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Critical branching diffusions: Proper normalization and conditioned limit

by

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RÉSUMÉ. — Ce travail concerne les processus de branchement markoviens critiques comprenant un ensemble général de types et possédant un semi-groupe de moments uniformément primitif au sens de [*3*]. Pour démontrer le théorème limite exponentiel pour un tel processus, il faut qu'un paramètre dépendant de la fonctionnelle du deuxième moment factoriel soit fini et positif. En permettant à ce paramètre d'être infini et en utilisant une fonctionnelle linéaire particulièrement dépendant de la probabilité de survie comme normalisation, on obtient un théorème limite conditionnel avec des conditions nécessaires et suffisantes. On montre également, que si cette condition n'est pas satisfaite, il n'existe pas de normalisation qui donne une loi limite non dégénérée et propre. Les résultats s'appliquent aux processus à *d* types avec un temps discret ou continu, aux processus de branchement soumis à une diffusion simple ou multiple sur les domaines bornés ainsi qu'à d'autres modèles.

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0. INTRODUCTION

Consider a critical Bienaymé-Galton-Watson process $\{Z_n\}$. If the second factorial moment 2μ of its offspring distribution is positive and finite, the process becomes eventually extinct, $\mathbf{P}(Z_n > 0) \sim (\mu n)^{-1}$, $n \to \infty$, and $\mathbf{P}(((\mu n)^{-1}Z_n \geq \lambda) \mid Z_n > 0) \to e^{-\lambda}$, $n \to \infty$, $\lambda \geq 0$. In case $\mu = \infty$ there is a necessary and sufficient condition on the tail of the offspring distribution under which $\mathbf{P}(\mathbf{P}(Z_n > 0)Z_n \geq \lambda \mid Z_n > 0)$ converges, as $n \to \infty$, to a non-trivial distribution function; see [*12*], [*13*], [*15*], and for the extension of this result to d-type processes [*1*], [*2*], [*14*]. In an unpublished part of his doctoral dissertation, which he kindly made available to us, R. S. Slack has given a negative answer to the question, whether there exists a normalization, necessarily different from $\mathbf{P}(Z_n > 0)$, which leads to a proper non-degenerate limit, if this condition is not satisfied. The argument is long and cumbersome.

In this paper we treat critical branching processes with a general set of types, whose moment semigroup is uniformly primitive in the sense of [*3*]. This setting includes positively regular d-type processes with discrete or continuous parameter, simple or multigroup branching diffusions on bounded domains with various boundary conditions, and related models. The exponential limit theorem in this general framework is standard. For its final form see [*4*], [*5*]. It requires that a certain parameter involving the second factorial moment functional be positive and finite. We now let this parameter be infinite and prove a corresponding limit theorem with necessary and sufficient conditions. We also show in general that up to a trivial factor a certain linear functional of the survival probability is the only normalization leading to a proper, non-degenerate limit distribution.

The paper is organized as follows. In section 1 we define the model, and in section 2 we present our results, deferring the proofs of the three theorems to later sections. In section 3 we reformulate the crucial tail condition appearing in the theorems in terms of processes with a branching law. Sections 4, 5 and 6 contain the proofs of the theorems, and in the appendix we collect without proof several results on slowly or regularly varying functions. Some basic relations and conditions, we keep referring to throughout the paper, are labelled in a suggestive way by letters instead of numbers. Except condition (S), which is introduced in section 2, all of these are defined in section 1. Results in the appendix are referred to as A.1 to A.4.

1. SETTING

The setting is the same as in [*4*]. Let $(X, \mathfrak{A})$ be a measurable space, $\mathscr{B}$ the Banach algebra of all bounded, $\mathfrak{A}$-measurable, complex-valued functions ξ on X with supremum-norm $\|\xi\|$, $\mathscr{B}_+$ the cone of non-negative functions in $\mathscr{B}$,

$$\mathscr{S} := \{ \xi \in \mathscr{B} : \|\xi\| < 1 \}, \qquad \overline{\mathscr{S}} := \{ \xi \in \mathscr{B} : \|\xi\| \leq 1 \},$$
$$\mathscr{S}_+ := \mathscr{S} \cap \mathscr{B}_+, \qquad \overline{\mathscr{S}}_+ := \overline{\mathscr{S}} \cap \mathscr{B}_+ .$$

Let $X^{(n)}$, $n \geq 1$, be the symmetrization of the direct product of n disjoint copies of X and $X^{(0)} := \{ \theta \}$ with some extra point θ. Define

$$\hat{X} := \bigcup_{n=0}^{\infty} X^{(n)},$$

and let $\hat{\mathfrak{A}}$ be the σ-algebra on $\hat{X}$ induced by $\mathfrak{A}$.

For $\xi \in \overline{\mathscr{S}}$ set

$$\begin{aligned} \tilde{\xi}(\hat{x}) &:= 1 ; & \hat{x} &= \theta , \\ &:= \prod_{\nu=1}^{n} \xi(x_\nu) ; & \hat{x} &= \langle x_1, \ldots, x_n \rangle \in X^{(n)}, \qquad n \in \mathbb{N} , \end{aligned}$$

and suppose to be given a *branching transition function* with *type-space* $(X, \mathfrak{A})$, that is, a transition function $\mathbf{P}_t(\hat{x}, \hat{A}) \mid \hat{X} \otimes \hat{\mathfrak{A}}$ with parameter set $T = \mathbb{N}$, or $T = (0, \infty)$, whose generating functional,

$$F_t(\hat{x}, \xi) := \int_X \tilde{\xi}(\hat{y}) \mathbf{P}_t(\hat{x}, d\hat{y}) ; \qquad t \in T, \quad \hat{x} \in \hat{X}, \quad \xi \in \overline{\mathscr{S}},$$

satisfies the *branching condition*,

$$\begin{aligned} F_t(\hat{x}, \xi) &= 1 ; & \hat{x} &= \theta \\ &= \prod_{\nu=1}^{n} F_t(\langle x_\nu \rangle, \xi) ; & \hat{x} &= \langle x_1, \ldots, x_n \rangle \in X^{(n)}, \quad n \in \mathbb{N} , \end{aligned} \tag{F.1}$$

for $t \in T$ and $\xi \in \overline{\mathscr{S}}$. Defining $F_t : \overline{\mathscr{S}} \to \overline{\mathscr{S}}$ by

$$F_t[\,\cdot\,](x) := F_t(\langle x \rangle, \cdot\,) ; \qquad x \in X ,$$

it follows from (F.1) and the Markov property that

(F.2) $$F_{t+s}[\xi] = F_t[F_s[\xi]]; \qquad t, s \in T, \quad \xi \in \overline{\mathscr{S}}.$$

That is, $\{ F_t \}_{t \in T}$ is a semigroup, the so-called *generating semigroup*.

Let $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ $\forall x \in X$, and define for $\xi \in B$

$$\hat{x}[\xi] := 0; \qquad \hat{x} = \theta,$$

$$:= \sum_{\nu=1}^{n} \xi(x_\nu); \qquad \hat{x} = \langle x_1, \ldots, x_n \rangle \in X^{(n)}, \qquad n \in \mathbb{N}.$$

Suppose that for $\xi = \mathbf{1}$ and thus all $\xi \in \mathscr{B}$

$$(M_t\xi)(x) := \int_{\hat{X}} \hat{x}[\xi] \mathbf{P}_t(\langle x \rangle, d\hat{x})$$

is bounded as a function of $x \in X$ for every fixed $t \in T$. Noting that

$$(M_t\xi)(x) = \lim_{\varepsilon \uparrow 1} \left. \frac{\partial F_t[\varepsilon \mathbf{1} + \lambda \xi](x)}{\partial \lambda} \right|_{\lambda = 0},$$

it follows by (F.2) that $\{ M_t \}_{t \in T}$ is a semigroup of linear-bounded operators on $\mathscr{B}$. We assume that the following condition is satisfied:

(M) *The moment semigroup* $\{ M_t \}_{t \in T}$ *can be represented as*

$$M_t = \rho^t P + \Delta_t, \qquad t \in T,$$

where $\rho \in (0, \infty)$,

$$P\xi = \Phi^*[\xi]\phi, \qquad \xi \in \mathscr{B},$$

with $\Phi^* : \mathscr{B} \to \mathbb{C}$ *non-negative, linear-bounded,* $\phi \in \mathscr{B}_+$, *and* $\Phi^*[\phi] = 1$, *further* $\Delta_t : \mathscr{B} \to \mathscr{B}$ *such that for all* $t \in T$

$$P\Delta_t\xi = \Delta_t P\xi = \mathbf{0}, \qquad \xi \in \mathscr{B},$$

$$-\alpha_t P\xi \leq \Delta_t \xi \leq \alpha_t P\xi, \qquad \xi \in \mathscr{B}_+,$$

with $\alpha_. : T \to \mathbb{R}_+$ *satisfying*

$$\rho^{-t}\alpha_t \downarrow 0, \qquad t \uparrow \infty.$$

Given $M_t : \mathscr{B} \to \mathscr{B}$, there exists a mapping $R_t(\cdot)[\cdot] : \overline{\mathscr{S}} \otimes \mathscr{B} \to \mathscr{B}$, sequentially continuous respective the product topology on bounded regions, non-increasing in the first variable and linear-bounded in the second, such that

(RM) $$\mathbf{0} = R_t(\mathbf{1})\xi \leq R_t(\eta)\xi \leq M_t\xi, \qquad (\eta, \xi) \in \overline{\mathscr{S}}_+ \otimes \mathscr{B}_+,$$

(FM) $$\mathbf{1} - F_t[\eta] = M_t[\mathbf{1} - \eta] - R_t(\eta)[\mathbf{1} - \eta], \qquad \eta \in \overline{\mathscr{S}}.$$

cf. [*4*] [*5*]. Keeping $t \in \mathrm{T}$ and $\eta \in \overline{\mathcal{S}}_+$ fixed, $\mathrm{R}_t(\eta)$ is by definition a linear-bounded operator on $\mathcal{B}$. We assume that the following condition is satisfied:

(R) *For every* $t \in \mathrm{T}$ *there exists a mapping* $g_t : \overline{\mathcal{S}}_+ \to \mathcal{B}$ *such that*

$$\mathrm{R}_t(\xi)[\mathbf{1} - \xi] = g_t[\xi]\rho^t\Phi^*[\mathbf{1} - \xi]\phi\,, \qquad \xi \in \overline{\mathcal{S}}_+\,,$$

$$\lim_{\|\mathbf{1}-\xi\|\to 0} \| g_t[\xi] \| = 0\,.$$

The problem of existence and construction of Markov branching processes, i. e., Markov processes with a branching transition function, has been extensively treated in [*6*], see also [*10*] and section 3 below. The conditions (M) and (R) are empty, if X is finite and M_t primitive. This is a direct consequence of Perron's theorem on primitive positive matrices. For infinite X both, (M) and (R), are non-trivial: A simple counter-example is homogeneous branching Brownian motion on $\mathbb{R}$. However, for branching diffusions on bounded domains, (M) has been verified quite generally, cf. [*3*], [*5*]. Conditions implying (R) for processes which can be constructed from a Markov process on X, a termination density, and a branching kernel, can be found together with branching diffusion examples in [*4*], [*5*]. Again, see also section 3 below.

2. RESULTS

It is tacitly assumed that (M) and (R) are satisfied with $\rho = 1$. Then

$$\mu := (2t)^{-1}\Phi^*\left[\int_{\hat{\mathrm{X}}} \{ \hat{x}[\phi]^2 - \hat{x}[\phi^2] \} \mathbf{P}_t(\langle \cdot \rangle, d\hat{x})\right]$$

is constant as a function of $t \in \mathrm{T}$, $0 \leq \mu \leq \infty$, cf. [*4*]. Noting that $\mathbf{P}_t(\hat{x}, \{\theta\}) = \mathrm{F}_t(\hat{x}, \mathbf{0})$ and that, in consequence of (F.1), $\mathrm{F}_t[\mathbf{0}]$ is non-decreasing in t, define

$$q(x) := \lim_{t\to\infty} \mathbf{P}_t(\langle x \rangle, \{\theta\})\,, \qquad x \in \mathrm{X}\,.$$

Quite generally either $\mu = 0$ and $\mathbf{P}_t(\langle x \rangle, \mathrm{X}^{(1)}) \equiv 1$, that is, $q = \mathbf{0}$, or $0 < \mu \leq \infty$ and $q = \mathbf{1}$. If $q = \mathbf{1}$ and $\mu < \infty$, it is well-known that

$$\mathbf{P}_t(\hat{x}, \hat{\mathrm{X}} \setminus \{\theta\}) \sim (\mu t)^{-1}\hat{x}[\phi]\,, \qquad t \to \infty\,,$$

and for $\xi \in \mathcal{B}_+$

$$\frac{\mathbf{P}_t(\hat{x}, \{ \hat{y} \in \hat{\mathrm{X}} : \{ \hat{y}[\xi] \leq \lambda t \} \cap (\hat{\mathrm{X}} \setminus \{\theta\}))}{\mathbf{P}_t(\hat{x}, \hat{\mathrm{X}} \setminus \{\theta\})} \underset{t\to\infty}{\longrightarrow} 1 - e^{-(\mu\Phi^*[\xi])^{-1}\lambda}, \qquad \lambda \geq 0\,,$$

cf. [*4*]. We are now interested in the case that $\mu = \infty$.

We begin with some preparatory results. Define

$$\mathbf{Q}_t(\hat{x}, \hat{\mathrm{A}}) := \frac{\mathbf{P}_t(\hat{x}, \hat{\mathrm{A}} \cap (\hat{\mathrm{X}} \setminus \{\theta\}))}{\mathbf{P}_t(\hat{x}, \hat{\mathrm{X}} \setminus \{\theta\})},$$

and let $(c_t)_{t\in\mathrm{T}} \subset (0, \infty)$ be such that $c_t \to 0$, as $t \to \infty$. For $\xi \in \mathscr{B}_+$ the Laplace-Stieltjes transform of $\mathbf{Q}_t(\hat{x}, \{c_t\hat{y}[\xi] \leq \lambda\})$ is given by

$$\Psi_t^{\hat{x},\xi}(u) = 1 - \frac{1 - \mathrm{F}_t(\hat{x}, e^{-c_t\xi u})}{1 - \mathrm{F}_t(\hat{x}, \mathbf{0})}, \qquad u \geq 0 .$$

PROPOSITION. — *The limit*

$$\Psi_\infty^{\hat{x},\xi}(u) := \lim_{t\to\infty} \Psi_t^{\hat{x},\xi}(u), \qquad u > 0,$$

exists for all $\hat{x} \in \hat{\mathrm{X}} \setminus \{\theta\}$ *if and only if*

$$1 - \Psi^\xi(u) := \lim_{t\to\infty} \frac{\Phi^*[\mathbf{1} - \mathrm{F}_t[e^{-c_t\xi u}]]}{\Phi^*[\mathbf{1} - \mathrm{F}_t[\mathbf{0}]]}, \qquad u > 0,$$

exists, and if these limits exist, then

$$\Psi_\infty^{\hat{x},\xi}(u) = \Psi^\xi(u), \qquad \hat{x} \in \hat{\mathrm{X}} \setminus \{\theta\}, \quad u > 0 . \tag{2.1}$$

The proof is immediate from (F.1) and the following lemma.

LEMMA. — *For every* $t \in \mathrm{T}$ *there exists a mapping* $h_t : \overline{\mathscr{S}}_+ \to \mathscr{B}$ *such that*

$$\begin{aligned} &\mathbf{1} - \mathrm{F}_t[\eta] = (\mathbf{1} + h_t[\eta])\Phi^*[\mathbf{1} - \mathrm{F}_t[\eta]]\phi, \qquad \eta \in \overline{\mathscr{S}}_+, \\ &\lim_{t\to\infty} \| h_t[\eta] \| = 0 \qquad \textit{uniformly on } \overline{\mathscr{S}}_+, \end{aligned} \tag{2.2}$$

where $\Phi^*[\mathbf{1} - \mathrm{F}_t[\eta]] > 0$ *for all* $t \in \mathrm{T}$ *and* $\eta \in \overline{\mathscr{S}}_+ \cap \{\Phi^*[\mathbf{1} - \eta] > 0\}$.

For the proof of this lemma see [4], [5]. Define

$$a_t := \Phi^*[\mathrm{P}_t(\langle\cdot\rangle, \hat{\mathrm{X}} \setminus \{\theta\})] = \Phi^*[\mathbf{1} - \mathrm{F}_t[0]], \qquad t > 0,$$

and note that $a_{t-\delta}/a_t \to 1$, as $t \to \infty$, by (F.2), (FM), (M) and (R) with $\rho = 1$, and $q = \mathbf{1}$.

PROPOSITION. — *If* Ψ^ξ *with* $c_t \equiv a_t$ *exists for some* $\xi \in \mathscr{B}_+$ *with* $\Phi^*[\xi] > 0$, *then it exists for all* $\xi \in \mathscr{B}_+$ *and*

$$\Psi^\xi(u) = \Psi^\phi(\Phi^*[\xi]u), \qquad u > 0 . \tag{2.3}$$

Proof. — Fix $u > 0$. Using (F.2), (FM), (RM) and (M) with $\rho = 1$, we can choose for every $\varepsilon > 0$ a $\delta \in \mathrm{T}$ such that for all $t \in \mathrm{T} \cap (\delta, \infty)$,

$$\begin{aligned} a_t^{-1}\Phi^*[\mathbf{1} - \mathrm{F}_t[e^{-ua_t\xi}]] &= a_t^{-1}\Phi^*[\mathbf{1} - \mathrm{F}_{t-\delta}[\mathbf{1} - (\mathbf{1} - \mathrm{F}_\delta[e^{-ua_t\xi}])]] \\ &= (a_{t-\delta}/a_t)a_{t-\delta}^{-1}\Phi^*[\mathbf{1} - \mathrm{F}_{t-\delta}[\mathbf{1} - (1 + \varepsilon)\Phi^*[\xi]ua_t\phi]]. \end{aligned}$$

Similarly, drawing also on (R), we can find for every $\varepsilon > 0$ a $\delta \in \mathrm{T}$ and a $t_0 > \delta$ such that for all $t \in \mathrm{T} \cap (t_0, \infty)$

$$a_t^{-1}\Phi^*[\mathbf{1} - \mathrm{F}_t[e^{-ua_t\xi}]] \geq (a_{t-\delta}/a_t)a_{t-\delta}^{-1}\Phi^*[\mathbf{1} - \mathrm{F}_{t-\delta}[\mathbf{1} - (1 - \varepsilon)\Phi^*[\xi]ua_t\phi]] .$$

Since by assumption, $a_t^{-1}\Phi^*[\mathbf{1} - \mathrm{F}_t[e^{-ua_t\xi}]]$ converges to a limit continuous in $u > 0$, further $a_{t-\delta}/a_t \to 1$, as $t \to \infty$, it follows that

$$a_t^{-1}\Phi^*[\mathbf{1} - \mathrm{F}_t[\mathbf{1} - u\Phi^*[\xi]a_t\phi]]$$

converges to the same limit. Now reverse the argument, using again the continuity in n and the fact that ξ enters only through the numerical factor $\Phi^*[\xi]$ of u. $\square$

From now on $q = \mathbf{1}$ and $\mu = \infty$. A crucial role will then by played by the following condition:

(S) *For some* $\varepsilon \in \mathrm{T}$ *and all* $t \in \mathrm{T} \cap (0, \varepsilon)$

$$\Phi^*[\mathrm{R}_t(1 - s\phi)\phi] = s^\alpha \mathrm{L}_t(s) , \qquad s \in [0, \|\phi\|^{-1}] ,$$

with $0 < \alpha \leq 1$, *independent of* t, *and* $\mathrm{L}_t(s)$ *slowly varying as* $s \downarrow 0$.

Using (F.2), (FM), (RM), (M) and (R) with $\rho = 1$,

$$\Phi^*[\mathrm{R}_2(\mathbf{1} - s\phi)\phi] = \Phi^*[\mathrm{R}_t(\mathbf{1} - s\phi)\phi] + \Phi^*\left[\mathrm{R}_t(\mathrm{F}_t[\mathbf{1} - s\phi])\frac{\mathbf{1} - \mathrm{F}_t[\mathbf{1} - s\phi]}{s}\right],$$

$$(\mathbf{1} - g_t[\mathbf{1} - s\phi])s\phi \leq \mathbf{1} - \mathrm{F}_t[\mathbf{1} - s\phi] \leq s\phi . \tag{2.4}$$

Hence, if (S) holds for some $\varepsilon \in \mathrm{T}$, it is satisfied for all $\varepsilon \in \mathrm{T}$, and by the uniform convergence property for slowly varying functions,

$$\lim_{s \downarrow 0} \frac{\Phi^*[\mathrm{R}_{t'}(\mathbf{1} - s\phi)\phi]}{\Phi^*[\mathrm{R}_t(\mathbf{1} - s\phi)\phi]} = \frac{t'}{t}, \qquad t, t' \in \mathrm{T} .$$

THEOREM 1. — *If* (S) *is satisfied, then for all* $\hat{x} \neq \theta$

$$\mathbf{P}_t(\hat{x}, \hat{\mathrm{X}} \setminus \{\theta\}) \sim t^{-1/\alpha}\mathrm{L}^*(t)\hat{x}[\phi] , \qquad t \to \infty , \tag{2.5}$$

where L^* *is slowly varying at infinity, and*

$$\mathbf{Q}_t(\hat{x}, \{a_t\hat{y}[\phi] \leq \lambda\}) \underset{t \to \infty}{\longrightarrow} \mathrm{D}(\lambda) , \qquad \lambda \geq 0 , \tag{2.6}$$

where D *has the Laplace-Stieltjes transform*

$$\Psi(u) = 1 - u(1 + u^\alpha)^{-1/\alpha} , \qquad u > 0 .$$

The proof will be given in Section 4 below. Setting

$$\xi = \sum_{\nu=1}^{n} u_\nu 1_{\mathrm{A}_\nu}$$

with some measurable decomposition $\{ A_\nu \}_{1 \le \nu \le n}$ of X and applying (2.3), we immediately obtain the following:

COROLLARY. — *If* (S) *is satisfied, then for any measurable decomposition* $\{ A_\nu \}_{1 \le \nu \le n}$ *of* X

$$(2.7) \quad \mathbf{Q}_t(\hat{x}, \{ a_t \hat{y}[1_{A_\nu}] \le \lambda_\nu ; \nu = 1, \ldots, n \}) \underset{t \to \infty}{\longrightarrow} D \left(\min_{1 \le \nu \le n} \{ \lambda_\nu / \Phi^*[1_{A_\nu}] \} \right), \quad \lambda \ge 0, \quad \nu = 1, \ldots, n,$$

with D *as in Theorem 1.*

More intuitively, if $\{ \hat{x}_t, \mathbf{P}^{\hat{x}} \}$ is a Markov process in $(\hat{X}, \hat{\mathfrak{A}})$ with transition probability $\mathbf{P}_t$, then (2.7) means that

$$a_t(\hat{x}_t[1_{A_1}], \ldots, \hat{x}_t[1_{A_n}]) \mid \hat{x}_t \neq \theta \underset{t \to \infty}{\overset{d}{\longrightarrow}} (\Phi^*[1_{A_1}], \ldots, \Phi^*[1_{A_n}])W$$

where W has the d. f. D.

THEOREM 2. — *If for some* $\xi \in \mathscr{B}_+$ *with* $\Phi^*[\xi] > 0$ *and all* $x \in X$

$$\mathbf{Q}_t(\langle x \rangle, \{ a_t \hat{y}[\xi] \le \lambda \}) \underset{t \to \infty}{\overset{d}{\longrightarrow}} D_\xi(\lambda), \qquad \lambda > 0$$

where D_ξ *is non-degenerate, admittedly defective, then* (S) *is satisfied.*

The proof will be given in Section 5 below. Suppose now that $\mu = \infty$ with (S) *not* satisfied. Does there exist a normalization $(c_t)_{t \in T}$ leading to a non-trivial, proper conditional limit d. f. ?

THEOREM 3. — *Suppose for some* $(c_t)_{t \in T}$, *for all* $x \in X$

$$\mathbf{Q}_t(\langle x \rangle, \{ c_t \hat{y}[\phi] \le \lambda \}) \underset{t \to \infty}{\overset{d}{\longrightarrow}} D_\phi(\lambda), \qquad \lambda > 0,$$

where D_ϕ *is proper and non-degenerate at zero, then*

$$c_t \sim C a_t, \qquad t \to \infty,$$

with $0 < C < \infty$.

The proof is to be found in Section 6.

3. CONDITION (S) IN TERMS OF THE BRANCHING LAW

Throughout this section $T = (0, \infty)$. If $\{ F_t \}$ is constructed from a transition semigroup $\{ T_t \}$ on $\mathscr{B}$, a termination density k, and a branching kernel π, there exist not only sufficient conditions for (M) and (R) in terms of these components, but also an explicit representation of μ, cf. [*3*], [*4*], [*5*]. We now derive a corresponding form of (S).

Let $\{ T_t \}$ be a transition semigroup on $\mathscr{B}$, not necessarily conservative, let $k \in \mathscr{B}_+$, and define $\{ T_t^0 \}$ as the transition semigroup obtained as the unique solution of the semigroup perturbation equation

$$T_t^0 = T_t - \int_0^t T_s k T_{t-s}^0 ds .$$

Let $\pi = \pi(x, \hat{A})$ be a stochastic kernel on $X \otimes \hat{\mathfrak{A}}$ with bounded first moment operator m,

$$(m\xi)(x) := \int_{\hat{X}} \hat{x}[\xi]\pi(x, d\hat{x}), \qquad \xi \in \mathscr{B}, \quad x \in X,$$

and define f as the generating mapping of π,

$$f[\eta](x) := \int_{\hat{X}} \pi(x, d\hat{x})\tilde{\eta}(\hat{x}), \qquad x \in X, \quad \eta \in \overline{\mathscr{S}}.$$

Since

$$\| f[\eta] - f[\xi] \| \leq \| m\mathbf{1} \| \, \| \eta - \xi \|,$$

there exists exactly one set of mappings $F_t : \overline{\mathscr{S}} \to \mathscr{B}$, $t \in T$, satisfying the non-linear perturbation equation

$$(3.1) \quad \mathbf{1} - F_t[\eta] = T_t^0(\mathbf{1} - \eta) + \int_0^t T_s^0 k(\mathbf{1} - f[F_{t-s}[\eta]])ds, \qquad \eta \in \overline{\mathscr{S}},$$

Using the semigroup property of $\{ T_t^0 \}$,

$$\begin{aligned} \mathbf{1} - F_{t+s}[\eta] &= T_t^0 T_s^0(\mathbf{1} - \eta) + \int_t^{t+s} T_u^0 k(\mathbf{1} - f[F_{t+s-u}[\eta]])du \\ &\quad + \int_0^t T_u^0 k(\mathbf{1} - f[F_{t+s-u}[\eta]])du \\ &= T_t^0 \left\{ T_s^0(\mathbf{1} - \eta) + \int_0^s T_v^0 k(\mathbf{1} - f[F_{s-v}[\eta]])dv \right\} \\ &\quad + \int_0^t T_u^0 k(\mathbf{1} - f[F_{t+s-u}[\eta]])du . \end{aligned}$$

Hence, by uniqueness, $\{ F_t \}$ is a semigroup. Representing $F_t[\eta]$ as the limit of the iteration sequence of (3.1), beginning with $\mathbf{0}$, it is easily verified that

(*a*) for every finite, measurable decomposition $\{ A_\nu \}_{1 \leq \nu \leq n}$ of X, $F_t[\Sigma_{\nu=1}^n 1_{A_\nu} s_\nu]$ is analytic in $(s_1, \ldots, s_n)$, $| s_\nu | < 1$, with non-negative coefficients,

(*b*) $F_t[\eta]$ is sequentially continuous in η with respect to the product topology on $\overline{\mathscr{S}}$.

Given (*b*), (*a*) is equivalent to

(a') F_t is analytic on $\mathcal{S}$ with non-negative Fréchet derivatives on $\mathcal{S}_+$. Finally, again by uniqueness, $F_t[\mathbf{1}] = \mathbf{1}$ for all $t \in T$.

Let us call a semigroup on $\overline{\mathcal{S}}$ with fixed point $\mathbf{1}$ and properties (a) (or (a')) and (b) a *pre-generating semigroup.* Clearly, every generating semigroup is a pre-generating semigroup. For the reverse some restrictions on $(X, \mathfrak{A})$ are needed to ensure the applicability of standard extension procedures in the construction of measures on $(\hat{X}, \hat{\mathfrak{A}})$. The topological structure assumed in the subsequent remark is sufficient. However, leaving aside technical convenience and conditions coming in via verification of (M), all we really need in this paper is a pre-generating semigroup.

Remark. — The above construction of $\{ F_t \}$ is to be interpreted as follows. Assume that $(X, \mathfrak{A})$ is a locally compact Hausdorff space with countable open base and Borel algebra. Let $\{ T_t \}$ be the transition semigroup of a right-continuous strong Markov process $\{ x_t \}$ on $(X, \mathfrak{A})$ with life time τ and trap ∂, possessing left limits for $t < \tau$. In particular, $\{ x_t \}$ could be a diffusion with (or without) absorbing barriers. Then $\{ T_t^0 \}$ is the transition semigroup of a process $\{ x_t^0 \}$ obtained from $\{ x_t \}$ by curtailing the life time with termination density k, using a second trap Δ in case of termination before τ. Given that k and m are bounded, $\{ x_t^0 \}$ and π uniquely determine a conservative, right-continuous strong Markov process on $(\hat{X}, \hat{\mathfrak{A}})$ constructed to the following intuitive rules: All particles at a time move independently of each other, each according to $\{ x_t \}$. A particle hitting ∂ disappears, a particle hitting Δ is instantaneously replaced by a population of new particles according to $\pi(x_{\tau_\Delta -}, \cdot)$, where $x_{\tau_\Delta -}$ is the left limit of the path at the hitting time τ_Δ of Δ. The transition function of this process has the branching property, and writing down the strong Markov property respective the time of first absorption or branching yields (3.1) for the corresponding generating semigroup, cf. [*6*], [*10*].

In analogy to (FM) we expand

$$\mathbf{1} - f[\eta] = m[\mathbf{1} - \eta] - r(\eta)[\mathbf{1} - \eta], \qquad \eta \in \overline{\mathcal{S}},$$

with $\mathbf{0} = r(\mathbf{1})\xi \leq r(\eta)\xi \leq m\xi$, $(\eta, \xi) \in \overline{\mathcal{S}}_+ \otimes \mathcal{B}_+$, and the same continuity properties of r as stated in section 1 for R_t.

PROPOSITION. — *Let $\{ F_t \}$ be a pre-generating semigroup determined by a system $[T_t, k, \pi]$ with bounded k and m. Suppose that* (M) *is satisfied and that there exist constants c and c^* for which $km\phi \leq c\phi$ and*

$$\Phi^*[km\xi] \leqslant c^*\Phi^*[\xi], \qquad \xi \in \mathcal{B}_+ . \tag{3.2}$$

Then (S) *is equivalent to*

$$\Phi^*[kr(\mathbf{1} - s\phi)\phi] = s^{\alpha}\mathrm{L}(s)\,, \tag{3.3}$$

where L(*s*) *is slowly varying as* $s \downarrow 0$, *in fact*

$$\lim_{s \downarrow 0} \frac{\mathrm{L}_t(s)}{\mathrm{L}(s)} = t\,, \qquad t > 0\,. \tag{3.4}$$

For the proof we need the following two facts:

(1) Linearization of (3.1) in the fixed point $\eta = \mathbf{1}$ results in the linear perturbation equation.

$$\mathrm{M}_t = \mathrm{T}_t^0 + \int_0^t \mathrm{T}_s^0 km\mathrm{M}_{t-s}ds \tag{3.5}$$

for the moment semigroup $\{ \mathrm{M}_t \}$. Again, by boundedness of *km*, the solution of (3.5) is unique.

(2) The assumptions of the Proposition imply that (R) is satisfied uniformly in $t \in [a, b]$ for any fixed $a > 0$ and $b < \infty$, cf. [*4*], [*5*].

Proof of the proposition. — By (3.1) and (3.5), $\mathrm{R}_t(\mathbf{1} - u\phi)\phi$ solves

$$w_t = w_t^1 + \int_0^t \mathrm{T}_s^0 kmw_{t-s}ds\,,$$

$$w_t^1 := \int_0^t \mathrm{T}_s^0 kr(\mathrm{F}_{t-s}[\mathbf{1} - u\phi])\frac{\mathbf{1} - \mathrm{F}_{t-s}[\mathbf{1} - u\phi]}{u}ds\,.$$

By (3.5), $\mathrm{T}_s^0\xi \leq \mathrm{M}_s\xi$, so that $\Phi^*[\mathrm{T}_t^0\xi] \leqslant \Phi^*[\xi]$, $\xi \in \mathscr{B}_+$. Hence, using $r(\eta)\xi \leq m\xi$, $(\eta, \xi) \in \overline{\mathscr{S}}_+ \otimes \mathscr{B}_+$, and (3.2),

$$\Phi^*[w_t^1] \leq \Phi^*[w_t] \leq e^{c^*t} \sup_{0 < s \leq t} \Phi^*[w_s^1]\,, \tag{3.6}$$

and using the last inequality of (2.4),

$$\sup_{0 < s \leq t} \Phi^*[w_s^1] \leq t\Phi^*[kr(\mathbf{1} - u\phi)\phi]\,. \tag{3.7}$$

From (3.5) and (3.2), $\Phi^*[\mathrm{T}_s^0\xi] \geq (1 - c^*s)\Phi^*[\xi]$, $\xi \in \mathscr{B}_+$. Set

$$\gamma_{\varepsilon,t}(u) := \sup_{\varepsilon < s \leq t} \| g_s[\mathbf{1} - u\phi] \|\,.$$

Then, using the first inequality of (2.4),

$$\Phi^*[w_t^l] \geq (1 - \gamma_{\varepsilon,t}(u))(1 - c^*t)(t - \varepsilon)\Phi^*[kr(\mathbf{1} - (1 - \gamma_{\varepsilon,t}(u))u\phi)\phi]\,. \tag{3.8}$$

Assuming (S), set $\gamma^v_{\varepsilon,t} := \sup_{u \leq v} \gamma_{\varepsilon,t}(u)$. From (3.6-3.8)

$$\frac{\Phi^*[kr(\mathbf{1} - \lambda u\phi)\phi]}{\Phi^*[kr(\mathbf{1} - u\phi)\phi]} \geq \frac{t-\varepsilon}{t}\frac{1-c^*t}{e^{c^*t}}(1-\gamma_{\varepsilon,t}(u))\frac{\Phi^*[\mathrm{R}_t(\mathbf{1} - \lambda u\phi)\phi]}{\Phi^*[\mathrm{R}_t(\mathbf{1} - u(1-\gamma^v_{\varepsilon,t})^{-1}\phi)\phi]}$$
$$= \left(\frac{\lambda}{u}(1-\gamma^v_{\varepsilon,t})\right)^\alpha \frac{\mathrm{L}_t(\lambda u)}{\mathrm{L}_t((1-\gamma^v_{\varepsilon,t})^{-1}u)}.$$

Choose t, ε, and v in this order. It follows that $\liminf_{u\downarrow 0}$ (l. h. s.) $\geq \lambda^\alpha$. Similarly, $\limsup_{u\downarrow 0}$ (l. h. s.) $\leq \lambda^\alpha$. That is, (S) implies (3.3).

Now assume (3.3) and define $\gamma^0_t(s) := 1$,

$$\gamma^n_t(s) := \gamma^{n-1}_t(s)(1 - \| g_{2^{-n}t}[\mathbf{1} - \gamma^{n-1}_t(s)s\phi] \|), \qquad n \in \mathbb{N}.$$

Then by induction, using (2.4) and (3.6-3.8),

$$\frac{\Phi^*[\mathrm{R}_t(\mathbf{1} - \lambda s\phi)\phi]}{\Phi^*[\mathrm{R}_t(\mathbf{1} - s\phi)\phi]} \leq \frac{2^{-n}te^{c^*2^{-n}t}}{(1-\gamma_{\varepsilon,2^{-n}t}(s))(1-c^*2^{-n}t)(2^{-n}t-\varepsilon)\gamma^n_t(s)}$$
$$\times \frac{\Phi^*[kr(\mathbf{1} - \lambda s\phi)\phi]}{\Phi^*[kr(\mathbf{1} - (1-\gamma_{\varepsilon,2^{-n}t}(\gamma^n(s)s))\gamma^n_t(s)s\phi)\phi]}$$

Fixing t, n, ε in this order leads to $\limsup_{s\downarrow 0}$ (l. h. s.) $\leq \lambda^\alpha$. Similarly, $\liminf_{s\downarrow 0}$ (l. h. s.) $\geq \lambda^\alpha$. That is, (3.3) implies (S).

The same estimates also yield (3.4). □

4. PROOF OF THEOREM 1

LEMMA. — *Given* (S),

$$(4.1) \qquad t\Phi^*[\mathrm{R}_\varepsilon(\mathbf{1} - a_t\phi)\phi] \to \frac{\varepsilon}{\alpha}, \qquad t \to \infty.$$

Proof. — It suffices to prove that for every $\varepsilon \in \mathrm{T}$

$$\lim_{\mathbb{N}\ni n\to\infty} n\Phi^*[\mathrm{R}_\varepsilon(\mathbf{1} - a_{n\varepsilon}\phi)\phi] = \frac{1}{\alpha}.$$

The continuous-parameter result then follows by the monotonicity of a_t in t and that of $\mathrm{R}_\varepsilon(\xi)$ in ξ. Without loss of generality $\varepsilon = 1$. Let

$$\Lambda(s) := \Phi^*[\mathrm{R}_1(\mathbf{1} - s\phi)\phi],$$
$$\Delta_n := \Phi^*[\mathrm{R}_1(\mathrm{F}_n[\mathbf{0}])(\mathbf{1} - \mathrm{F}_n[\mathbf{0}])].$$

Then using (FM), $a_{n+1} = a_n - a_n\Delta_n$, and by the mean value theorem

$$\Lambda(a_n) - \Lambda(a_{n+1}) = a_n\Delta_n\Lambda'(a_n - \theta_n a_n\Delta_n)$$

with some θ_n, $0 < \theta_n < 1$. Thus

$$\frac{1}{\Lambda(a_{n+1})} - \frac{1}{\Lambda(a_n)} = \frac{a_n\Delta_n\Lambda'(a_n - \theta_n a_n\Delta_n)}{\Lambda(a_n)\Lambda(a_n - a_n\Delta_n)} = \mathrm{A}_n\mathrm{B}_n\mathrm{C}_n\mathrm{D}_n\,,$$

$$\mathrm{A}_n := \frac{a_n}{a_n - \theta_n a_n\Delta_n}, \qquad \mathrm{C}_n := \frac{\Lambda(a_n - \theta_n a_n\Delta_n)}{\Lambda(a_n - a_n\Delta_n)},$$

$$\mathrm{B}_n := \frac{(a_n - \theta_n a_n\Delta_n)\Lambda'(a_n - \theta_n a_n\Delta_n)}{\Lambda(a_n - \theta_n a_n\Delta_n)}, \qquad \mathrm{D}_n := \frac{\Delta_n}{\Lambda(a_n)}.$$

Clearly, $\mathrm{A}_n \to 1$. Using (S) and A.1, $\mathrm{B}_n \to \alpha$. By the uniform convergence property of slowly varying functions, $\mathrm{C}_n \to 1$, and by (2.2), $\mathrm{D}_n \to 1$. Hence

$$\Lambda(a_{n+1})^{-1} - \Lambda(a_n)^{-1} \to \alpha\,.$$

Cesaro summation completes the proof. □

Proof of theorem 1. — From (4.1) and (S)

$$a_{\lambda t}/a_t \sim \lambda^{-1/\alpha}[\mathrm{L}(a_{\lambda t})/\mathrm{L}(a_t)]^{-1/\alpha}, \qquad t \to \infty\,.$$

It suffices to consider $\lambda > 1$. By the monotonicity of a_t

$$1 \geq a_{\lambda t}/a_t \geq a_{[\lambda t]+1}/a_{[t]} = \prod_{\nu=[t]}^{[\lambda t]} a_{\nu+1}/a_\nu = \prod_{\nu=[t]}^{[\lambda t]} (1 - \Delta_\nu)\,.$$

In the notation of the preceding proof, $\Delta_\nu \sim \Lambda(a_\nu) \sim (\alpha\nu)^{-1}$, $\nu \to \infty$, so that for sufficiently large t

$$a_{\lambda t}/a_t \geq (1 - (\alpha[t])^{-1})^{[\lambda t]+1} \geq c > 0$$

with c independent of t. Hence, by the uniform convergence property, $\mathrm{L}(a_{\lambda t})/\mathrm{L}(a_t) \to 1$, thus $a_{\lambda t}/a_t \sim \lambda^{-1/\alpha}$, i. e., $a_t = t^{-1/\alpha}\mathrm{L}^*(t)$, L^* slowly varying at infinity. In case $\mathrm{T} = \mathbb{N}$ we have made use of A.2 at this point. Recalling (F.1), we have (2.5).

We now prove (2.6). Fix $\lambda > 0$ and set

$$\phi_t := e^{-a_t\phi\lambda}, \qquad t > 0\,.$$

As we have seen in Section 2, it suffices to show that

$$(4.2) \qquad a_t^{-1}\Phi^*[\mathbf{1} - \mathrm{F}_t[\phi_t]] \to 1 - \Psi(\lambda).$$

Pick $\varepsilon_1, \varepsilon_2 \in (0, 1)$. Since $0 < a_t \downarrow 0$, we can choose $u(t) \to \infty$, as $t \to \infty$, such that

$$(4.3) \qquad a_{u(t)} \leq \lambda a_t(1 + \varepsilon_1)/(1 - \varepsilon_2) \leq a_{u(t)-1}.$$

By (2.2) for sufficiently large t

$$(4.4) \qquad (1 - \varepsilon_2)a_t\phi \leq \mathbf{1} - \mathrm{F}_t[\mathbf{0}].$$

Using (FM), (RM), and $1 - e^{-x} \sim x(1 + O(x))$, $x \downarrow 0$,

$$(4.5) \qquad \mathbf{1} - \lambda(1 - \varepsilon_1)a_t\phi \leq \mathrm{F}_1[\phi_t].$$

From the second inequality in (4.3) and (4.4)

$$\lambda(1 + \varepsilon_1)a_t\phi \leq (1 - \varepsilon_2)a_{u(t)-1}\phi \leq \mathbf{1} - \mathrm{F}_{u(t)-1}[\mathbf{0}].$$

From this and (4.5)

$$(4.6) \qquad \mathbf{1} - \mathrm{F}_1[\phi_t] \leq \lambda(1 + \varepsilon_1)a_t\phi \leq \mathbf{1} - \mathrm{F}_{u(t)-1}[\mathbf{0}].$$

Similarly, there is some $v(t) \to \infty$, $t \to \infty$, such that for sufficiently large t

$$a_{v(t)+1} \leq \lambda a_t(1 - \varepsilon_1)/(1 + \varepsilon_2) \leq a_{v(t)},$$
$$\mathbf{1} - \mathrm{F}_{v(t)}[\mathbf{0}] \leq (1 + \varepsilon_2)a_v\phi,$$
$$\mathrm{F}_1[\phi_t] \leq \mathbf{1} - \lambda(1 - \varepsilon_1)a_t\phi,$$

and from this

$$(4.7) \qquad \mathbf{1} - \mathrm{F}_1[\phi_t] \geq \mathbf{1} - \mathrm{F}_{v(t)+1}[\mathbf{0}].$$

Combining (4.6) and (4.7), we arrive at

$$(4.8) \qquad \mathbf{1} - \mathrm{F}_{v(t)+t+1}[\mathbf{0}] \leq \mathbf{1} - \mathrm{F}_t[\phi_t] \leq \mathbf{1} - \mathrm{F}_{u(t)+t-1}[\mathbf{0}].$$

Using (FM) and (2.2),

$$a_u/a_{u-1} = \mathbf{1} - \Phi^*[\mathrm{R}_1(\mathrm{F}_{u-1}[\mathbf{0}])(\mathbf{1} - \mathrm{F}_{u-1}[\mathbf{0}])/a_{u-1}] \to 1, \qquad u \to \infty.$$

Hence, by (4.3),

$$a_{u(t)}/a_t \to \lambda(1 + \varepsilon_1)/(1 - \varepsilon_2), \qquad t \to \infty.$$

From this, by (4.1) and the uniform convergence property of slowly varying functions,

$$(4.9) \qquad t/u(t) \to \lambda^\alpha[(1 + \varepsilon_1)/(1 - \varepsilon_2)]^\alpha, \qquad t \to \infty.$$

Also by (4.1)

(4.10) $a_{u(t)+t-1}/a_t \sim [(u(t)+t-1)/t]^{-1/\alpha}[\mathrm{L}(a_{u(t)+t-1})/\mathrm{L}(a_t)]^{-1/\alpha}, \qquad t \to \infty .$

Proceeding as in the first part of the proof, $\mathrm{L}(a_{u(t)+t-1})/\mathrm{L}(a_t) \to 1$, so that, by (4.1), (4.9),

$$a_{u(t)+t-1}/a_t \to (1 + \lambda^{-\alpha}[(1-\varepsilon_2)/(1+\varepsilon_1)]^{\alpha})^{-1/\alpha} .$$

Similarly,

$$a_{v(t)+t+1}/a_t \to (1 + \lambda^{-\alpha}[(1+\varepsilon_2)/(1-\varepsilon_1)]^{\alpha})^{-1/\alpha} .$$

Applying Φ^* to (4.8), then using the last two relations, and finally letting $\varepsilon_1, \varepsilon_2 \to 0$ yields (4.2), thus completing the proof. □

5. PROOF OF THEOREM 2

PROPOSITION. — *The hypothesis of Theorem 2 implies*

(5.1) $$a_t = t^{-1/\alpha}\mathrm{L}_1(t), \qquad t \in \mathrm{T},$$

where $0 < \alpha \leq 1$ *and* L_1 *is slowly varying at infinity.*

Proof. — *Step 1.* — We show that

(5.2) $$\liminf_{\mathbb{N}\ni n\to\infty} n(1 - a_{n+1}/a_n) \geq 1 .$$

Let $\lambda \in [0, 1]$ and for convenience $\|\phi\| = 1$. Define

$$\mathrm{A}(\lambda) := \Phi^*[\mathbf{1}] - \Phi^*[\mathrm{F}_1[\mathbf{1} - \phi + \lambda\phi]] - 1 + \lambda ,$$
$$\mathrm{B}(\lambda) := -(1-\lambda)/\mathrm{A}(\lambda) .$$

Since the derivative $\mathrm{A}'(\lambda)$ is concave,

$$\frac{1}{2}(1-\lambda)\mathrm{A}'(\lambda) \leq \mathrm{A}(\lambda) \leq (1-\lambda)\mathrm{A}'(\lambda) ,$$

and from this

$$0 \leq (1-\lambda)\mathrm{B}'(\lambda)/\mathrm{B}(\lambda) \leq 1 .$$

That is, $\mathrm{B}(\lambda)$ is non-decreasing and $(1-\lambda)\mathrm{B}(\lambda)$ is non-increasing. Hence, for $0 \leq \lambda_1 < \lambda_2 < 1$,

$$0 \leq \mathrm{B}(\lambda_2) - \mathrm{B}(\lambda_1) \leq \mathrm{B}(\lambda_1)(\lambda_2 - \lambda_1)/(1-\lambda_2) .$$

With $\lambda_1 := 1 - a_n$ and $\lambda_2 := 1 - a_{n+1}$

$$0 \leq \mathrm{B}(1 - a_{n+1}) - \mathrm{B}(1 - a_n) = \frac{a_n}{a_{n+1}} \cdot \frac{a_n - a_{n-1}}{a_n - \Phi^*[\mathbf{1} - \mathrm{F}_1[\mathbf{1} - a_n\phi]]}$$

By (FM) and (2.2), $a_n/a_{n+1} \to 1$. Furthermore, for $\eta, \xi \in \overline{\mathcal{S}}_+$ such that $\eta \leq \xi$, we have

$$0 \leq 1 - \frac{\Phi^*[F_1[\xi] - F_1[\eta]]}{\Phi^*[\xi - \eta]} \leq \frac{\Phi^*[R_1(\eta)(\xi - \eta)]}{\Phi^*[\xi - \eta]},$$

so that

$$(5.3) \quad \frac{a_n - a_{n+1}}{a_n - \Phi^*[\mathbf{1} - F_1[\mathbf{1} - a_n\phi]]} = \frac{\Phi^*[F_1[\mathbf{1} - a_n\phi] - F_n[\mathbf{0}]]}{\Phi^*[\mathbf{1} - a_n\phi - F_{n-1}[\mathbf{0}]]} \cdot \frac{\Phi^*[F_n[\mathbf{0}] - F_{n-1}[\mathbf{0}]]}{\Phi^*[F_{n+1}[\mathbf{0}] - F_n[\mathbf{0}]]} \to 1 .$$

Hence,

$$\limsup_{\mathbb{N} \ni n \to \infty} n^{-1} B(1 - a_n) \leq 1 .$$

Recalling the definition of $B(\lambda)$ and once more applying (5.3) yields (5.2).

Step 2. — We show (5.1) with some α not necessarily in (0, 1]. According to A.3 it suffices to show that $a_{nk}/a_n \to c_k > 0$ for every $k \in \mathbb{N}$. We proceed by induction. That is, we assume $a_{nj}/a_n \to c_j > 0$ and proceed to $a_{n(j+1)}/a_n \to c_{j+1} > 0$. Define

$$\psi_n(t) := a_n^{-1}\Phi^*[1 - F_n[e^{-ta_n\phi}]] .$$

By (2.2) there is for every $\varepsilon > 0$ an n_0 such that for $n \geq n_0$

$$e^{-r_n a_n \phi} \leq F_{nj}[\mathbf{0}] \leq e^{-t_n a_n \phi},$$

$$r_n := (1 + \varepsilon)a_{nj}/a_n , \qquad t_n := (1 - \varepsilon)a_{nj}/a_n .$$

That is,

$$(5.4) \qquad \psi_n(t_n) \leq a_{n(j+1)}/a_n \leq \psi_n(r_n) .$$

By hypothesis and the Proposition preceding the statement of Theorem 1, $\psi_n(t)$ converges, as $n \to \infty$, to a function $\psi(t)$ continuous in $t > 0$. Since

$$0 \leq \psi_n(r) - \psi_n(t) \leq a_n^{-1}\Phi^*[e^{-ra_n\phi} - e^{-ta_n\phi}] \leq t - r ,$$

the family $\{ \psi_n \}$ is equicontinuous, so that the convergence is uniform on compact t-intervals not containing 0. Hence, letting $n \to \infty$ in (5.4) yields

$$\psi((1 - \varepsilon)c_j) \leq \liminf_{n \to \infty} \frac{a_{n(j+1)}}{a_n} \leq \psi((1 + \varepsilon)c_j) .$$

Now let $\varepsilon \to 0$, and recall the non-degeneracy assumption.

Step 3. — It remains to show $\alpha \in (0, 1]$. By Step 1 it suffices to verify

$$(5.5) \qquad n(1 - a_{n+1}/a_n) \to 1/\alpha , \qquad n \to \infty .$$

Using (FM),

$$0 \leq \Phi^*[\mathrm{F}_{n+1}[\mathbf{0}] - \mathrm{F}_n[\mathbf{0}]] = \Phi^*[\mathrm{R}_1(\mathrm{F}_n[\mathbf{0}])(\mathbf{1} - \mathrm{F}_n[\mathbf{0}])] \downarrow 0 .$$

Hence, if $\lambda > 1$,

$$a_n - a_{[\lambda n]+1} = \sum_{\nu=n}^{[\lambda n]} (a_\nu - a_{\nu+1}) \leq \left(\lambda - 1 + \frac{1}{n}\right) n(a_n - a_{n+1}) ,$$

so that, by Step 2,

$$\liminf_{n\to\infty} n(1 - a_{n+1}/a_n) \geq \frac{1 - \lambda^{-1/\alpha}}{\lambda - 1} \to \frac{1}{\alpha}, \qquad \lambda \downarrow 1 .$$

Similarly, with $\lambda \uparrow 1$,

$$\limsup_{n\to\infty} n(1 - a_{n+1}/a_n) \leq 1/\alpha . \qquad \square$$

Proof of theorem 2. — It suffices to derive (S) with $t = 1$. The proof is the same for any other $t \in \mathrm{T}$. For $0 < s < a_1$ we can choose $\lambda = \lambda(s) \in \mathrm{T}$ such that

$$a_{\lambda+1} < s \leq a_\lambda .$$

Then

$$a_{\lambda+1}^{-1}\Phi^*[\mathrm{F}_1[\mathbf{1} - a_{\lambda+1}\phi] - (\mathbf{1} - a_{\lambda+1}\phi)] \leq \Phi^*[\mathrm{R}_1(\mathbf{1} - s\phi)\phi]$$
$$\leq a_\lambda^{-1}\Phi^*[\mathrm{F}_1[\mathbf{1} - a_\lambda\phi] - (\mathbf{1} - a_\lambda)\phi] .$$

Multiplying through by λ, applying (5.3) and (5.5), then letting $s \downarrow 0$ leads to

(5.6) $$\lambda(s)\Phi^*[\mathrm{R}_1(\mathbf{1} - s\phi)\phi] \to 1/\alpha , \qquad s \to 0 .$$

From the preceding Proposition

$$1/t \sim a_t^\alpha \mathrm{L}_1(t)^{-\alpha} , \qquad t \to \infty .$$

From this, by A.4,

$$1/t \sim a_t^\alpha \mathrm{L}_2(1/a_t) , \qquad t \to \infty ,$$

where L_2 is slowly varying at infinity. By definition of λ, $a_{\lambda(s)}/s \to 1, s \to 0$. Thus, by the uniform convergence property, $\mathrm{L}_2(1/a_\lambda) \sim \mathrm{L}_2(1/s)$, $s \to 0$. Substituting $t = \lambda(s)$ yields

$$1/\lambda(s) \sim s^\alpha \mathrm{L}_2(1/s) , \qquad s \to \infty .$$

Combined with (5.6), this completes the proof. $\square$

6. PROOF OF THEOREM 3

The assumption is that for some $(c_t)_{t\in T} \subset \mathbb{R}_+$

$$(6.1) \qquad \frac{\Phi^*[1 - F_t[e^{-c_t\phi u}]]}{\Phi^*[1 - F_t[0]]} \underset{t\to\infty}{\longrightarrow} 1 - \Psi^*(u), \qquad u > 0,$$

where Ψ^* is the Laplace-Stieltjes transform of a proper d. f., non-degenerate at zero. We shall repeatedly use the fact that the convergence in (6.1) is automatically uniform on bounded u-intervals.

The aim is to prove $c_t/a_t \to \gamma$ where $\gamma > 0$. Since we can assume without loss of generality that c_t is monotone, we may restrict ourselves to the case $T = \mathbb{N}$. In fact, if $T = \mathbb{R}\setminus\{0\}$ and c_t is monotone,

$$\frac{c_{[t]}}{a_{[t]+1}} \cdot \frac{a_{[t]+1}}{a_{[t]}} \leq \frac{c_t}{a_t} \leq \frac{c_{[t]}}{a_{[t]}} \cdot \frac{a_{[t]}}{a_{[t]+1}},$$

so that, if $c_{[t]}/a_{[t]} \to \gamma$, also $c_t/a_t \to \gamma$.

LEMMA 1. — *Suppose for some $m, n : \mathbb{N} \to \mathbb{N}$ with $m(k), n(k) \to \infty$, as $k \to \infty$,*

$$a_{n+m}/a_n \to K, \qquad k \to \infty.$$

Then

$$\Phi^*[1 - F_n[e^{-a_m\phi}]]/a_n \to K, \qquad k \to \infty,$$

and vice versa.

Proof. — It suffices to prove the explicitly stated direction. Clearly $K \in [0, 1]$. First let $K \in (0, 1)$. We are done, if we can show $c_n^{-1}a_m \to \gamma$, where γ is the (unique) solution of $1 - \Psi^*(\gamma) = K$. Suppose, for some subsequence $\{k'\} \subset \{k\}$, $c_{n(k')}^{-1}a_{m(k')} \to \lambda \neq \gamma$. If $\gamma < \lambda \leq \infty$, then, using (2.2), for some $\varepsilon > 0$ and sufficiently large k',

$$-c_{n(k')} \log F_{m(k')}[0] \geq (\gamma + \varepsilon)\phi,$$

so that

$$a_{n(k')}^{-1}a_{n(k')+m(k')} \geq a_{n(k')}\Phi^*[1 - F_{n(k')}[e^{-c_{n(k')}(\gamma+\varepsilon)\phi}]] \to 1 - \Psi^*(\gamma + \varepsilon)$$

and thus $K \geq 1 - \Psi^*(\gamma + \varepsilon)$. On the other hand, $K = 1 - \Psi^*(\gamma)$, so that $\Psi^*(\gamma + \varepsilon) \geq \Psi^*(\gamma)$, which is impossible. Similarly the assumption that $0 \leq \lambda < \gamma$ leads to $\Psi^*(\gamma - \varepsilon) \leq \Psi^*(\gamma)$, which again is impossible. Hence, $\lambda = \gamma$.

Next let $K = 0$. Then it suffices to show that $c_n^{-1}a_m \to 0$. Suppose the latter is not the case. Then, for some $\varepsilon > 0$ and some subsequence

$\{k'\} \subset \{k\}$, $c_{n(k')}^{-1} a_{m(k')} \geq \varepsilon > 0$, and thus, by (2.2), for some $\delta > 0$ and sufficiently large k',

$$- c_{n(k')}^{-1} \log \mathrm{F}_{m(k')}[\mathbf{0}] \geq \delta\phi > 0 \,,$$

that is,

$$a_{n(k')}^{-1} a_{n(k')+m(k')} \geq a_{n(k')}^{-1} \Phi^* [\mathbf{1} - \mathrm{F}_{n(k')}[e^{-c_{n(k')}\delta\phi}]] \to 1 - \Psi^*(\delta) > 0 \,.$$

Since the quotient on the left tends to $\mathrm{K} = 0$, we have a contradiction. Hence, $c_n^{-1} a_m \to 0$.

The case $\mathrm{K} = 1$ is handled similarly. □

Fix $k : \mathbb{N} \to \mathbb{N}$ such that

$$\Phi^* [\mathrm{F}_{k(n)}[\mathbf{0}]] \leq e^{-c_n} \leq \Phi^* [\mathrm{F}_{k(n)+1}[\mathbf{0}]] \,.$$

Then $a_{k(n)} \sim c_n$, $n \to \infty$, and in view of (6.1),

$$a_n^{-1} a_{n+k(n)} \to 1 - \Psi^*(1) := c \,, \qquad n \to \infty \,. \tag{6.2}$$

LEMMA 2. — *There exists a sequence $n' : \mathbb{N} \to \mathbb{N}$ with $n'(n) \to \infty$, as $n \to \infty$, such that $a_{n'(n)}^{-1} c_{n'(n)}$ tends to a finite positive number, as $n \to \infty$.*

Proof. — Define

$$1 - \delta_n := \frac{1 - \| h_{n+k}[\mathbf{0}] \|}{1 + \| h_n[\mathbf{0}] \|} \,, \qquad n \in \mathbb{N} \,.$$

Then, using (2.2) and convexity,

$$\begin{aligned} a_{n+2k} &\geq \Phi^* [\mathbf{1} - \mathrm{F}_k[\mathbf{1} - a_n^{-1} a_{n+k}(1 - \delta_n)(\mathbf{1} - \mathrm{F}_n[\mathbf{0}])]] \\ &\geq a_n^{-1} a_{n+k}(1 - \delta_n) a_{n+k} \,, \end{aligned}$$

so that

$$\begin{aligned} \frac{a_{n+k}}{a_n} \geq \frac{a_{n+2k}}{a_n} &= \frac{a_{n+2k}}{a_{n+k}} \cdot \frac{a_{n+k}}{a_n} \\ &\geq (1 - \delta_n) \left(\frac{a_{n+k}}{a_n} \right)^2 \to c^2 \,, \qquad n \to \infty \,. \end{aligned}$$

Hence, there exist $d \in [c^2, c]$ and $n' : \mathbb{N} \to \mathbb{N}$, $n'(n) \to \infty$, $n \to \infty$, such that

$$a_{n'}^{-1} a_{n'+2k(n')} \to d \,, \qquad n \to \infty \,.$$

Then, by Lemma 1, as $n \to \infty$,

$$a_{n'}^{-1} \Phi^* [\mathbf{1} - \mathrm{F}_{n'}[e^{-a_{2k(n')}\phi}]] \to d \,,$$

and from this, by (6.1),

$$a_{2k(n')} \sim c_{n'} \Psi^{*-1}(1 - d) \sim a_{k(n')} \Psi^{*-1}(1 - d) := a \,,$$

i. e., $a_{k(n')}^{-1} a_{2k(n')} \to a$. From this, again by Lemma 1,

$$a_{k(n')}^{-1} \Phi^* [\mathbf{1} - \mathrm{F}_{k(n')}[e^{-a_{k(n')}\phi}]] \to a \,,$$

which, by (6.1), implies

$$a_{k(n')} \sim c_{k(n')}\Psi^{*-1}(1-a), \qquad n \to \infty . \qquad \square$$

Lemma 2 implies that c_n/a_n can tend neither to 0 nor to ∞. In addition,

$$\begin{aligned} a_n^{-1}\Phi^*[1-e^{-c_n u\phi}] &= a_n^{-1}\Phi^*[\mathrm{M}_t[1-e^{-c_n u\phi}]] \\ &\geq a_n^{-1}\Phi^*[1-\mathrm{F}_n[e^{-c_n u\phi}]] \to 1-\Psi^*(u), \end{aligned}$$

so that

$$\liminf_{n\to\infty} \frac{c_n}{a_n} \geq \frac{1-\Psi^*(u)}{u} > 0 \qquad \forall u .$$

Assume the c_n to be normalized in such a way that

$$\liminf_{n\to\infty} \frac{c_n}{a_n} = 1 ,$$

so that, in particular, $\mathrm{M} := -\Psi^{*\prime}(0+) \leq 1$. If $c_n/a_n \nrightarrow 1$ there exist a $1 < \mathrm{K} \leq \infty$ and $i : \mathbb{N} \to \mathbb{N}$, $i(n) \to \infty$, such that $c_i/a_i \to \mathrm{K}$.

LEMMA 3. — *If there exist* $\mathrm{K} > 1$ *and* $i : \mathbb{N} \to \mathbb{N}$ *with* $i(n) \to \infty, n \to \infty$, *such that* $c_i/a_i \to \mathrm{K}$, $n \to \infty$, *then*

$$\limsup_{n\to\infty} \frac{c_n}{a_n} = \infty .$$

Proof. — Under the hypothesis of the lemma

$$\begin{aligned} a_{2i} &\sim a_i^{-1}\Phi^*[1-\mathrm{F}_i[e^{-a_i\phi}]] \\ &\sim a_i^{-1}\Phi^*[1-\mathrm{F}_i[e^{-c_i\phi/\mathrm{K}}]] \to 1-\Psi^*(\mathrm{K}^{-1}) . \end{aligned}$$

Defining

$$g_1(u) \equiv g(u) := 1-\Psi^*(u/\mathrm{K}) ,$$
$$g_{\mathrm{N}+1}(u) := g(g_{\mathrm{N}}(u)) , \qquad \mathrm{N} \in \mathbb{N} ,$$

it follows by induction that, for every N,

$$a_{(\mathrm{N}+1)i}/a_i \to g_{\mathrm{N}}(1) := g_{\mathrm{N}} , \qquad n \to \infty .$$

Since g is concave, non-decreasing with $g(0) = 0$, and $g'(0+) = \mathrm{M}/\mathrm{K} < 1$, we have

$$g_{\mathrm{N}} \downarrow 0 , \qquad \mathrm{N} \uparrow \infty ,$$

$$\begin{aligned} 1 > \mathrm{M}/k \geq g_{\mathrm{N}+1}/g_{\mathrm{N}} &= \lim_{n\to\infty} (a_{(\mathrm{N}+1)i}/a_{\mathrm{N}i}) \\ &= \lim_{n\to\infty} a_{\mathrm{N}i}^{-1}\Phi^*[1-\mathrm{F}_{\mathrm{N}i}[e^{-a_i\phi}]] \\ &= \lim_{n\to\infty} a_{\mathrm{N}i}^{-1}\Phi^*[1-\mathrm{F}_{\mathrm{N}i}[e^{-c_i\phi/\mathrm{K}}]] . \end{aligned}$$

Using (6.1), this implies

$$c_i/c_{\mathrm{N}i} \to \mathrm{K}\Psi^{*-1}(1-g_{\mathrm{N}+1}/g_{\mathrm{N}}) ,$$

so that

$$\begin{aligned} \frac{c_{\mathrm{N}i}}{a_{\mathrm{N}i}} = \frac{c_{\mathrm{N}i}}{c_i} \cdot \frac{a_i}{a_{\mathrm{N}i}} \cdot \frac{c_i}{a_i} &\underset{n\to\infty}{\sim} [\mathrm{K}\Psi^{*-1}(1-g_{\mathrm{N}+1}/g_{\mathrm{N}})]^{-1} \cdot g_{\mathrm{N}-1}^{-1} \cdot \mathrm{K} \\ &\geq [\Psi^{*-1}(1-\mathrm{M}/\mathrm{K})]^{-1} \cdot g_{\mathrm{N}-1}^{-1} \underset{\mathrm{N}\to\infty}{\longrightarrow} \infty . \qquad \square \end{aligned}$$

LEMMA 4. — *Under the hypothesis of Theorem 3*

$$c_{n+1}/c_n \to 1, \qquad n \to \infty.$$

Proof. — Using (6.2), Lemma 1, and the fact that $a_{n+1}/a_n \to 1$, as $n \to \infty$,

$$\begin{aligned} c &= \lim_{n\to\infty} (a_{n+1+k(n+1)}/a_{n+1}) = \lim_{n\to\infty} (a_{n+k(n+1)}/a_n) \\ &= \lim_{n\to\infty} a_n^{-1}\Phi^*[1 - F_n[e^{-a_{k(n+1)}\phi}]] \\ &= \lim_{n\to\infty} a_n^{-1}\Phi^*[1 - F_n[e^{-c_{n+1}\phi}]], \end{aligned}$$

and from this, by (6.1), $c_{n+1}/c_n \to \Psi^{*-1}(1-c) = 1$. □

LEMMA 5. — *Let $(i(n))_{n\in\mathbb{N}}$ be a subsequence of $(n)_{n\in\mathbb{N}}$ such that $c_i/a_i \to K < \infty$ as $n \to \infty$. If K is sufficiently large, then*

$$\lim_{n\to\infty} \frac{c_{2i}}{a_{2i}} > K.$$

Proof. — In the proof of Lemma 3 we had

$$\lim_{n\to\infty} \frac{a_{(N+1)i}}{a_{Ni}} \leq \frac{M}{K} \leq \frac{1}{K}.$$

Setting N = 2, 3

$$\begin{aligned} K^{-2} \geq g_3/g_1 &= \lim_{n\to\infty} (a_{4i}/a_{2i}) \\ &= \lim_{n\to\infty} a_{2i}^{-1}\Phi^*[1 - F_{2i}[e^{-a_{2i}\phi}]]. \end{aligned}$$

Hence, by (6.1),

$$(6.3) \qquad a_{2i}/c_{2i} \to\; : K^* \geq \Psi^{*-1}(1 - K^{-2}), \qquad n \to \infty.$$

Suppose $K^* > (2K)^{-1}$. Then, by the monotonicity and concavity of Ψ^*,

$$\frac{1 - \Psi^*(K^*)}{(2K)^{-1}} > \frac{1 - \Psi^*((2K)^{-1})}{(2K)^{-1}} > 1 - \Psi^*(1) = c,$$

so that, by (6.3), $c < 2/K$. For sufficiently large K this is impossible. Hence $K^* \leq (2K)^{-1}$ and thus

$$\lim_{n\to\infty} (c_{2i}/a_{2i}) \geq 2K > K. \qquad \square$$

LEMMA 6. — *Suppose $c_n/a_n \to 1$, as $n \to \infty$, and $K > 2/c$. Then there exist subsequences $(j(n))_{n\in\mathbb{N}}$, $(l(n))_{n\in\mathbb{N}}$, and $(m(n))_{n\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that*

$$j(n) \leq l(n) \leq m(n) \leq 2j(n), \qquad n \in \mathbb{N}$$
$$c_j/a_j \to K, \qquad c_l/a_l \to 1, \qquad c_m/a_m \to K, \qquad n \to \infty.$$

Proof. — Set $\gamma_n := c_n/a_n$, $n \in \mathbb{N}$. Since $a_{n+1}/a_n \to 1$ and, by Lemma 4, also $c_{n+1}/c_n \to 1$, as $n \to \infty$, there exists an increasing sequence of integers $M(n)$ such that for $N \geq M(n)$

$$|\gamma_{N+1}/\gamma_N - 1| \leq 2^{-n}/K .$$

Since $\limsup \gamma_n = \infty$, there exist $N(n) \geq M(n)$ such that $\gamma_{N(n)} > K$, and since $\liminf \gamma_n = 1$, there exist $l(n) > N(n)$ such that

$$|\gamma_{l(n)} - 1| \leq 2^{-n}/K < K , \qquad n \in \mathbb{N} .$$

Let $j(n) - 1$ be the largest integer smaller $l(n)$ such that $\gamma_{j(n)-1} \geq K$. Then

$$K > \gamma_{j(n)} = (\gamma_{j(n)}/\gamma_{j(n)-1})\gamma_{j(n)-1} \geq K(1 - 2^{-n}/K) = K - 2^{-n},$$

that is,

$$\gamma_{j(n)} = c_{j(n)}/a_{j(n)} \to K , \qquad n \to \infty .$$

By Lemma 5 this implies $\gamma_{2j(n)} > K$ for sufficiently large n. By definition of $j(n)$, any integer $m \in [j(n), l(n)]$ satisfies $\gamma_m < K$, so that $2j(n) > l(n)$. Now let $m(n)-1$ be the smallest integer larger than $l(n)$ such that $\gamma_{m(n)-1} < K$. Then $l(n) < m(n) \leq 2j(n)$ and

$$K \leq \gamma_{m(n)} = (\gamma_{m(n)}/\gamma_{m(n)-1})\gamma_{m(n)-1} \leq K(1 + 2^{-n}/K) = K + 2^{-n},$$

that is,

$$\gamma_{m(n)} = c_{m(n)}/a_{m(n)} \to K , \qquad n \to \infty . \qquad \square$$

Proof of theorem 3. — Suppose $c_n/a_n \nrightarrow 1$ and let K, $l(n)$, and $m(n)$ be as in Lemma 6. Then, since $a_{n+1}/a_n \to 1$, as $n \to \infty$, and

$$m(n) \leq 2l(n) \leq 2m(n) \qquad \forall n \in \mathbb{N} ,$$

$$\begin{aligned}(a_{2l(n)}/a_{l(n)})^4 &= \prod_{\nu=0}^{l(n)} (a_{l(n)+\nu+1}/a_{l(n)+\nu})^4 \\ &\leq (a_{2l(n)+1}/a_{2l(n)})^{4l(n)} \leq (a_{2l(n)+1}/a_{2l(n)})^{2m(n)} \leq (a_{2m(n)+1}/a_{2m(n)})^{2m(n)} \\ &\leq \prod_{\nu=0}^{2m(n)-1} a_{2m(n)+\nu+1}/a_{2m(n)+\nu} = a_{4m(n)}/a_{2m(n)} .\end{aligned}$$

From $c_{l(n)}/a_{l(n)} \to 1$, as in the proof of Lemma 3, $a_{2l(n)}/a_{l(n)} \to 1 - \Psi^*(1) = c$. From $c_{m(n)}/a_{m(n)} \to K$, on the other hand, $a_{4m(n)}/a_{2m(n)} \to g_3/g_2 \leq K^{-2}$. Hence, $c^4 \leq K^{-2}$. Since we can choose K arbitrarily large, this contradicts $c > 0$. Thus $c_n/a_n \to 1$, as $n \to \infty$. $\square$

APPENDIX

SOME FACTS ABOUT REGULAR VARIATION

We compile here for convenience, and without proofs, useful properties of regularly varying functions which we have used. In addition to the references cited in each instance, there is a thorough treatment with proofs in the book by E. Seneta [*11*].

DEFINITIONS. — A measurable real function $L(x)$ defined on $[a, \infty)$ is called slowly varying at infinity if it is positive and for each $\lambda > 0$

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1 . \tag{1}$$

A measurable real function $R(x)$ is regularly varying at infinity with exponent $\alpha \in (-\infty, \infty)$ if it is of the form $x^{\alpha}L(x)$ for some slowly varying L.

It is enough for (1) to hold for λ in an open interval, and more over the convergence is uniform for λ in any finite interval $[a, b]$, $0 < a < b$, [*11*] (uniform convergence property). We say that $R(x)$ is slowly (regularly) varying at 0 if $R(1/x)$ is slowly (regularly) varying at ∞, and by translation of the origin we can define slow (regular) variation at any point.

A.1. (Lamperti [*8*]). Suppose that $f(x) > 0$ and $f'(x)$ exists for $x \in (0, c)$, $c > 0$. Then

$$\lim_{x \to 0} \frac{xf'(x)}{f(x)} = \alpha \tag{2}$$

implies

$$f(x) = x^{\alpha}L(x) \tag{3}$$

where L is slowly varying at 0. Conversely, if (3) holds and $f'(x)$ exists and is monotone near 0, then (2) is true.

A.2. (Rubin and Vere-Jones [*9*]). Suppose that $f(x)$ is non-decreasing for $x \in (0, c)$ and $\{ \theta_n \}$ is a sequence of positive reals tending to 0 in such a way that $\theta_n/\theta_{n+1} < C < \infty$. If

$$\lim_{\mathbb{N} \ni n \to \infty} \frac{f(\lambda\theta_n)}{f(\theta_n)} = \lambda^{\alpha}$$

for all λ in some bounded interval then $f(x)$ is regularly varying with exponent α.

A.3. (Slack [*13*]). Suppose that $f(t)$ is decreasing to 0 as $t \to \infty$, $f(n+1)/f(n) \to 1$ as $\mathbb{N} \ni n \to \infty$, and for each integer $k \geq 2$

$$\lim_{\mathbb{N} \ni n \to \infty} \frac{f(kn)}{f(n)}$$

exists and is positive. Then $f(t)$ is regularly varying at ∞.

A.4. (Kohlbecker [*7*], Lemma 3). Suppose that L is slowly varying at ∞, $0 < \alpha < 1$, and u is sufficiently large. Then there exists an asymptotically ($u \to \infty$) unique function S_u such that

$$u = (1/S_u)^{1/\alpha}L(1/S_u),$$
$$S_u = u^{-\alpha}L^*(u), \qquad u \to \infty,$$

where L^* is slowly varying at ∞.

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