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Quantum structures in Galilei general relativity

by

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ABSTRACT. – We give a necessary and sufficient condition on the existence of quantum structures on a curved spacetime with absolute time, and classify these structures. We refer to the geometric approach to quantum mechanics on a Galilei general relativistic background, as formulated by Jadczyk and Modugno. These results are analogous to those of geometric quantisation, but they involve the topology of spacetime, rather than the topology of the configuration space. © Elsevier, Paris

Key words: Galilei general relativity, particle mechanics, jets, non-linear connections, cosymplectic forms, quantisation, cohomology.

RÉSUMÉ. – Nous donnons une condition nécessaire et suffisante sur l'existence de structures quantiques sur un espace-temps courbe avec temps absolu et classifions ces structures. Nous nous référons au formalisme géométrique de la mécanique quantique sur un espace-temps galiléen développé par Jadczyk et Modugno. Ces résultats sont analogues à ceux de la quantification géométrique, mais ils dépendent de la topologie de l'espace-temps plutôt que de la topologie de l'espace de configuration. © Elsevier, Paris

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INTRODUCTION

The problem of finding a covariant formulation of quantum mechanics has been challenged by many authors. One way to solve this problem is to start from a covariant formulation of classical mechanics and to give a covariant procedure of quantisation. There are several formulations of classical mechanics and the quantisation procedure based on a curved spacetime with absolute time (see, for example, [4, 5, 15, 17, 24, 25, 26]). These formulations can inspire interesting investigations of the case of Einstein's general relativity (see, for example, [20]).

Here, we consider a recent formulation of Galilei classical and quantum mechanics based on jets, connections and cosymplectic forms due to Canarutto, Jadczyk and Modugno [2, 9, 10] (see also [27]). This approach presents analogies with geometric quantisation [14, 23, 7, 22, 31] but important novelties as well. In a few words, spacetime is a fibred manifold equipped with a vertical metric, a gravitational connection and an electromagnetic field; these structures produce naturally a cosymplectic form. Moreover, quantum mechanics is formulated on a line bundle over spacetime equipped with a connection whose curvature is proportional to the above form.

This formulation is manifestly covariant, due to the use of intrinsic techniques on manifolds. Moreover, it reduces to standard quantisation in the flat case, hence it recovers all standard examples of quantum mechanics. In particular, the standard examples of geometric quantisation (i.e. harmonic oscillator and hydrogen atom) are recovered in an easier way. Another interesting feature of the above formulation is that it can be extended to Einstein's general relativity [12, 13, 28, 29].

The existence of quantum structures of the above quantisation procedure is an important problem. In this paper, we give a theorem of Kostant-Souriau type (see, for instance, [14, 23, 7, 19]), which states a topological necessary and sufficient existence condition on the spacetime and the cosymplectic form. Also, we give a classification theorem for quantum structures. As one could expect, the results are analogous to those of geometric quantisation, but involve the topology of spacetime, rather than the topology of the configuration space.

Finally, we illustrate the above formulation and results by means of some examples. As a consequence, we recover a result of [27] in a simpler way.

Now, we are going to assume the fundamental spaces of units of measurement and constants.

The theory of *unit spaces* has been developed in [9, 10] to make the independence of classical and quantum mechanics from scales explicit. Unit spaces are defined similarly to vector spaces, but using the abelian semigroup $\mathbb{R}^+$ instead of the field of real numbers $\mathbb{R}$. In particular, *positive* unit spaces are defined to be 1-dimensional (over $\mathbb{R}^+$) unit spaces. It is possible to define n-th tensorial powers and n-th roots of unit spaces. Moreover, if $\mathbb{P}$ is a positive unit space and $p \in \mathbb{P}$, then we denote by $1/p \in \mathbb{P}^*$ the dual element. Hence, we can set $\mathbb{P}^{-1} := \mathbb{P}^*$. In this way, we can introduce rational powers of unit spaces.

We assume the following unit spaces.

–$\mathbb{T}$, the oriented one-dimensional vector space of *time intervals*;

–$\mathbb{L}$, the positive unit space of *length units*;

–$\mathbb{M}$, the positive unit space of *mass units*.

The positively oriented component of $\mathbb{T}$ (which is a positive unit space) is denoted by $\mathbb{T}^+$. A positively oriented non-zero element $u_0 \in \mathbb{T}^+$ (or $u^0 \in (\mathbb{T}^+)^{-1}$) represents a *time unit of measurement*, a *charge* is represented by an element $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$, and a *particle* is represented by a pair (m, q), where m is a mass and q is a charge. A tensor field with values into mixed rational powers of $\mathbb{T}$, $\mathbb{L}$, $\mathbb{M}$ is said to be *scaled*. We assume the *Planck's constant* $\hbar \in (\mathbb{T}^+)^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

We end this introduction by assuming manifolds and maps to be C^∞.

1. CLASSICAL STRUCTURES

In this section we present an overview of Galilei's general relativity, as formulated in [9, 10], together with some results of [19].

ASSUMPTION G.1. – We assume the *spacetime* to be a fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$, where $\dim \boldsymbol{E} = 4$, $\dim \boldsymbol{T} = 1$, and $\boldsymbol{T}$ is an affine space associated with an oriented vector space $\mathbb{T}$. □

We will denote with (x^0, y^i) a fibred chart on $\boldsymbol{E}$ adapted to a time unit of measurement $u_0 \in \mathbb{T}$. We will deal with the tangent and vertical bundles $T\boldsymbol{E}$ and $V\boldsymbol{E} := \ker Tt \subset T\boldsymbol{E}$ on $\boldsymbol{E}$. We denote by (∂_0, ∂_i), (d^0, d^i) and $(\check{d}^i)$ the local bases of vector fields on $\boldsymbol{E}$, of 1-forms on $\boldsymbol{E}$ and of sections of the dual bundle $V^*\boldsymbol{E} \to \boldsymbol{E}$ induced by an adapted chart. Latin indices $i, j, \dots$ will denote space-like coordinates, Greek indices $\lambda, \mu, \dots$ will denote spacetime coordinates.

We will deal also with the first jet bundle $t_0^1 : J_1\boldsymbol{E} \to \boldsymbol{E}$, i.e. the space of equivalence classes of sections having a first-order contact at a certain point. The charts induced on $J_1\boldsymbol{E}$ by an adapted chart on $\boldsymbol{E}$ are denoted by (x^0, y^i, y^i_0); the local vector fields and forms of $J_1\boldsymbol{E}$ induced by (y^i_0) are denoted by (∂^0_i) and (d^i_0), respectively.

We recall the natural inclusion $D : J_1\boldsymbol{E} \to T^*\boldsymbol{T} \otimes T\boldsymbol{E}$ over $\boldsymbol{E}$, whose coordinate expression is $D = d^0 \otimes (\partial_0 + y^i_0 \partial_i)$. We have the complementary map $\mathrm{id} - D := \theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}$. The map θ yields the inclusion $\theta^* : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} V^*\boldsymbol{E} \to J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E}$ over $J_1\boldsymbol{E}$, sending the local basis $(\check{d}^i)$ into $\theta^i := (d^i - y^i_0 d^0)$. See [18] for more details about jets and the related natural maps.

A *motion* is defined to be a section $s : \boldsymbol{T} \to \boldsymbol{E}$.

An *observer* is defined to be a section

$$o : \boldsymbol{E} \to J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}.$$

An observer o can be regarded as a scaled vector field on $\boldsymbol{E}$ whose integral curves are motions; hence o yields a local fibred splitting $\boldsymbol{E} \to \boldsymbol{T} \times \boldsymbol{P}$, where $\boldsymbol{P}$ is a set of integral curves of o. An observer is said to be *complete* if the above splitting is a global splitting of $\boldsymbol{E}$. An observer o can also be regarded as a connection on the fibred manifold $\boldsymbol{E} \to \boldsymbol{T}$. Accordingly, we define the translation fibred isomorphism $\nabla[o]$ associated with o

$$\nabla[o] : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E} : \sigma \mapsto \nabla[o](\sigma) := \sigma - o(t^1_0(\sigma)).$$

We have the coordinate expressions $o = u^0 \otimes (\partial_0 + o^i_0\, \partial_i)$, $\nabla[o] = (y^i_0 - o^i_0)\, d^0 \otimes \partial_i$. A fibred chart (x^0, y^i) is said to be *adapted* to an observer o if it is adapted to local splitting of $\boldsymbol{E}$ induced by o, i.e. $o^i_0 = 0$.

Next, we consider additional structures on our fibred manifold given by special types of metrics and connections.

ASSUMPTION G.2. – We assume the spacetime to be endowed with a scaled vertical Riemannian metric

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E}). \quad \square$$

We have the coordinate expression $g = g_{ij} \check{d}^i \otimes \check{d}^j$, with $g_{ij} \in C^\infty(\boldsymbol{E}, \mathbb{L}^2 \otimes \mathbb{R})$. We denote by $\bar{g}$ the contravariant metric induced by g. An observer o is said to be *isometric* if $L_o g = 0$.

DEFINITION 1.1. – *We define a* spacetime connection *to be a torsion free linear connection*

$$K : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E}$$

on $TE \to E$, such that

$$\nabla\, dt = 0. \quad \square$$

The coordinate expression of a spacetime connection K is of the type

$$(1) \qquad K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i \dot{\partial}_i), \qquad K_\lambda{}^i := K_\lambda{}^i{}_0 \dot{x}^0 + K_\lambda{}^i{}_j \dot{y}^j,$$

where $K_\lambda{}^i{}_\mu \in C^\infty(E)$.

We can prove [10] that any spacetime connection K yields an affine connection $\Gamma[K]$ on J_1E. The coordinate expression of such a correspondence is $\Gamma_\lambda{}^i{}_\mu = K_\lambda{}^i{}_\mu$. Moreover, the connection $\Gamma[K]$ yields a connection

$$(2) \qquad \gamma[K] := D \lrcorner\, \Gamma[K] : J_1E \to \mathbb{T}^* \otimes TJ_1E$$

on the fibred manifold $J_1E \to T$. We have the coordinate expression

$$(3)\ \gamma[K] = u^0 \otimes (\partial_0 + y_0^i \partial_i + \gamma^i \partial_i^0), \quad \gamma^i := K_h{}^i{}_k y_0^h y_0^k + 2K_h{}^i{}_0 y^h + K_0{}^i{}_0.$$

We can interpret a spacetime connection K through an observer o. We have the splitting of $\nabla[K]o$ into its symmetric and antisymmetric part, namely $\nabla[K]o = 1/2\ (\Sigma[o] + \Phi[o])$, with coordinate expressions

$$\Sigma[o] = -2u^0 \otimes (\Gamma_{0j0}\, d^0 \odot d^j + \Gamma_{ij0}\, d^i \odot d^j)\ ,$$
$$\Phi[o] = -2u^0 \otimes (\Gamma_{0j0}\, d^0 \wedge d^j + \Gamma_{ij0}\, d^i \wedge d^j)\ .$$

So, o allows us to split the spacetime connection K into the triple $(\check{K}, \check{\Sigma}[o], \Phi[o])$, where $\check{K}$ and $\check{\Sigma}[o]$ are the restrictions of K and $\Sigma[o]$ to the fibers of $E \to T$. The correspondence

$$K \mapsto (\check{K}, \check{\Sigma}[o], \Phi[o])$$

turns out to be a bijection between the set of spacetime connections and the set of triples constituted by a linear connection on the fibres of $E \to T$, a scaled symmetric 2-form on the vertical bundle and a scaled antisymmetric 2-form on spacetime. By the way, this proves the existence of spacetime connections.

A spacetime connection K is said to be *metric* if $\nabla[K']g = 0$, where K' is the restriction of K to the vertical bundle $VE \to E$. The metric g does not determine completely a metric spacetime connection K, as in the Einstein case; this is due to the degeneracy of g. More precisely, $\check{K}$ coincides with the Riemannian connection induced by g on the fibres of $\boldsymbol{E} \to \boldsymbol{T}$ and any observer o yields the equality $\check{\Sigma}[o] = g \circ (L_o \bar{g})$. In coordinates adapted to an observer o the above two conditions read, respectively, as

$$(4) \quad K_{ihj} = -\frac{1}{2}(\partial_i g_{hj} + \partial_j g_{hi} - \partial_h g_{ij}) , \qquad K_{0ij} + K_{0ji} = -\partial_0 g_{ij} .$$

Remark 1.1. – The choice of an observer yields a bijection between the set of metric spacetime connections and the set of scaled 2-forms of the type

$$\Phi : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{L}^2) \otimes \overset{2}{\wedge} T^*\boldsymbol{E} .$$

This fact implies the existence of metric spacetime connections. □

Now, we introduce a geometric object which plays a key role in the formulation of field equations and of classical and quantum mechanics.

DEFINITION 1.2. – *Let K be a spacetime connection. The 2-form*

$$\Omega := \nu_{\Gamma[K]} \overline{\wedge} \theta : J_1\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{L}^2) \otimes \overset{2}{\wedge} T^* J_1\boldsymbol{E},$$

where $\overline{\wedge}$ is the wedge product followed by a-metric contraction, and $\nu_{\Gamma[K]}$ is the vertical projection complementary to $\Gamma[K]$, is said to be the fundamental 2-form *on $J_1\boldsymbol{E}$ induced by g and K.* □

In what follows, we will indicate the dependence of Ω on the spacetime connection by the symbol $\Omega[K]$. It has been proved that $\Omega[K]$ is the unique scaled 2-form on $J_1\boldsymbol{E}$ which is naturally induced by g and K [11]. We have the coordinate expression

$$(5) \qquad \Omega[K] = g_{ij} u^0 \otimes \left(d_0^i - \left(K_{\lambda}{}^{i}{}_{k} y_0^k + K_{\lambda}{}^{i}{}_{\circ}\right) d^\lambda\right) \wedge \theta^j$$

Now, we postulate the gravitational field as a spacetime connection, and we postulate the electromagnetic field as a 2-form on $\boldsymbol{E}$[1]. Next, we state the field equations. Then, we will see how to encode the two fields into a single spacetime connection.

ASSUMPTION G.3. – We assume that $\boldsymbol{E}$ is endowed with a spacetime connection $K^\natural$, the *gravitational field*, and with a scaled 2-form $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \overset{2}{\wedge} T^*\boldsymbol{E}$, the *electromagnetic field*. □

ASSUMPTION G.4. – We assume that $K^\natural$ and F fulfill the following *first field equations*:

$$d\Omega[K^\natural] = 0, \qquad dF = 0. \quad \square$$

[1] This formulation of the electromagnetic field is very similar to those of [1, 16]

Any particle (m, q) allows us to realise the coupling between the gravitational and electromagnetic field as

$$\Omega := \Omega[K^\natural] + \frac{q}{2m} F. \tag{6}$$

The form Ω turns out to be closed, i.e. $d\Omega = 0$. Moreover, one can see that Ω is non degenerate, i.e. $dt \wedge \Omega \wedge \Omega \wedge \Omega \neq 0$.

DEFINITION 1.3. – We say Ω to be the *cosymplectic form* of $\boldsymbol{E}$ associated with g, K and dt. □

It can be seen [9, 10] that there exists a unique spacetime connection K such that $\Omega = \Omega[K]$. The coefficients of K turns out to be

$$K_h{}^i{}_k = K^\natural{}_h{}^i{}_k \,, \qquad K_0{}^i{}_k = K^\natural{}_0{}^i{}_k + \frac{q}{2m} F^i{}_k \,, \tag{7}$$
$$K_0{}^i{}_0 = K^\natural{}_0{}^i{}_0 + \frac{q}{m} F^i{}_0 \,.$$

We can interpret the first field equation through an observer o [9, 10]. It is easy to prove (e.g., in adapted coordinates) that $\Phi[o] = 2o^*\Omega[K]$. Then, the first field equation is equivalent to the system

$$\nabla[K']g = 0, \qquad d\Phi[o] = 0. \tag{8}$$

So, K is a metric spacetime connection and $\Phi[o]$ is closed.

Now, we show that the closed 2-form $\Omega[K]$ admits a distinguished class of potentials. We will see that these potentials play a key role in the quantisation procedure. We note that the constant $m/\hbar$ yields the non-scaled 2-form $m/\hbar\, \Omega[K]$; it is natural to search the potentials of this 2-form.

Let o be an observer. We define the *kinetic energy* and the *kinetic momentum form*, respectively, as

$$k[o] := 1/2\, \frac{m}{\hbar} g \circ (\nabla[o], \nabla[o]) : J_1\boldsymbol{E} \to \mathbb{T}^*,$$
$$p[o] := \theta^* \lrcorner\ \frac{m}{\hbar} g^\flat \circ \nabla[o] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}. \quad \square$$

Moreover, we denote by $\alpha[o] : \boldsymbol{E} \to T^*\boldsymbol{E}$ a generic local potential of $m/\hbar\, \Phi[o]$, according to $2d\alpha[o] = m/\hbar\, \Phi[o]$.

THEOREM 1.1. – *Let o be an observer. Then the local section*

$$\tau := k[o] + p[o] + \alpha[o] : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \subset T^*J_1\boldsymbol{E}$$

is a local potential of $m/\hbar\,\Omega[K]$, according to $2d\tau = m/\hbar\,\Omega[K]$. Moreover, any other potential τ' of $m/\hbar\,\Omega[K]$ with values in $T^*\boldsymbol{E}$ is of the above type, and the form

$$\tau - \tau' : \boldsymbol{E} \to T^*\boldsymbol{E}$$

is a closed form on $\boldsymbol{E}$, rather than on $J_1\boldsymbol{E}$.

For a proof, see [19]. We have the coordinate expression

$$\tau = -\frac{m}{2\hbar}\, u^0 g_{ij} y_0^i y_0^j\, d^0 + \frac{m}{\hbar} u^0 g_{ij} y_0^j\, d^i + \alpha_\lambda\, d^\lambda \tag{9}$$

DEFINITION 1.4. – A (local) potential of $m/\hbar\,\Omega[K]$ of the above type is said to be a *Poincaré-Cartan form* associated with $m/\hbar\,\Omega[K]$. □

Remark 1.2. – We will not discuss the second field equations in detail. For example, in the vacuum we assume the second field equations

$$r[K^\natural] = 0, \qquad \operatorname{div}^\natural F = 0$$

where $r[K^\natural]$ is the Ricci tensor of $K^\natural$, and $\operatorname{div}^\natural$ is the covariant divergence operator induced by $K^\natural$ (see [2, 9, 10] for a more complete discussion). □

Remark 1.3. – The law of particle motion [2, 9, 10] for a motion s is assumed to be

$$\nabla[\gamma[K]] j_1 s := T j_1 s - \gamma \circ j_1 s = 0,$$

with coordinate expression

$$\partial^2_{00} s^i - (K_h{}^i{}_k \circ s)\partial_0 s^h \partial_0 s^k - 2(K_0{}^i{}_k \circ s)\partial_0 s^k - (K_0{}^i{}_0 \circ s) = \frac{q}{m}\left(F^i{}_0 \circ s + F^i{}_k \circ s \partial_0 s^k\right) .$$

The connection γ can be regarded (up to a time scale) as a vector field on $J_1\boldsymbol{E}$, hence a motion fulfilling the above equation is just an integral curve of γ. Moreover, it can be easily seen that γ fulfills $i_\gamma\Omega = 0$; in [2, 9, 10] it is proved that γ is the unique connection on $J_1\boldsymbol{E} \to \boldsymbol{T}$, which is projectable on D and whose components γ^i are second order polynomials in the variables y_0^i, fulfilling the above equation. In other words, the law of particle motion is given by the foliation $\ker(\Omega[K])$. □

Remark 1.4. – It is proved [19] that the form $\Omega[K]$ induces naturally an intrinsic Euler-Lagrange morphism $\mathcal{E}$ whose Euler-Lagrange equations are equivalent to the the law of particle motion. The morphism $\mathcal{E}$ is locally variational, and there exists a distinguished class of Lagrangians which induce $\mathcal{E}$, and whose Poincaré-Cartan forms [6] turn out to be the Poincaré-Cartan forms associated with $m/\hbar\,\Omega[K]$. □

2. QUANTUM STRUCTURES

A covariant formulation of the quantisation of classical mechanics of one scalar particle has been developed starting from the above classical theory in [9, 10].

In this section, we recall the geometric structures which allow to formulate the quantisation procedure of [9, 10], namely the *quantum bundle* and the *quantum connection*. Then, we will present a necessary and sufficient condition for the existence of quantum structures on a given background, together with a classification theorem. Those results are inspired from the analogous results in geometric quantisation [14, 23, 7]. Finally, we analyse some examples of applications to exact solutions [28].

In this section, we denote by $\Omega \equiv \Omega[K]$ the cosymplectic form on $\boldsymbol{E}$ associated with g and K. Also, we assume a particle (m, q).

Quantum bundle and quantum connection

DEFINITION 2.1. – A *quantum bundle* is defined to be a complex line bundle $\boldsymbol{Q} \to \boldsymbol{E}$ on spacetime, endowed with a Hermitian metric h. □

Two complex line bundles $\boldsymbol{Q}$, $\boldsymbol{Q}'$ on $\boldsymbol{E}$ are said to be *equivalent* if there exists an isomorphism of complex line bundles $f : \boldsymbol{Q} \to \boldsymbol{Q}'$ on $\boldsymbol{E}$. If $\boldsymbol{Q}$, $\boldsymbol{Q}'$ are equivalent Hermitian complex line bundles, then $\boldsymbol{Q}$, $\boldsymbol{Q}'$ are also isometric, due to the fact that the fibres have complex dimension 1.

Let us denote by $\mathcal{L}(\boldsymbol{E})$ the set of equivalence classes of (Hermitian) complex line bundles. Then $\mathcal{L}(\boldsymbol{E})$ has a natural structure of abelian group with respect to complex tensor product, and there exists a natural abelian group isomorphism $\mathcal{L}(\boldsymbol{E}) \to H^2(\boldsymbol{E}, \mathbb{Z})$ [7, 30].

Quantum histories are represented by sections $\psi : \boldsymbol{E} \to \boldsymbol{Q}$. We denote by

$$\mathbf{i} : \boldsymbol{Q} \to V\boldsymbol{Q} \simeq \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} : q \mapsto (q, q)$$

the *Liouville form* on $\boldsymbol{Q}$.

DEFINITION 2.2. – A *quantum connection* is defined to be a connection C on the bundle $J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$ fulfilling the properties:

(i) C is Hermitian;

(ii) C is universal (see [10, 18] for a definition);

(iii) the curvature $R[\mathrm{C}]$ of C fulfills:

$$R[\mathrm{C}] = i\frac{m}{\hbar}\Omega \otimes \mathbf{i}. \quad \square$$

The universality is equivalent to the fact that C is a family of connections on $\boldsymbol{Q} \to \boldsymbol{E}$ parametrised by observers. We remark that the first field equation $d\Omega = 0$ turns out to be equivalent to the Bianchi identity for a quantum connection C.

DEFINITION 2.3. – A pair $(\boldsymbol{Q}, \mathrm{C})$ is said to be a *quantum structure*. Two quantum structures $(\boldsymbol{Q}_1, \mathrm{C}_1)$, $(\boldsymbol{Q}_2, \mathrm{C}_2)$, are said to be *equivalent* if there exists an equivalence $f : \boldsymbol{Q}_1 \to \boldsymbol{Q}_2$ which maps C_1 into C_2. □

As we will see in next section, in general not any quantum bundle admits a quantum structure. We say a quantum bundle $\boldsymbol{Q}$ to be *admissible* if there exists a quantum structure $(\boldsymbol{Q}, \mathrm{C})$. We denote by

$$\mathcal{QB} \subset \mathcal{L}(\boldsymbol{E}) \tag{10}$$

the set of equivalence classes of admissible quantum bundles.

Let $[\boldsymbol{Q}] \in \mathcal{QB}$. Then we define $\mathcal{QS}[\boldsymbol{Q}]$ to be the set of equivalence classes of quantum structures having quantum bundles in the equivalence class $[\boldsymbol{Q}]$. If $[\boldsymbol{Q}'] \in \mathcal{QB}$ and $[\boldsymbol{Q}] \neq [\boldsymbol{Q}']$, then $\mathcal{QS}[\boldsymbol{Q}]$ and $\mathcal{QS}[\boldsymbol{Q}']$ are clearly disjoint. So, we define

$$\mathcal{QS} := \bigsqcup_{[\boldsymbol{Q}] \in \mathcal{QB}} \mathcal{QS}[\boldsymbol{Q}] \tag{11}$$

to be the set of equivalence classes of quantum structures.

The task of the rest of the paper is to analyse the structures of $\mathcal{QB}$ and $\mathcal{QS}$. To this aim, we devote the final part of this subsection to some technical result.

THEOREM 2.1. – *For any star-shaped open subset $\boldsymbol{U} \subset \boldsymbol{E}$, chosen a trivialisation of $\boldsymbol{Q}$ over $\boldsymbol{U}$, we have*

$$\mathrm{C} = \mathrm{C}^{\|}_{\boldsymbol{U}} + i\tau_{\boldsymbol{U}} \otimes \mathbf{i}$$

where $\mathrm{C}^{\|}_{\boldsymbol{U}}$ is the local flat connection induced by the trivialisation, and $\tau_{\boldsymbol{U}}$ is a distinguished choice (induced by C*) of a Poincaré-Cartan form over $\boldsymbol{U}$ associated with $m/\hbar\,\Omega$.*

Proof. – In fact, the coordinate expression of a Hermitian connection C on $J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$ is

$$\mathrm{C} = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + i\mathrm{C}_\lambda d^\lambda \otimes \mathbf{i} + i\mathrm{C}^0_i d^i_0 \otimes \mathbf{i}\,,$$

and universality is expressed by $\mathrm{C}^0_i = 0$. The result follows from the coordinate expression of $R[\mathrm{C}]$. Q.E.D.

Now, we study the change of the coordinate expression of a quantum connection C with respect to a change of chart. Let $\boldsymbol{U}_1, \boldsymbol{U}_2 \subset \boldsymbol{E}$ be two star-shaped open subsets such that $\boldsymbol{U}_1 \cap \boldsymbol{U}_2 \neq \emptyset$, and b_1, b_2 be two local bases for sections $\boldsymbol{Q}$ over $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, respectively. Suppose that we have the change of base expressions

$$b_1 = c_{12} b_2 = \exp(2\pi i f_{12}) b_2 \ ,$$

with $c_{12} : \boldsymbol{U}_1 \cap \boldsymbol{U}_2 \to U(1) \subset \mathbb{C}$ and $f_{12} : \boldsymbol{U}_1 \cap \boldsymbol{U}_2 \to \mathbb{R}$. Let

$$\mathrm{C} = \mathrm{C}_1^{\shortparallel} + i\tau_1 \otimes \mathbf{i} = \mathrm{C}_2^{\shortparallel} + i\tau_2 \otimes \mathbf{i}$$

on $\boldsymbol{U}_1 \cap \boldsymbol{U}_2$. Then, it follows from a coordinate computation that

$$(12) \qquad \mathrm{C}_1^{\shortparallel} - \mathrm{C}_2^{\shortparallel} = -2\pi i df_{12} \otimes \mathbf{i} \qquad\qquad \tau_1 - \tau_2 = 2\pi df_{12}$$

Remark 2.1. – We have introduced the quantum structures on a Galilei general relativistic background. We could proceed by defining an algebra of quantisable functions, a quantum Lagrangian (which yields the generalised Schroedinger equation) and an algebra of quantum operators [9, 10]. □

Existence of quantum structures

In this subsection we give a necessary and sufficient condition for the existence of a quantum bundle and a quantum connection. A fundamental role is played by the properties of the Poincaré-Cartan form.

We follow a presentation of the Kostant-Souriau theorem [14, 23] given in [7]. See also [19, 29]. We denote the Čech cohomology of $\boldsymbol{E}$ with values in $\mathbb{Z}$ ($\mathbb{R}$) by $H^*(\boldsymbol{E}, \mathbb{Z})$ ($H^*(\boldsymbol{E}, \mathbb{R})$). We recall that the inclusion $i : \mathbb{Z} \to \mathbb{R}$ yields a group morphism

$$(13) \qquad i : H^2(\boldsymbol{E}, \mathbb{Z}) \to H^2(\boldsymbol{E}, \mathbb{R})$$

which is not necessarily an injective morphism. We also recall the natural isomorphism $H^*(\boldsymbol{E}, \mathbb{R}) \to H^*_{\text{de Rham}}(\boldsymbol{E})$.

We observe that there is a (not natural) isomorphism $H^*(J_1\boldsymbol{E}, \mathbb{R}) \simeq H^*(\boldsymbol{E}, \mathbb{R})$, due to the topological triviality of the fibers of $J_1\boldsymbol{E} \to \boldsymbol{E}$. Anyway, due to the properties of the Poincaré-Cartan form, the closed form Ω yields naturally a class in $H^2(\boldsymbol{E}, \mathbb{R})$.

LEMMA 2.1. – *The class* $[m/\hbar\, \Omega] \in H^2_{de\ Rham}(J_1\boldsymbol{E})$ *yields a class*

$$[qs] \equiv [qs]\big([\frac{m}{\hbar}\Omega]\big) \in H^2(\boldsymbol{E}, \mathbb{R}) \ .$$

Proof. – Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a *good cover* of $\boldsymbol{E}$, i.e. an open cover in which any finite intersection is either empty or diffeomorphic to $\mathbb{R}^n$. For any $i \in I$ choose a Poincaré-Cartan form τ_i of Ω on the tubular neighbourhood $t_0^{1^{-1}}(U_i)$. For any $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, by virtue of theorem 1.1, choose a potential $2\pi f_{ij} \in C^\infty(\boldsymbol{E})$ of the closed one-form $\tau_i - \tau_j$ on $\boldsymbol{E}$. We define a 2-cochain qs as follows: for each $i, j, k \in I$ such that $U_i \cap U_j \cap U_k \neq \emptyset$ let $(qs)_{ijk} := f_{ij} + f_{jk} - f_{ik}$. It is easily proved that qs is closed and the class $[qs]$ depends only on the class $[m/\hbar\,\Omega]$. Q.E.D.

THEOREM 2.2. – *The following conditions are equivalent.*

(i) *There exists a quantum structure* $(\boldsymbol{Q}, \mathrm{C})$.

(ii) *The cohomology class* $[qs] \in H^2(\boldsymbol{E}, \mathbb{R})$ *determined by the (de Rham class of the) closed scaled 2-form* Ω *lies in the subgroup*

$$[qs] \in i(H^2(\boldsymbol{E}, \mathbb{Z})) \subset H^2(\boldsymbol{E}, \mathbb{R}).$$

Proof. – Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a good cover of $\boldsymbol{E}$. Suppose that the second condition holds. Then, we observe that the morphism $i : H^2(\boldsymbol{E}, \mathbb{Z}) \to H^2(\boldsymbol{E}, \mathbb{R})$ is given as $i([qs]) = [i(qs)]$, where $(i(qs))_{ijk} := i((qs)_{ijk})$ for each $i, j, k \in I$ with $U_i \cap U_j \cap U_k \neq \emptyset$.

Hence, there exist functions f_{ij}, f_{jk}, f_{ik} like in the above Lemma such that $(f_{ij} + f_{jk} - f_{ik}) \in \mathbb{Z}$. Let us set

$$c_{ij} : \boldsymbol{U}_i \cap \boldsymbol{U}_j \to U(1) : x \mapsto \exp 2\pi i f_{ij}$$

We have $c_{ij} c_{jk} = c_{ik}$, hence c_{ij} is the cocycle of an isomorphism class in $\mathcal{L}(\boldsymbol{E})$.

Moreover, we have

$$\tau_i - \tau_j = 2\pi d f_{ij} = \frac{1}{i} \frac{dc_{ij}}{c_{ij}};$$

hence, the one-forms $i\tau_i \otimes \mathbf{i}$ yield a global quantum connection.

Conversely, if the first condition holds, we use theorem 2.1 and equation (12) with respect to the the trivialisation over the good cover. In this way, the functions f_{ij} give rise to the constant functions $f_{ij} + f_{jk} - f_{ik}$ with values in $\mathbb{Z}$, hence to a class $[qs] \in i(H^2(\boldsymbol{E}, \mathbb{Z}))$. Q.E.D.

Classification of quantum structures

In this subsection we will use a more general definition of affine space rather than the usual one. Namely, the triple $(A, G, \cdot)$ is defined to be an *affine space A associated with the group G* if A is a set, G is a group and $\cdot$ is a free and transitive right action of G on A. Note that to every $a \in A$ the map $r_a : G \to A : g \mapsto ag$ is a bijection.

We start by assuming that the existence condition is satisfied by the spacetime.

ASSUMPTION Q.1. – We assume that the cosymplectic 2-form Ω fulfills the following *integrality condition*:

$$[qs] \equiv [qs]\big([\frac{m}{\hbar}\Omega]\big) \in i(H^2(\boldsymbol{E}, \mathbb{Z})) \subset H^2(\boldsymbol{E}, \mathbb{R}) \ . \quad \square$$

The first classification result shows the structure of $\mathcal{QB}$.

THEOREM 2.3. – *The set $\mathcal{QB} \subset \mathcal{L}(\boldsymbol{E})$ of quantum bundles compatible with Ω is the set*

$$i^{-1}([qs]) \subset H^2(\boldsymbol{E}, \mathbb{Z}),$$

hence $\mathcal{QB}$ has a natural structure of affine space associated with the abelian group $\ker i \subset H^2(\boldsymbol{E}, \mathbb{Z})$.

Proof. – The first part of the statement comes directly from the proof of the existence theorem; the rest of the statement is trivial. Q.E.D.

Let $[\boldsymbol{Q}, \mathrm{C}], [\boldsymbol{Q}', \mathrm{C}'] \in \mathcal{QS}[\boldsymbol{Q}]$, $f : \boldsymbol{Q} \to \boldsymbol{Q}'$ be an equivalence and f_* be the induced map on connections. Then we have

$$\mathrm{C}' - f_*\mathrm{C} = -2\pi i D \otimes \mathbf{i}, \tag{14}$$

where D is a closed 1-form on $\boldsymbol{E}$, and $[\boldsymbol{Q}, \mathrm{C}] = [\boldsymbol{Q}', \mathrm{C}']$ if and only if

$$D = \frac{1}{2\pi i}\frac{dc}{c},$$

where $c : \boldsymbol{E} \to U(1)$.

LEMMA 2.2. – *There exists an abelian group isomorphism*

$$H^1(\boldsymbol{E}, \mathbb{Z}) \to \left\{ \frac{1}{2\pi i}\frac{dc}{c} | c : \boldsymbol{E} \to U(1) \right\} .$$

Proof. – Using a procedure similar to the proof of the existence theorem we can prove that

$$\left\{ \frac{1}{2\pi i} \frac{dc}{c} | c : \boldsymbol{E} \to U(1) \right\}$$

is isomorphic to $i(H^1(\boldsymbol{E}, \mathbb{Z}))$. A standard argument [30] shows that $i : H^1(\boldsymbol{E}, \mathbb{Z}) \to H^1(\boldsymbol{E}, \mathbb{R})$ is an injective morphism. Q.E.D.

Now, we are able to classify the (inequivalent) quantum structures having equivalent quantum bundles.

THEOREM 2.4. – *Let* $[\boldsymbol{Q}] \in \mathcal{QB}$. *Then the set* $\mathcal{QS}[\boldsymbol{Q}]$ *has a natural structure of affine space associated with the abelian group*

$$H^1(\boldsymbol{E}, \mathbb{R}) / H^1(\boldsymbol{E}, \mathbb{Z}) .$$

If $[D] \in H^1(\boldsymbol{E}, \mathbb{R}) / H^1(\boldsymbol{E}, \mathbb{Z})$ *and* $[\boldsymbol{Q}, \mathrm{C}] \in \mathcal{QS}[\boldsymbol{Q}]$, *then the affine space operation is defined by*

$$[\boldsymbol{Q}, \mathrm{C}] \cdot [D] := [\boldsymbol{Q}, \mathrm{C} - 2\pi i D \otimes \mathrm{i}] .$$

The structure of the set $\mathcal{QS}$ is easily recovered from its definition and the above two theorems. Let us set

$$p : \mathcal{QS} \to \mathcal{QB} : [\boldsymbol{Q}, \mathrm{C}] \mapsto [\boldsymbol{Q}] ; \tag{15}$$

p is a surjective map.

THEOREM 2.5. – *There exists a pair of bijections* $(B, \overline{B})$ *such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{QS} & \xrightarrow{B} & H^1(\boldsymbol{E}, \mathbb{R}) / H^1(\boldsymbol{E}, \mathbb{Z}) \times \ker i \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ \mathcal{QB} & \xrightarrow{\overline{B}} & \ker i \end{array}$$

In concrete applications it is preferable to express the product group in the above diagram in a more compact way. A standard cohomological argument [7, 30] yields the exact sequence

$$0 \to H^1(\boldsymbol{E}, \mathbb{R}) / H^1(\boldsymbol{E}, \mathbb{Z}) \to H^1(\boldsymbol{E}, U(1)) \xrightarrow{\delta_1} \ker i \to 0,$$

where δ_1 is the Bockstein morphism. So, for every equivalence class $[\boldsymbol{Q}] \in \ker i$ the set ${\delta_1}^{-1}([\boldsymbol{Q}])$ has a natural structure of affine space associated with $H^1(\boldsymbol{E}, \mathbb{R}) / H^1(\boldsymbol{E}, \mathbb{Z})$.

COROLLARY 2.1. – *The set of quantum structures is in bijection with the abelian group $H^1(\boldsymbol{E}, U(1))$. If $\boldsymbol{E}$ is simply connected, then there exists only one equivalence class of quantum structures.*

Proof. – The first assertion is due to the structure of the map δ_1. The last assertion follows from the natural isomorphism $H^1(\boldsymbol{E}, U(1)) \simeq \mathrm{Hom}(\pi_1(\boldsymbol{E}), U(1))$. Q.E.D.

Examples of quantum structures

From a physical viewpoint, it is interesting to study concrete examples of Galilei general relativistic classical spacetimes, and investigate the existence and classification of quantum structures over such spacetimes.

Example 2.1 *(Newtonian spacetimes).* – The Newtonian spacetime has been introduced in [10], here we will just recall the basic facts. We assume further hypotheses on $\boldsymbol{E}$, g, $K^\natural$ and F.

– We assume that $\boldsymbol{E}$ is an affine space associated with a vector space $\bar{\boldsymbol{E}}$, and t is an affine map associated with a linear map $\bar{t} : \bar{\boldsymbol{E}} \to \mathbb{T}$.

It turns out that $V\boldsymbol{E} \simeq \boldsymbol{E} \times \boldsymbol{P} \to \boldsymbol{E}$ is a trivial bundle, where $\boldsymbol{P} := \ker \bar{t}$.

– We assume that g is a scaled metric $g \in \mathbb{L}^2 \otimes \boldsymbol{P}^* \otimes \boldsymbol{P}^*$ on the vector space $\boldsymbol{P}$.

Denote by $K^{\|}$ the natural flat connection on $\boldsymbol{E}$ induced by the affine structure. Clearly, we have $\nabla[K^{\|\prime}]g = 0$, where $K^{\|\prime}$ is the restriction of $K^{\|}$ to the bundle $V\boldsymbol{E} \to \boldsymbol{E}$.

–We assume that the restrictions $K^{\natural\prime}$ and $K^{\|\prime}$ of $K^\natural$ and $K^{\|}$ to the bundle $V\boldsymbol{E} \to \boldsymbol{E}$ coincide.

–We assume $F = 0$.

The above hypotheses imply

$$K^\natural = K^{\|} + dt \otimes dt \otimes N,$$

where $N : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes V\boldsymbol{E}$, with coordinate expression $N = N_0{}^i{}_0\, u^0 \otimes u^0 \otimes \partial_i$. Hence, the law of particle motion takes the form

$$\nabla[\gamma[K^{\|}]]j_1 s = N \circ s\,;$$

in coordinates, $\partial_{00} s^i = N^i \circ s$.

In this case, $\boldsymbol{E}$ is topologically trivial, hence $H^2(\boldsymbol{E}, \mathbb{R}) = \{0\}$, so that the integrality condition is fulfilled. Corollary 2.1 implies that there is only one equivalence class of quantum structures. □

Example 2.2 *(Spherically symmetric exact solution)*. – In [27] we found that, in the case $F = 0$, under spherical symmetry assumptions on $\boldsymbol{E}$, g and $K^\natural$, there exists a unique Galilei classical spacetime. Also, we found a unique equivalence class of quantum structures under spherical symmetry hypotheses.

Here, we repeat some of the constructions and give a simplified and improved version of the results obtained so far.

– We assume that $\boldsymbol{E} \to \boldsymbol{T}$ is a bundle, and $s : \boldsymbol{T} \to \boldsymbol{E}$ is a global section.

– We assume that each fibre of $\boldsymbol{E} \to \boldsymbol{T}$ endowed with the restriction of g is a complete, spherically symmetric Riemannian manifold in the sense of [27].

The above assumptions on $\boldsymbol{E}$ and g imply that $\boldsymbol{E} \to \boldsymbol{T}$ is (not naturally) isometric to $\boldsymbol{T} \times \boldsymbol{P} \to \boldsymbol{T}$, where $\boldsymbol{P}$ is a Euclidean vector space, the isometry being provided by a complete isometric observer. Let $\boldsymbol{E}' := \boldsymbol{E} \setminus s(\boldsymbol{T})$. Then we have the natural splitting $\boldsymbol{E}' \simeq \mathbb{L} \times \boldsymbol{S}$, where $\mathbb{L}$ represents the distance from the origin in each fibre and $\boldsymbol{S} \to \boldsymbol{T}$ is a bundle whose fibres represent space-like directions. Now, by recalling remark 1.1, we implement the intuitive idea of spherically symmetric gravitational field as follows.

– We assume that there exists a complete isometric observer o (which is said to be a *spherically symmetric observer*) such that $\Phi[o]$ is a scaled 2-form on $\mathbb{L} \times \boldsymbol{T}$.

As a solution of the field equations in the vacuum we obtain a unique spherically symmetric gravitational field $K^\natural$ defined on $\boldsymbol{E}'$. Namely, we have

$$K^\natural = K^{||} + dt \otimes dt \otimes N^\natural,$$

where $K^{||}$ is the natural flat connection on $\boldsymbol{E}'$ and $N^\natural = \frac{\boldsymbol{k}}{r^2}$ where $\boldsymbol{k} : \boldsymbol{T} \to \mathbb{T}^{-2} \otimes \mathbb{L}^3$ and r is the space-like distance from the origin. Moreover, we have the coordinate expression

$$\Phi[o] = 2o^*\Omega[K^\natural] = -2u^0 \otimes \frac{\boldsymbol{k}}{r^2} d^0 \wedge d^r.$$

By the way, there exists a unique spherically symmetric observer.

We can easily generalise the above result to the case $F \neq 0$, obtaining a Coulomb-like field.

Now, by a comparison with the classical Newton's law of gravitation, if we assume that the field $K^\natural$ is generated by a particle (m, q), then we can assume $\boldsymbol{k} = -km$, where $k \in (\mathbb{T}^+)^{-2} \otimes \mathbb{L}^3 \otimes \mathbb{M}^*$ is the *gravitational coupling constant* (the minus sign is chosen in order to have an attractive force).

We choose the global potential $\alpha[o] = +k(u_0)m/(\hbar r)\, d^0$ of the form $m/\hbar\Phi[o]$, according to $2d\alpha[o] = m/\hbar\Phi[o]$, and introduce the quantum connection

$$\mathrm{C} := \mathrm{C}^{\parallel} + i\tau\mathbf{i},$$

on the quantum bundle $\boldsymbol{E} \times \mathbb{C}$, where

$$\tau = -\frac{m}{2\hbar}\, u^0 g_{ij} y_0^i y_0^j d^0 + \frac{m}{\hbar}\, u^0 g_{ij} y_0^j dy^i + \frac{k(u_0)m}{\hbar r} d^0.$$

The integrality condition is clearly fulfilled, and, by Corollary 2.1, there is a unique equivalence class of quantum connections.

The result is the same obtained in [27], but with weaker hypotheses. There, we made hypotheses of spherical symmetry on the quantum bundle and the quantum connection; here, we did not need this. Moreover, the results can be easily generalised to the case of a spherically symmetric (Coulomb) field $F \neq 0$. □

Example 2.3 *(Dirac's monopole).* – Usually, Dirac's monopole is defined to be a certain type of magnetic field on Minkowski spacetime. Here, we define and study it in the Galilei's case.

– We assume on $\boldsymbol{E}$ and g the same hypotheses of the Newtonian case; moreover, we assume that $\boldsymbol{E}$ is endowed with the flat gravitational field $K^{\natural} \equiv K^{\parallel}$.

– We assume an *inertial motion* $s : \boldsymbol{T} \to \boldsymbol{E}$, i.e. a motion which is also an affine map.

The motion s induces a complete isometric observer o by means of the translations of $\boldsymbol{E}$, hence an isometric splitting $\boldsymbol{E} \simeq \boldsymbol{T} \times \boldsymbol{P}$, where $\boldsymbol{P}$ is a Euclidean vector space, and an isometric splitting

$$\boldsymbol{E}' := \boldsymbol{E} \setminus \{s(\boldsymbol{T})\} \to \mathbb{T} \times \mathbb{L} \times S,$$

where $\mathbb{L}$ represents the distance from the origin and S is the space of directions. The manifold S has a natural metric such that any $l \in \mathbb{L}$ yields an isometry of S with the unit sphere in $\boldsymbol{P}$. The scaled multiples of the volume form ν on S are natural candidates of electromagnetic field.

– We assume a particle (m, q). Moreover, we assume the magnetic field

$$F := \mu\nu : S \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \overset{2}{\wedge} T^*S\ , \tag{16}$$

where $\mu \in \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}$ is assumed to be the *magnetic charge* of the monopole.

The coordinate expression of F with respect to polar coordinates turns out to be

$$F = \mu \sin \vartheta d^{\vartheta} \wedge d^{\varphi}.$$

Of course, we have $dF = 0$, and a direct computation shows that $\operatorname{div}^{\natural} F = 0$.

We have

$$[\frac{m}{\hbar}\Omega] = [\frac{m}{\hbar}\Omega[K^{\|}] + \frac{m}{\hbar}\frac{q}{2m}F] = [\frac{q\mu}{2\hbar}\nu]. \tag{17}$$

A computation [8, p. 164] shows that if $(q\mu)/\hbar \in \mathbb{Z}$ then $[\frac{m}{\hbar}\Omega]$ fulfills the integrality condition.

Being spacetime topologically equivalent to $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, Corollary 2.1 yields that for any $(q\mu)/\hbar \in \mathbb{Z}$ there exists a unique equivalence class of quantum structures compatible with F.

It is interesting to note that if $\mu' \in \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}$ with $\mu \neq \mu'$ and $(q\mu')/\hbar \in \mathbb{Z}$, then the respective quantum bundles are not isomorphic; the same holds by considering another particle (m', q'), with $q \neq q'$ and $(q'\mu)/\hbar \in \mathbb{Z}$. In particular, if $q \neq 0$, then the class of quantum bundles compatible with F is not the trivial class. So, this is a first example of non-trivial quantum structure on a spacetime with absolute time.

We remark that there exists a purely gravitational example of such a non-trivial situation provided by the nonrelativistic limit of the Taub-NUT solution, and leading to quantisation of mass [3]. □

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