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Asymptotic observables for N-body Stark Hamiltonians

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ABSTRACT. – We prove the existence of some asymptotic observables for N-body Stark Hamiltonians, and study their spectral properties, in particular, their relation to the spectrum of the Hamiltonian H. © Elsevier, Paris

Key words: Asymptotic observables N-body Stark Hamiltonians, N-body scattering theory, asymptotic completeness, spectral property.

RÉSUMÉ. – Nous prouvons l'existence de quelques observables asymptotiques pour le hamiltonien de Stark à N-corps, et étudions leurs caractères spectraux, surtout leur relation avec le spectre hamiltonien H. © Elsevier, Paris

1. INTRODUCTION

In this paper, we study some asymptotic observables for N-body Stark Hamiltonians.

We consider a system of N particles moving in a given constant electric field $\mathcal{E} \in \mathbf{R}^d$, $\mathcal{E} \neq 0$. Let m_j, e_j and $r_j \in \mathbf{R}^d$, $1 \leq j \leq N$, denote the mass, charge and position vector of the j-th particle, respectively. The N particles under consideration are supposed to interact with one another through the pair potentials $V_{jk}(r_j - r_k)$, $1 \le j < k \le N$. Then the total Hamiltonian for such a system is described by

$$\tilde{H} = \sum_{1 \le j \le N} \left\{ -\frac{1}{2m_j}\Delta_{r_j} - e_j \mathcal{E} \cdot r_j \right\} + V,$$

where $\xi \cdot \eta = \sum_{j=1}^{d} \xi_j \eta_j$ for ξ, $\eta \in \mathbf{R}^d$ and the interaction V is given as the sum of the pair potentials

$$V = \sum_{1 \le j < k \le N} V_{jk}(r_j - r_k).$$

As usual, we consider the Hamiltonian $\tilde{H}$ in the center-of-mass frame. We introduce the metric $\langle r, \tilde{r} \rangle = \sum_{j=1}^{N} m_j r_j \cdot \tilde{r}_j$ for $r = (r_1, \ldots, r_N)$ and $\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_N) \in \mathbf{R}^{d \times N}$. We use the notation $|r| = \langle r, r \rangle^{1/2}$. Let X and $X_{\rm cm}$ be the configuration spaces equipped with the metric $\langle \cdot, \cdot \rangle$, which are defined by

$$X = \left\{ r \in \mathbf{R}^{d \times N} \,\middle|\, \sum_{1 \le j \le N} m_j r_j = 0 \right\},$$

$$X_{\rm cm} = \left\{ r \in \mathbf{R}^{d \times N} \,\middle|\, r_j = r_k \text{ for } 1 \le j < k \le N \right\}.$$

These two subspaces are mutually orthogonal. We denote by $\pi : \mathbf{R}^{d \times N} \to X$ and $\pi_{\rm cm} : \mathbf{R}^{d \times N} \to X_{\rm cm}$ the orthogonal projections onto X and $X_{\rm cm}$, respectively. For $r \in \mathbf{R}^{d \times N}$, we write $x = \pi r$ and $x_{\rm cm} = \pi_{\rm cm} r$, respectively. Let $E \in X$ and $E_{\rm cm} \in X_{\rm cm}$ be defined by

$$E = \pi\left(\frac{e_1}{m_1}\mathcal{E}, \ldots, \frac{e_N}{m_N}\mathcal{E}\right), \quad E_{\rm cm} = \pi_{\rm cm}\left(\frac{e_1}{m_1}\mathcal{E}, \ldots, \frac{e_N}{m_N}\mathcal{E}\right),$$

respectively. Then the total Hamiltonian $\tilde{H}$ is decomposed into $\tilde{H} = H \otimes Id + Id \otimes T_{\rm cm}$, where Id is the identity operator, H is defined by

$$H = -\frac{1}{2}\Delta - \langle E, x \rangle + V \qquad \text{on } L^2(X),$$

$T_{\rm cm}$ denotes the free Hamiltonian $T_{\rm cm} = -\Delta_{\rm cm}/2 - \langle E_{\rm cm}, x_{\rm cm} \rangle$ acting on $L^2(X_{\rm cm})$, and Δ (resp. $\Delta_{\rm cm}$) is the Laplace-Beltrami operator on X (resp. $X_{\rm cm}$). We assume that $|E| \ne 0$. This is equivalent to saying that $e_j/m_j \ne e_k/m_k$ for at least one pair (j, k). Then H is called an N-body Stark Hamiltonian in the center-of-mass frame.

A non-empty subset of the set $\{1, \dots, N\}$ is called a cluster. Let C_j, $1 \le j \le m$, be clusters. If $\bigcup_{1 \le j \le m} C_j = \{1, \dots, N\}$ and $C_j \cap C_k = \emptyset$ for $1 \le j < k \le m$, $a = \{C_1, \dots, C_m\}$ is called a cluster decomposition. We denote by $\#(a)$ the number of clusters in a. We denote by $\mathcal{A}$ the set of cluster decompositions. We let a, $b \in \mathcal{A}$. If b is obtained as a refinement of a, that is, if each cluster in b is a subset of a cluster in a, we say $b \subset a$, and its negation is denoted by $b \not\subset a$. We note that $a \subset a$ is regarded as a refinement of a itself. If, in particular, b is a strict refinement of a, that is, if $b \subset a$ and $b \ne a$, this relation is denoted by $b \subsetneq a$. We denote by $\alpha = (j, k)$ the $(N-1)$–cluster decomposition $\{(j,k), (1), \dots, (\widehat{j}), \dots, (\widehat{k}), \dots, (N)\}$.

Next we define the two subspaces X^a and X_a of X as

$$X^a = \left\{ r \in X \;\middle|\; \sum_{j \in C} m_j r_j = 0 \text{ for each cluster } C \text{ in } a \right\},$$
$$X_a = \left\{ r \in X \;\middle|\; r_j = r_k \text{ for each pair } \alpha = (j,k) \subset a \right\}.$$

We note that X^α is the configuration space for the relative position of j-th and k-th particles. Hence we can write $V_\alpha(x^\alpha) = V_{jk}(r_j - r_k)$. These spaces are mutually orthogonal and span the total space $X = X^a \oplus X_a$, so that $L^2(X)$ is decomposed as the tensor product $L^2(X) = L^2(X^a) \otimes L^2(X_a)$. We also denote by $\pi^a : X \to X^a$ and $\pi_a : X \to X_a$ the orthogonal projections onto X^a and X_a, respectively, and write $x^a = \pi^a x$ and $x_a = \pi_a x$ for a generic point $x \in X$. The intercluster interaction I_a is defined by

$$I_a(x) = \sum_{\alpha \not\subset a} V_\alpha(x^\alpha),$$

and the cluster Hamiltonian

$$H_a = H - I_a = -\frac{1}{2}\Delta - \langle E, x \rangle + V^a, \quad V^a(x^a) = \sum_{\alpha \subset a} V_\alpha(x^\alpha),$$

governs the motion of the system broken into non-interacting clusters of particles. Let $E^a = \pi^a E$ and $E_a = \pi_a E$. Then the operator H_a acting on $L^2(X)$ is decomposed into

$$H_a = H^a \otimes Id + Id \otimes T_a \qquad \text{on } L^2(X^a) \otimes L^2(X_a),$$

where H^a is the subsystem Hamiltonian defined by

$$H^a = -\frac{1}{2}\Delta^a - \langle E^a, x^a \rangle + V^a \qquad \text{on } L^2(X^a),$$

T_a is the free Hamiltonian defined by

$$T_a = -\frac{1}{2}\Delta_a - \langle E_a, x_a \rangle \qquad \text{on } L^2(X_a),$$

and Δ^a (resp. Δ_a) is the Laplace-Beltrami operator on X^a (resp. X_a). By choosing the coordinates system of X, which is denoted by $x = (x^a, x_a)$, appropriately, we can write $\Delta^a = |\nabla^a|^2$ and $\Delta_a = |\nabla_a|^2$, where $\nabla^a = \partial_{x^a} = \partial/\partial x^a$ and $\nabla_a = \partial_{x_a} = \partial/\partial x_a$ are the gradients on X^a and X_a, respectively. We note that we denote by x^a (resp. x_a) a vector in X^a (resp. X_a) as well as the coordinates system of X^a (resp. X_a). We write $p = -i\nabla$, $p^a = -i\nabla^a$ and $p_a = -i\nabla_a$.

We now state the precise assumption on the pair potentials. Let c be a maximal element of the set $\{a \in \mathcal{A} \mid E^a = 0\}$ with respect to the relation $\subset$. As is easily seen, such a cluster decomposition uniquely exists and it follows that $E^\alpha = 0$ if $\alpha \subset c$, and $E^\alpha \neq 0$ if $\alpha \not\subset c$. Thus the potential V_α with $\alpha \not\subset c$ (resp. $\alpha \subset c$) describes the pair interaction between two particles with $e_j/m_j \neq e_k/m_k$ (resp. $e_j/m_j = e_k/m_k$). If, in particular, $e_j/m_j \neq e_k/m_k$ for any $j \neq k$, then c becomes the N-cluster decomposition. We make different assumptions on V_α according as $\alpha \not\subset c$ or $\alpha \subset c$. We assume that $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$ is a real-valued function and has the decay property

$(V.1)$ $\quad |\partial^\beta_{x^\alpha} V_\alpha(x^\alpha)| = O(|x^\alpha|^{-(\rho'+|\beta|)}), \quad \alpha \subset c, \quad \rho' > 0,$

$(V.2)$ $\quad |\partial^\beta_{x^\alpha} V_\alpha(x^\alpha)| = O(|x^\alpha|^{-(\rho+|\beta|/2)}), \quad \alpha \not\subset c, \quad \rho > 0,$

$(V.3)$ $\quad |\partial^\beta_{x^\alpha} V_\alpha(x^\alpha)| = O(|x^\alpha|^{-(\rho+\mu|\beta|)}), \quad \alpha \not\subset c, \quad \rho, \mu > 0$ with $\rho + \mu > 1$.

We should note that we may allow that the potentials have some local singularities, in particular, Coulomb singularities if $d \geq 3$ (see [HMS1]). But, for the simplicity of the argument below, we do not deal with the singularities. Under this assumption, all the Hamiltonians defined above are essentially self-adjoint on C_0^∞. We denote their closures by the same notations. Throughout the whole exposition, the notations c, ρ', ρ and μ are used with the meanings described above. We make some remarks about potentials. For $\alpha \subset c$, if $\rho' > 1$ (resp. $0 < \rho' \leq 1$), V_α is called a short-range (resp. long-range) potential. For $\alpha \not\subset c$, if $\rho > 1/2$ (resp. $0 < \rho \leq 1/2$), V_α is called a short-range (resp. long-range) potential. If we consider the problem of the asymptotic completeness for long-range N-body Stark Hamiltonians, we should study the Dollard-type (resp. Graf-type) modified wave operators under the assumptions $(V.1)$ and $(V.2)$ (resp. $(V.1)$ and $(V.3)$) (cf. [A1], [AT1-2], [Gr2], [JO], [JY], [HMS2] and [W1-2]).

We assume that $a \subset c$. Then the subsystem Hamiltonian H^a does not have the Stark effect, that is, $E^a = 0$. Hence it may have bound states in $L^2(X^a)$. We denote by $\sigma_{pp}(H^a)$ the pure point spectrum of H^a, and define $\mathcal{T}_a = \bigcup_{b \subsetneq a} \sigma_{pp}(H^b)$ and $\mathcal{E}_a = \bigcup_{b \subset a} \sigma_{pp}(H^b)$. We note that $\sigma_{pp}(H^a) = \{0\}$ if $\#(a) = N$. We also denote the direction of E by $\omega = E/|E|$ and write $z = \langle x, \omega \rangle$. We should note that $z = \langle x_a, \omega \rangle$ because of $\omega^a = 0$. We set

$$X_{\|} = \{x \in X \mid x = \gamma\omega \text{ for } \gamma \in \mathbf{R}\}, \quad X_{\perp} = X \ominus X_{\|},$$

$x_{\|} = z\omega \in X_{\|}$ and $x_{\perp} = x - x_{\|} \in X_{\perp}$, and write $x_{a,\perp} = \pi_a x_{\perp}$. Then we can write $x_a = (x_{a,\perp}, x_{\|})$. We also write $\xi_a = (\xi_{a,\perp}, \xi_{\|})$ for the coordinates dual to $x_a = (x_{a,\perp}, x_{\|})$ and denote by $p_a = -i\nabla_a = (p_{a,\perp}, p_{\|})$ the corresponding velocity operator. If we write $\partial_{\|} = \omega\partial_z$, we see that $p_{\|} = -i\partial_{\|}$ and $p_{a,\perp} = p_a - p_{\|}$. Let I_a^c be the intercluster interaction obtained from H^c:

$$I_a^c(x) = I_a^c(x^c) = \sum_{\alpha \subset c,\, \alpha \not\subset a} V_\alpha(x^\alpha).$$

For N-body long-range scattering, some asymptotic observables are very useful for showing the asymptotic completeness for the systems without the Stark effect. In particular, the asymptotic energy has been used by Enss [E], Sigal-Soffer [SS1-2], Dereziński [D2] and Gérard [G], and the asymptotic velocity has been used by Enss [E], Dereziński [D1-2] and Zielinski [Z]. Especially, Dereziński [D2] studied the spectral properties of the asymptotic energy and the asymptotic velocity, too. We concern ourselves with the asymptotic observables for N-body Stark Hamiltonians.

We now formulate the results obtained in this paper. We use the following convention for smooth cut-off functions F with $0 \le F \le 1$, which is often used throughout the discussion below. For sufficiently small $\delta > 0$, we define

$$\begin{aligned} F(s \le d) &= 1 \quad \text{for } s \le d - \delta, \quad = 0 \quad \text{for } s \ge d, \\ F(s \ge d) &= 1 \quad \text{for } s \ge d + \delta, \quad = 0 \quad \text{for } s \le d, \\ F(s = d) &= 1 \quad \text{for } |s - d| \le \delta, \quad = 0 \quad \text{for } |s - d| \ge 2\delta \end{aligned}$$

and $F(d_1 \le s \le d_2) = F(s \ge d_1)F(s \le d_2)$. The choice of $\delta > 0$ does not matter to the argument below, but we sometimes write F_δ for F when we want to clarify the dependence on $\delta > 0$.

THEOREM 1.1. – *Suppose that V satisfies (V.1), and (V.2) or (V.3). Let $f \in C_\infty(X)$, $C_\infty(X)$ being the space of continuous functions on X*

vanishing at infinity. Then the following strong limits exist:

$$s - \lim_{t\to\pm\infty} e^{itH} f\left(\frac{p - Et}{t}\right) e^{-itH} = f(0), \tag{1.1}$$

$$s - \lim_{t\to\pm\infty} e^{itH} f\left(\frac{x - \frac{E}{2}t^2}{t^2}\right) e^{-itH} = f(0). \tag{1.2}$$

This result implies that $\||p - Et|e^{-itH}\psi\| = o(|t|)$ and $\||x - Et^2/2|e^{-itH}\psi\| = o(|t|^2)$ as $t \to \pm\infty$ for $\psi \in \mathcal{D}$, where $\mathcal{D}$ is some appropriate dense set of $L^2(X)$ (see [Gr2] for the two-body case). This fact was pointed out by [A2] in the term of propagation estimates. In particular, (1.2) implies that the particles asymptotically concentrate in any conical neighborhood of E, and this fact has played an important role for the proof of the asymptotic completeness for long-range N-body Stark Hamiltonians given by [A1], [AT1-2] and [HMS2]. Theorem 1.1 can be proved by the results of [A2]. The following theorem is a refinement of the above properties.

THEOREM 1.2. – *Suppose that* V *satisfies* (V.1), *and* (V.2) *or* (V.3). *Let* $f_1 \in C_\infty(X_\perp)$, $f_2 \in C_\infty(X^c)$, $g_1 \in C_\infty(X_\|)$ *and* $g_2 \in C_\infty(X_c)$. *Then the following strong limits exist:*

$$s - \lim_{t\to\pm\infty} e^{itH} f_1\left(\frac{x_\perp}{t}\right) e^{-itH}, \tag{1.3}$$

$$s - \lim_{t\to\pm\infty} e^{itH} f_2\left(\frac{x^c}{t}\right) e^{-itH}, \tag{1.4}$$

$$s - \lim_{t\to\pm\infty} e^{itH} g_1(p_\| - Et) e^{-itH}, \tag{1.5}$$

$$s - \lim_{t\to\pm\infty} e^{itH} g_1\left(\frac{x_\| - \frac{E}{2}t^2}{t}\right) e^{-itH}, \tag{1.6}$$

$$s - \lim_{t\to\pm\infty} e^{itH} g_2(p_c - Et) e^{-itH}, \tag{1.7}$$

$$s - \lim_{t\to\pm\infty} e^{itH} g_2\left(\frac{x_c - \frac{E}{2}t^2}{t}\right) e^{-itH}. \tag{1.8}$$

(1.5) (*resp.* (1.7)) *equals* (1.6) (*resp.* (1.8)). *Moreover, there exists a unique vector in* $X_\perp$ *(resp.* X^c*) of commuting self-adjoint operators* $P_\perp^\pm(H)$ *(resp.* $P^{c,\pm}(H)$*) such that* (1.3) (*resp.* (1.4)) *equals* $f_1(P_\perp^\pm(H))$ *(resp.* $f_2(P^{c,\pm}(H))$*).* $P_\perp^\pm(H)$ *and* $P^{c,\pm}(H)$ *commute with* H.

This result implies that $\| |p_c - Et| e^{-itH}\psi\| \leq C_\psi$ and $\| |x - Et^2/2| e^{-itH}\psi\| \leq C'_\psi |t|$ as $t \to \pm\infty$ for some positive constants C_ψ and C'_ψ. In particular, we should note that the asymptotic velocity $P_\perp^\pm(H)$ perpendicular to the vector E exists and commutes with H. Here we may construct the asymptotic observables by following the argument of Dereziński [D1-2]: Denoting (1.3) by $\gamma(f_1)$, for any open set $\Theta \subset X_\perp$, we define

$$E_\Theta \equiv \sup\left\{\gamma(f_1) \;\middle|\; f_1 \in C_\infty(X_\perp),\, 0 \leq f_1 \leq 1,\, \operatorname{supp} f_1 \subset \Theta\right\}.$$

Clearly, E_Θ are orthogonal projections that satisfy

$$E_{\Theta_1} E_{\Theta_2} = E_{\Theta_1 \cap \Theta_2}$$

for any open sets Θ_1 and Θ_2. As usual, we may extend the definition of E_Θ to arbitrary Borel subsets $\Theta \subset X_\perp$. We obtain a map

$$\Theta \mapsto E_\Theta$$

defined for every Borel subset $\Theta \subset X_\perp$ that satisfies the following conditions:

(1) E_Θ is an orthogonal projection,

(2) $E_\emptyset = 0$,

(3) if $\Theta = \bigcup_{n=1}^\infty \Theta_n$ and for $j \neq k$ we have $\Theta_j \cap \Theta_k = \emptyset$, then

$$E_\Theta = s - \lim_{N \to \infty} \sum_{n=1}^{N} E_{\Theta_n},$$

(4) $E_{\Theta_1} E_{\Theta_2} = E_{\Theta_1 \cap \Theta_2}$.

Since, as usual, for any Borel function g on $X_\perp$, we may define the integral

$$\int g(x_\perp)\, dE(x_\perp),$$

we may construct the asymptotic velocity $P_\perp^\pm(H)$ as

$$P_\perp^\pm(H) \equiv \int x_\perp\, dE(x_\perp).$$

The notion of the asymptotic velocity is very useful for showing the asymptotic completeness for N-body long-range scattering without the Stark effect, and, in fact, J.Dereziński [D2] constructed the asymptotic velocity and used it to prove the problem. Also, he showed some properties of it, in particular, the relation between the asymptotic energy and it. He also showed the existence of the asymptotic "intercluster momentum" $D_a^{\pm}(H_{M,a,W})$ for the time-dependent Hamiltonian $H_{M,a,W}(t) = -\Delta/2 + V^a(x) + W(t,x)$, which is defined by

$$s - \lim_{t \to \pm\infty} U_{M,a,W}(t)^* g(p_a) U_{M,a,W}(t)$$

for $g \in C_\infty(X_a)$, where $U_{M,a,W}(t)$ is the propagator generated by $H_{M,a,W}(t)$. He studied the relation between the asymptotic velocity and the asymptotic "intercluster momentum". The property that (1.5) (resp. (1.7)) equals (1.6) (resp. (1.8)) is an analogue of his results. However, both for N-body Schrödinger operators and for N-body Stark Hamiltonians, we have not known the existence of the asymptotic "innercluster momentum" yet: For example, in the case of N-body Schrödinger operators, the asymptotic "innercluster momentum" should be defined by

$$s - \lim_{t \to \pm\infty} U_{M,a,W}(t)^* g(p^a) U_{M,a,W}(t)$$

for $g \in C_\infty(X^a)$. Thus we here consider the asymptotic velocity and "intercluster momentum" only.

Of course, in the way similar to the above one, we may construct the asymptotic velocity "$P_{\|}^{\pm}(H)$" parallel to E by virtue of Theorem 1.2. But it is easily seen that "$P_{\|}^{\pm}(H)$" cannot commute with H since $E \neq 0$. Then we need some alternative asymptotic observables for H to study the spectral properties of H in terms of the asymptotic observables for H. Now we consider some asymptotic energy for H. The following result is an analogue of Dereziński's result for N-body Stark Hamiltonians, but we have to require that V satisfies $(V.1)$, and $(V.2)$ with $\rho > 1/2$ or $(V.3)$.

THEOREM 1.3. – *Suppose that V satisfies $(V.1)$, and $(V.2)$ with $\rho > 1/2$ or $(V.3)$. Let $h \in C_\infty(\mathbf{R})$. Then there exist the following strong limits:*

$$s - \lim_{t \to \pm\infty} e^{itH} h(T_{\|}) e^{-itH}, \tag{1.9}$$

$$s - \lim_{t \to \pm\infty} e^{itH} h(T_c) e^{-itH}, \tag{1.10}$$

where $T_{\|} = p_z^2/2 - |E|z$. Moreover, there exists a unique self-adjoint operator $T_{\|}^{\pm}$ (resp. $T_c^{\pm}$) such that (1.9) *(resp.* (1.10)*) equals $h(T_{\|}^{\pm})$ (resp. $h(T_c^{\pm})$). $P_{\perp}^{\pm}(H)$ (resp. $P^{c,\pm}(H)$), $T_{\|}^{\pm}$ (resp. $T_c^{\pm}$) and H are mutually* *commutative. They have the following properties:*

$$\sigma(H, P_\perp^\pm(H), T_\|^\pm)$$
$$= \bigcup_{a\subset c} \{(\lambda, \xi_{a,\perp}, \lambda_\|) \mid \lambda = \frac{1}{2}\xi_{a,\perp}^2 + \lambda_\| + \tau,\ \xi_{a,\perp} \in X_{a,\perp}, \lambda_\| \in \mathbf{R},\ \tau \in \mathcal{E}_a\}, \tag{1.11}$$

$$\sigma(H, P^{c,\pm}(H), T_c^\pm)$$
$$= \bigcup_{a\subset c} \{(\lambda, \xi_a^c, \lambda_c) \mid \lambda = \frac{1}{2}(\xi_a^c)^2 + \lambda_c + \tau,\ \xi_a^c \in X_a^c,\ \lambda_c \in \mathbf{R},\ \tau \in \mathcal{E}_a\}, \tag{1.12}$$

where $X_{a,\perp} = X_\perp \ominus X^a$ *and* $X_a^c = X^c \ominus X^a$.

In §4, we will state a result analogous to this result under the assumption that V satisfies $(V.1)$ and $(V.2)$ with $0 < \rho \leq 1/2$.

The idea of the proofs of Theorems 1.2 and 1.3 is as follows: The key fact used in order to prove Theorems 1.2 and 1.3 is Theorem 3.3 (see §3.). This theorem is also the key fact for proving the asymptotic completeness for N-body Stark Hamiltonians in [AT2]. It implies that we can replace the propagator e^{-itH} generated by the full Hamiltonian H by the propagator $U_c(t)$ generated by the appropriate time-dependent Hamiltonian $H_c(t)$. Then we may change the original problem into the one in the frame accelerated by E (the moving frame). Consequently, we have only to study the asymptotic observables for the time-dependent N-body Schrödinger operator $H_{M,c}(t)$, which were studied by Dereziński [D2].

Throughout this paper, we consider the case when $t \to \infty$. Other cases can be treated similarly.

The plan of this paper is as follows: In §2, we collect the known results to be used in later sections. In §3, we prove Theorems 1.2 and 1.3. In §4, under the assumption that V satisfies $(V.1)$ and $(V.2)$ with $0 < \rho \leq 1/2$, we study a property analogous to the one of Theorem 1.3.

2. KNOWN RESULTS

In this section, we collect the known results to be used in later sections. First, we recall the spectral properties of N-body Stark Hamiltonians, which has been studied by Herbst-Møller-Skibsted [HMS1]. We use the following notations throughout this paper. Let $\omega = E/|E|$ be the direction of E. We denote the coordinate $z \in \mathbf{R}$ by $z = \langle x, \omega\rangle$, so that H is written as $H = -\Delta/2 - |E|\, z + V$. Let $A = \langle \omega, p\rangle = -i\partial_z$. We should note that

$$\langle z\rangle^{-1/2}\partial_j(H+i)^{-1},\ \langle z\rangle^{-1}\partial_j\partial_k(H+i)^{-1} : L^2(X) \to L^2(X)$$

are bounded, where ∂_j and ∂_k are any components of ∇.

THEOREM 2.1. – *Suppose that V satisfies* (V.1), *and* (V.2) *or* (V.3). *Then*

(1) H has no bound states.

(2) Let $0 < \sigma < |E|$. Then one can take $\delta > 0$ so small (uniformly in $\lambda \in \mathbf{R}$) that

$$F_\delta(H = \lambda) i[H, A] F_\delta(H = \lambda) \geq \sigma F_\delta(H = \lambda)^2. \tag{2.1}$$

Next we recall the almost analytic extension method due to Helffer and Sjöstrand [HeSj], which is useful in analyzing operators given by functions of self-adjoint operators. For two operators B_1 and B_2, we define

$$ad_{B_1}^0(B_2) = B_2, \quad ad_{B_1}^n(B_2) = [ad_{B_1}^{n-1}(B_2), B_1], \quad n \geq 1.$$

For $m \in \mathbf{R}$, let $\mathcal{F}^m$ be the set of functions $f \in C^\infty(\mathbf{R})$ such that

$$|f^{(k)}(s)| \leq C_k \langle s \rangle^{m-k}, \quad k \geq 0.$$

If $f \in \mathcal{F}^m$ with $m \in \mathbf{R}$, then there exists $\tilde{f} \in C^\infty(\mathbf{C})$ such that $\tilde{f}(s) = f(s)$ for $s \in \mathbf{R}$, $\operatorname{supp} \tilde{f}(\zeta) \subset \{\zeta \in \mathbf{C} : |Im\, \zeta| \leq \tilde{d}(1 + |Re\, \zeta|)\}$ for some $\tilde{d} > 0$ and

$$|\bar{\partial}_\zeta \tilde{f}(\zeta)| \leq C_M \langle \zeta \rangle^{m-1-M} |Im\, \zeta|^M, \quad M \geq 0.$$

Such a function $\tilde{f}(\zeta)$ is called an almost analytic extension of f. Let B be a self-adjoint operator. If $f \in \mathcal{F}^{-m}$ with $m > 0$, then $f(B)$ is represented by

$$f(B) = \frac{i}{2\pi} \int_{\mathbf{C}} \bar{\partial}_\zeta \tilde{f}(\zeta)(B - \zeta)^{-1}\, d\zeta \wedge d\bar{\zeta}.$$

For $f \in \mathcal{F}^m$ with $m \in \mathbf{R}$, we have the following formulas of the asymptotic expansion of the commutator:

$$\begin{aligned} [B_1, f(B)] &= \sum_{n=1}^{M-1} \frac{(-1)^{n-1}}{n!} ad_B^n(B_1) f^{(n)}(B) + R_M \\ &= \sum_{n=1}^{M-1} \frac{1}{n!} f^{(n)}(B) ad_B^n(B_1) + R'_M, \end{aligned}$$

$$R_M = \frac{1}{2\pi i} \int_{\mathbf{C}} \bar{\partial}_\zeta \tilde{f}(\zeta)(B - \zeta)^{-1} ad_B^M(B_1)(B - \zeta)^{-M}\, d\zeta \wedge d\bar{\zeta},$$

$$R'_M = \frac{(-1)^{M+1}}{2\pi i} \int_{\mathbf{C}} \bar{\partial}_\zeta \tilde{f}(\zeta)(B - \zeta)^{-M} ad_B^M(B_1)(B - \zeta)^{-1}\, d\zeta \wedge d\bar{\zeta},$$

R_M is bounded if there exists k such that $m + k < M$ and $ad_B^M(B_1)(B + i)^{-k}$ is bounded. Similarly, R'_M is bounded if there exists k such that

$m + k < M$ and $(B+i)^{-k} ad_B^M(B_1)$ is bounded. For the proof, see [G]. We use the above formulas frequently.

Next, we state the known results about asymptotic observables for N-body Hamiltonians without the Stark effect, which we will frequently use to prove Theorems 1.2 and 1.3. The results were obtained by Dereziński [D2] and used for showing the asymptotic completeness for N-body long-range scattering (see Sect. 4 of [D2]).

Let H_M be an N-body Hamiltonian without the Stark effect:

$$H_M = -\frac{1}{2}\Delta + V \text{ on } L^2(X), \quad V(x) = \sum_{\alpha} V_\alpha(x^\alpha),$$

where each $V_\alpha(x^\alpha)$ satisfies $(V.1)$. Then, as in §1, we define the cluster Hamiltonian $H_{M,a}$ and subsystem Hamiltonian H_M^a, $a \in \mathcal{A}$, as follows:

$$H_{M,a} = -\frac{1}{2}\Delta + V^a \text{ on } L^2(X), \quad V^a(x) = \sum_{\alpha \subset a} V_\alpha(x^\alpha),$$

$$H_M^a = -\frac{1}{2}\Delta^a + V^a \text{ on } L^2(X^a).$$

Here we recall that $V^a(x) = V^a(x^a)$. We introduce a time-dependent potential $W(t,x)$ which is a smooth real-valued function on $\mathbf{R} \times X$ such that

$$|\partial_x^\beta W(t,x)| \leq C_\beta \langle t\rangle^{-(\sigma+|\beta|)}, \quad t \geq 1 \tag{2.2}$$

for some $\sigma > 0$. Then we define time-dependent Hamiltonians

$$H_{M,W}(t) = H_M + W(t,x),$$
$$H_{M,a,W}(t) = H_{M,a} + W(t,x).$$

We denote by $U_{M,W}(t)$ (resp. $U_{M,a,W}(t)$) the propagator generated by $H_{M,W}(t)$ (resp. $H_{M,a,W}(t)$), where we say that $U(t)$ is the propagator generated by $H(t)$ if $\{U(t)\}_{t\geq 1}$ is a family of unitary operators such that for $\psi \in D(H(1))$, $\psi_t = U(t)\psi$ is a strong solution of $id\psi_t/dt = H(t)\psi_t$, $\psi_1 = \psi$.

The following theorem was proved by Dereziński [D2] (see Theorems 4.1, 4.2 and 4.3 of [D2]). The proof is based on the Graf's idea [Gr1], but we omit it.

THEOREM 2.2. – (1) *For any $h \in C_\infty(\mathbf{R})$, the following strong limits exist:*

$$s - \lim_{t\to\infty} U_{M,W}(t)^* h(H_M) U_{M,W}(t), \tag{2.3}$$

$$s - \lim_{t\to\infty} U_{M,a,W}(t)^* h(H_{M,a}) U_{M,a,W}(t), \tag{2.4}$$

$$s - \lim_{t\to\infty} U_{M,a,W}(t)^* h(H_M^a) U_{M,a,W}(t). \tag{2.5}$$

There exists a unique self-adjoint operator $H^+_{M,W}$ *(resp.* $H^+_{M,a,W}$, $H^{a,+}_{M,W}$*) such that* (2.3) *(resp.* (2.4), (2.5)*) equals* $h(H^+_{M,W})$ *(resp.* $h(H^+_{M,a,W})$, $h(H^{a,+}_{M,W})$*).*

(2) *For any* $g \in C_\infty(X_a)$, *there exist*

$$s-\lim_{t\to\infty} U_{M,a,W}(t)^* g(p_a) U_{M,a,W}(t), \tag{2.6}$$

$$s-\lim_{t\to\infty} U_{M,a,W}(t)^* g\Big(\frac{x_a}{t}\Big) U_{M,a,W}(t), \tag{2.7}$$

and they equal each other. There exists a unique vector in X_a *of commuting self-adjoint operators* $D^+_a(H_{M,a,W})$ *such that the limits* (2.6) *and* (2.7) *equal* $g(D^+_a(H_{M,a,W}))$. *Moreover,* $D^+_a(H_{M,a,W})$ *and* $H^+_{M,a,W}$ *commute, and*

$$H^+_{M,a,W} = H^{a,+}_{M,W} + \frac{1}{2}(D^+_a(H_{M,a,W}))^2. \tag{2.8}$$

(3) *Let* $J \in C_b(X)$, $C_b(X)$ *being the space of bounded continuous functions on* X. *Then there exists*

$$s-\lim_{t\to\infty} U_{M,W}(t)^* J\Big(\frac{x}{t}\Big) U_{M,W}(t). \tag{2.9}$$

There exists a unique vector in X *of commuting self-adjoint operators* $P^+(H_{M,W})$ *such that the limit* (2.9) *equals* $J(P^+(H_{M,W}))$. *Moreover,* $P^+(H_{M,W})$ *and* $H^+_{M,W}$ *commute, and*

$$D^+_a(H_{M,a,W}) = P^+(H_{M,a,W})_a. \tag{2.10}$$

(4) *When* $W(t,x) \equiv 0$,

$$E_{\{0\}}(P^+(H)) = E^{pp}(H). \tag{2.11}$$

Here $E_\Theta(P)$ *is the spectral projection of a vector in* X *of commuting self-adjoint operators* P *onto a Borel subset* Θ *of* X, *and* $E^{pp}(H)$ *is the eigenprojection of* H.

(5)

$$\sigma(H^+_{M,W}, P^+(H_{M,W})) = \bigcup_{a\in\mathcal{A}} \Big\{(\lambda,\xi_a) \;\Big|\; \lambda = \frac{1}{2}\xi_a^2 + \tau,\ \xi_a \in X_a,\ \tau \in \mathcal{E}_a\Big\}. \tag{2.12}$$

3. PROOF OF THEOREMS 1.2 AND 1.3

In this section, we prove Theorems 1.2 and 1.3. First we assume that V satisfies $(V.1)$, and $(V.2)$ or $(V.3)$. We begin with stating the propagation estimates for the propagator e^{-itH}, which were obtained by [AT2] (see Propositions 3.1, 3.2, 3.5 and 3.7 of [AT2]). We omit the proof.

PROPOSITION 3.1. – *Let $h \in C_0^\infty(\mathbf{R})$.*

(1) *Then there exists $M \gg 1$ dependent on h such that for $\psi \in L^2(X)$,*

$$\int_1^\infty \frac{dt}{t}\left\| F\left(\frac{\langle x\rangle}{t^2} = M\right) h(H) e^{-itH}\psi \right\|^2 \leq C\,\|\psi\|^2, \tag{3.1}$$

and, for $\psi \in \mathcal{S}(X)$, $\mathcal{S}(X)$ being the Schwartz space on X,

$$\int_1^\infty \frac{dt}{t}\left\| F\left(\frac{\langle x\rangle}{t^2} \geq M\right) h(H) e^{-itH}\psi \right\|^2 < \infty. \tag{3.2}$$

(2) *Let $0 < \nu < |E|$ and $L > 0$. Then for any $\psi \in L^2(X)$,*

$$\int_1^\infty \frac{dt}{t}\left\| F\left(-L \leq \frac{z}{t^2} \leq \frac{\nu}{2}\right) h(H) e^{-itH}\psi \right\|^2 \leq C\,\|\psi\|^2. \tag{3.3}$$

(3) *Let M be as in (1) and ν be as in (2). Fix $\epsilon_1 > 0$ and $r > 0$. Assume that $q \in S_0(X) = \{q \in C^\infty(X) \mid |\partial_x^\beta q(x)| \leq C_\beta \langle x\rangle^{-|\beta|}\}$ vanishes in $\Gamma(\omega, \epsilon_1, r) = \{x \in X \mid \langle \omega, x/|x|\rangle \geq 1 - \epsilon_1, |x| > r\}$, where $\omega = E/|E|$. Then*

$$\int_1^\infty \frac{dt}{t}\left\| F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right) F\left(\frac{\langle x\rangle}{t^2} \leq M\right) q h(H) e^{-itH}\psi \right\|^2 \leq C\,\|\psi\|^2. \tag{3.4}$$

(4) *Let M, ν and $q \in S_0(X)$ be as above. Let $\Phi(t)$ denote one of the following three operators*

$$F\left(\frac{\langle x\rangle}{t^2} \geq M\right),\quad F\left(\frac{z}{t^2} \leq \frac{\nu}{2}\right),\quad F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right) F\left(\frac{\langle x\rangle}{t^2} \leq M\right) q.$$

Then

$$s - \lim_{t\to\infty} \Phi(t) h(H) e^{-itH} = 0.$$

By taking account of this proposition and following the argument of [AT2], we introduce an auxiliary time-dependent Hamiltonian $H_c(t)$ which approximates the full Hamiltonian H:

Let $q_c \in S_0(X)$ be such that $q_c = 1$ in $\Gamma(\omega, \epsilon_1, |E|/3)$, and $q_c = 0$ outside $\Gamma(\omega, 2\epsilon_1, |E|/4)$. Let $\tilde{q}_c \in S_0(X)$ be such that $\tilde{q}_c = 1$ in $\Gamma(\omega, 2\epsilon_1, |E|/4)$, and $\tilde{q}_c = 0$ outside $\Gamma(\omega, 3\epsilon_1, |E|/5)$. By definition, it follows that $\tilde{q}_c q_c = q_c$. We define

$$\varphi_c(t,x) = F\left(\frac{\langle x\rangle}{t^2} \le M\right) F\left(\frac{z}{t^2} \ge \frac{|E|}{3}\right) q_c(x), \tag{3.5}$$

$$W_c(t,x) = W_c(t, x^c, x_c) = F\left(\frac{z}{t^2} \ge \frac{|E|}{4}\right) \tilde{q}_c(x) I_c(x). \tag{3.6}$$

We should note that $\varphi_c(t,x) I_c(x) = \varphi_c(t,x) W_c(t,x)$. By the assumption $(V.2)$ or $(V.3)$, W_c obeys the estimate

$$|\partial_t^m \partial_x^\beta W_c(t,x)| \le C_{m\beta} \langle t\rangle^{-m} (\langle t\rangle + \langle x\rangle^{1/2})^{-(2\rho+|\beta|)}, \quad t \ge 1. \tag{3.7}$$

Then we define the time-dependent Hamiltonian

$$H_c(t) = H_c + W_c(t,x), \tag{3.8}$$

and denote by $U_c(t)$ the propagator generated by $H_c(t)$, that is, $\{U_c(t)\}_{t\ge 1}$ is a family of unitary operators such that for $\psi \in D(H_c(1))$, $\psi_t = U_c(t)\psi$ is a strong solution of $i d\psi_t/dt = H_c(t)\psi_t$, $\psi_1 = \psi$.

Then we have the following proposition which is an analogue of Proposition 3.1 for the propagator $U_c(t)$. The result was obtained by [AT2] (see Propositions 4.1-4.4 of [AT2]). We omit the proof.

PROPOSITION 3.2. – *Let $h \in C_0^\infty(\mathbf{R})$.*

(1) *The there exists $M \gg 1$ dependent on h such that for $\psi \in L^2(X)$,*

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{\langle x\rangle}{t^2} = M\right) h(H_c(t)) U_c(t)\psi \right\|^2 \le C\,\|\psi\|^2, \tag{3.9}$$

and, for $\psi \in \mathcal{S}(X)$,

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{\langle x\rangle}{t^2} \ge M\right) h(H_c(t)) U_c(t)\psi \right\|^2 < \infty. \tag{3.10}$$

(2) *Let $0 < \nu < |E|$ and $L > 0$. Then for any $\psi \in L^2(X)$,*

$$\int_1^\infty \frac{dt}{t} \left\| F\left(-L \le \frac{z}{t^2} \le \frac{\nu}{2}\right) h(H_c(t)) U_c(t)\psi \right\|^2 \le C\,\|\psi\|^2. \tag{3.11}$$

(3) *Let M be as in (1) and ν be as in (2). Fix $\epsilon_1 > 0$ and $r > 0$. Assume that $q \in S_0(X)$ vanishes in $\Gamma(\omega, \epsilon_1, r)$. Then*

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{z}{t^2} \ge \frac{\nu}{2}\right) F\left(\frac{\langle x\rangle}{t^2} \le M\right) q h(H_c(t)) U_c(t)\psi \right\|^2 \le C\,\|\psi\|^2. \tag{3.12}$$

(4) *Let M, ν and $q \in S_0(X)$ be as above. Let $\Phi(t)$ denote one of the following three operators*

$$F\left(\frac{\langle x\rangle}{t^2} \geq M\right), \quad F\left(\frac{z}{t^2} \leq \frac{\nu}{2}\right), \quad F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right)F\left(\frac{\langle x\rangle}{t^2} \leq M\right)q.$$

Then

$$s - \lim_{t\to\infty} \Phi(t)h(H_c(t))U_c(t) = 0.$$

If we have the above two propositions, we can prove the following theorem and its corollary, which are the key facts for showing the asymptotic completeness for N-body Stark Hamiltonians in [AT2] (see Theorems 4.5 and 4.6 of [AT2]). We omit the proof.

THEOREM 3.3. – *Let the notations be as above. Then there exist the following strong limits*

$$\Omega_c \equiv s - \lim_{t\to\infty} e^{itH}U_c(t), \tag{3.13}$$

$$\Omega_c^* = s - \lim_{t\to\infty} U_c(t)^* e^{-itH}. \tag{3.14}$$

COROLLARY 3.4. – (Asymptotic clustering) *Let the notation be as above. Then for $\psi \in L^2(X)$, there exists $\psi_c \in L^2(X)$ such that as $t \to \infty$,*

$$e^{-itH}\psi = U_c(t)\psi_c + o(1). \tag{3.15}$$

Now we prove Theorem 1.2. For this sake, we introduce a family of the unitary operators $\{T(t)\}_{t\in\mathbf{R}}$ on $L^2(X)$ as follows: For $u(x) \in L^2(X)$, we define

$$(T(t)u)(x) = e^{it|E|z - it^3|E|^2/6}u\left(x - \frac{E}{2}t^2\right). \tag{3.16}$$

We also introduce the time-dependent Hamiltonian

$$H_{M,c}(t) = H_{M,c} + W_c\left(t, x^c, x_c + \frac{E}{2}t^2\right) = H_{M,c} + W_{M,c}(t), \tag{3.17}$$

where we recall that $H_{M,c} = -\Delta/2 + V^c(x)$ acts on $L^2(X)$ and does not have the Stark effect. We denote by $U_{M,c}(t)$ the propagator generated by $H_{M,c}(t)$, where $U_{M,c}(1) = Id$. The family of transformations $\{T(t)\}_{t\in\mathbf{R}}$ was introduced by Jensen-Yajima [JY], by which Stark Hamiltonians are transformed into Hamiltonians without constant electric fields (see also [AH] and [H]). In fact, we see by the argument similar to [JY] that

$$U_c(t) = T(t)U_{M,c}(t)T(1)^{-1}. \tag{3.18}$$

This representation has played an important role to prove the asymptotic completeness of the Dollard-type modified wave operators for N-body Stark Hamiltonians in [AT2]. We should note that the observables $p_\perp$, $x_\perp/t$, p^c and x^c/t which we consider in Theorem 1.2 do not undergo a change under the transformation $T(t)$. We also note that for $f \in C_\infty(X)$,

$$T(t)^{-1} f\left(x - \frac{E}{2}t^2\right)T(t) = f(x). \tag{3.19}$$

By virtue of the relations (3.18) and (3.19), we have only to apply Theorem 2.2 to the propagator $U_{M,c}(t)$ in order to prove the existence of the asymptotic velocities (1.3) and (1.4), and of the limits (1.6) and (1.8), since the time-dependent potential $W_{M,c}(t)$ satisfies the estimate (2.2) with $\sigma = 2\rho$ by virtue of (3.7). It is sufficient to show that (1.3) and (1.6) exist.

Proof of the existence of (1.3) and (1.6). – By Theorem 3.3, we have only to show that there exists the strong limit

$$s - \lim_{t\to\infty} U_c(t)^* f\left(\frac{x - \frac{E}{2}t^2}{t}\right)U_c(t) \tag{3.20}$$

for $f((x - Et^2/2)/t) = f_1(x_\perp/t)$ with $f_1 \in C_\infty(X_\perp)$ in the case for proving the existence of (1.3), or for $f((x-Et^2/2)/t) = g_1((x_\| - Et^2/2)/t)$ with $g_1 \in C_\infty(X_\|)$ in the case for proving the existence of (1.6). If we obtain the limit (3.20), the limits (1.3) and (1.6) can be written as

$$\begin{aligned} &s - \lim_{t\to\infty} e^{itH} f\left(\frac{x - \frac{E}{2}t^2}{t}\right)e^{-itH} \\ =&\Omega_c\left(s - \lim_{t\to\infty} U_c(t)^* f\left(\frac{x - \frac{E}{2}t^2}{t}\right)U_c(t)\right)\Omega_c^*, \end{aligned}$$

and, hence, we see that there exist (1.3) and (1.6). Now, by (3.18) and (3.19), the limit (3.20) can be written as

$$s - \lim_{t\to\infty} T(1)U_{M,c}(t)^* f\left(\frac{x}{t}\right)U_{M,c}(t)T(1)^{-1}. \tag{3.21}$$

Thus, by applying Theorem 2.2, we see that the limit (3.21) exists, that is, (3.20) exists. In particular, the asymptotic velocities $P_\perp^+(H)$ and $P^{c,+}(H)$ exist, and they commute with H. □

Next we prove the existence of the limits (1.5) and (1.7). Obviously, we have only to show the existence of (1.7). We need the following lemma.

Proof of the existence of (1.7). – It is sufficient to prove that for any $g_2 \in C_0^\infty(X_c)$, there exists (1.7). The Heisenberg derivative of $g_2(p_c - Et)$ is calculated as follows: By (3.7),

$$\begin{aligned}\mathbf{D}_{H_c(t)} g_2(p_c - Et) =& \frac{d}{dt} g_2(p_c - Et) + i[H_c(t), g_2(p_c - Et)] \\ =& i[W_c(t,x), g_2(p_c - Et)] = O(t^{-(1+2\rho)}).\end{aligned}$$

Thus, by using Cook's method, we see that (1.7) exists. □

Next we prove that (1.7) equals (1.8). We need the following lemma.

LEMMA 3.5. – *Let $\psi \in \mathcal{S}(X)$. Then as $t \to \infty$,*

$$\left\| \left(x_c - p_c t + \frac{E}{2} t^2 \right) U_c(t)\psi \right\| = O(t^{\max(0,1-2\rho)}). \tag{3.22}$$

Proof. – The Heisenberg derivative of $x_c - p_c t + Et^2/2$ is

$$\mathbf{D}_{H_c(t)} \left(x_c - p_c t + \frac{E}{2} t^2 \right) = t \nabla_c W_c(t,x) = O(t^{-2\rho}).$$

Thus, by integration, we have (3.22). □

Proof of (1.7)=(1.8). – We have only to show that for any $g_2 \in C_0^\infty(X_c)$, (1.7) equals (1.8). By a calculus of pseudodifferential operators, we have

$$\begin{aligned} & g_2\left(\frac{x - \frac{E}{2}t^2}{t} \right) - g_2(p_c - Et) \\ =& \int_0^1 \left\langle \nabla_c g_2 \left(\theta \frac{x - \frac{E}{2}t^2}{t} + (1-\theta)(p_c - Et) \right), \frac{x_c - p_c t + \frac{E}{2}t^2}{t} \right\rangle d\theta \\ & + \frac{i}{2t} \int_0^1 \Delta_c g_2 \left(\theta \frac{x - \frac{E}{2}t^2}{t} + (1-\theta)(p_c - Et) \right) d\theta. \end{aligned}$$

Thus, by Lemma 3.5, we see that for $\psi \in \mathcal{S}(X)$,

$$\left\| \left(g_2 \left(\frac{x - \frac{E}{2}t^2}{t} \right) - g_2(p_c - Et) \right) U_c(t)\psi \right\| = O(t^{\max(-1,-2\rho)}).$$

This implies that (1.7) equals (1.8). □

Now we prove Theorem 1.3. Here we assume that V satisfies $(V.1)$ and $(V.2)$ with $\rho > 1/2$. The case where V satisfies $(V.1)$ and $(V.3)$ can also be proved similarly. First we prove the existence of the limits (1.9) and (1.10). Then we obtain the existence of the asymptotic energies $T_{\|}^{+}$ and T_c^{+} by the similar argument to the one of Dereziński [D2]. Obviously, it is sufficient to prove that (1.10) exists.

Proof of the existence of (1.10). – We have only to show that for any $h \in C_0^\infty(\mathbf{R})$, (1.10) exists. We shall prove the existence of the following limit:

$$s - \lim_{t\to\infty} U_c(t)^* h(T_c) U_c(t). \tag{3.23}$$

If we have the limit (3.23), the limit (1.10) can be written as

$$\begin{aligned} &s - \lim_{t\to\infty} e^{itH} h(T_c) e^{-itH} \\ =&\Omega_c\Big(s - \lim_{t\to\infty} U_c(t)^* h(T_c) U_c(t)\Big)\Omega_c^*, \end{aligned}$$

and, by Theorem 3.3, we see that (1.10) exists. Since T_c commute with H_c, the Heisenberg derivative of $h(T_c)$ is

$$\mathbf{D}_{H_c(t)} h(T_c) = i[W_c(t,x), h(T_c)].$$

By using the almost analytic extension method and the fact that $\langle z\rangle^{-1/2} p_c h'(T_c)$ is bounded, we have, by virtue of (3.7),

$$\mathbf{D}_{H_c(t)} h(T_c) = O(t^{-2\rho}).$$

Since $2\rho > 1$, by using Cook's method, we see that (3.23) exists. □

Taking account of that $x_\perp/t$ (resp. x^c/t) commute with $T_{\|}$ (resp. T_c), we see that $P_\perp^{+}(H)$ (resp. $P^{c,+}(H)$) commute with $T_{\|}^{+}$ (resp. T_c^{+}). Also, by using the argument similar to the one for showing the intertwining property of the wave operators, we have $T_{\|}^{+}$ and T_c^{+} commute with H. Thus we are interested in the joint spectrum of those commuting self-adjoint operators.

We introduce the new time-dependent Hamiltonian

$$H_{c,G}(t) = H_c + W_{c,G}(t,x^c), \quad W_{c,G}(t,x^c) = W_c\Big(t, x^c, \frac{E}{2}t^2\Big), \tag{3.24}$$

and denote by $U_{c,G}(t)$ the propagator generated by $H_{c,G}(t)$. Since we may write

$$H_{c,G}(t) = H_G^c(t)\otimes Id + Id\otimes T_c, \quad H_G^c(t) = H^c + W_{c,G}(t,x^c), \tag{3.25}$$

we should note that, denoting by $U_G^c(t)$ the propagator generated by $H_G^c(t)$, we may write

$$U_{c,G}(t) = U_G^c(t) \otimes e^{-i(t-1)T_c}. \tag{3.26}$$

We also note that, by virtue of (3.7), $W_{c,G}(t)$ satisfies the estimate

$$|\partial_t^m \partial_{x^c}^\beta W_{c,G}(t,x^c)| \le C_{m\beta}\langle t\rangle^{-m}(\langle t\rangle + \langle x^c\rangle^{1/2})^{-(2\rho+|\beta|)}. \tag{3.27}$$

Now we shall replace $U_c(t)$ by $U_{c,G}(t)$.

LEMMA 3.6. – *Let $\psi \in \mathcal{S}(X)$. Then as $t \to \infty$,*

$$\|(p_c - Et)U_c(t)\psi\| = O(1), \tag{3.28}$$

$$\left\|\left(x_c - \frac{E}{2}t^2\right)U_c(t)\psi\right\| = O(t), \tag{3.29}$$

$$\|(p_c - Et)U_{c,G}(t)\psi\| = O(1), \tag{3.30}$$

$$\left\|\left(x_c - \frac{E}{2}t^2\right)U_{c,G}(t)\psi\right\| = O(t). \tag{3.31}$$

Proof. – Since the Heisenberg derivatives of $p_c - Et$ are

$$\mathbf{D}_{H_c(t)}(p_c - Et) = O(t^{-(1+2\rho)}), \quad \mathbf{D}_{H_{c,G}(t)}(p_c - Et) = 0,$$

we have (3.28) and (3.30) by integration. Also, the Heisenberg derivatives of $x_c - Et^2/2$ are

$$\mathbf{D}_{H_c(t)}\left(x_c - \frac{E}{2}t^2\right) = p_c - Et, \quad \mathbf{D}_{H_{c,G}(t)}\left(x_c - \frac{E}{2}t^2\right) = p_c - Et.$$

By integration, we have (3.29) and (3.31), by virtue of (3.28) and (3.30). □

PROPOSITION 3.7. – *Suppose that V satisfies (V.1), and (V.2) with $\rho > 1/2$ or (V.3). Then there exist the following strong limits:*

$$s - \lim_{t\to\infty} U_c(t)^* U_{c,G}(t), \tag{3.32}$$

$$s - \lim_{t\to\infty} U_{c,G}(t)^* U_c(t). \tag{3.33}$$

Proof. – We have only to prove that for any $\psi \in \mathcal{S}(X)$, the limits (3.32) and (3.33) exist. We prove the existence of (3.33) only. We may show the existence of (3.32) similarly. Since

$$\frac{d}{dt}(U_{c,G}(t)^* U_c(t)\psi) = U_{c,G}(t)^* i(W_{c,G}(t,x^c) - W_c(t,x))U_c(t),$$

$$W_{c,G}(t,x^c) - W_c(t,x)$$

$$= -\int_0^1 \left\langle \nabla_c W_c\left(t, x^c, \theta x_c + (1-\theta)\frac{E}{2}t^2\right), x_c - \frac{E}{2}t^2\right\rangle d\theta,$$

we have, by virtue of (3.7) and Proposition 3.7,

$$\frac{d}{dt}(U_{c,G}(t)^* U_c(t)\psi) = O(t^{-2\rho}).$$

Since $2\rho > 1$, by using Cook's method, we see that (3.33) exists. □

Combining Theorem 3.3 with Proposition 3.7, we have the following proposition.

PROPOSITION 3.8. – *Suppose that V satisfies $(V.1)$, and $(V.2)$ with $\rho > 1/2$ or $(V.3)$. Then there exist the following strong limits:*

$$\Omega_{c,G} \equiv s - \lim_{t\to\infty} e^{itH} U_{c,G}(t), \tag{3.34}$$

$$\Omega_{c,G}^* = s - \lim_{t\to\infty} U_{c,G}(t)^* e^{-itH}. \tag{3.35}$$

Proof of (1.12). – Now we write

$$\begin{aligned} f_2(P^{c,+}(H)) =& s - \lim_{t\to\infty} e^{itH} f_2\left(\frac{x^c}{t}\right) e^{-itH} \\ =& \Omega_{c,G}\left(s - \lim_{t\to\infty} U_{c,G}(t)^* f_2\left(\frac{x^c}{t}\right) U_{c,G}(t)\right)\Omega_{c,G}^* \\ =& \Omega_{c,G}\left\{\left(s - \lim_{t\to\infty} U_G^c(t)^* f_2\left(\frac{x^c}{t}\right) U_G^c(t)\right) \otimes Id\right\}\Omega_{c,G}^* \\ =& \Omega_{c,G} f_2(P^{c,+}(H_G^c))\Omega_{c,G}^*, \\ h(H^{c,+}) =& s - \lim_{t\to\infty} e^{itH} h(H^c) e^{-itH} \\ =& \Omega_{c,G}\left(s - \lim_{t\to\infty} U_{c,G}(t)^* h(H^c) U_{c,G}(t)\right)\Omega_{c,G}^* \\ =& \Omega_{c,G}\left\{\left(s - \lim_{t\to\infty} U_G^c(t)^* h(H^c) U_G^c(t)\right) \otimes Id\right\}\Omega_{c,G}^* \\ =& \Omega_{c,G}(h(H_G^{c,+}) \otimes Id)\Omega_{c,G}^*. \end{aligned}$$

Noting that $H^c = H_M^c$, we may apply Theorem 2.2. Thus we have

$$\begin{aligned} \sigma(H^{c,+}, P^{c,+}(H)) =& \sigma(H_G^{c,+}, P^{c,+}(H_G^c)) \\ =& \bigcup_{a\subset c} \{(\lambda^c, \xi_a^c) \mid \lambda^c = \frac{1}{2}(\xi_a^c)^2 + \tau,\ \xi_a^c \in X_a^c,\ \tau \in \mathcal{E}_a\}. \end{aligned} \tag{3.36}$$

Moreover, we shall prove the existence of the asymptotic energy H_c^+: For $h \in C_\infty(\mathbf{R})$,

$$h(H_c^+) = s - \lim_{t\to\infty} e^{itH} h(H_c) e^{-itH}. \tag{3.37}$$

We have only to prove that for $h \in C_0^\infty(\mathbf{R})$, the limit (3.37) exists. For this sake, we show that the following strong limit exists:

$$s - \lim_{t\to\infty} U_{c,G}(t)^* h(H_c) U_{c,G}(t). \tag{3.38}$$

The Heisenberg derivative of $h(H_c)$ is

$$\mathbf{D}_{H_{c,G}(t)} h(H_c) = i[W_{c,G}(t, x^c), h(H_c)] = O(t^{-2\rho}),$$

where we used the fact that $\langle z\rangle^{-1/2}\nabla^c h'(H_c)$ is bounded. Since $2\rho > 1$, by Cook's method, we see that (3.38) exists. Then we may write the limit (3.37) as

$$h(H_c^+) = \Omega_{c,G}\Big(s - \lim_{t\to\infty} U_{c,G}(t)^* h(H_c) U_{c,G}(t)\Big)\Omega_{c,G}^*,$$

and thus we see that H_c^+ exists. Taking account of the fact $H_c = H^c \otimes Id + Id \otimes T_c$, we also obtain that for $h \in C_\infty(\mathbf{R})$,

$$h(H_c^+) = h(H^{c,+} + T_c^+). \tag{3.39}$$

Also, by virtue of (3.26), we have

$$h(T_c^+) = \Omega_{c,G} h(T_c) \Omega_{c,G}^*,$$

and, hence, we see that $\sigma(T_c^+) = \sigma(T_c) = \mathbf{R}$. Combining this fact with (3.36) and (3.39), we obtain

$$\begin{aligned} &\sigma(H_c^+, P^{c,+}(H), T_c^+) \\ &= \bigcup_{a\subset c} \Big\{(\lambda, \xi_a^c, \lambda_c) \;\Big|\; \lambda = \frac{1}{2}(\xi_a^c)^2 + \lambda_c + \tau,\ \xi_a^c \in X_a^c,\ \lambda_c \in \mathbf{R},\ \tau \in \mathcal{E}_a\Big\}. \end{aligned} \tag{3.40}$$

Finally we prove that for $h \in C_\infty(\mathbf{R})$,

$$h(H) = h(H_c^+). \tag{3.41}$$

If we have (3.41), (1.12) follows from (3.40). We have only to prove that for any $h \in C_0^\infty(\mathbf{R})$ and $\psi = h_1(H)\psi \in L^2(X)$ with $h_1 \in C_0^\infty(\mathbf{R})$,

$$h(H)\psi = \lim_{t\to\infty} e^{itH} h(H_c) e^{-itH}\psi. \tag{3.42}$$

We define $\varphi_c(t,x)$ associated with h_1 as in (3.5). Then, by virtue of Proposition 3.1, the right-hand side of (3.42) may be written as

$$\begin{aligned}&\lim_{t\to\infty} e^{itH}h(H_c)e^{-itH}\psi\\&=\lim_{t\to\infty} e^{itH}h(H_c)\varphi_c(t,x)e^{-itH}\psi\\&=\lim_{t\to\infty} e^{itH}\varphi_c(t,x)h(H)e^{-itH}\psi\\&=\lim_{t\to\infty} e^{itH}h(H)e^{-itH}\psi=h(H)\psi,\end{aligned}$$

where we used the fact that $h(H_c)\varphi_c(t,x)-\varphi_c(t,x)h(H)=O(t^{-2\rho})$. Thus the proof of (1.12) is completed. □

Proof of (1.11). – By Theorem 1.2, we know that (1.5) equals (1.6), and (1.7) equals (1.8). From this fact, we have for $g_3\in C_\infty(X_{c,\perp})$,

$$g_3(P^+_{c,\perp}(H))=s-\lim_{t\to\infty} e^{itH}g_3(p_{c,\perp})e^{-itH}=\Omega_{c,G}g_3(p_{c,\perp})\Omega^*_{c,G},$$

and thus we see that $\sigma(P^+_{c,\perp}(H))=X_{c,\perp}$ and $h(T^+_c)=h(T^+_{\|}+(P_{c,\perp}(H))^2/2)$. Therefore, (1.11) follows from this fact, (1.12) and $X_{a,\perp}=X^c_a\oplus X_{c,\perp}$. □

4. LONG-RANGE CASE

In this section, we prove an analogue of Theorem 1.3 under the assumption that V satisfies $(V.1)$ and $(V.2)$ with $0<\rho\le 1/2$. The result which we want to show is the following theorem.

THEOREM 4.1. – *Suppose that V satisfies $(V.1)$ and $(V.2)$ with $0<\rho\le 1/2$. Then*

$$\begin{aligned}\sigma(H,H^+_{c,\perp},P^+_\perp(H))=\bigcup_{a\subset c}\{(\lambda,\lambda_{c,\perp},\xi_{a,\perp})\mid\lambda&\\=\lambda_{c,\perp}+\lambda^a,\ \lambda_{c,\perp}=\frac{1}{2}(\xi_{a,\perp})^2+\tau,&\\\lambda^a\in\mathbf{R},\ \xi_{a,\perp}\in X_{a,\perp},\ \tau\in\mathcal{E}_a\},&\end{aligned}\qquad(4.1)$$

where $H_{c,\perp}=H^c\otimes Id+Id\otimes T_{c,\perp}$ and $T_{c,\perp}=(p_{c,\perp})^2/2$.

REMARQUE 4.2. – Under the assumption of Theorem 4.1, we do not know whether the asymptotic energies $T_{\|}^{+}$ and T_c^{+} exist or not. But, the results of [A2] roughly say that along the time evolution and as $t \to \infty$,

$$|p_{\|} - Et| \leq C, \left|x_{\|} - p_{\|}t + \frac{E}{2}t^2\right| \leq C\chi(t),$$
$$|p_c - Et| \leq C, \left|x_c - p_c t + \frac{E}{2}t^2\right| \leq C\chi(t),$$

where $\chi(t) = t^{1-2\rho}$ if $0 < \rho < 1/2$, and $\chi(t) = \log t$ if $\rho = 1/2$. Hence, if we see that

$$\left|x_{\|} - p_{\|}t + \frac{E}{2}t^2\right| = O(\chi(t)), \left|x_c - p_c t + \frac{E}{2}t^2\right| = O(\chi(t)) \qquad \text{(A)}$$

hold, we have that along the time evolution and as $t \to \infty$,

$$T_{\|} = \frac{1}{2}(p_{\|} - Et)^2 - \left\langle E, x_{\|} - p_{\|}t + \frac{E}{2}t^2\right\rangle = O(\chi(t)),$$
$$T_c = \frac{1}{2}(p_c - Et)^2 - \left\langle E, x_c - p_c t + \frac{E}{2}t^2\right\rangle = O(\chi(t)),$$

and this seems to imply that the asymptotic energies $T_{\|}^{+}$ and T_c^{+} do not exist. We think that this may be caused by the slowly decreasing of the first derivatives of the intercluster potential $I_c(x)$. But we do not know whether (A) hold or not, that is, the estimates of [A2] are optimal or not.

To prove Theorem 4.1, we need some propositions and lemmas.

PROPOSITION 4.3. – *Suppose that $a \subset c$, $\Theta \subset Z_{a,\perp} \equiv X_{a,\perp} \setminus \cup_{b \not\subset a, b \subset c} X_{b,\perp}$ is a compact set of $Z_{a,\perp}$, and $J \in C_0^\infty(Z_{a,\perp})$ satisfying that $J = 1$ on Θ. Put $\tilde{W}_a(t, x_a) = W_c(t, x_a) + J(x_{a,\perp}/t) I_a^c(x_{a,\perp})$. Then for $t \geq 1$*

$$|\partial_t^m \partial_{x_a}^\beta \tilde{W}_a(t, x_a)| \leq C_{m\beta} \langle t \rangle^{-(m+|\beta|+\min(\rho',2\rho))}. \qquad (4.2)$$

Now define the time-dependent Hamiltonian $H_{a,\tilde{W}_a}(t)$ as $H_{a,\tilde{W}_a}(t) = H_a + \tilde{W}_a(t, x_a)$ and denote by $U_{a,\tilde{W}_a}(t)$ the propagator generated by $H_{a,\tilde{W}_a}(t)$. Then there exist the asymptotic velocity $P_\perp^+(H_{a,\tilde{W}_a})$ and the strong limits

$$\Omega_{a\Theta}^{+} \equiv s - \lim_{t\to\infty} U_{a,\tilde{W}_a}(t)^* e^{-itH} E_\Theta(P_\perp^+(H)), \qquad (4.3)$$

$$s - \lim_{t\to\infty} e^{itH} U_{a,\tilde{W}_a}(t) E_\Theta(P_\perp^+(H_{a,\tilde{W}_a})), \qquad (4.4)$$

where $E_\Theta(P)$ is the spectral projection of a vector in X of commuting self-adjoint operators P onto a Borel subset Θ of X. The limit (4.4) equals $(\Omega^+_{a\Theta})^*$. Moreover the following relations hold.

$$\Omega^+_{a\Theta}P^+_\perp(H)E_\Theta(P^+_\perp(H))(\Omega^+_{a\Theta})^* = P^+_\perp(H_{a,\tilde W_a})E_\Theta(P^+_\perp(H_{a,\tilde W_a})), \quad (4.5)$$

$$\Omega^+_{a\Theta}HE_\Theta(P^+_\perp(H))(\Omega^+_{a,\Theta})^* = H^+_{a,\tilde W_a}E_\Theta(P^+_\perp(H_{a,\tilde W_a})), \quad (4.6)$$

$$\Omega^+_{a\Theta}H^+_{c,\perp}E_\Theta(P^+_\perp(H))(\Omega^+_{a,\Theta})^* = H^+_{a,\perp,\tilde W_a}E_\Theta(P^+_\perp(H_{a,\tilde W_a})), \quad (4.7)$$

where $H_{a,\perp} = H^a \otimes Id + Id \otimes T_{a,\perp}$ and $T_{a,\perp} = (p_{a,\perp})^2/2$.

Proof. – The proof of this proposition is essentially similar to that of Proposition 4.7 of [D2].

In §3, we proved the existence of the strong limit (3.20), which implies the existence of the asymptotic velocity $P^+_\perp(H_{c,W_c})$. Combining this with Theorem 3.3, we have

$$E_\Theta(P^+_\perp(H)) = \Omega_c E_\Theta(P^+_\perp(H_{c,W_c}))\Omega^*_c, \quad (4.8)$$

by the definition of the asymptotic velocities. Since we easily see that (4.2) holds, we can prove the existence of the asymptotic velocity $P^+_\perp(H_{a,\tilde W_a})$ in the way similar to that of proving the existence of $P^+_\perp(H_{c,W_c})$. Now we shall show the existence of the strong limits

$$s-\lim_{t\to\infty} U_{a,\tilde W_a}(t)^*U_c(t)E_\Theta(P^+_\perp(H_{c,W_c})), \quad (4.9)$$

$$s-\lim_{t\to\infty} U_c(t)^*U_{a,\tilde W_a}(t)E_\Theta(P^+_\perp(H_{a,\tilde W_a})), \quad (4.10)$$

which implies that by noting that by virtue of (4.8),

$$\begin{aligned}\Omega^+_{a\Theta} &= s-\lim_{t\to\infty} U_{a,\tilde W_a}(t)^*e^{-itH}\Omega_c E_\Theta(P^+_\perp(H_{c,W_c}))\Omega^*_c \\ &= s-\lim_{t\to\infty} U_{a,\tilde W_a}(t)^*U_c(t)E_\Theta(P^+_\perp(H_{c,W_c}))\Omega^*_c,\end{aligned}$$

there exist the strong limits (4.3) and (4.4). We prove the existence of (4.9) only. The existence of (4.10) may be proved similarly.

Put $\tilde W_{M,a}(t,x_a) = W_c(t,x_a+Et^2/2) + J(x_{a,\perp}/t)I^c_a(x_{a,\perp})$ and define the time- dependent Hamiltonian $H_{M,a,\tilde W_{M,a}}(t)$ as $H_{M,a,\tilde W_{M,a}}(t) = H_{M,a} + \tilde W_{M,a}(t)$. We denote by $U_{M,a,\tilde W_{M,a}}(t)$ the propagator generated by $H_{M,a,\tilde W_{M,a}}(t)$. By the argument similar to [JY], we see that

$$U_{a,\tilde W_a}(t) = T(t)U_{M,a,\tilde W_{M,a}}(t)T(1)^{-1}. \quad (4.11)$$

Here we used the fact that $T(t)^{-1}(x_\perp/t)T(t) = x_\perp/t$. Then by (3.18) and (4.11), we can rewrite (4.9) as

$$\begin{aligned} &s - \lim_{t\to\infty} T(1)U_{M,a,\tilde{W}_{M,a}}(t)^* U_{M,c}(t)T(1)^{-1}E_\Theta(P_\perp^+(H_{c,W_c})) \\ =&s - \lim_{t\to\infty} T(1)U_{M,a,\tilde{W}_{M,a}}(t)^* U_{M,c}(t)E_\Theta(P_\perp^+(H_{M,c,W_{M,c}}))T(1)^{-1}. \end{aligned} \tag{4.12}$$

Here we used the fact that

$$\begin{aligned} E_\Theta(P_\perp^+(H_{c,W_c})) &= T(1)E_\Theta(P_\perp^+(H_{M,c,W_{M,c}}))T(1)^{-1} \\ &= E_\Theta(P_\perp^+(H_{M,c,W_{M,c}})), \end{aligned}$$

which follows from the fact that $T(t)^{-1}(x_\perp/t)T(t) = x_\perp/t$ and the definition of the asymptotic velocities $P_\perp^+(H_{c,W_c})$ and $P_\perp^+(H_{M,c,W_{M,c}})$. Hence we have only to prove the existence of the strong limit

$$s - \lim_{t\to\infty} U_{M,a,\tilde{W}_{M,a}}(t)^* U_{M,c}(t)E_\Theta(P_\perp^+(H_{M,c,W_{M,c}})). \tag{4.13}$$

Since the time-dependent Hamiltonians which we have to consider now do not have the Stark effect, this may be shown by the argument quite similar to that of the proof of Proposition 4.7 of [D2]. Therefore (4.9) exists.

The other statements except (4.6) and (4.7) follow from the intertwining relation

$$\Omega_{a\Theta}^+ = s - \lim_{t\to\infty} E_\Theta(P_\perp^+(H_{a,\tilde{W}_a}))U_{a,\tilde{W}_a}(t)^* e^{-itH}, \tag{4.14}$$

which is seen by definition.

We shall show that (4.6) and (4.7) hold. To prove (4.6), we first prove the existence of the asymptotic energy H_{c,W_c}^+. We have only to prove the existence of $h(H_{c,W_c}^+)$ for any $h \in C_0^\infty(\mathbf{R})$. For $h \in C_0^\infty(\mathbf{R})$, we can rewrite $h(H_{c,W_c}^+)$ as

$$h(H_{c,W_c}^+) = s - \lim_{t\to\infty} U_c(t)^* h(H_c(t))U_c(t), \tag{4.15}$$

by using the fact that $h(H_c) - h(H_c(t)) = O(t^{-2\rho})$. The existence of (4.15) follows from the fact that for $h \in C_0^\infty(\mathbf{R})$, the Heisenberg derivative $\mathbf{D}_{H_c(t)}h(H_c(t))$ is $O(t^{-(1+2\rho)})$. Similarly, we may prove the existence of the asymptotic energy $H_{a,\tilde{W}_a}^+$, since the Heisenberg derivative $\mathbf{D}_{H_{a,\tilde{W}_a}(t)}h(H_{a,\tilde{W}_a}(t))$ is $O(t^{-(1+\min(\rho',2\rho))})$ for $h \in C_0^\infty(\mathbf{R})$.

Next we prove the relation

$$h(H) = \Omega_c h(H^+_{c,W_c})\Omega^*_c. \tag{4.16}$$

for $h \in C_\infty(\mathbf{R})$. We have only to prove (4.16) for $h \in C_0^\infty(\mathbf{R})$. We should note that for $h \in C_0^\infty(\mathbf{R})$,

$$h(H_c(t))\varphi_c(t,x) - \varphi_c(t,x)h(H) = O(t^{-1}), \tag{4.17}$$

by virtue of the almost analytic extension method (see the proof of Theorem 4.5 of [AT2]). We have for any $h \in C_0^\infty(\mathbf{R})$ and $\psi = h_1(H)\psi \in L^2(X)$ with some $h_1 \in C_0^\infty(\mathbf{R})$,

$$\begin{aligned} h(H)\psi &= \lim_{t\to\infty} e^{itH}h(H)e^{-itH}\psi \\ &= \lim_{t\to\infty} e^{itH}h(H)\varphi_c(t,x)e^{-itH}\psi \\ &= \lim_{t\to\infty} e^{itH}\varphi_c(t,x)h(H_c(t))e^{-itH}\psi \\ &=\Omega_c\Big\{\lim_{t\to\infty} U_c(t)^*\varphi_c(t,x)h(H_c(t))U_c(t)\Big\}\Omega^*_c\psi \\ &=\Omega_c\Big\{\lim_{t\to\infty} U_c(t)^*h(H_c(t))U_c(t)\Big\}\Omega^*_c\psi = \Omega_c h(H^+_{c,W_c})\Omega^*_c\psi \end{aligned}$$

by virtue of Propositions 3.1 and 3.2. Since the set $\{\psi \in L^2(X) \mid \psi = h_1(H)\psi$ forsome $h_1 \in C_0^\infty(\mathbf{R})\}$ is dense in $L^2(X)$, we see that (4.16) holds. Hence, by virtue of (4.16), we have only to prove the relation

$$\Omega^+_{a\Theta}\Omega_c H^+_{c,W_c}E_\Theta(P^+_\perp(H_{c,W_c}))\Omega^*_c(\Omega^+_{a,\Theta})^* = H^+_{a,\tilde{W}_a}E_\Theta(P^+_\perp(H_{a,\tilde{W}_a})). \tag{4.18}$$

In virtue of (4.14), we may write $\Omega^+_{a\Theta}\Omega_c$ and $\Omega^*_c(\Omega^+_{a\Theta})^*$ as

$$\Omega^+_{a\Theta}\Omega_c = s - \lim_{t\to\infty} E_\Theta(P^+_\perp(H_{a,\tilde{W}_a}))U_{a,\tilde{W}_a}(t)^*U_c(t), \tag{4.19}$$
$$\Omega^*_c(\Omega^+_{a\Theta})^* = s - \lim_{t\to\infty} U_c(t)^*U_{a,\tilde{W}_a}(t)E_\Theta(P^+_\perp(H_{a,\tilde{W}_a})). \tag{4.20}$$

Take $\tilde{J} \in C_0^\infty(Z_{a,\perp})$ such that $J\tilde{J} = \tilde{J}$ and $\tilde{J} = 1$ on Θ. Any vector in $Ran\, E_\Theta(P^+_\perp(H_{a,\tilde{W}_a}))$ can be approximated by vectors $\phi \in \mathcal{S}(X)$ such that

$$\phi = \lim_{t\to\infty} U_{a,\tilde{W}_a}(t)^*\tilde{J}\Big(\frac{x_{a,\perp}}{t}\Big)U_{a,\tilde{W}_a}(t)\phi.$$

Hence we have only to consider (4.18) for such ϕ. Here we need the following lemma.

LEMMA 4.4. – *Let $\phi \in \mathcal{S}(X)$. Then as $t \to \infty$,*

$$\|p_\perp U_{a,\tilde{W}_a}(t)\phi\| = O(1), \tag{4.21}$$

$$\|x_\perp U_{a,\tilde{W}_a}(t)\phi\| = O(t). \tag{4.22}$$

Proof. – First we are concerned with $p_{a,\perp}$ and $x_{a,\perp}$. Since the Heisenberg derivative $\mathbf{D}_{H_{a,\tilde{W}_a}(t)} p_{a,\perp}$ is $O(t^{-(1+\min(\rho',2\rho))})$, (4.21) with replacing $p_\perp$ by $p_{a,\perp}$ is obtained by integration. Since the Heisenberg derivative $\mathbf{D}_{H_{a,\tilde{W}_a}(t)} x_{a,\perp}$ is $p_{a,\perp}$, we have (4.22) with replacing $x_\perp$ by $x_{a,\perp}$, by integrating (4.21) in t.

Next we consider the statements of the lemma associated with p^a and x^a. We note that $p^a f(H^a)$ is bounded for $f \in \mathcal{F}^{-1/2}$, and $g(H^a)\phi \in L^2(X)$ for $g \in \mathcal{F}^{1/2}$ and $\phi \in \mathcal{S}(X)$. Thus we first prove

$$\|g(H^a) U_{a,\tilde{W}_a}(t)\phi\| = O(1) \tag{4.23}$$

for $g \in \mathcal{F}^{1/2}$, which implies (4.21) with replacing $p_\perp$ by p^a holds. Since the Heisenberg derivative $\mathbf{D}_{H_{a,\tilde{W}_a}(t)} g(H^a)$ is 0, we have (4.23) by integration. Moreover, since the Heisenberg derivative $\mathbf{D}_{H_{a,\tilde{W}_a}(t)} x^a$ is p^a, we have (4.22) with replacing $x_\perp$ by x^a. □

Continuation of the proof of Proposition 4.3. – For $h \in C_0^\infty(\mathbf{R})$, we shall compute $h(H^+_{a,\tilde{W}_a}) E_\Theta(P^+_\perp(H_{a,\tilde{W}_a}))\phi$ with above ϕ. We put $\approx{W}_a(t,x) \equiv W_c(t,x) + J(x_{a,\perp}/t) I_a^c(x) = W_c(t,x) + J(x_{a,\perp}/t) I_a^c(x^c)$, which satisfies

$$|\partial_t^m \partial_x^\beta \approx{W}_a(t,x)| \le C_{m\beta} \langle t\rangle^{-(m+|\beta|+\min(\rho',2\rho))} \tag{4.24}$$

for $t \ge 1$. By noting that $x^a \in X_\perp$ and using Lemma 4.4 and the facts that $\tilde{J}(x_{a,\perp}/t)\{I_a^c(x) - J(x_{a,\perp}/t) I_a^c(x)\} = 0$ and that $\approx{W}_a(t,x) - \tilde{W}_a(t,x_a) = x^a \int_0^1 \nabla_{x^a} \approx{W}_a(t, x_a + \theta x^a)\, d\theta$, we have

$$\Big\{h(H_{a,\tilde{W}_a}(t)) \tilde{J}\Big(\frac{x_{a,\perp}}{t}\Big) - \tilde{J}\Big(\frac{x_{a,\perp}}{t}\Big) h(H_{a,\tilde{W}_a}(t))\Big\} U_{a,\tilde{W}_a}(t)\phi = O(t^{-1}), \tag{4.25}$$

$$\Big\{h(H_{a,\tilde{W}_a}(t)) \tilde{J}\Big(\frac{x_{a,\perp}}{t}\Big) - \tilde{J}\Big(\frac{x_{a,\perp}}{t}\Big) h(H_c(t))\Big\} U_{a,\tilde{W}_a}(t)\phi = O(t^{-\min(1,\rho',2\rho)}) \tag{4.26}$$

by virtue of the almost analytic extension method. (4.25) implies that $h(H^+_{a,\tilde W_a})$ commutes with $E_\Theta(P^+_\perp(H_{a,\tilde W_a}))$. Hence, by using (4.19), (4.20) and (4.26), we have

$$\begin{aligned} &h(H^+_{a,\tilde W_a})E_\Theta(P^+_\perp(H_{a,\tilde W_a}))\phi \\ =&\lim_{t\to\infty} E_\Theta(P^+_\perp(H_{a,\tilde W_a}))U_{a,\tilde W_a}(t)^* h(H_{a,\tilde W_a}(t))\tilde J\Big(\frac{x_{a,\perp}}{t}\Big)U_{a,\tilde W_a}(t)\phi \\ =&\lim_{t\to\infty} E_\Theta(P^+_\perp(H_{a,\tilde W_a}))U_{a,\tilde W_a}(t)^* \tilde J\Big(\frac{x_{a,\perp}}{t}\Big)h(H_c(t))U_{a,\tilde W_a}(t)\phi \\ =&\Omega^+_{a\Theta}\Omega_c h(H^+_{c,W_c})\Omega^*_c(\Omega^+_{a\Theta})^*\phi, \end{aligned}$$

where we used the fact that $\tilde J = 1$ on Θ. This implies that (4.18) holds. As for proving (4.7), we show the existence of the asymptotic energies $H^+_{c,\perp}$ and $H^+_{a,\perp,\tilde W_a}$ only, since the proof of (4.7) can be quite similar to that of (4.6). We prove the existence of $H^+_{c,\perp}$ only. That of $H^+_{a,\perp,\tilde W_a}$ can be shown similarly. We write

$$\begin{aligned} h(H^+_{c,\perp}) =& s-\lim_{t\to\infty} e^{itH}h(H_{c,\perp})e^{-itH} \\ =&\Omega_c\Big\{s-\lim_{t\to\infty} U_c(t)^*h(H_{c,\perp})U_c(t)\Big\}\Omega^*_c \\ =&\Omega_c T(1)\Big\{s-\lim_{t\to\infty} U_{M,c}(t)^*h(H_{c,\perp})U_{M,c}(t)\Big\}T(1)^{-1}\Omega^*_c \end{aligned}$$

for $h \in C_\infty(\mathbf{R})$. By noting that $H_{c,\perp} = H_{M,c,\perp}$ and applying Theorem 2.2, we see that the asymptotic energy $H^+_{c,\perp}$ exists. The proof of Proposition 4.3 is completed. □

Now we shall introduce the propagators which can approximate the propagators $U_{a,\tilde W_a}(t)$, $a \subset c$, asymptotically. In the following, we follows the argument in [AT2].

We first note that the propagators $U_{a,\tilde W_a}(t)$, $a \subset c$, can be decomposed as

$$U_{a,\tilde W_a}(t) = e^{-i(t-1)H^a} \otimes \hat U_{a,\tilde W_a}(t) \quad \text{on } L^2(X^a)\otimes L^2(X_a), \tag{4.27}$$

where $\hat U_{a,\tilde W_a}(t)$ is the propagator generated by the time-dependent Hamiltonian
$\hat H_{a,\tilde W_a}(t) \equiv T_a + \tilde W_a(t, x_a)$ on $L^2(X_a)$. We should note that $E \in X_a$ for $a \subset c$. Then we shall introduce the propagators by which the propagators $\hat U_{a,\tilde W_a}(t)$ can be approximated asymptotically.

We construct an approximate solution to the Hamilton-Jacobi equation

$$\partial_t S(t,\xi_a) + \langle E, \nabla_{\xi_a} S(t,\xi_a)\rangle = \frac{1}{2}|\xi_a|^2 + \tilde W_a(t, \nabla_{\xi_a} S(t,\xi_a)) \tag{4.28}$$

associated with $\hat{H}_{a,\tilde{W}_a}(t)$. Without loss of generality, we assume that $1/\min(\rho', 2\rho)$ is not an integer. Set $L = [1/\min(\rho', 2\rho)]$, so that $(L+1)\min(\rho', 2\rho) > 1$. We first define $K_0(t, \xi_a)$ by

$$K_0(t, \xi_a) = \frac{1}{2}|\xi_a|^2 t - \frac{|E|}{2}\langle \omega, \xi_a \rangle t^2 + \frac{|E|^2}{6} t^3.$$

Then K_0 satisfies

$$\partial_t K_0 + \langle E, \nabla_{\xi_a} K_0 \rangle = \frac{1}{2}|\xi_a|^2.$$

We further define $K_j(t, \xi_a)$, $1 \le j \le L$, for $t \ge 1$ inductively as the solution to

$$\partial_t K_j + \langle E, \nabla_{\xi_a} K_j \rangle = F_{j-1}(t, \xi_a), \qquad K_j(1, \xi_a) = 0,$$

where

$$F_j(t, \xi_a) = \tilde{W}_a\left(t, \sum_{m=0}^{j} \nabla_{\xi_a} K_m(t, \xi_a)\right) - \tilde{W}_a\left(t, \sum_{m=0}^{j-1} \nabla_{\xi_a} K_m(t, \xi_a)\right).$$

LEMMA 4.5.

$$\partial_{\xi_a}^{\beta} K_j(t, \xi_a) = O(t^{1-j\min(\rho', 2\rho)}), \quad 1 \le j \le L,$$

uniformly in ξ_a.

Proof. – The proof is quite similar to that of Lemma 6.1 in [AT2]. The lemma is easily verified by induction. The solution K_j is given by

$$K_j(t, \xi_a) = \int_1^t F_{j-1}(s, (s-t)E + \xi_a)\, ds.$$

In particular, we have

$$K_1(t, \xi_a) = \int_1^t \tilde{W}_a\left(s, \left(\frac{s^2}{2} - st\right)E + s\xi_a\right) ds$$

and hence K_1 obeys the estimates in the statement of the lemma by (4.2). Assume that K_m, $1 \le m \le j-1$, satisfies the estimates in the lemma. Then it follows that $\partial_{\xi_a}^{\beta} F_{j-1}(t, \xi_a) = O(t^{-j\min(\rho', 2\rho)})$. This proves that K_j also satisfies the desired estimates and the proof is completed. □

The approximate solution $S(t,\xi_a)$ to the equation (4.28) is now defined by

$$S(t,\xi_a)=\sum_{j=0}^{L}K_j(t,\xi_a),\qquad t\geq 1. \tag{4.29}$$

Then we have for $t\geq 1$,

$$\partial_t S+\langle E,\nabla_{\xi_a}S\rangle-\frac{1}{2}|\xi_a|^2-\tilde{W}_a(t,\nabla_{\xi_a}S)=-F_L(t,\xi_a),$$

and hence it follows from Lemma 4.5 that

$$\partial_{\xi_a}^{\beta}\left\{\partial_t S+\langle E,\nabla_{\xi_a}S\rangle-\frac{1}{2}|\xi_a|^2-\tilde{W}_a(t,\nabla_{\xi_a}S)\right\}=O(t^{-(L+1)\min(\rho',2\rho)}) \tag{4.30}$$

uniformly in ξ_a. We also consider the time-dependent Hamiltonian

$$\check{H}_a(t)=T_a+\check{W}_a(t,p_a),\quad \check{W}_a(t,\xi_a)=\tilde{W}_a(t,(\nabla_{\xi_a}S)(t,\xi_a)),$$

and denote by $\check{U}_a(t)$ the propagator generated by $\check{H}_a(t)$. We put $Y(t,\xi_a)=(\nabla_{\xi_a}S)(t,\xi_a+Et)$ and $\bar{W}_a(t,\xi_a)=\tilde{W}_a(t,Y(t,\xi_a))$. Then $\check{U}_a(t)$ is explicitly represented by

$$\check{U}_a(t)=e^{-i(t-1)T_a}e^{-i\int_1^t\bar{W}_a(s,p_a)\,ds}. \tag{4.31}$$

LEMMA 4.6. – *Let the notations be as above and $\psi\in\mathcal{S}(X_a)$. Then as $t\to\infty$,*

$$\|(x_a-(\nabla_{\xi_a}S)(t,p_a))\hat{U}_{a,\tilde{W}_a}(t)\psi\|=O(1), \tag{4.32}$$
$$\|(x_a-(\nabla_{\xi_a}S)(t,p_a))\check{U}_a(t)\psi\|=O(1). \tag{4.33}$$

Proof. – Let $\Phi(t)=x_a-(\nabla_{\xi_a}S)(t,p_a)$. We calculate the Heisenberg derivative $\mathbf{D}_{\hat{H}_{a,\tilde{W}_a}(t)}\Phi(t)$. We write $\hat{H}_{a,\tilde{W}_a}(t)=\check{H}_a(t)+\tilde{W}_a(t,x_a)-\check{W}_a(t,p_a)$. It follows from (4.30) that

$$\begin{aligned}\mathbf{D}_{\check{H}_a(t)}\Phi(t)&=\Phi'(t)+i[\check{H}_a(t),\Phi(t)]\\&=(\nabla_{\xi_a}F_L)(t,p_a)=O(t^{-(L+1)\min(\rho',2\rho)}).\end{aligned}$$

Noting that $-(L+1)\min(\rho',2\rho)<-1$, (4.33) is proved by this estimate. We also have

$$i[\tilde{W}_a(t,x_a)-\check{W}_a(t,p_a),\Phi(t)]=O(t^{-(1+\min(\rho',2\rho))})\Phi(t)+O(t^{-(1+\min(\rho',2\rho))})$$

by a simple calculus of pseudodifferential operators. Hence the Heisenberg derivative $\mathbf{D}_{\hat{H}_{a,\tilde{W}_a}(t)}\Phi(t)$ takes the form

$$\mathbf{D}_{\hat{H}_{a,\tilde{W}_a}(t)}\Phi(t)$$
$$= O(t^{-(1+\min(\rho',2\rho))})\Phi(t) + O(t^{-(1+\min(\rho',2\rho))}) + O(t^{-(L+1)\min(\rho',2\rho)}).$$

This yields

$$\|\Phi(t)\hat{U}_{a,\tilde{W}_a}(t)\psi\| \leq C\left\{1 + \int_1^t s^{-(1+\min(\rho',2\rho))}\|\Phi(s)\hat{U}_{a,\tilde{W}_a}(s)\psi\|\,ds\right\}$$

and hence (4.32) follows immediately from the Gronwall inequality. □

By Lemma 4.6, we have the following proposition.

PROPOSITION 4.7. – *There exist the strong limits*

$$s - \lim_{t\to\infty} \hat{U}_{a,\tilde{W}_a}(t)^*\check{U}_a(t), \qquad s - \lim_{t\to\infty} \check{U}_a(t)^*\hat{U}_{a,\tilde{W}_a}(t).$$

Proof. – By a simple calculus of pseudodifferential operators, we have

$$\begin{aligned}\tilde{W}_a(t,x_a) - \check{W}_a(t,p_a) =& O(t^{-(1+\min(\rho',2\rho))})(x_a - (\nabla_{\xi_a}S)(t,p_a)) \\ &+ O(t^{-(1+\min(\rho',2\rho))}).\end{aligned}$$

Combining this fact with Lemma 4.6, we have the proposition. □

Now we define the time-dependent Hamiltonian by $\tilde{H}_a(t) = T_a + \bar{W}_a(t,p_{a,\perp})$, and denote by $\tilde{U}_a(t)$ the propagator generated by $\tilde{H}_a(t)$. Since T_a commutes with $\bar{W}_a(t,p_{a,\perp})$, $\tilde{U}_a(t)$ is explicitly represented by

$$\tilde{U}_a(t) = e^{-i(t-1)T_a}e^{-i\int_1^t \bar{W}_a(s,p_{a,\perp})\,ds}. \tag{4.34}$$

We shall replace $\check{U}_a(t)$ by $\tilde{U}_a(t)$. We need the following proposition.

PROPOSITION 4.8. – *There exist the strong limits*

$$s - \lim_{t\to\infty} \check{U}_a(t)^*\tilde{U}_a(t), \qquad s - \lim_{t\to\infty} \tilde{U}_a(t)^*\check{U}_a(t).$$

Proof. – By virtue of (4.31) and (4.34), we have only to prove that as $t \to \infty$, $\int_1^t\{\bar{W}_a(s,\xi_a) - \bar{W}_a(s,\xi_{a,\perp})\}\,ds$ converges locally uniformly in ξ_a. We write

$$\bar{W}_a(s,\xi_a) - \bar{W}_a(s,\xi_{a,\perp}) = \int_0^1 (\partial_z\bar{W}_a)(s,\tau\xi_{\|} + \xi_{a,\perp})\langle\xi_{\|},\omega\rangle\,d\tau.$$

We note that $\partial_t\{Y(t,\xi_a)\} = (\partial_t\nabla_{\xi_a}S)(t,\xi_a + Et) + |E|(\partial_z\nabla_{\xi_a}S)(t,\xi_a + Et) = |E|(\partial_z\nabla_{\xi_a}S)(t,\xi_a+Et)+\xi_a+O(t^{-\min(\rho',2\rho)})$ as $t \to \infty$ uniformly in ξ_a, by the definition of $Y(t, \xi_a)$ (see also the proof of Lemma 4.5). Then we see that as $s \to \infty$,

$$\begin{aligned}
&\partial_s\{\bar{W}_a(s, \tau\xi_{\|} + \xi_{a,\perp})\} \\
=&(\partial_s\tilde{W}_a)(s, Y(s, \tau\xi_{\|} + \xi_{a,\perp})) \\
&+ \langle(\nabla_{x_a}\tilde{W}_a)(s, Y(s, \tau\xi_{\|} + \xi_{a,\perp})), \partial_s\{Y(s, \tau\xi_{\|} + \xi_{a,\perp})\}\rangle \\
=&|E|\langle(\nabla_{x_a}\tilde{W}_a)(s, Y(s, \tau\xi_{\|} + \xi_{a,\perp})), (\partial_z\nabla_{\xi_a}S)(s, \tau\xi_{\|} + \xi_{a,\perp} + Es)\rangle \\
&+ O(s^{-(1+\min(\rho', 2\rho))})
\end{aligned}$$

holds locally uniformly in ξ_a and uniformly in $0 \leq \tau \leq 1$. Here we used the fact that $(\nabla_{x_a}\tilde{W}_a)(s, Y(s, \tau\xi_{\|} + \xi_{a,\perp})) = O(s^{-(1+\min(\rho', 2\rho))})$ and $(\partial_s\tilde{W}_a)(s, Y(s, \tau\xi_{\|} + \xi_{a,\perp})) = O(s^{-(1+\min(\rho', 2\rho))})$ hold locally uniformly in ξ_a and uniformly in $0 \leq \tau \leq 1$. Hence we see that

$$\begin{aligned}
&(\partial_z\bar{W}_a)(s, \tau\xi_{\|} + \xi_{a,\perp})\langle\xi_{\|}, \omega\rangle \\
=&\langle(\nabla_{x_a}\tilde{W}_a)(s, Y(s, \tau\xi_{\|} + \xi_{a,\perp})), (\partial_z\nabla_{\xi_a}S)(s, \tau\xi_{\|} + \xi_{a,\perp} + Es)\rangle\langle\xi_{\|}, \omega\rangle \\
=&\frac{\langle\xi_{\|}, \omega\rangle}{|E|}\partial_s\{W_a(s, \tau\xi_{\|} + \xi_{a,\perp})\} + O(s^{-(1+\min(\rho', 2\rho))})
\end{aligned}$$

holds locally uniformly in ξ_a and uniformly in $0 \leq \tau \leq 1$. This implies the proposition. □

Combining the above two propositions, we have the following proposition.

PROPOSITION 4.9. – *There exist the strong limits*

$$s - \lim_{t\to\infty} \check{U}_{a,\tilde{W}_a}(t)^*\tilde{U}_a(t), \qquad s - \lim_{t\to\infty} \tilde{U}_a(t)^*\check{U}_{a,\tilde{W}_a}(t).$$

Here we define the time-dependent Hamiltonian $\bar{H}_a(t)$ by

$$\bar{H}_a(t) = H_a + \bar{W}_a(t, p_{a,\perp}) = H^a \otimes Id + Id \otimes \tilde{H}_a(t)$$

on $L^2(X) = L^2(X^a) \otimes L^2(X_a)$, and denote by $\bar{U}_a(t)$ the propagator generated by $\bar{H}_a(t)$. Then, by noting (4.27), the following proposition is an immediate consequence of Proposition 4.9.

PROPOSITION 4.10. – *There exist the strong limits*

$$\bar{\Omega}_a \equiv s - \lim_{t\to\infty} U_{a,\tilde{W}_a}(t)^*\bar{U}_a(t), \tag{4.35}$$

$$\bar{\Omega}_a^* = s - \lim_{t\to\infty} \bar{U}_a(t)^*U_{a,\tilde{W}_a}(t). \tag{4.36}$$

By virtue of Proposition 4.10, we have the existence of the asymptotic velocity $P_\perp^+(H_{a,\bar{W}_a})$ and asymptotic energies $H^+_{a,\bar{W}_a}$ and $H^+_{a,\perp,\bar{W}_a}$ which satisfy

$$P_\perp^+(H_{a,\bar{W}_a}) = \bar{\Omega}_a^* P_\perp^+(H_{a,\tilde{W}_a})\bar{\Omega}_a, \tag{4.37}$$

$$H_{a,\bar{W}_a} = \bar{\Omega}_a^* H^+_{a,\tilde{W}_a}\bar{\Omega}_a. \tag{4.38}$$

$$H_{a,\perp,\bar{W}_a} = \bar{\Omega}_a^* H^+_{a,\perp,\tilde{W}_a}\bar{\Omega}_a. \tag{4.39}$$

Thus we shall consider the relation between $P_\perp^+(H_{a,\bar{W}_a})$, $H^+_{a,\bar{W}_a}$ and $H^+_{a,\perp,\bar{W}_a}$. Here we shall introduce the "conditional" asymptotic energy $T^+_{\|,\bar{W}_a}$ as follows:

PROPOSITION 4.11. – *Let the notations be as above. Let $h \in C_\infty(\mathbf{R})$. Then there exists the strong limit*

$$s - \lim_{t\to\infty} \bar{U}_a(t)^* h(T_\|)\bar{U}_a(t). \tag{4.40}$$

Moreover, there exists a unique self-adjoint operator $T^+_{\|,\bar{W}_a}$ such that (4.40) equals $h(T^+_{\|,\bar{W}_a})$. $P_\perp^+(H_{a,\bar{W}_a})$, $H^+_{a,\bar{W}_a}$, $H^+_{a,\perp,\bar{W}_a}$ and $T^+_{\|,\bar{W}_a}$ are mutually commutative, and satisfy

$$H^+_{a,\bar{W}_a} = H^+_{a,\perp,\bar{W}_a} + T^+_{\|,\bar{W}_a}, \tag{4.41}$$

$$H^+_{a,\perp,\bar{W}_a} = H^a + \frac{1}{2}(P^+_{a,\perp}(H_{a,\bar{W}_a}))^2. \tag{4.42}$$

Proof. – As for the existence of (4.40), it is sufficient to prove that for $h \in C_0^\infty(\mathbf{R})$. Since $\mathbf{D}_{\bar{H}_a(t)}h(T_\|) = 0$, we see that (4.40) exists for $h \in C_0^\infty(\mathbf{R})$. In fact, we have

$$T^+_{\|,\bar{W}_a} = T_\|. \tag{4.43}$$

The mutual commutativity of $P_\perp^+(H_{a,\bar{W}_a})$, $H^+_{a,\bar{W}_a}$, $H^+_{a,\perp,\bar{W}_a}$ and $T^+_{\|,\bar{W}_a}$ is trivial by their definitions. Now we introduce the asymptotic energy $T^+_{a,\bar{W}_a}$ as follows: For $h \in C_\infty(\mathbf{R})$,

$$h(T^+_{a,\bar{W}_a}) = s - \lim_{t\to\infty} \bar{U}_a(t)^* h(T_a)\bar{U}_a(t). \tag{4.44}$$

There exists $T^+_{a,\bar{W}_a}$ since $\mathbf{D}_{\bar{H}_a(t)}h(T_a) = 0$ for $h \in C_0^\infty(\mathbf{R})$, and we see that by their definitions, $T^+_{a,\bar{W}_a}$ commutes with $P_\perp^+(H_{a,\bar{W}_a})$, $H^+_{a,\bar{W}_a}$, $H^+_{a,\perp,\bar{W}_a}$ and $T^+_{\|,\bar{W}_a}$, and that

$$H^+_{a,\bar{W}_a} = H^{a,+}_{\bar{W}_a} + T^+_{a,\bar{W}_a} = H^a + T^+_{a,\bar{W}_a}, \tag{4.45}$$

where we used the fact that

$$\bar{U}_a(t) = e^{-i(t-1)H^a} \otimes \tilde{U}_a(t). \tag{4.46}$$

In the other way, by using (4.11) and the fact that $T(1)(x_\perp/t)T(1)^{-1} = x_\perp/t$, we have

$$P_\perp^+(H_{a,\bar{W}_a}) = T(1)P_\perp^+(H_{a,M,\tilde{W}_{M,a}})T(1)^{-1} = P_\perp^+(H_{a,M,\tilde{W}_{M,a}}). \tag{4.47}$$

Combining this with Theorem 2.2 (2) and using the fact that $T(1)p_{a,\perp}T(1)^{-1} = p_{a,\perp}$, we obtain

$$P_{a,\perp}^+(H_{a,\tilde{W}_a}) = D_{a,\perp}^+(H_{a,\tilde{W}_a}), \tag{4.48}$$

where $D_{a,\perp}^+(H_{a,\bar{W}_a})$ is defined by

$$g(D_{a,\perp}^+(H_{a,\bar{W}_a})) = s - \lim_{t\to\infty} U_{a,\bar{W}_a}(t)^* g(p_{a,\perp}) U_{a,\bar{W}_a}(t)$$

for $g \in C_\infty(X_{a,\perp})$. Moreover, we see that the asymptotic observable $D_{a,\perp}^+(H_{a,\bar{W}_a})$ exists since $\mathbf{D}_{\bar{H}_a(t)} g(p_{a,\perp}) = 0$ for $g \in C_0^\infty(X_{a,\perp})$. By their definitions, $D_{a,\perp}^+(H_{a,\bar{W}_a})$ satisfies

$$D_{a,\perp}^+(H_{a,\bar{W}_a}) = \bar{\Omega}_a^* D_{a,\perp}^+(H_{a,\tilde{W}_a})\bar{\Omega}_a.$$

Combining this with (4.37) and (4.48), we have

$$P_{a,\perp}^+(H_{a,\bar{W}_a}) = D_{a,\perp}^+(H_{a,\bar{W}_a}). \tag{4.49}$$

Then by noting that $T_a = T_\| + (p_{a,\perp})^2/2$ and that H^a, $T_\|$ and $(p_{a,\perp})^2/2$ are mutually commutative, we obtain

$$\begin{aligned} T_{a,\bar{W}_a}^+ &= T_{\|,\bar{W}_a}^+ + \frac{1}{2}(D_{a,\perp}^+(H_{a,\bar{W}_a}))^2 \\ &= T_{\|,\bar{W}_a}^+ + \frac{1}{2}(P_{a,\perp}^+(H_{a,\bar{W}_a}))^2, \end{aligned} \tag{4.50}$$

$$H_{a,\perp,\bar{W}_a}^+ = H^a + \frac{1}{2}(P_{a,\perp}^+(H_{a,\bar{W}_a}))^2, \tag{4.51}$$

by virtue of (4.49). Combining (4.45) with (4.50) and (4.51), we have (4.41) and (4.42). □

By virtue of Proposition 4.10, we may rewrite Proposition 4.11 in terms of the asymptotic observables for the propagator $U_{a,\bar{W}_a}(t)$ as follows:

PROPOSITION 4.12. – *Let the notations be as above. Let $h \in C_\infty(\mathbf{R})$. Then there exists the strong limit*

$$s - \lim_{t\to\infty} U_{a,\tilde{W}_a}(t)^* h(T_\|) U_{a,\tilde{W}_a}(t). \tag{4.52}$$

Moreover, there exists a unique self-adjoint operator $T^+_{\|,\tilde{W}_a}$ such that (4.52) equals $h(T^+_{\|,\tilde{W}_a})$. $P^+_\perp(H_{a,\tilde{W}_a})$, $H^+_{a,\tilde{W}_a}$, $H^+_{a,\perp,\tilde{W}_a}$ and $T^+_{\|,\tilde{W}_a}$ are mutually commutative, and satisfy

$$H^+_{a,\tilde{W}_a} = H^+_{a,\perp,\tilde{W}_a} + T^+_{\|,\tilde{W}_a}, \tag{4.53}$$

$$H^+_{a,\perp,\tilde{W}_a} = H^a + \frac{1}{2}(P^+_{a,\perp}(H_{a,\tilde{W}_a}))^2. \tag{4.54}$$

Here we need the known results for N-body systems without the Stark effect, which were obtained by Dereziński [D2] (see Lemmas 4.10 and 4.12 of [D2]).

LEMMA 4.13. – *Let the notations be as in §2. For any $a \in \mathcal{A}$,*

$$X_a \times \mathbf{R} \cap \sigma(P^+(H_{M,a,W}), H^{a,+}_{M,W}) \subset X_a \times \mathcal{E}_a,$$
$$\sigma(P^+(H_{M,a,W}), H^{a,+}_{M,W}) \supset X_a \times (\sigma_{pp}(H^a_M) \setminus \mathcal{T}_a).$$

Completion of the proof of Theorem 4.1. – We shall use Proposition 4.3 to reduce the statements for H to those for the time-dependent Hamiltonians $H_{a,\tilde{W}_a}(t)$. We should note that

$$\bigcup_{a\subset c} Z_{a,\perp} = X_\perp. \tag{4.55}$$

By virtue of (4.11), we have

$$H^{a,+}_{M,\tilde{W}_{M,a}} = H^a,$$

where we used the fact $T(t)^{-1}H^aT(t) = H^a = H^a_M$, $a \subset c$. By using this, (4.43), (4.47), Proposition 4.12 and Lemma 4.13, and noting that $\sigma(T_\|) = \mathbf{R}$, we obtain

$$\sigma(H, H^+_{c,\perp}, P^+_\perp(H)) \subset \bigcup_{a\subset c} \times \left\{(\lambda, \lambda_{c,\perp}, \xi_{a,\perp}) \mid \lambda = \lambda_{c,\perp} + \lambda^a, \lambda_{c,\perp} = \frac{1}{2}(\xi_{a,\perp})^2 + \tau, \lambda^a \in \mathbf{R}, \xi_{a,\perp} \in X_{a,\perp}, \tau \in \mathcal{E}_a\right\}, \tag{4.56}$$

$$\sigma(H, H_{c,\perp}^{+}, P_{\perp}^{+}(H)) \supset \bigcup_{a \subset c} \times \Big\{ (\lambda, \lambda_{c,\perp}, \xi_{a,\perp}) \;\Big|\; \lambda = \lambda_{c,\perp} + \lambda^a, \lambda_{c,\perp} = \frac{1}{2}(\xi_{a,\perp})^2 + \tau,$$
$$\lambda^a \in \mathbf{R}, \xi_{a,\perp} \in Z_{a,\perp}, \tau \in \sigma_{pp}(H^a) \setminus \mathcal{T}_a \Big\}. \qquad (4.57)$$

But the closure of the right-hand side of (4.57) is the right-hand side of (4.56). This implies the theorem. □

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