(\mathrm{A}_{r,j})\,|e^{i\langle t-s,\lambda_{r,j}\rangle}-1|^{p'}\right)^{1/p'}$$

where

$$\xi_j = \frac{\tilde{\mathrm{M}}(\mathrm{A}_{r,j})}{m^{1/p}(\mathrm{A}_{r,j})}.$$

(Note that for the last inequality in (1.36) we use the simple observation that $\sup_{1\le k\le n} k^{1/p'}(|\eta_k|)^* \le \left(\sum_{k=1}^{n}(|\eta_k|^*)^{p'}\right)^{1/p'} \le \left(\sum_{k=1}^{\infty}|\eta_k|^{p'}\right)^{1/p'}$ for any sequence of real numbers $\{\eta_k\}$.) Note that by (1.28)

$$t^p\mathrm{P}(|\xi_j|>t)\le \mathrm{C}_p.$$

Therefore

$$(1.37)\qquad \mathrm{E}\left(\sum_{j=1}^{\infty}|\xi_j|^{p'}m^{p'/p}(\mathrm{A}_{r,j})\,|e^{i\langle t-s,\lambda_{r,j}\rangle}-1|^{p'}\right)^{1/p'}$$
$$\le \mathrm{C}'_p\left(\sum_{j=1}^{\infty}m(\mathrm{A}_{r,j})\,|e^{i\langle t-s,\lambda_{r,j}\rangle}-1|^{p}\right)^{1/p}$$

and

$$\text{(1.38)} \qquad E\left(\sum_{j=1}^{\infty} m^{2/p}(A_{r,j}) \mid \xi_j \mid^2\right)^{1/2} \leq C''\left(\sum_{j=1}^{\infty} m(A_{r,j})\right).$$

One obtains (1.38) from (2.4) of Lemma 2.1 [4]; (1.34) is obtained in a similar fashion.

Let E_ε denote expectation with respect to $\{ \varepsilon_j \}$. By Lemma 1.1, (1.2 *a*) and (1.36) we have

$$E_\varepsilon \{ \sup_{t \in T} \mid \tilde{Y}^r(t) \mid \} \leq C_{q'}\left[J_{q'}\left(\left(\sum_{j=1}^{\infty} \mid \xi_j \mid^{p'} m^{p'/p}(A_{r,j}) \mid e^{i\langle t-s, \lambda_{r,j}\rangle} - 1 \mid^{p'}\right)^{1/p'}\right) + \left(\sum_{j=1}^{\infty} \mid \xi_j \mid^{p'} m^{p'/p}(A_{r,j})\right)^{1/p'} + \left(\sum_{j=1}^{\infty} \mid \xi_j \mid^2 m^{2/p}(A_{r,j})\right)^{1/2}\right].$$

Taking expectation with respect to $\{ \xi_j \}$ and using (1.37) and (1.38) we get (1.34). (We also use the fact that for a random metric ρ_ω,

$$E_\omega J_{q'}(\rho_\omega) \leq C_{q'} [J_{q'}(E_\omega \rho_\omega) + E_\omega \sup_{s,t \in T} \tilde{\rho}_\omega(s, t)],$$

see Lemma 3.1 [8]). This completes the proof of the theorem.

The next theorem is a version of Theorem 1.8 for the more general class of stochastic processes defined in (1.30) and (1.31).

THEOREM 1.9. — Fix p, $1 < p < 2$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Let (T, τ) be a compact metric space and m be a positive symmetric measure supported on the unit ball of C(T) such that

$$\int_B \| x \|_\tau^p m(dx) < \infty,$$

where $\| \ \|_\tau$ is defined in (1.20). Assume that

$$J_{q'}(\tau) < \infty$$

for some $2 < q' < q$. Let M be a random measure satisfying (1.26), (1.28) for this p and control measure m. Then the stochastic integral

$$\tilde{Z} = \int x d\tilde{M}$$

(see (1.30) and (1.31)), has a version with continuous sample paths and is in the domain of normal attraction of the p-stable process

$$Z = \int x dM$$

where M is an independently scattered p-stable measure with control measure m.

Proof. — This theorem improves Theorem 4.4 [*4*] just as Theorem 1.8 improved Theorem 4.1 [*4*]. The necessary changes are similar to those in the proof of Theorem 1.8.

REMARKS 1.10. — We don't know whether Theorems 1.8 and 1.9 are valid with $J_q(\ )$ replacing $J_{q'}(\ )$. When $p = 2$ this is the case (see [*2*]). The place we need q' instead of q is in the second inequality in (1.36) so that we can obtain (1.37). If the $\{ \xi_j \}$ were independent we wouldn't have to do this since we could use Theorem 3.3 and Corollary 3.8 [*8*] to obtain

$$E\left\| \sum_{j=1}^{\infty} \varepsilon_j (e^{i\langle t,\lambda_{r,j}\rangle} - e^{i\langle s,\lambda_{r,j}\rangle}) M(A_{r,j}) \right\|_{\psi_q} \leq C_p^1 \left(\sum_{j=1}^{\infty} m(A_{r,j}) \mid e^{i\langle t-s,\lambda_{r,j}\rangle} - 1 \mid^p \right)^{1/p}.$$

(This is what we do in Section 3 [*8*]).

2. NECESSARY CONDITIONS FOR THE CONTINUITY OF p-STABLE RANDOM FOURIER SERIES

Let G be a compact Abelian group with dual group Γ. Let θ be a canonical p-stable real valued random variable, i. e. $E \exp iu\theta = \exp - \mid u \mid^p$, $0 < p \leq 2$. Let $\{ \theta_\gamma \}$ be i. i. d. copies of θ (i. e. we index the i. i. d. copies of θ by the countable set Γ) and let $\{ a_\gamma \}$ be complex numbers. Fix a p, $1 \leq p \leq 2$ and assume that $\sum_{\gamma\in\Gamma} \mid a_\gamma \mid^p < \infty$. We will be concerned with the random Fourier series

$$\sum_{\gamma\in\Gamma} a_\gamma \theta_\gamma \gamma(t), \qquad t \in G. \tag{2.1}$$

A subset Λ of Γ is called a Sidon set if there exists a constant $\chi > 0$ such that for all sequences $\{ a_\gamma \}$

$$\sup_{t\in G} \left| \sum_{\gamma\in\Lambda} a_\gamma \gamma(t) \right| \geq \chi \sum_{\gamma\in\Lambda} | a_\gamma | .$$

χ is called the Sidon constant of Λ. Now let $\{ A_j \}_{j\in J}$ be disjoint subsets of Γ. We call $\{ A_j \}_{j\in J}$ a Sidon partition if there is a constant $K > 0$ such that any set $\{ \gamma_j \mid j \in J \}$ with $\gamma_j \in A_j$ for each $j \in J$ is a Sidon set with Sidon constant K.

The next theorem extends Theorem VII.1.6 [7] from the case $p = 2$ to $1 \leq p \leq 2$. It is actually a simple consequence of Theorem VII.1.6 [7].

THEOREM 2.1. — Fix a p, $1 \leq p \leq 2$ and assume that the series in (2.1) converges uniformly a. s. (or, equivalently, has a version with continuous sample paths). Then for any Sidon partition $\{ A_j \}_{j\in J}$ of Γ we must have

$$\sum_{j\in J} \Big(\sum_{\gamma\in A_j} | a_\gamma |^p \Big)^{1/p} < \infty \qquad \text{for} \qquad 1 < p \leq 2 \tag{2.2}$$

or, for $p = 1$, we must have

$$\sum_{j\in J} \Big(\sum_{\gamma\in A_j} | a_\gamma | \, L\Big(\sum_{\gamma\in A_j} | a_\gamma | \Big)^{-1} \Big) < \infty , \tag{2.3}$$

where $Lx = \max(1, \log x)$.

Proof. — Fix p, $1 \leq p < 2$. Without loss of generality we will assume $\sum_{\gamma\in\Gamma} | a_\gamma |^p = 1$. Let m be a probability measure on Γ such that $m(\{ \gamma \}) = | a_\gamma |^p$. We now recall a useful representation of p-stable processes that was used in [8]. Let $\{ Y_k \}$ be i. i. d. copies of Y where $P(Y \geq \lambda) = e^{-\lambda}$, $\lambda \geq 0$ and define $\Gamma_j = Y_1 + \ldots + Y_j$. Let $\{ \varepsilon_k \}$ be a Rademacher sequence independent of $\{ Y_k \}$. Consider v as a random variable with values in Γ distributed according to m and let $\{ v_k \}$ be i. i. d. copies of v. The series

$$\sum_{j=1}^{\infty} \varepsilon_j (\Gamma_j)^{-1/p} v_j(t), \qquad t \in G \tag{2.4}$$

is equivalent to the series in (2.1), (i. e. they have the same finite joint distributions).

We can now prove the theorem. Fix $\{ \Gamma_j \}$ and $\{ v_j \}$ and consider (2.4) as a Rademacher series. If (2.1) converges uniformly a. s. then (2.4) must converge for $\{ \Gamma_j \}$, $\{ v_j \}$ in a set of measure 1. Therefore, by Theorem 1.6, Chapter VII [7] we must have

$$\sum_{j \in J} \left(\sum_{k=1}^{\infty} (\Gamma_k)^{-2/p} I_{[v_k \in A_j]} \right)^{1/2} < \infty , \tag{2.5}$$

on a set of measure 1. It follows from Lemma 1.4 [8] that

$$\sum_{k=1}^{\infty} (\Gamma_k)^{-2/p} I_{[v_k \in A_j]} \overset{\mathscr{D}}{=} c_{p/2} \theta_j \left(\sum_{\gamma \in A_j} | a_\gamma |^p \right)^{2/p}$$

where θ_j is a non-negative canonical $p/2$ stable real valued random variable, $c_{p/2}$ is a constant and $\overset{''\mathscr{D}''}{=}$ denotes equal in distribution. Furthermore $\{ \theta_j \}$ are i. i. d. (This follows from Lemma 1.4 [8] by using characteristic functions; see the proof of Lemma 1.5 [8] as well). Therefore the convergence a. s. of (2.5) is equivalent to the convergence a. s. of

$$\sum_{j \in J} | \theta_j |^{1/2} \left(\sum_{\gamma \in A_j} | a_\gamma |^p \right)^{1/p} . \tag{2.6}$$

Using the Three Series Theorem we see that for $1 < p < 2$ (2.6) converges a. s. if and only if (2.2) holds, while for $p = 1$, (2.6) converges uniformly a. s. if and only if (2.3) holds. Considering the relation between (2.5) and (2.6) we have proved the theorem when $1 \leq p < 2$. The case $p = 2$ is Theorem 1.6 [7].

We will now extend Theorem 2.1 to locally compact Abelian groups. This will enable us to obtain necessary conditions for the existence of continuous p-stable random Fourier transforms (i. e. stochastic integrals on the character group). For the remainder of this paper we take G to be a locally compact Abelian group, Γ its dual group and $K \subset G$ a compact, symmetric, neighborhood of the identity in G. Following [8], Definition 6.1, a subset $\Lambda = \{ \gamma_n \mid n \in N \}$ is called a topological Sidon set with respect to $K \subset G$ if there exists a constant $C > 0$ such that

$$\forall n \in \mathbb{N}, \quad \forall \{ a_i \} \in \mathbb{C}^n, \qquad \sum_{i=1}^{n} | \alpha_i | \leq C \sup_{t \in K} \left| \sum_{i=1}^{n} \alpha_i \gamma_i \right| . \tag{2.7}$$

Now, let $\{ A_j \mid j \in J \}$ be disjoint subsets of Γ. We say that $\{ A_j \}$ is a topological Sidon partition with respect to $K \subset G$ if all subsets $\{ \gamma_j \mid j \in J \}$ with $\gamma_j \in A_j$ for each $j \in J$ are topological Sidon sets satisfying (2.7).

Let m be a σ-finite positive measure on Γ. Fix $1 \leq p \leq 2$ and let M be an independently scattered random p-stable measure on Γ with control measure m (i. e. M satisfies (1.15) and (1.16)). The stochastic integral, referred to just before (1.17) will now be denoted

$$(2.8) \qquad \int_\Gamma \gamma_t M(d\gamma), \qquad t \in K.$$

THEOREM 2.2. — Let $1 \leq p \leq 2$. Suppose that the stochastic integral in (2.8) has a version with continuous paths, which we will also denote by (2.8). Then for each topological Sidon partition $\{ A_j \mid j \in J \}$ we must have

$$(2.9) \qquad \sum_{j \in J} m^{1/p}(A_j) < \infty \qquad \text{for} \qquad 1 < p \leq 2$$

and, for $p = 1$, we must have

$$(2.10) \qquad \sum_{j \in J} m(A_j) L(m(A_j)^{-1}) < \infty$$

Note that Theorem 2.2 contains Theorem 2.1. In order to prove Theorem 2.2 we will need the following lemma which is a generalization of Theorem 1.6, page 131 [7].

LEMMA 2.3. — Let $\{ A_j \}$ be a topological Sidon partition of Γ with respect to $K \subset G$ and let Γ' be any countable subset of Γ. Let $\{ a_\gamma \}$ be complex numbers. Then we must have

$$(2.11) \qquad \sum_{j \in J} \left(\sum_{\gamma \in A_j \cap \Gamma'} | a_\gamma |^2 \right)^{1/2} \leq B \left[J_2(\rho) + \left(\sum_{\gamma \in \Gamma'} | a_\gamma |^2 \right)^{1/2} \right]$$

where $\rho = \rho(s, t) = \left(\sum_{\gamma \in \Gamma'} | a_\gamma |^2 \, | \gamma(t) - \gamma(s) |^2 \right)^{1/2}$, $s, t \in K$, $J_2(\rho)$ is as defined just before Lemma 1.1 but with respect to the compact pseudo-metric space (K, ρ) and B is a constant independent of $\{ a_\gamma \}$ and Γ'.

Proof. — To prove this lemma we extend the proof of Theorem VII.1.6 in [7]: Let $\{ \gamma_j \mid j \in J \}$ be a topological Sidon set with respect to K and with constant C. Then there exists a constant β depending only on C and K such that for all finitely supported scalar sequences $\{ \lambda_j \}$ we have

$$\| \Sigma \lambda_j \gamma_j \|_{L^{\psi_2}(dm_K)} \leq \beta (\Sigma | \lambda_j |^2)^{1/2} \tag{2.12}$$

where we have denoted by m_K Haar measure restricted to K and by $L^{\psi_2}(dm_K)$ the corresponding Orlicz space as defined in § 1 (this definition still makes sense if P is not a probability measure, equivalently, we can always normalize so that $m(K)=1$.) Note that (2.12) extends (1.29), Chapter VII [7]. To verify (2.12) it suffices to show that there exists a constant β' such that for all $p > 2$

$$\| \Sigma \lambda_j \gamma_j \|_{L^p(dm_K)} \leq \beta' \sqrt{p} (\Sigma | \lambda_j |^2)^{1/2} , \tag{2.13}$$

(cf. e. g. [7], p. 90). This can be shown by following a classical argument of Rudin, which we sketch for the convenience of the reader. Since $\{ \gamma_j \mid j \in J \}$ satisfies (2.7) then, *a fortiori*, for all choices of $\{ \delta_j \}$, with $\delta_j = \pm 1$

$$\Sigma \alpha_j \delta_j \leq c \| \Sigma \alpha_j \gamma_j \|_{C(K)} .$$

Therefore, the linear map $\Sigma \alpha_j \gamma_j \to \Sigma \alpha_j \delta_j$ can be extended to a linear map $\xi : C(K) \to \mathbb{C}$ by the Hahn Banach Theorem such that $\| \xi \|_{C(K)^*} \leq C$ and $\forall j \in J$, $\langle \xi, \gamma_j \rangle = \delta_j$. Therefore for each choice of signs $\delta = \{ \delta_j \}_{j \in J}$ with $\delta_j = \pm 1$, there exists a measure μ_δ supported by K such that

$$\int_K \gamma_j(x) \mu_\delta(dx) = \delta_j, \qquad \forall j \in J \tag{2.14}$$

and

$$\| \mu_\delta \| \leqslant C .$$

Now let

$$f = \Sigma \varepsilon_j \lambda_j \gamma_j$$

and

$$g = \Sigma \lambda_j \gamma_j .$$

We have

$$g(x) = \int_K f(x + t) \mu_\delta(dt) .$$

By the convexity of the L^p-norm

$$\left(\int_K | g(x) |^p dm_K(x) \right)^{1/p} \leq \| \mu_\delta \| \sup_{t \in K} \left(\int_K | f(x + t) |^p dm_K(x) \right)^{1/p} \tag{2.15}$$
$$\leq C \left(\int_{K+K} | f(x) |^p dm(x) \right)^{1/p} .$$

Finally we average the last term of (2.15) over all choices of signs $\{\delta_j\}$ and obtain, by Khintchine's Inequality,

$$\left(\int_K |g(x)|^p dm_K(x)\right)^{1/p} \leq \beta'\sqrt{p}(\Sigma|\lambda_j|^2)^{1/2},$$

where β' is a constant depending only on K and C. This establishes (2.13) and consequently (2.12).

Using (2.12) in place of (1.29) p. 131 [7] we can obtain, in place of (1.28) p. 131 [7]

$$\left\|\sum_{\gamma\in\Gamma'} a_\gamma \gamma\right\|_{L^{\psi_2}(dm_K)} \leq \beta\left(\sum_{j\in J}\left(\sum_{\gamma\in A_j\cap\Gamma'} |a_\gamma|^2\right)^{1/2}\right). \tag{2.16}$$

We now consider theorem 1.6, p. 131 [7]. Replacing (1.28) in [7] by (2.16) and using $\{A_j \cap \Gamma'\}_{j\in J}$ in place of $\{\Gamma_j\}_{j\in J}$ we have that

$$P\left[\frac{|\tilde{X}_t - \tilde{X}_s|}{C\rho(s,t)} > u\right] \leq 2\exp(-u^2/2)$$

for some constant C, where $\tilde{X}_t$ is defined on page 132 [7]. It then follows from Theorem 3.1, page 25 [7] that

$$\sum_j\left(\sum_{\gamma\in A_j\cap\Gamma'} |a_\gamma|^2\right)^{1/2} = \sum_{\gamma\in\Gamma'} |a_\gamma b_\gamma| \tag{2.17}$$
$$\leq E_\omega \sup_{t\in K} |X_t^2(\omega)| \leq \beta[J_2(\rho) + \hat{\rho} + E_\omega\tilde{X}_0(\omega)].$$

(The unfamiliar notation in (2.17) is from [7].) By definition

$$\hat{\rho} \leq \sqrt{2}\left(\sum_{\gamma\in\Gamma'} |a_\gamma|^2\right)^{1/2}.$$

Also since

$$\tilde{X}_0(\omega) = \sum_{\gamma\in\Gamma'} |a_\gamma b_\gamma|\,\gamma(\omega)$$

it follows from (2.12), with $b_\gamma = a_\gamma \Big/ \left(\sum_{\gamma\in A_j\cap\Gamma'} |a_\gamma|^2\right)^{1/2}$ that

$$E|\tilde{X}_0(\omega)| \leq \beta_1\left(\sum_{\gamma\in\Gamma'} |a_\gamma|^2\right)^{1/2}.$$

Here β and β_1 are constants independent of $\{ a_\gamma \}$ and Γ'. These final remarks and (2.17) complete the proof of Lemma 2.3.

Proof of Theorem 2.2. — We first consider the case $p=2$. Without loss of generality we can take the control measure m that determines the stochastic integral in (2.8) to be a probability measure. Let $\{ \gamma_k \}_{k=1}^{\infty}$ be an i. i. d. sequence of random variables with values in Γ each one distributed according to m. Let us consider one particular realization of this sequence and denote it by $\{ \gamma_k(\omega) \}_{k=1}^{\infty}$, ($\omega \in \Omega$, the infinite product space generated by m). Consider

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \gamma_k(\omega) .$$

By (2.11) we have

$$\sum_{j \in J} \left(\frac{1}{n} \sum_{k=1}^{n} I_{[\gamma_k(\omega) \in A_j]} \right)^{1/2} \leq B [J_2(\rho(\omega)) + 1]$$

where

$$\rho(\omega) = \rho(s, t ; \omega) = \left(\frac{1}{n} \sum_{k=1}^{n} | \gamma_k(t ; \omega) - \gamma_k(s ; \omega) |^2 \right)^{1/2}$$

Note that

$$E\rho(\omega) \leq \left(\int_\Gamma | \gamma_t - \gamma_s |^2 m(d\gamma) \right)^{1/2} \equiv \tilde{\rho} .$$

Using this along with Lemma 3.6, p. 37 [7] and (1.7), p. 52 [7] (or more generally Lemma 2.3, p. 22 [7]), we have

$$E \sum_{j \in J} \left(\frac{1}{n} \sum_{k=1}^{n} I_{[\gamma_k(\omega) \in A_j]} \right)^{1/2} \leq B' [J_2(\tilde{\rho}) + 1]$$

for some absolute constant B' (i. e. B' may depend on G and K but it does not depend on $\{ A_j \}$). It follows by the Dudley-Fernique theorem (see [7] for references) that (2.8) has a version with continuous paths iff $J_2(\tilde{\rho}) < \infty$. Thus we get

$$E \sum_{j \in J} \left(\frac{1}{n} \sum_{k=1}^{n} I_{[\gamma_k(\omega) \in A_j]} \right)^{1/2} \leq C \tag{2.18}$$

for some constant C independent of n. By the strong law of large numbers (2.18) implies (2.9) when $p=2$.

The proof in the cases $1 \leq p < 2$ is easier; it is essentially the same as the proof of Theorem 2.1. The series in (2.4) (for $t \in K$) is equivalent to (2.8) when $\{ v_j \}$ are i. i. d. according to the new control measure m. If we use (2.11) in place of (1.29), p. 131 [7] we get

$$\sum_{j \in J} \left(\sum_{k=1}^{n} (\Gamma_k)^{-2/p} I_{[v_k \in A_j]} \right)^{1/2} < \infty$$

on a set of measure 1 with respect to $\{ \Gamma_k \}$ and $\{ v_k \}$. As in Theorem 2.1

$$\sum_{k=1}^{\infty} (\Gamma_k)^{-2/p} I_{[v_k \in A_j]} \overset{\mathcal{D}}{=} c_{p/2} \theta_j (m(A_j))^{2/p},$$

for $c_{p/2}$ and θ_j as in Theorem 2.1. The result now follows as in Theorem 2.1.

As an application of Theorem 2.2 we obtain a result of Nisio [9]. Let $G = R$ and $K = [-1, 1]$. Consider

$$\text{(2.19)} \qquad \int_\Gamma \gamma_t M(d\gamma), \qquad t \in K$$

where M has control measure m, which we take to be symmetric, in the sense that $m(\{ e^{i\lambda t} \mid \lambda \in (0, a) \}) = m(\{ e^{i\lambda t} \mid \lambda \in (-a, 0) \})$. Let

$$A_j = \{ e^{i\lambda t} \mid \lambda \in [2^{2j}, 2^{2j+1}) \}$$
$$B_j = \{ e^{i\lambda t} \mid \lambda \in [2^{2j+1}, 2^{2j+2}) \}$$

Both $\{ A_j \}_{j=0}^{\infty}$ and $\{ B_j \}_{j=0}^{\infty}$ are topological Sidon partitions. Using Theorem 2.2 first with $\{ A_j \}$ and then with $\{ B_j \}$ we see that if (2.19) has a version with continuous paths then necessarily

$$\text{(2.20)} \qquad \sum_{j=0}^{\infty} m^{1/2}(\{ e^{i\lambda t} \mid \lambda \in [2^i, 2^{j+1}] \}) < \infty$$

Now since (2.16) has a version with continuous paths if and only if

$$\text{(2.21)} \qquad \mathrm{Re}\left(\int_\Gamma \gamma_t M(d\gamma) \right), \qquad t \in K$$

has a version with continuous paths (see Theorem 1.3, p. 10 [7]) we get Nisio's result: If the real valued stationary Gaussian process (2.21) has a version with continuous paths then (2.20) must hold.

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