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A unified approach to constrained mechanical systems as implicit differential equations

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ABSTRACT. – A unified approach to the Lagrangian description of (time-independent) constrained mechanical systems is provided through a technique generating implicit differential equations on T^*Q from 1-forms defined on the total space of any fibre bundle over TQ. © Elsevier, Paris

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RÉSUMÉ. – On propose une approche unifiée de la description Lagrangienne des systèmes mécaniques avec contraintes (sans dépendance explicite de temps). Cette approche est basée sur une technique permettant d'engendrer des équations différentielles implicites sur T^*Q à partir de 1-formes définies sur l'espace global d'un fibré arbitraire de classe C^∞ au-dessus de TQ. © Elsevier, Paris

1. INTRODUCTION

(i) The subject of the present article is basically a new geometric technique for generating a unified global formulation of the various differential equations one encounters in time-independent *Lagrangian* dynamics.

In the local coordinate formalism, the equations we are referring to are the classical Lagrange equations of the motion of a mechanical system acted upon by conservative or nonconservative force fields and subject to linear or nonlinear nonholonomic constraints, all of which arise in implicit form from d'Alembert's principle of virtual work [1].

In the global geometric formalism, the corresponding equations should then be expected to arise in implicit form from a suitable expression of the above principle, their reducibility to explicit form (i.e., to vectors fields on some carrier space) being expected to hold only under special conditions.

As a matter of fact, the vector field approach has extensively been adopted in the literature when dealing geometrically with the various problems of Lagrangian dynamics (see [4,5,7–12,19,20] and references therein), giving rise to a number of ad hoc theories which fail to show a joint origin from a unique mechanical principle.

So our program is to obtain a geometric formulation of such a unifying principle as an implicit differential equation, and then to discuss its reducibility to explicit form.

(ii) The technique we shall adopt arises from the following considerations.

The dynamics of a conservative system described by a (regular or irregular) Lagrangian function L on the tangent bundle TQ of its configuration space Q, is globally governed—as is well known [16, 18]—by the implicit differential equation on the cotangent bundle T^*Q generated by the differential dL, i.e., the submanifold D_L of TT^*Q defined as the image of $\alpha^{-1} \circ dL$ (α being the canonical diffeomorphism of TT^*Q onto T^*TQ).

In a previous paper [2], D_L has been shown to be actually equivalent to the implicit second-order Euler–Lagrange equation on Q (submanifold of the second tangent bundle $T^2Q \subset TTQ$) deduced, through a geometrized variational calculus, from Hamilton's principle.

In a subsequent paper [3], D_L has been embodied in a geometric treatment—concerning the implicit differential equation D_θ on T^*Q generated by any 1-form θ on TQ—which encompassies the dynamics of nonconservative systems.

In the present paper, D_θ will be embodied in a more general geometric treatment—concerning the implicit differential equation D on T^*Q generated by any 1-form Θ on the total space of a fibre bundle ρ over TQ

(Section 3)—which encompassies the dynamics of constrained systems as well.

(iii) The matter proceeds as follows.

Under a suitable hypothesis on a "Legendre" mapping associated with Θ (Section 4), D will be shown to be fully equivalent to an implicit second-order equation E on Q, whose solution curves in Q—the motions of (Q, Θ)—will generally have to obey a distinguished nonholonomic constraint, given in the form of a submanifold C of TQ and containing the whole sequence of "intrinsic" constraint subsets extracted from the equation itself through the integrability algorithm [17] (general version of the algorithm first proposed in [11,12] for irregular Lagrangian systems and then extended in [13,14] to linearly constrained systems).

In the case $\Theta = \rho^*\theta + \Phi$ (Section 5) E will be given a presymplectic formulation, which will prove to be able to describe two different types of constrained systems [22]: in the first type, the additional 1-form Φ plays the merely geometrical role of introducing a constraint submanifold C with the only purpose of selecting, from the motions of a "free" system (Q, θ), those whose velocities belong to C; in the second type, Φ plays the truly mechanical role of introducing a constraint force field, as well as a constraint submanifold C, with the purpose of altering the dynamics of (Q, θ) and keeping the dynamically possible velocities in C.

For $\theta = dL + F$ (Section 6)—L being any regular or irregular Lagrangian function and F (semibasic) any external force field—the above presymplectic equation E will be given a Lagrangian formulation, which will correspond to a principle of virtual work characterizing the dynamically possible motions as those, consistent with the constraint submanifold C, along which the sum of inertial, external and constraint forces annihilates—i.e., does no work in—all the virtual displacements (vertical vectors tangent to TQ).

If the constraint force field is conceived as the annihilator of a given vector bundle of "admissible" virtual displacements (Section 7), then E will correspond to d'Alembert's principle, requiring that, along the dynamically possible motions, the sum of inertial and external forces should annihilate the admissible virtual displacements (in absence of external forces, that will prove to be equivalent to Hölder's variational principle). Necessary and sufficient conditions for such an equation to be reducible to explicit form, will be worked out.

The classical theory [1]—as well as its geometric setting [4,8,9,19, 20,22]—of linear and nonlinear nonholonomic constraints, will then be obtained as a particular case (Section 8).

The above geometric formulation of d'Alembert dynamics will also be illustrated by showing how nicely both classical and relativistic particle dynamanics can be set in the scheme (Section 9).

The coordinate expression of the main points of the theory will finally be displayed (Section 10).

Further developments leading to a Hamilton–Dirac formulation of the theory, will be the object of a forthcoming paper.

2. PRELIMINARIES

Here is a list of the main geometric tools we shall adopt in the sequel.

(i) Let M be a smooth manifold.

The tangent and cotangent bundle projections onto M will be denoted by $\tau_M : TM \to M$ and $\pi_M : T^*M \to M$, respectively.

If $\psi : M \to N$ is a smooth mapping, $T\psi : TM \to TN$ is the tangent mapping of ψ, and $\psi^* : \Lambda N \to \Lambda M$ the pull-back of the exterior algebra of M into that of N by ψ.

The Liouville 1-form on T^*M will be denoted by $\vartheta_M : T^*M \to T^*T^*M : \xi \to \vartheta_M(\xi) := \xi \circ T_\xi \pi_M$.

If $\psi : M \to N$ is a smooth bundle, $V\psi := \{x \in TM \mid T\psi(x) = 0\}$ is the vertical bundle to ψ and $V^0\psi := \{\xi \in T^*M \mid \langle x \mid \xi\rangle = 0, \forall x \in V\psi$ such that $\tau_M(x) = \pi_M(\xi)\}$ is its annihilator. There is a bundle morphism

$$\begin{array}{ccc} V^0\psi & \xrightarrow{\;\varpi_\psi\;} & T^*N \\ {\scriptstyle \pi_M|_{V^0\psi}}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow[\;\psi\;]{} & N \end{array}$$

such that, for any $m \in M$, its restriction $\xi \in V^0_m\psi \mapsto \varpi_\psi(\xi) \in T^*_{\psi(m)}N$—defined by $\xi = \varpi_\psi(\xi) \circ T_m\psi$—is a linear isomorphism.

(ii) Basic *tangent derivations* of ΛM (see [16,18]) are the following.

Let $i_T : \Lambda M \to \Lambda TM$ be the τ_M-derivation of degree-1 which vanishes on $\Lambda^0 M$ and acts on any $\theta \in \Lambda^1 M$ by putting, for any $x \in TM$, $(i_T\theta)(x) := i_x\theta = \langle x|\theta\rangle$ (where inner product i_x is defined by the usual paring $\langle\,|\,\rangle$ between vectors and forms). Hence it follows that i_T acts on any $\omega \in \Lambda^2 M$ by $(i_T\omega)(x) := i_x\omega \circ T_x\tau_M$.

From i_T one also draws a τ_M-derivation of zero degree, given by $d_T := i_T d + d i_T$ (where d denotes the exterior derivative of both ΛM and ΛTM), satisfying, for any $\psi : M \to N$, $d_T\psi^* = (T\psi)^* d_T$.

(iii) The key role in the *geometry of a tangent bundle* $M = TQ$ (see [5,7,10]) is played by the vertical lifting $\nu : TQ \times_Q TQ \to TTQ$, whose restriction ν_v to the fibre $\{v\} \times T_q Q \equiv T_q Q$ over any $v \in TQ$ (with $q := \tau_Q(v)$) maps isomorphically $T_q Q$ onto its own tangent space at v, i.e., $V_v \tau_Q$.

On the one hand, ν transforms the tangent mapping of τ_Q into the almost-tangent structure $S : TTQ \to TTQ$ defined, for any $v \in TQ$, by $S_v := S_{|T_v TQ} := \nu_v \circ T_v \tau_Q$.

On the other hand, ν transforms the identity mapping of TQ into the dilation vector field $\Delta : TQ \to TTQ$ defined, at any $v \in TQ$, by $\Delta(v) := \nu_v(v)$.

Then the vertical tangent bundle $V\tau_Q$ can be described as the set of all vectors $x \in TTQ$ satisfying $S(x) = 0$.

The second tangent bundle $T^2 Q$, defined as the set of all vectors $x \in TTQ$ satisfying $T\tau_Q(x) = \tau_{TQ}(x)$, is also characterized by $S(x) = \Delta(\tau_{TQ}(x))$.

The horizontal (or semibasic) cotangent bundle $V^0 \tau_Q$ is finally characterized as the set of all covectors $\xi \in T^*TQ$ satisfying $i_S \xi := \xi \circ S_{\pi_{TQ}(\xi)} = 0$.

The above adjoint operator $i_S : T^*TQ \to T^*TQ$ also defines a derivation of zero degree of ΛTQ vanishing on $\Lambda^0 TQ$, from which one draws another derivation of degree 1 given by $d_S := i_S d - d i_S$.

Finally recall the canonical diffeomorphism $\alpha : TT^*Q \to T^*TQ$ characterized by $\pi_{TQ} \circ \alpha = T\pi_Q$ and $d_T \vartheta_Q = \alpha^* \vartheta_{TQ}$ (see [18]), whose inverse α^{-1} takes any $\xi \in T^*TQ$ attached at $\pi_{TQ}(\xi) =: v$ onto an image $\alpha^{-1}(\xi) \in TT^*Q$ attached at $\tau_{T^*Q}(\alpha^{-1}(\xi)) = \xi \circ \nu_v$.

3. CONSTRAINED DYNAMICS

The matter has a pattern of increasingly specialized theories, each one sketched in a separate section.

The first concerns a geometric technique which generates—on cotangent bundles—a type of implicit differential equation exhibiting a distinguished nonholonomic constraint.

(i) Let Q be a smooth manifold and Θ a 1-form on the total space Y of a fibre bundle over TQ.

If $\rho : Y \to TQ$ denotes the bundle projection of Y onto TQ, let

$$\Sigma := \{ y \in Y \mid \Theta(y) \in V^0 \rho \}$$

be the *critical subset* associated with Θ, and put

$$\widehat{\Theta} := \varpi_\rho \circ \Theta_{|_\Sigma} : \Sigma \to T^*TQ.$$

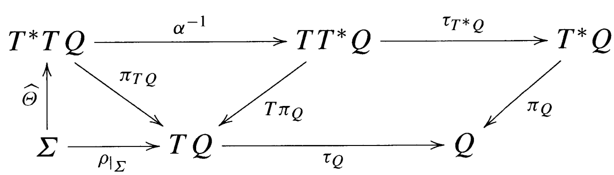
Then define the *evolution operator*

$$\mathcal{E} := \alpha^{-1} \circ \widehat{\Theta} : \Sigma \to TT^*Q$$

and the *Legendre mapping*

$$\mathcal{L} := \tau_{T^*Q} \circ \mathcal{E} : \Sigma \to T^*Q.$$

From the commutativity of diagram



it follows that

$$T\pi_Q \circ \mathcal{E} = \rho_{|\Sigma} \tag{3.1}$$

and

$$\pi_Q \circ \mathcal{L} = \tau_Q \circ \rho_{|\Sigma}. \tag{3.2}$$

(ii) Now consider the implicit first-order differential equation $D \subset TT^*Q$ on T^*Q generated by Θ through

$$D := \mathcal{E}(\Sigma).$$

A smooth curve k in T^*Q is an *integral curve* of D, if its tangent lifting $\dot{k}$ satisfies $\operatorname{Im} \dot{k} \subset D$.

In that case, one has

$$\dot{k} = \mathcal{E} \circ \varsigma. \tag{3.3}$$

for some curve ς in Σ and then, in view of (3.1),

$$T\pi_Q \circ \dot{k} = \rho \circ \varsigma. \tag{3.4}$$

A smooth curve γ in Q will be called a *base integral curve* of D, if $\gamma = \pi_Q \circ k$ for some integral curve k.

In that case, owing to (3.3) and (3.4), one has

$$k = \mathcal{L} \circ \varsigma \tag{3.5}$$

and

$$\dot{\gamma} = \rho \circ \varsigma. \tag{3.6}$$

From (3.6) it follows that each base integral curve γ of D obeys the nonholonomic constraint

$$\operatorname{Im} \dot{\gamma} \subset \rho(\Sigma).$$

Therefore, if the *constraint subset*

$$C := \rho(\Sigma)$$

is a proper subset of TQ, D will be said to be a *constrained dynamics*.

In the sequel, C will be assumed to be a submanifold of TQ.

4. SECOND-ORDER DYNAMICS

If the Legendre mapping $\mathcal{L}$ is projectable by ρ, then D will be shown to be fully equivalent to an implicit second-order equation E.

The sequences of constraint subsets extracted from D and E, respectively, will be related to each other.

(i) Assume $\mathcal{L}$ to be constant on each "fibre" of $\rho_{|\Sigma}$, i.e.,

$$\mathcal{L} = \mathcal{L}_C \circ \rho_{|\Sigma}, \tag{4.1}$$

and

$$\mathcal{L}_C : C \to T^*Q$$

to be a smooth mapping.

In view of (3.2), $\mathcal{L}_C$ will satisfy

$$\pi_Q \circ \mathcal{L}_C = \tau_{Q|C}$$

and then Θ will be said to define a *Legendre morphism* $\mathcal{L}_C$ from $\tau_{Q_{|C}}$ to π_Q.

In that case, owing to (3.5), (3.6) and (4.1), each integral curve k of D is exactly the *Legendre lifting* of the corresponding base integral curve γ, i.e.,

$$k = \mathcal{L}_C \circ \dot{\gamma}, \tag{4.2}$$

which establishes a one-to-one correspondence between the integral curves and the base integral curves of D.

Therefore D behaves like a second-order differential equation on Q, whose actual unknown is γ—smooth curve in Q—and whose solutions are the base integral curves. Such solutions, owing to (4.2), are characterized by

$$\mathrm{Im}(\mathcal{L}_C \circ \dot{\gamma})^{\cdot} \subset D.$$

As far as the study of the solution curves is concerned, D can actually be replaced by a genuine implicit second-order differential equation $E \subset T^2 Q$ on Q, given by

$$E := T^2 Q \cap T\mathcal{L}_C^{-1}(D) \subset TC.$$

Indeed the integral curves of E are exactly the tangent liftings of its own base integral curves, which are in turn characterized by

$$\mathrm{Im}\, \ddot{\gamma} \subset E,$$

i.e.,

$$\mathrm{Im}(T\mathcal{L}_C \circ \ddot{\gamma}) = \mathrm{Im}(\mathcal{L}_C \circ \dot{\gamma})^{\cdot} \subset D.$$

D and E are then to be regarded as equivalent equations, since they admit the same base integral curves (and their respective integral curves are bijectively related to one another by $\mathcal{L}_C$).

The base integral curves will be referred to as the *motions* of (Q, Θ).

(ii) In conformity with the integrability algorithm developed in [17], put, for any positive integer n,

$$K_n := \tau_{T^*Q}(D \cap TK_{n-1})$$

(where $K_0 := T^*Q$ and TK_n denotes the set of all vectors tangent to smooth curves of T^*Q lying in K_n).

$\{K_n\}$ is a decreasing sequence of subsets of T^*Q, all containing the *motion subset* $K^{(i)}$ swept by the integral curves of D.

Each K_n is then an *intrinsic* constraint imposed by equation D on the Legendre liftings of the motions of (Q, Θ).

Now let $\{C_n\}$ be the decreasing sequence of subsets of TQ, all containing the motion subset $C^{(i)}$ of E, likewise extracted from the latter equation.

Each C_n is then an intrinsic constraint imposed by equation E on the tangent liftings of the motions of (Q, Θ).

$\{C_n\}$ turns out to be related to $\{K_n\}$ as follows.

First notice that, from

$$E \subset TC$$

one draws

$$C_n \subset C.$$

Then, since $\mathcal{L}_C(C_1) = \mathcal{L}_C(\tau_{TQ}(E)) = \tau_{T^*Q}(T\mathcal{L}_C(E))$, from $T\mathcal{L}_C(E) \subset D$ it follows that $\mathcal{L}_C(C_1) \subset K_1$.

As a consequence one has $T\mathcal{L}_C(E_1) \subset D_1$, whence $\mathcal{L}_C(C_2) \subset K_2$.

In conclusion

$$\mathcal{L}_C(C_n) \subset K_n.$$

As to $C^{(i)}$ and $K^{(i)}$, since $\mathcal{L}_C$ bijectively relates the integral curves of E to those of D, one gets the stronger result

$$\mathcal{L}_C\big(C^{(i)}\big) = K^{(i)}.$$

5. PRESYMPLECTIC DYNAMICS

If Θ differs from a "free" 1-form $\rho^*\theta$ by a "constraint" term Φ, then E will be given a presymplectic formulation.

The two different types of "external" constraints discussed in [22, p. 20], turn out to be included in our theory.

(i) Let

$$\Theta = \rho^*\theta + \Phi,$$

where θ is any 1-form on TQ and Φ is a 1-form on Y whose critical subset coincides with $\Sigma_0 := \{y \in Y \mid \Phi(y) \in V^0\rho_0,\ \rho_0 := \tau_Q \circ \rho\}$.

The critical subset of Θ is then $\Sigma = \Sigma_0$.

We shall show that Θ defines a Legendre morphism.

Put

$$\widehat{\Phi} := \varpi_\rho \circ \Phi_{|\Sigma}$$

and

$$\widetilde{\Phi} := \varpi_{\rho_0} \circ \Phi_{|\Sigma} = \varpi_{\tau_Q} \circ \varpi_\rho \circ \Phi_{|\Sigma} = \varpi_{\tau_Q} \circ \widehat{\Phi}.$$

For any $y \in \Sigma$, one has

$$\widehat{\Phi}(y) = \widetilde{\Phi}(y) \circ T_{\rho(y)}\tau_Q,$$

whence

$$\widehat{\Phi}(y) \circ \nu_{\rho(y)} = 0. \tag{5.1}$$

For any $y \in \Sigma$, one also has

$$\Theta(y) = \theta\big(\rho(y)\big) \circ T_y\rho + \widehat{\Phi}(y) \circ T_y\rho,$$

whence

$$\widehat{\Theta}(y) = \theta\big(\rho(y)\big) + \widehat{\Phi}(y). \tag{5.2}$$

As a consequence of (5.1) and (5.2),

$$\begin{aligned}\mathcal{L}(y) &= \tau_{T^*Q}\big(\mathcal{E}(y)\big) = \tau_{T^*Q}\big(\alpha^{-1}\big(\widehat{\Theta}(y)\big)\big) = \widehat{\Theta}(y) \circ \nu_{\rho(y)}\\ &= \theta\big(\rho(y)\big) \circ \nu_{\rho(y)} + \widehat{\Phi}(y) \circ \nu_{\rho(y)} = \theta\big(\rho(y)\big) \circ \nu_{\rho(y)}\\ &= \tau_{T^*Q}\big(\alpha^{-1}\big(\theta\big(\rho(y)\big)\big)\big) = \tau_{T^*Q}\big(\mathcal{E}_\theta\big(\rho(y)\big)\big)\\ &= \mathcal{L}_\theta\big(\rho(y)\big),\end{aligned}$$

where $\mathcal{L}_\theta := \tau_{T^*Q} \circ \mathcal{E}_\theta$ is the Legendre morphism from τ_Q to π_Q deduced from the evolution operator $\mathcal{E}_\theta := \alpha^{-1} \circ \theta$ associated with θ.

So we obtain

$$\mathcal{L} = \mathcal{L}_\theta \circ \rho_{|\Sigma},$$

which shows that Θ defines a Legendre morphism given by

$$\mathcal{L}_C = \mathcal{L}_{\theta|C}. \tag{5.3}$$

(ii) In view of (5.3), the second-order equation generated by Θ is

$$E = TC \cap \widehat{E}$$

with

$$\widehat{E} := T^2Q \cap T\mathcal{L}_\theta^{-1}(D).$$

It will be given a presymplectic formulation as follows.

For any $x \in TTQ$, the condition

$$T\mathcal{L}_\theta(x) \in D$$

reads

$$T\mathcal{L}_\theta(x) = \mathcal{E}(y) = \alpha^{-1}\big(\widehat{\Theta}(y)\big)$$

or, owing to (5.2),

$$\alpha\big(T\mathcal{L}_\theta(x)\big) = \theta\big(\rho(y)\big) + \widehat{\Phi}(y)$$

for some $y \in \Sigma$, which—recalling (3.1)—will have to satisfy

$$\rho(y) = T\pi_Q\big(\mathcal{E}(y)\big) = T\pi_Q\big(T\mathcal{L}_\theta(x)\big) = T\tau_Q(x).$$

For any $x \in T^2Q$, the above condition then reads

$$\alpha\big(T\mathcal{L}_\theta(x)\big) - \theta\big(\tau(x)\big) = \widehat{\Phi}(y), \qquad \rho(y) = \tau(x)$$

with $\tau := \tau_{TQ}|_{T^2Q} = T\tau_Q|_{T^2Q}$.

On the other hand—as will be shown in Appendix A(i)—if $x \in T^2Q$, then

$$\alpha\big(T\mathcal{L}_\theta(x)\big) - \theta\big(\tau(x)\big) = \eta_\theta\big(\tau(x)\big) - i_x\omega_\theta, \tag{5.4}$$

where $\omega_\theta := -di_S\theta$ and $\eta_\theta := di_\Delta\theta - \theta$ are the *presymplectic* 2*-form* and the *energy* 1*-form* associated with θ.

For any $x \in T^2Q$, the above condition then reads

$$\eta_\theta\big(\tau(x)\big) - i_x\omega_\theta = \widehat{\Phi}(y), \qquad \rho(y) = \tau(x).$$

Therefore

$$\widehat{E} = \big\{x \in T^2Q \mid \eta_\theta\big(\tau(x)\big) - i_x\omega_\theta = \widehat{\Phi}(y),\ \rho(y) = \tau(x)\big\}. \tag{5.5}$$

Remark that $\tau(\widehat{E}) \subset C$ and then $\widehat{E}$—as a differential equation—is fully equivalent to E.

(iii) Now a few comments are in order.

Let us first consider $\widehat{\Phi} = 0$.

In that case, the dynamics associated with (Q, Θ) reduces to

$$D = \mathcal{E}_\theta(C) \subset D_\theta \quad \text{and} \quad E = TC \cap E_\theta,$$

where $D_\theta := \mathcal{E}_\theta(TQ)$ and

$$\begin{aligned} E_\theta :&= T^2Q \cap T\mathcal{L}_\theta^{-1}(D_\theta) \\ &= \{x \in T^2Q \mid \eta_\theta(\tau(x)) - i_x\omega_\theta = 0\} \end{aligned}$$

represent the dynamics associated with (Q, θ) (see [3]).

Thus, Φ does not actually affect the dynamics of (Q, θ), its only role being that of introducing a "geometric" element—the constraint subset C—with the aim, if $C \neq TQ$, of selecting, from the motions of (Q, θ) (i.e., the base integral curves of D_θ or E_θ), those whose tangent liftings lie in C. If $C = TQ$, one has $D = D_\theta$, $E = E_\theta$ and then (Q, Θ) is indistinguishable from (Q, θ)—which will be called a *free* system, as no constraint is *a priori* imposed on its motions.

Let us now turn to $\widehat{\Phi} \neq 0$.

In that case, Φ does affect the dynamics of the free system, its role being that of introducing not only a constraint subset C, but also a new "mechanical" element—the *constraint force field* $\widehat{\Phi}$ (or $\widetilde{\Phi}$)—altering the dynamics of the free system as is shown by Eq. (5.5). If $C \neq TQ$, $\widehat{\Phi}$ is just the force one empirically expects is needed in order to maintain the constraint.

6. LAGRANGE DYNAMICS

If θ differs from a "Lagrangian" 1-form dL by a "force field" F, then E will be given a Lagrangian formulation corresponding to a principle of virtual work.

(i) Let

$$\Theta = \rho^*\theta + \Phi$$

be as in Section 5 and assume $i_S\theta$ to be d_S-exact, i.e.,

$$i_S\theta = d_S L, \tag{6.1}$$

for some smooth *Lagrangian function* L on TQ. That amounts to saying $\theta = dL + F$ with $i_S F = 0$.

The above splitting of θ is determined up to a gauge choice given by $(L, F) \mapsto (L - \tau_Q^* U,\ F + \tau_Q^*\, dU)$, U being an arbitrary smooth function on Q. As a consequence, when we refer to a gauge (L, F), F—if nonnull—will be assumed to be nonexact.

With reference to a gauge (L, F), equation E can be reformulated as follows.

From

$$\omega_\theta = \omega_L, \qquad \eta_\theta = dE_L - F$$

with

$$\omega_L := -dd_S L, \qquad E_L := \Delta L - L$$

and putting

$$[L] : T^2Q \to V^0\tau_Q : x \mapsto dE_L\big(\tau(x)\big) - i_x\omega_L,$$

one obtains

$$\widehat{E} = \big\{x \in T^2Q \mid [L](x) = F\big(\tau(x)\big) + \widehat{\Phi}(y),\ \rho(y) = \tau(x)\big\}.$$

If we put

$$\widetilde{[L]} := \varpi_{\tau_Q} \circ [L] \ :\ T^2Q \to T^*Q$$

and

$$\widetilde{F} := \varpi_{\tau_Q} \circ F \ :\ TQ \to T^*Q,$$

we can write

$$\widehat{E} = \big\{x \in T^2Q \mid \widetilde{[L]}(x) = \widetilde{F}\big(\tau(x)\big) + \widetilde{\Phi}(y),\ \rho(y) = \tau(x)\big\}.$$

(ii) The base integral curves of E—or $\widehat{E}$—are characterized by

$$\widetilde{[L]} \circ \ddot{\gamma} = \widetilde{F} \circ \dot{\gamma} + \widetilde{\Phi} \circ \varsigma$$

for some curve ς in Σ satisfying $\rho \circ \varsigma = \ddot{\gamma}$.

That is the law of the motion of a constrained mechanical system described by a Lagrangian L and possibly acted upon by an *external force field* $\widetilde{F}$.

If $\widetilde{\Phi} = 0$, the motions of the system are then conceived as those which, obeying the constraint $\operatorname{Im}\dot{\gamma} \subset C$, possibly deviate from the *comparison* or *inertial motions*—characterized by *Euler–Lagrange equation* $\widetilde{[L]} \circ \ddot{\gamma} = 0$ or, equivalently, by Hamilton's variational principle (see [2])—in that their *inertial force* $-\widetilde{[L]} \circ \ddot{\gamma}$ is balanced by a *nonconservative* force $\widetilde{F} \circ \dot{\gamma}$ (see [3]).

Remark that any other admissible gauge choice would lead to different specifications of the (conventional) notions of inertia and force, without altering of course the (observable) class of motions.

If $\widetilde{\Phi} \neq 0$, the motions of the system are those maintained on the constraint subset C by any possible constraint force $\widetilde{\Phi} \circ \varsigma$ which, coupled to $\widetilde{F} \circ \dot{\gamma}$, still balances $-\widetilde{[L]} \circ \ddot{\gamma}$.

The above law, owing to the covector nature of the force fields therein involved, just corresponds to the *principle of virtual work*

$$\langle u \mid \widetilde{\Phi} \circ s(t) + \widetilde{F} \circ \dot{\gamma}(t) - \widetilde{[L]} \circ \ddot{\gamma}(t) \rangle = 0, \quad \forall u \in V_{\dot{\gamma}(t)} \tau_Q \equiv T_{\gamma(t)} Q,$$

requiring that—at any instant t of the time interval where γ is defined—the total work done by the inertial, external and constraint forces should vanish in all the *virtual displacements* (the vector bundle of such displacements being given by $V\tau_Q$, identified by the vertical lifting ν with $TQ \times_Q TQ$).

7. D'ALEMBERT DYNAMICS

If the constraint subset C is the set of zeros of some functional constraints and the constraint force field $\widetilde{\Phi}$ is the set of annihilators of a vector subbundle of "admissible" virtual displacements, then E will be shown to correspond to d'Alembert's principle (and, in absence of external forces, to Hölder's variational principle).

Conditions for such an equation to be reducible to explicit form, will be worked out.

(i) Put

$$Y = TQ \times \mathbb{R}^m$$

and denote the projection of Y onto TQ (respectively, $\mathbb{R}^m$) by ρ (respectively, σ). From $\pi_Y = \pi_{TQ} \times \pi_{\mathbb{R}^m}$ (where $\pi_{\mathbb{R}^m}$ is the projection of $\mathbb{R}^m \times \mathbb{R}^m$ onto the first factor), one obtains that a 1-form Φ on Y splits up into a couple (Φ_1, Φ_2) formed by a section Φ_1 of π_{TQ} along ρ and a section Φ_2 of $\pi_{\mathbb{R}^m}$ along σ.

Now let

$$\Theta = \rho^* \theta + \Phi,$$

θ being a 1-form on TQ of type (6.1) and $\Phi = (\Phi_1, \Phi_2)$ a 1-form on Y defined, for any $(v, \lambda) \in Y$, by

$$\Phi_1(v, \lambda) := \lambda A(v) \in T_v^* TQ,$$
$$\Phi_2(v, \lambda) := \big(\lambda, B(v)\big) \in T_\lambda^* \mathbb{R}^m \equiv \{\lambda\} \times \mathbb{R}^m,$$

where $A = (A^a)$ is an m-tuple of linearly independent—if nonnul—semibasic 1-forms on TQ (consequently $\lambda A := \lambda_a A^a$) and $B = (B_b)$ is an m-tuple of independent real-valued smooth functions on TQ.

Θ is a 1-form of the type taken into consideration in Section 6.

The critical subset is $\Sigma = B^{-1}(0) \times \mathbb{R}^m$ and then the constraint subset is $C = B^{-1}(0)$.

B then plays the role of a system of *functional constraints*.

For any $(v, \lambda) \in \Sigma$, one has

$$\widehat{\Phi}(v, \lambda) = \lambda A(v)$$

and then

$$\widetilde{\Phi}(v, \lambda) = \lambda \widetilde{A}(v),$$

where $\widetilde{A} := \varpi_{\tau_Q} \circ A := (\varpi_{\tau_Q} \circ A^a)$.

Thus, for each $v \in C$, the constraint forces $\widetilde{\Phi}(v, \lambda)$ sweep the subspace

$$\{\lambda \widetilde{A}(v)\}_{\lambda \in \mathbb{R}^m} \subset T^*_{\tau_Q(v)} Q$$

which is the annihilator of

$$\operatorname{Ker} \widetilde{A}(v) \subset T_{\tau_Q(v)} Q.$$

The vector subbundle $\mathcal{A} \subset V\tau_Q$ image through ν of

$$\bigcup_{v \in TQ} (\{v\} \times \operatorname{Ker} \widetilde{A}(v)) \subset TQ \times_Q TQ$$

will be called the bundle of the *admissible* virtual displacements.

With reference to a gauge (L, F), the equation $E = TC \cap \widehat{E}$ is expressed by

$$\widehat{E} = \{x \in T^2Q \mid \widetilde{[L]}(x) = \widetilde{F}(\tau(x)) + \lambda \widetilde{A}(\tau(x)),\ B(\tau(x)) = 0,\ \lambda \in \mathbb{R}^m\}$$

(where the real numbers $\lambda = (\lambda_a)$ are known as *Lagrange multipliers*).

The base integral curves of E are then characterized by

$$B \circ \dot{\gamma} = 0 \tag{7.1}$$

and

$$\widetilde{[L]} \circ \ddot{\gamma} = \widetilde{F} \circ \dot{\gamma} + \lambda \widetilde{A} \circ \dot{\gamma} \tag{7.2}$$

for some curve λ in $\mathbb{R}^m$.

The latter condition is equivalent to d'Alembert's principle

$$\langle u \mid \widetilde{F} \circ \dot{\gamma}(t) - \widetilde{[L]} \circ \ddot{\gamma}(t) \rangle = 0, \qquad \forall u \in \mathcal{A}_{\dot{\gamma}(t)} \equiv \operatorname{Ker} \widetilde{A}(\dot{\gamma}(t)), \tag{7.3}$$

requiring that—at any instant t of the time interval where γ is defined—the total work done by the inertial and external forces should vanish in all the admissible virtual displacements.

(ii) In the case $F = 0$, d'Alembert's principle can be given a variational interpretation (see also [2]).

Let $\mathcal{I}_{t_1}^{t_2}(\gamma) = \int_{t_1}^{t_2} L \circ \dot{\gamma}\, \mathrm{d}t$ be the action integral of a smooth curve $\gamma : I \subset \mathbb{R} \to Q$ in $[t_1, t_2] \subset I$.

If $\{\varphi_s\}_{s \in \mathbb{R}}$ is a one-parameter group of transformations of Q, the first variation of the action $s \mapsto \mathcal{I}_{t_1}^{t_2}(\varphi_s \circ \gamma)$—starting from $\varphi_0 \circ \gamma = \gamma$—is given by

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathcal{I}_{t_1}^{t_2}(\varphi_s \circ \gamma)\bigg|_{s=0} = \int_{t_1}^{t_2} \langle Z \mid dL \rangle \circ \dot{\gamma}\, \mathrm{d}t$$

(Z being the tangent lifting to TQ of the infinitesimal generator ζ of the group).

Recall that—as Z and ζ are τ_Q-related to each other—

$$\langle \zeta \circ \gamma \mid \widetilde{[L]} \circ \ddot{\gamma} \rangle = \langle Z \circ \dot{\gamma} \mid [L] \circ \ddot{\gamma} \rangle \tag{7.4}$$

and that—as will be shown in Appendix A(ii)—

$$\langle Z \circ \dot{\gamma} | [L] \circ \ddot{\gamma} \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \langle SZ \circ \dot{\gamma} \mid dL \circ \dot{\gamma} \rangle - \langle Z \circ \dot{\gamma} \mid dL \circ \dot{\gamma} \rangle. \tag{7.5}$$

Moreover, for each $t \in I$,

$$(SZ)_{\dot{\gamma}(t)} = \nu_{\dot{\gamma}(t)} \circ T_{\dot{\gamma}(t)} \tau_Q (Z_{\dot{\gamma}(t)}) = \nu_{\dot{\gamma}(t)}(\zeta_{\gamma(t)}).$$

So, if $\zeta_{\gamma(t_1)} = 0$ and $\zeta_{\gamma(t_2)} = 0$, one has

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathcal{I}_{t_1}^{t_2}(\varphi_s \circ \gamma)\bigg|_{s=0} = -\int_{t_1}^{t_2} \langle \zeta \circ \gamma \mid \widetilde{[L]} \circ \ddot{\gamma} \rangle\, \mathrm{d}t.$$

As a consequence, γ satisfies *Hölder's principle*, requiring

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathcal{I}_{t_1}^{t_2}(\varphi_s \circ \gamma)\bigg|_{s=0} = 0$$

for each $[t_1, t_2] \subset I$ and each group $\{\varphi_s\}_{s\in\mathbb{R}}$ with infinitesimal generator ζ such that $\zeta_{\gamma(t_i)} = 0$ for $i = 1, 2$ and $\zeta_{\gamma(t)} \in \operatorname{Ker} \widetilde{A}(\dot\gamma(t))$ for all $t \in I$ (see, e.g., [1, p. 18] and [22, p. 22]), iff

$$\widetilde{[L]} \circ \ddot\gamma = \lambda \widetilde{A} \circ \dot\gamma$$

for some curve λ of $\mathbb{R}^m$.

(iii) Clearly, as far as the above geometric formulation of d'Alembert principle is concerned, any special assumption on the Lagrangian function—such as regularity, requiring that the vector bundle morphism $^\flat : TTQ \to T^*TQ : x \mapsto i_x\omega_L$ should admit an inverse morphism $^\sharp$—would be totally irrelevant.

Regularity will only play a role, when reducibility to explicit form of the dynamics is dealt with, as will now be shown.

$E \subset TC$ will be said to be reducible to explicit or normal form on C, if there exists a (unique) vector field Γ on C such that

$$E = \operatorname{Im} \Gamma.$$

Remarkable is that not even in the case of L being a regular Lagrangian is the equation E reducible to explicit form, unless suitable conditions on (A, B) are fulfilled.

Firstly we shall deal with the case of (A, B) being a purely geometric constraint, i.e., $A = 0$.

In that case, we have

$$E = TC \cap E_\theta = TC \cap \operatorname{Im} X \quad \text{with } X := {}^\sharp dE_L - {}^\sharp F.$$

As a consequence, if—and only if—the Lie derivatives $XB := (XB_b)$ vanish on C (i.e., X is tangent to C), one has $E = \operatorname{Im}\Gamma$, Γ being the vector field on C given by $\Gamma := X_{|C}$.

Now we shall deal with the case of (A, B) being a mechanical constraint, i.e., $A \neq 0$.

In that case, E can be described as the set of all vectors $x \in T^2Q$ attached at $v := \tau(x) \in C$ such that, for some $\lambda \in \mathbb{R}^m$,

$$x = (X - \lambda Z)(v) \quad \text{and} \quad x(B) = (XB - \lambda ZB)(v) = 0$$

with $Z := (Z^a) := ({}^\sharp A^a)$ and $ZB := (Z^a B_b)$.

As a consequence, if—and only if—the $m \times m$ function matrix $\mathcal{M} := ZB$ is nonsingular at every point of C, one has $E = \operatorname{Im}\Gamma$, Γ being the vector field on C given by $\Gamma := X_{|C} - \Lambda Z_{|C}$ with $\Lambda : C \to \mathbb{R}^m : v \mapsto (XB(v))(\mathcal{M}(v))^{-1}$.

Finally remark by contrast that, even in the case of an irregular Lagrangian, the equation E might be reducible to explicit form (an example—relativistic particle dynamics—will be shown in Section 9(ii)).

8. NONHOLONOMIC DYNAMICS

The classical approach to nonholonomic functional constraints will now be obtained as a particular case.

Holonomic constraints will also be included.

(i) The main point underlying the theory in Section 7(i) (and shared by Dazord's approach [6]) is that assigning the constraints means assigning the bundle $\mathcal{A}$ of admissible virtual displacements and the manifold C constraining the motions—described by A and B, respectively.

Special theories can then be obtained by assuming suitable links between A and B.

The classical theory (see, e.g., Arnold et al. [1, pp. 16–17]) can be deduced through the assumption that, to nonholonomic functional constraints B with linearly independent fibre derivatives $d_S B$, there correspond admissible virtual displacements defined by

$$A = d_S B.$$

From the above assumption it follows that, at any $v \in T_q Q \cap C$, the space $\mathcal{A}_v$ of admissible virtual displacements—i.e., the image through ν_v of Ker $\tilde{A}(v)$, with $\tilde{A}(v) = d_v B \circ \nu_v$—is

$$V_v \tau_Q \cap \operatorname{Ker} d_v B = T_v(T_q Q) \cap T_v C,$$

whose elements actually correspond to the mechanical notion of infinitesimal displacements consistent with the constraints (see Whittaker [21, p. 215] and Marle [15, p. 299]).

In the standard case of linear constraints $B = i_T \beta + \tau_Q^* g$—$\beta$ and g being m-tuples of linearly independent 1-forms and real-valued smooth functions on Q, respectively—one gets (see Woodhouse [22, pp. 20–23])

$$A = \tau_Q^* \beta.$$

In particular,

$$A = \tau_Q^* df$$

—f being a submersion of Q onto $\mathbb{R}^m$—will correspond to integrable constraints $B = d_T f$ which only allow of motions lying on the leaves of f.

In any case, under the hypotheses of regular Lagrangian and normalizable dynamics, the explicit equation obtained in Section 7(iii) corresponds to the main results displayed in the current literature (see, e.g., Vershik and Fadeev [19,20], Cariñena and Rañada [4], de Léon and de Diego [8,9]).

(ii) Holonomic constraints, which allow of motions lying on just one leaf of a submersion $f\colon Q \to \mathbb{R}^m$, correspond to $A = \tau_Q^* df$, $B = \tau_Q^* f$.[1]

In that case, putting $Q_0 := f^{-1}(0)$ and recalling that $T^2Q \cap TTQ_0 = T^2Q_0$, equation $\widehat{E}$ turns out to be equivalent to

$$E_0 := \{x \in T^2Q_0 \mid [L](x) = F(\tau(x)) + (\lambda \tau_Q^* df)(\tau(x)),\ \lambda \in \mathbb{R}^m\}.$$

Recall that (for any $x \in T^2Q$ and putting $v := \tau(x)$, $q := \tau_Q(v)$) one has $[L](x) - F(v) \in V_v^0 \tau_Q$ and then $[L](x) - F(v) = p \circ T_v \tau_Q$ with $p \in T_q^* Q$.

Therefore, the condition

$$[L](x) - F(v) = (\lambda \tau_Q^* df)(v) = (\lambda\, df(q)) \circ T_v \tau_Q$$

reads $p = \lambda\, df(q)$, or $p_{|T_q Q_0} = 0$ that is, $p \circ T_v(i \circ \tau_{Q_0}) = 0$ (with $i : Q_0 \hookrightarrow Q$).

Now consider $\theta_0 := j^*\theta$ (with $j := Ti : TQ_0 \hookrightarrow TQ$), i.e., $\theta_0 = dL_0 + F_0$, where $L_0 := j^*L$ and $F_0 := j^*F$.

For any $x \in T^2Q_0$, one has

$$\begin{aligned}[L_0](x) - F_0(v) &= ([L](x) - F(v)) \circ T_v j = p \circ T_v \tau_Q \circ T_v j \\ &= p \circ T_v(\tau_Q \circ j) = p \circ T_v(i \circ \tau_{Q_0})\end{aligned}$$

and then the above condition reads

$$[L_0](x) - F_0(v) = 0.$$

In conclusion we obtain

$$E_0 = \{x \in T^2Q_0 \mid [L_0](x) = F_0(\tau(x))\}$$

which is nothing but the second-order equation on Q_0 generated by θ_0.

---

[1] If $\theta = dL$, that amounts to saying $\Theta = dG$, where $G : TQ \times \mathbb{R}^m \to \mathbb{R} : (v, \lambda) \mapsto L(v) + \lambda f(\tau_Q(v))$ is a Morse family of functions on the fibres of ρ.

9. PARTICLE DYNAMICS

Classical and relativistic particle dynamics will now be framed in the above geometric formulation of d'Alembert dynamics.

(i) Let $Q := \mathcal{K}^n$ be the manifold of all the configurations of n particles in a frame of reference $\mathcal{K}$ (affine space modelled on a 3-dimensional vector space $\mathcal{K}'$ carrying a Euclidean inner product $\cdot$).

For each $i = 1, \ldots, n$, $r_i : Q \to \mathcal{K}$ will denote the projection of Q onto its ith factor and $r_i' : TQ \to \mathcal{K}'$ will be defined by putting, for each $q \in Q$, $r'_{i|T_qQ} := d_q r_i : T_q Q \to T_{r_i(q)}\mathcal{K} \equiv \mathcal{K}'$ (tangent mapping of r_i at q); moreover dr_i and dr_i' will denote the maps which take each $q \in Q$ to $d_q r_i$ and each $v \in TQ$ to $d_v r_i'$, respectively.

Then let $\Theta = \rho^*(dL + F) + \Phi$ be a 1-form on $TQ \times \mathbb{R}^m$ defined as follows.

L is the kinetic energy of the masses $m_i > 0$ associated with the particles, i.e.,

$$L := \frac{1}{2}\sum_{i=1}^{n} m_i r_i' \cdot r_i' : TQ \to \mathbb{R}.$$

F is the virtual work of the vector force fields $F_i : TQ \to \mathcal{K}'$ acting in $\mathcal{K}$ on the particles, i.e.,

$$\widetilde{F} := \sum_{i=1}^{n} F_i \cdot (dr_i \circ \tau_Q) : TQ \to T^*Q.$$

Φ corresponds to a couple (A, B) given as in Section 7 (or, in particular, Section 8).

The implicit second-order equation E generated by Θ is expressed, in terms of its base integral curves, by (7.1) and (7.2). The latter (i.e., d'Alembert's principle (7.3)) now reads as follows.

Owing to (7.4) and (7.5), for any vector field ζ on Q, we have

$$\begin{aligned}\langle \zeta \circ \gamma | \widetilde{[L]} \circ \ddot{\gamma} \rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle SZ \circ \dot{\gamma} \Big| \sum_{i=1}^{n} m_i (r_i' \circ \dot{\gamma}) \cdot (dr_i' \circ \dot{\gamma}) \right\rangle \\ &\quad - \left\langle Z \circ \dot{\gamma} \Big| \sum_{i=1}^{n} m_i (r_i' \circ \dot{\gamma}) \cdot (dr_i' \circ \dot{\gamma}) \right\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} m_i (r_i' \circ \dot{\gamma}) \cdot \langle SZ \circ \dot{\gamma} \mid dr_i' \circ \dot{\gamma} \rangle \right)\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{n} m_i \left(r_i' \circ \dot{\gamma}\right) \cdot \left\langle Z \circ \dot{\gamma} \mid dr_i' \circ \dot{\gamma} \right\rangle \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} m_i \frac{\mathrm{d}}{\mathrm{d}t}(r_i \circ \gamma) \cdot \langle \zeta \circ \gamma \mid dr_i \circ \gamma \rangle \right) \\
&\quad - \sum_{i=1}^{n} m_i \frac{\mathrm{d}}{\mathrm{d}t}(r_i \circ \gamma) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \langle \zeta \circ \gamma \mid dr_i \circ \gamma \rangle \\
&= \sum_{i=1}^{n} m_i \frac{\mathrm{d}^2}{\mathrm{d}t^2}(r_i \circ \gamma) \cdot \langle \zeta \circ \gamma \mid dr_i \circ \gamma \rangle \\
&= \left\langle \zeta \circ \gamma \,\middle|\, \sum_{i=1}^{n} m_i \frac{\mathrm{d}^2}{\mathrm{d}t^2}(r_i \circ \gamma) \cdot (dr_i \circ \gamma) \right\rangle,
\end{aligned}
$$

i.e., owing to the arbitrariness of ζ,

$$
\widetilde{[L]} \circ \ddot{\gamma} = \sum_{i=1}^{n} m_i \frac{\mathrm{d}^2}{\mathrm{d}t^2}(r_i \circ \gamma) \cdot (dr_i \circ \gamma).
$$

On the other hand, we also have

$$
\tilde{F} \circ \dot{\gamma} = \sum_{i=1}^{n} \left(F_i \circ \dot{\gamma}\right) \cdot (dr_i \circ \gamma).
$$

As a consequence, if we put δr_i to denote the map which takes each $v \in T_q Q$ to $\delta_v r_i := d_q r_i|_{\operatorname{Ker} \tilde{A}(v)}$, from (7.2) or (7.3) we reobtain the traditional expression of d'Alembert's principle

$$
\sum_{i=1}^{n} \left(F_i \circ \dot{\gamma} - m_i \frac{\mathrm{d}^2}{\mathrm{d}t^2}(r_i \circ \gamma) \right) \cdot \left(\delta r_i \circ \dot{\gamma}\right) = 0.
$$

(ii) Now let Q be the space-time manifold of General Relativity (endowed with a Lorentz metric describing a gravitational field).

The dynamically possible world lines—parametrized by proper time—of a test particle moving in the gravitational field and acted upon by an external (e.g., electromagnetic) force field, are the base integral curves of the implicit equation generated by the 1-form

$$
\Theta_1 := \rho^*(dK + \Psi) + \Phi
$$

defined on $Y := TQ \times \mathbb{R}$ as follows.

K is the regular Lagrangian associated with the Lorentz metric $\langle\ ,\ \rangle$ of Q, i.e.,

$$K := \tfrac{1}{2}\langle \mathrm{id}_{TQ}, \mathrm{id}_{TQ}\rangle.$$

Ψ is any semibasic 1-form on TQ, describing an external force field (e.g., $\Psi := i_T\mathbf{F}$ describes the action of an electromagnetic field $\mathbf{F} \in \Lambda_2 Q$ on a test particle of unit proper mass and unit electric charge).

Φ is the 1-form on Y characterized by the couple (A, B) given by $A = 0$, $B = 2K - 1$.

The second-order equation generated by Θ_1 is

$$E_1 = TC \cap \mathrm{Im}\, X$$

with $C = \{v \in TQ \mid \langle v, v\rangle = 1\}$ and $X = {}^\sharp dK - {}^\sharp\Psi$.

The base integral curves of E_1, characterized by

$$\langle \dot\gamma, \dot\gamma\rangle = 1 \quad \text{and} \quad \ddot\gamma = X \circ \dot\gamma,$$

are all of the base integral curves of X which admit normalized timelike tangent liftings.

X is tangent to C iff the function

$$\tfrac{1}{2}XB = XK = \langle X \mid dK\rangle = \langle X \mid i_X\omega_K + \Psi\rangle = \langle X \mid \Psi\rangle = \langle T\tau_Q \circ X \mid \widetilde{\Psi}\rangle$$
$$= \langle \mathrm{id}_{TQ} \mid \widetilde{\Psi}\rangle$$

—i.e., the *power* $\Pi_\Psi := \langle \mathrm{id}_{TQ} \mid \widetilde{\Psi}\rangle$ of Ψ—vanishes on C (that is the case, e.g., if Ψ is an electromagnetic force field, since—for any $v \in TQ$—

$$\Pi_\Psi(v) = \langle v \mid \widetilde{\Psi}(v)\rangle = \langle v \mid i_v\mathbf{F}\rangle = 0).$$

If—and only if—such a condition holds, the equation is reducible on C to explicit form

$$E_1 = \mathrm{Im}\, X_{|C}.$$

It is interesting to observe that the explicit equation $X_{|C}$ on C also arises from an irregular Lagrangian.

Indeed we shall show that it is generated by the 1-form

$$\Theta_2 := \rho^*(dL + F) + \Phi$$

defined on $\mathcal{T} \times \mathbb{R} \subset Y$—where $\mathcal{T} := \{v \in TQ \mid \langle v, v\rangle > 0\}$—by putting $L := \sqrt{2K}$ (irregular Lagrangian associated with the Lorentz metric) and

$$F := \frac{1}{\sqrt{2K}}\Psi$$

(with $\Pi_\Psi = 0$).

The second-order equation generated by Θ_2 is

$$E_2 = TC \cap \widehat{E}$$

with

$$\widehat{E} := \{x \in T^2Q \mid i_x\omega_L = dE_L(v) - F(v),\ v := \tau(x) \in C\}.$$

Recalling that $\Delta K = 2K$ and $d_S K = -i_\Delta \omega_K$, one obtains

$$\omega_L = \frac{1}{\sqrt{2K}}\left(\omega_K - \frac{1}{2K}\, dK \wedge i_\Delta \omega_K\right), \qquad E_L = 0.$$

As a consequence, the condition characterizing $\hat{E}$ reads

$$i_x\omega_K = i_{X(v)}\omega_K + g(x) i_{\Delta(v)}\omega_K$$

i.e.,

$$x = X(v) + g(x)\Delta(v)$$

with

$$g(x) := \frac{1}{2K(v)}\, d_T K(x).$$

On the other hand (recalling that $XK = \Pi_\Psi = 0$), for any

$$x = X(v) + a\Delta(v), \quad a \in \mathbb{R},$$

one has

$$g(x) = \left(\frac{1}{2K}\, XK + \frac{a}{2K}\, \Delta K\right)(v) = a.$$

So

$$\widehat{E} = \{x \in T^2Q \mid x = X(v) + a\Delta(v),\ a \in \mathbb{R},\ v := \tau(x) \in C\}.$$

Moreover $x = X(v) + a\Delta(v) \in TC$ iff $x(B) = 0$, i.e., $a = 0$.
Hence

$$E_2 = \mathrm{Im}\, X_{|C}.$$

10. COORDINATE EXPRESSION

It is instructive to describe the main points of our construction in local coordinates.

In our (standard) coordinate notation, indexes will be omitted.

(i) On Y—locally or globally isomorphic to $TQ \times \mathbb{R}^m$—we shall adopt bundle coordinates (q, v, λ), where ρ has coordinate expression given by

$$\rho : (q, v, \lambda) \in Y \to (q, v) \in TQ.$$

In such coordinates, from

$$\begin{aligned} \Theta &: (q, v, \lambda) \in Y \\ &\to \big(q, v, \lambda / \Theta_q(q, v, \lambda), \Theta_v(q, v, \lambda), \Theta_\lambda(q, v, \lambda)\big) \in T^*Y \end{aligned}$$

we deduce that $y \equiv (q, v, \lambda) \in \Sigma$, i.e., $\Theta(y) \in V^0\rho$, iff

$$\Theta_\lambda(q, v, \lambda) = 0,$$

and then

$$\begin{aligned} \widehat{\Theta} := \varpi_\rho \circ \Theta_{|\Sigma} &: (q, v, \lambda) \in \Sigma \\ &\to \big(q, v / \Theta_q(q, v, \lambda), \Theta_v(q, v, \lambda)\big) \in T^*TQ. \end{aligned}$$

Hence, recalling that $\alpha^{-1} : (q, v/r, s) \in T^*TQ \to (q, s/v, r) \in TT^*Q$ (see [18]), we obtain

$$\mathcal{E} := \alpha^{-1} \circ \widehat{\Theta} : (q, v, \lambda) \in \Sigma \to \big(q, \Theta_v(q, v, \lambda)/v, \Theta_q(q, v, \lambda)\big) \in TT^*Q.$$

So, for any $z \equiv (q, p/\dot{q}, \dot{p}) \in TT^*Q$, we have that $z \in D := \mathrm{Im}\,\mathcal{E}$ iff, for some $\lambda \in \mathbb{R}^m$,

$$\Theta_\lambda(q, \dot{q}, \lambda) = 0 \tag{10.1}$$

and

$$p = \Theta_v(q, \dot{q}, \lambda), \qquad \dot{p} = \Theta_q(q, \dot{q}, \lambda). \tag{10.2}$$

Now let $k \equiv (q, p)$—with $q = q(t)$, $p = p(t)$—be a smooth curve in the given coordinate domain of T^*Q, and $\dot{k} \equiv (\dot{q}, \dot{p})$—with $\dot{q} = \mathrm{d}q/\mathrm{d}t$, $\dot{p} = \mathrm{d}p/\mathrm{d}t$—its tangent lifting.

From the above coordinate description of D, it follows that k is an integral curve of D, i.e., $\operatorname{Im}\dot{k} \subset D$, iff, for some $\lambda = \lambda(t) \in \mathbb{R}^m$, the functions $(q(t), p(t))$ satisfy the system of first-order implicit differential equations given by (10.1) and (10.2).

As a consequence, the projection $\gamma := \pi_Q \circ k$ will be represented by functions $q(t)$ satisfying the system of mixed-order implicit differential equations given by (10.1) and

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\Theta_v(q, \dot{q}, \lambda) = \Theta_q(q, \dot{q}, \lambda) \tag{10.3}$$

which then locally characterize the base integral curves of D.

(ii) In the main case $\Theta = \rho^*\theta + \Phi$, described in Section 5, one has

$$\begin{aligned}
\Theta_q(q, v, \lambda) &= \theta_q(q, v) + \Phi_q(q, v, \lambda),\\
\Theta_v(q, v, \lambda) &= \theta_v(q, v) + \Phi_v(q, v, \lambda),\\
\Theta_\lambda(q, v, \lambda) &= \Phi_\lambda(q, v, \lambda),
\end{aligned}$$

and, at each point $(q, v, \lambda) \in \Sigma$,

$$\Theta_v(q, v, \lambda) = \theta_v(q, v).$$

As a consequence, Eqs. (10.1)–(10.3) read

$$\Phi_\lambda(q, \dot{q}, \lambda) = 0, \tag{10.4}$$

$$p = \theta_v(q, \dot{q}), \qquad \dot{p} = \theta_q(q, \dot{q}) + \Phi_q(q, \dot{q}, \lambda), \tag{10.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_v(q, \dot{q}) - \theta_q(q, \dot{q}) = \Phi_q(q, \dot{q}, \lambda), \tag{10.6}$$

and clearly exhibit a one-to-one correspondence between the integral curves and the base integral curves of D (i.e., the second-order-like behaviour of D).

(iii) For the above $\Theta = \rho^*\theta + \Phi$, the Legendre mapping

$$\mathcal{L} := \tau_{T^*Q} \circ \mathcal{E} : (q, v, \lambda) \in \Sigma \to \big(q, \Theta_v(q, v, \lambda)\big) \in T^*Q$$

turns out to be projectable by ρ, namely $\mathcal{L} = \mathcal{L}_\theta \circ \rho_{|\Sigma}$ with

$$\mathcal{L}_\theta : (q, v) \in TQ \to \big(q, \theta_v(q, v)\big) \in T^*Q.$$

Hence, for any $x \equiv (q, v/\dot{q}, \dot{v}) \in TTQ$,

$$T\mathcal{L}_\theta(x) \equiv \left(q, \theta_v(q,v)/\dot{q}, \frac{\partial \theta_v}{\partial q}\dot{q} + \frac{\partial \theta_v}{\partial v}\dot{v}\right) \in TT^*Q$$

(where the partial derivatives are evaluated at (q, v)).

So $x \in \widehat{E} := T^2Q \cap T\mathcal{L}_\theta^{-1}(D)$ iff, for some $\lambda \in \mathbb{R}^m$, the coordinates $(q, v/\dot{q}, \dot{v})$ satisfy

$$\dot{q} = v, \qquad \frac{\partial \theta_v}{\partial q}\dot{q} + \frac{\partial \theta_v}{\partial v}\dot{v} - \theta_q(q,v) = \Phi_q(q, v, \lambda) \tag{10.7}$$

as well as (10.4).

Now let $c \equiv (q, v)$—with $q = q(t)$, $v = v(t)$—be a smooth curve in the given coordinate domain of TQ, and $\dot{c} \equiv (\dot{q}, \dot{v})$—with

$$\dot{q} = \frac{\mathrm{d}q}{\mathrm{d}t}, \qquad \dot{v} = \frac{\mathrm{d}v}{\mathrm{d}t}$$

—its tangent lifting.

From the above coordinate description of $\widehat{E}$, it follows that c is an integral curve of $\widehat{E}$, i.e., $\mathrm{Im}\,\dot{c} \subset \widehat{E}$, iff, for some $\lambda(t) \in \mathbb{R}^m$, the functions $(q(t), v(t))$ satisfy Eqs. (10.4) and (10.7).

As a consequence, the projection $\gamma := \tau_Q \circ c$ will be represented by functions $q(t)$ satisfying Eqs. (10.4) and (10.6), which then locally characterize the base integral curves of $\widehat{E}$ as well.

The same local description of $\widehat{E}$ will now be obtained from the coordinate expression of the presymplectic formalism.

Standard computations show that $\omega_\theta := -di_S\theta$ has a block-matrix of components given by

$$\begin{bmatrix} \frac{\partial \theta_v}{\partial q} - \left(\frac{\partial \theta_v}{\partial q}\right)^T & \frac{\partial \theta_v}{\partial v} \\ -\left(\frac{\partial \theta_v}{\partial v}\right)^T & 0 \end{bmatrix}$$

As a consequence, for any $x \equiv (q, v/\dot{q}, \dot{v}) \in TTQ$, one has

$$(i_x\omega_\theta)_q = \left(\frac{\partial \theta_v}{\partial q}\right)^T \dot{q} - \frac{\partial \theta_v}{\partial q}\dot{q} - \frac{\partial \theta_v}{\partial v}\dot{v},$$

$$(i_x\omega_\theta)_v = \left(\frac{\partial \theta_v}{\partial v}\right)^T \dot{q}.$$

Moreover, from $\eta_\theta := di_\Delta\theta - \theta$, one draws

$$(\eta_\theta(\tau(x)))_q = \left(\frac{\partial\theta_v}{\partial q}\right)^T v - \theta_q(q, v),$$

$$(\eta_\theta(\tau(x)))_v = \left(\frac{\partial\theta_v}{\partial v}\right)^T v.$$

Hence

$$\left(\eta_\theta(\tau(x)) - i_x\omega_\theta\right)_q = \left(\frac{\partial\theta_v}{\partial q}\right)^T (v - \dot q) + \frac{\partial\theta_v}{\partial q}\dot q + \frac{\partial\theta_v}{\partial v}\dot v - \theta_q(q, v),$$

$$\left(\eta_\theta(\tau(x)) - i_x\omega_\theta\right)_v = \left(\frac{\partial\theta_v}{\partial v}\right)^T (v - \dot q).$$

On the other hand, one has

$$\widehat{\Phi} := \varpi_\rho \circ \Phi_\Sigma : (q, v, \lambda) \in \Sigma \to \left(q, v/\Phi_q(q, v, \lambda), 0\right) \in T^*TQ.$$

So $x \in \widehat{E}$—i.e., $x \in T^2Q$ and $\eta_\theta(\tau(x)) - i_x\omega_\theta = \widehat{\Phi}(y)$ for some $y \in \Sigma$ such that $\rho(y) = \tau(x)$—iff its coordinates $(q, v/\dot q, \dot v)$ satisfy Eqs. (10.4) and (10.7).

Moreover, if γ is a smooth curve in Q represented by functions $q = q(t)$, then $\eta_\theta \circ \dot\gamma - i_{\ddot\gamma}\omega_\theta$ is a section of $V^0\tau_Q$ along $\dot\gamma$ admitting components given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\theta_v(q, \dot q) - \theta_q(q, \dot q), 0\right).$$

As a consequence, we reobtain that γ is a base integral curve of $\widehat{E}$—i.e.,

$$\eta_\theta \circ \dot\gamma - i_{\ddot\gamma}\omega_\theta = \widehat{\Phi} \circ \sigma$$

for some curve σ in Σ such that $\rho \circ \sigma = \dot\gamma$—iff the functions $q(t)$ satisfy Eqs. (10.4) and (10.6).

(iv) If $\theta = dL + F$ with F semibasic, one has

$$\theta_q = \frac{\partial L}{\partial q} + F_q \quad \text{and} \quad \theta_v = \frac{\partial L}{\partial v}.$$

Eqs. (10.5)–(10.7) then read

$$p = \frac{\partial L}{\partial v}, \qquad \dot p = \frac{\partial L}{\partial q} + F_q(q, \dot q) + \Phi_q(q, \dot q, \lambda), \tag{10.8}$$

$$\dot{q} = v, \quad \left[\frac{\partial}{\partial q}\left(\frac{\partial L}{\partial v}\right)\right]\dot{q} + \left[\frac{\partial}{\partial v}\left(\frac{\partial L}{\partial v}\right)\right]\dot{v} - \frac{\partial L}{\partial q} = F_q(q, v) + \Phi_q(q, v, \lambda), \tag{10.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F_q(q, \dot{q}) + \Phi_q(q, \dot{q}, \lambda), \tag{10.10}$$

which are the familiar coordinate Lagrange equations meant as first-order on T^*Q, first-order on TQ and second-order on Q, respectively.

Such equations could also be obtained from the coordinate expression of the geometric Lagrangian formalism introduced in Section 6. It will suffice, e.g., to observe that, in the present case, the above coordinate presymplectic calculations would lead us to recognize the left- and the right-hand side of Eq. (10.10) just as the components of $\widetilde{[L]} \circ \ddot{\gamma}$ and $\tilde{F} \circ \dot{\gamma} + \tilde{\Phi} \circ \sigma$, respectively.

The choice of Φ adopted in Section 7 corresponds, in local coordinates, to

$$\Phi_q(q, v, \lambda) = \lambda A_q(q, v),$$
$$\Phi_v(q, v, \lambda) = 0,$$
$$\Phi_\lambda(q, v, \lambda) = B(q, v).$$

As a consequence, Eq. (10.4)—characterizing the constraint subset $C := \rho(\Sigma)$—reads

$$B(q, \dot{q}) = 0.$$

Moreover, in the right-hand side of Eqs. (10.8)–(10.10), the constraint forces are linear functions of the Lagrange multipliers with coefficients A_q which—as illustrated in Section 8—are classically related to the functional constraints B by

$$A_q = \frac{\partial B}{\partial v}$$

(in particular $A_q = \beta_q$ for linear constraints $B = i_T\beta + \tau_Q^* g$, and $A_q = \partial f/\partial q$ for integrable linear constraints $B = d_T f$).

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11. APPENDIX A

The proofs of formulae (5.4) and (7.5) appear in [3] and [2], respectively. They are restated here for the sake of completeness.

(i) In order to prove (5.4), we preliminarly recall that

$$i_S\theta = \mathcal{L}_\theta^* \vartheta_Q, \tag{11.1}$$

which holds since, for any $v \in TQ$,

$$\mathcal{L}_\theta(v) = \tau_{T^*Q}\big(\alpha^{-1}\big(\theta(v)\big)\big) = \theta(v) \circ \nu_v,$$

and then

$$\begin{aligned}(i_S\theta)(v) &= \theta(v) \circ S_v = \theta(v) \circ \nu_v \circ T_v\tau_Q = \mathcal{L}_\theta(v) \circ T_v\tau_Q \\ &= \mathcal{L}_\theta(v) \circ T_{\mathcal{L}_\theta(v)}\pi_Q \circ T_v\mathcal{L}_\theta = \vartheta_Q\big(\mathcal{L}_\theta(v)\big) \circ T_v\mathcal{L}_\theta \\ &= \big(\mathcal{L}_\theta^*\vartheta_Q\big)(v).\end{aligned}$$

Now put $\iota : T^2Q \hookrightarrow TTQ$ and $\tau := \tau_{TQ} \circ \iota$.

For any $x \in T^2Q$,

$$\begin{aligned}T_x\tau &= T_x(\tau_{TQ} \circ \iota) = T_x(T\tau_Q \circ \iota) = T_x(T\pi_Q \circ T\mathcal{L}_\theta \circ \iota) \\ &= T_x(\pi_{TQ} \circ \alpha \circ T\mathcal{L}_\theta \circ \iota) \\ &= T_{\alpha(T\mathcal{L}_\theta(x))}\pi_{TQ} \circ T_x(\alpha \circ T\mathcal{L}_\theta \circ \iota).\end{aligned}$$

As a consequence,

$$\begin{aligned}\alpha\big(T\mathcal{L}_\theta(x)\big) \circ T_x\tau &= \vartheta_{TQ}\big(\alpha\big(T\mathcal{L}_\theta(x)\big)\big) \circ T_x(\alpha \circ T\mathcal{L}_\theta \circ \iota) \\ &= \big((\alpha \circ T\mathcal{L}_\theta \circ \iota)^*\vartheta_{TQ}\big)(x).\end{aligned} \tag{11.2}$$

Moreover, owing to (11.1) and $\omega_\theta := -di_S\theta$,

$$\begin{aligned}(\alpha \circ T\mathcal{L}_\theta \circ \iota)^*\vartheta_{TQ} &= \iota^*T\mathcal{L}_\theta^*\alpha^*\vartheta_{TQ} = \iota^*T\mathcal{L}_\theta^*d_T\vartheta_Q = \iota^*d_T\mathcal{L}_\theta^*\vartheta_Q \\ &= \iota^*d_Ti_S\theta = \iota^*di_Ti_S\theta + \iota^*i_Tdi_S\theta \\ &= \tau^*di_\Delta\theta - \iota^*i_T\omega_\theta\end{aligned} \tag{11.3}$$

since $\iota^*i_Ti_S\theta = \tau^*i_\Delta\theta$—indeed, for any $x \in T^2Q$,

$$\begin{aligned}(\iota^*i_Ti_S\theta)(x) &= (i_Ti_S\theta)(x) = \langle x \mid i_S\theta\rangle = \langle S(x) \mid \theta\rangle = \big\langle \Delta\big(\tau(x)\big) \mid \theta\big\rangle \\ &= (i_\Delta\theta)\big(\tau(x)\big) = \big(\tau^*i_\Delta\theta\big)(x).\end{aligned}$$

From (11.2), (11.3) and $\eta_\theta := di_\Delta\theta - \theta$, we obtain

$$\begin{aligned}
&\big(\alpha\big(T\mathcal{L}_\theta(x)\big) - \theta\big(\tau(x)\big)\big) \circ T_x\tau = \big((\alpha \circ T\mathcal{L}_\theta \circ \iota)^* \vartheta_{TQ} - \tau^*\theta\big)(x) \\
&\quad = \big(\tau^*\eta_\theta - \iota^* i_T\omega_\theta\big)(x) \\
&\quad = \eta_\theta\big(\tau(x)\big) \circ T_x\tau - (i_T\omega_\theta)(x) \circ T_x\iota \\
&\quad = \eta_\theta\big(\tau(x)\big) \circ T_x\tau - i_x\omega_\theta \circ T_x\tau_{TQ} \circ T_x\iota \\
&\quad = \big(\eta_\theta\big(\tau(x)\big) - i_x\omega_\theta\big) \circ T_x\tau. \qquad (11.4)
\end{aligned}$$

Since $T_x\tau$ is surjective, (11.4) is equivalent to

$$\alpha\big(T\mathcal{L}_\theta(x)\big) - \theta\big(\tau(x)\big) = \eta_\theta\big(\tau(x)\big) - i_x\omega_\theta$$

which is our claim.

(ii) In order to prove (7.5), we preliminarly recall that, for any second-order field X on TQ (i.e., $SX = \Delta$),

$$S[X, Z] = 0 \qquad (11.5)$$

which holds since Z satisfies—as is well known—$[Z, \Delta] = 0$ and $[Z, S] = 0$, where $[Z, S]X := [Z, SX] - S[X, Z]$.

Now, for any X as above,

$$\begin{aligned}
&\langle Z \,|\, [L] \circ X\rangle = \langle Z \,|\, dE_L - i_X\omega_L\rangle \\
&\quad = \langle Z \,|\, dE_L\rangle + \langle X, Z \,|\, dd_SL\rangle \\
&\quad = \langle Z \,|\, dE_L\rangle + X\langle Z \,|\, d_SL\rangle - Z\langle X \,|\, d_SL\rangle - \langle [X, Z] \,|\, d_SL\rangle \\
&\quad = \langle Z \,|\, dE_L\rangle + X\langle SZ \,|\, dL\rangle - Z\langle SX \,|\, dL\rangle - \langle S[X, Z] \,|\, dL\rangle \\
&\quad = \langle Z \,|\, dE_L\rangle + X\langle SZ \,|\, dL\rangle - Z\langle \Delta \,|\, dL\rangle \quad \text{(owing to (11.5))} \\
&\quad = \langle Z \,|\, dE_L\rangle + X\langle SZ \,|\, dL\rangle - \langle Z \,|\, dE_L\rangle - \langle Z \,|\, dL\rangle \\
&\quad = X\langle SZ \,|\, dL\rangle - \langle Z \,|\, dL\rangle.
\end{aligned}$$

Putting again $\tau := \tau_{TQ} \circ \iota$ with $\iota : T^2Q \hookrightarrow TTQ$ and recalling that $\tau \circ X$ is an identity mapping, the above result also reads

$$\langle Z \circ \tau \,|\, [L]\rangle \circ X = \big(d_T\langle SZ \,|\, dL\rangle\big) \circ X - \langle Z \,|\, dL\rangle \circ \tau \circ X,$$

or, owing to the arbitrariness of X,

$$\langle Z \circ \tau \,|\, [L]\rangle = \big(d_T\langle SZ \,|\, dL\rangle\big) \circ \iota - \langle Z \,|\, dL\rangle \circ \tau.$$

As a consequence, along any smooth curve γ of Q, we obtain

$$\begin{aligned}\langle Z \circ \dot{\gamma} \mid [L] \circ \ddot{\gamma} \rangle &= \langle Z \circ \tau \circ \ddot{\gamma} \mid [L] \circ \ddot{\gamma} \rangle \\ &= \big(d_T \langle SZ \mid dL \rangle\big) \circ \ddot{\gamma} - \langle Z \mid dL \rangle \circ \dot{\gamma} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \langle SZ \circ \dot{\gamma} \mid dL \circ \dot{\gamma} \rangle - \langle Z \circ \dot{\gamma} \mid dL \circ \dot{\gamma} \rangle\end{aligned}$$

which is our claim.

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