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A continuous family of automata: the Ising automata

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A la mémoire de Claude Itzykson

ABSTRACT. – We discuss the concept of a continuous family of automata and examplify our theory with the family of Ising automata.

RÉSUMÉ. – Nous discutons le concept de famille continue d'automates et nous illustrons notre théorie par l'étude des automates attachés à une chaîne d'Ising.

1. INTRODUCTION

Physics is mostly concerned with discontinuous phenomena. Phase transitions have always played a central role in Statistical Mechanics and many a scientist, both among physicits and mathematicians, has defined and discussed models which insists on critical values of the parameters.

Our point of view is somehow opposite. The induced field of an inhomogeneous Ising chain can be described in terms of the so called

Ising automata which depend on the external field. The induced field varies continuously with the external field and therefore the family of Ising automata must be continuous. This may appear somewhat surprising since an automaton is a discrete object. How can a four state automaton, say, become a five state automaton in a continuous fashion? One of the goals of our article is to explain how this is possible.

We shall also discuss the energy of the Ising chain as well as the entropy *i.e.* the logarithm of the number of configurations in the fundamental state per site of the inhomogeneous chain. Our method is different from that of B. Derrida, J. Vannimenus and Y. Pomeau [2] since we take advantage of some of the results obtained in the first paragraphs *i.e.* automaton theory. Our results confirm theirs, and this is fortunate! Our sole intention in this last part of our work is to present a new approach.

2. AUTOMATA

Let E be a finite set with at least two elements. E is called the input set or alphabet and will be fixed throughout this section. An *automata* over E is by definition a triple $M = (\Sigma, A_0, f)$ where Σ is a finite set called the set of states, where $A_0 \in \Sigma$ is called the *initial state* and where finally f maps $\Sigma \times E$ into Σ; f is called the *next state function.*

For simplicity we denote $A\,\varepsilon := f\,(A,\,\varepsilon)$, $A \in \Sigma$, $\varepsilon \in E$. We also denote

$$A\,\varepsilon_1\,\varepsilon_2...\varepsilon_n := (...(A\,\varepsilon_1)\,\varepsilon_2...)\,\varepsilon_n.$$

An automaton can be represented as a graph. The states of the automaton are the nods or vertices of the graph and the map f can be seen as oriented edges (labelled arrows) which join A to $A\,\varepsilon$, $A \in \Sigma$, $\varepsilon \in E$.

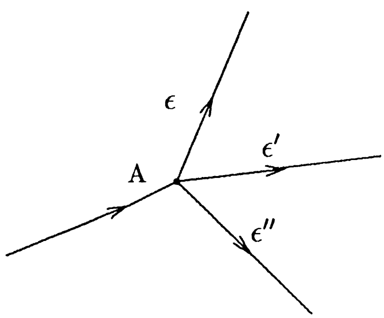

Fig. 1.

An automaton M is called *connected* if for any $A \in \Sigma$ $A \neq A_0$ there exists an input sequence $\varepsilon_1 \varepsilon_2 ... \varepsilon_n$ such that $A = A_0 \varepsilon_1 \varepsilon_2 ... \varepsilon_n$. In the following we only consider connected automata.

Let $M = (\Sigma, A_0, f)$ and $M' = (\Sigma', A'_0, f')$ be two automata. We say that M' is a *factor* of M if there exists a surjection $F : \Sigma \to \Sigma'$ such that $F(A_0) = A'_0$ and $F(A\varepsilon) = F(A)\varepsilon$ for all $A \in \Sigma$ and $\varepsilon \in E$. M and M' are said to be *isomorphic* if the above map F is a bijection. From now on we identify isomorphic automata.

The *product* $M \times M'$ of two automata is automaton $M \times M' = (\Sigma'', (A_0, A'_0), f \otimes f')$ where $(A_0, A'_0) \in \Sigma'' \subset \Sigma \times \Sigma'$ and where

$$f \otimes f'((A, A'), \varepsilon) := (f(A, \varepsilon), f'(A', \varepsilon)).$$

Σ'' is the "connected" component of $\Sigma \times \Sigma'$ containing (A_0, A'_0) with respect to $f \otimes f'$. It is clear that $M \times M'$ and $M' \times M$ are isomorphic and that $(M \times M') \times M''$ is isomorphic to $M \times (M' \times M'')$ so that the product is commutative and associative. Also $M \times M = M$. Quite obviously M and M' are factors of $M \times M'$.

Two automata M and M' are said to be *adjacent* if they have a nontrivial common factor where by nontrivial we mean an automaton which has at least two states.

3. AUTOMATA WITH OUTPUT

An automaton with output (also called sequential transducer) is an automaton $M = (\Sigma, A_0, f)$ together with an output function $\varphi : \Sigma \to \Theta$ where Θ is some given set. Given an output sequence $\varepsilon_1 \varepsilon_2 ... \varepsilon_n$ we define the output sequence $\delta_1 \delta_2 ... \delta_n$

$$\delta_i := \varphi(A_0 \varepsilon_1 \varepsilon_2 ... \varepsilon_i), \qquad i = 1, 2, ..., n,$$

and write

$$\delta_1 \delta_2 ... \delta_n = (M, \varphi)(\varepsilon_1 \varepsilon_2 ... \varepsilon_n).$$

Two automata with output (M, φ) and (M', φ') can be identified if $(M, \varphi) w = (M', \varphi') w$ for all input sequence w.

A *language* L attached to M is specified by an output function $\varphi : \Sigma \to \{0, 1\}$:

$$L = \{w \in E^* / \varphi(A_0 w) = 1\}.$$

Here E^* denotes the set of words on the alphabet E. Let $F = \varphi^{-1}(1) \subset \Sigma$. F is called the set of *final states*. An obviously equivalent way to define the language L is $L = \{w \in E^*/A_0\, w \in F\}$. A language is said to be *proper* if $L \neq \varnothing$ and $L \neq E^*$.

THEOREM 1. – *Let M and M′ be two automata. The following statements are equivalent.*

(1) *M and M′ are adjacent;*

(2) *There exist nonconstant output functions φ and φ' such that $\varphi(A_0) = \varphi'(A'_0)$ and such that for all input sequences $w \in E^*$*

$$(M,\, \varphi)\, w = (M',\, \varphi')\, w;$$

(3) *There exists a proper language L which is attached both to M and to M′.*

This theorem should be compared with classical results: *see* [3], Corollary 4.3, p. 42.

Proof. – $*(1) \Rightarrow (2)$. Assume that $M'' = (\Sigma'',\, A''_0,\, f'')$ is a nontrivial common factor of $M = (\Sigma,\, A_0,\, f)$ and of $M' = (\Sigma',\, A'_0,\, f')$ with factor maps $F : \Sigma \to \Sigma''$ and $F' : \Sigma' \to \Sigma''$. We consider F and F' as output functions. They are not constant since they are surjections onto Σ'' and since $\operatorname{card}(\Sigma'') \geq 2$. It is clear that $F(A_0) = F'(A'_0) = A''_0$. Now for all input sequence $w = \varepsilon_1\, \varepsilon_2 ... \varepsilon_n$

$$\begin{aligned}(M,\, F)\,(\varepsilon_1\, \varepsilon_2 ... \varepsilon_n) &= (A''_0\, \varepsilon_1)\,(A''_0\, \varepsilon_1\, \varepsilon_2) ... (A''_0\, \varepsilon_1\, \varepsilon_2 ... \varepsilon_n)\\ &= (M',\, F')\,(\varepsilon_1\, \varepsilon_2 ... \varepsilon_n).\end{aligned}$$

This establishes (2).

$*(2) \Rightarrow (3)$. Assume (2) with output functions $\varphi : \Sigma \to \Theta$ and $\varphi' : \Sigma' \to \Theta$. We can suppose that both φ and φ' are surjective and that $\operatorname{card}(\Theta) \geq 2$. Choose $\theta \in \Theta$ and define $F = \varphi^{-1}(\theta) \subset \Sigma$ and $F' = \varphi'^{-1}(\theta) \subset \Sigma'$. Clearly $\emptyset \neq F \neq \Sigma$ and $\emptyset \neq F' \neq \Sigma'$. By assumption, for all $w \in E^*$

$$(M,\, \varphi)\, w = (M',\, \varphi')\, w$$

and also $\varphi(A_0) = \varphi'(A'_0)$ so that the two languages

$$L = \{w \in E^*/A_0\, w \in F\}$$
$$L' = \{w \in E^*/A'_0\, w \in F'\}$$

coincide. Assertion (3) is now proved since L is proper.

$*(3) \Rightarrow (1)$. Assume (3) with L, F and F' such that $\emptyset \neq L \neq E^*$ and

$$L = \{w \in E^*/A_0\, w \in F\} = \{w \in E^*/A'_0\, w \in F'\}.$$

Let $x \in E^*$ be a word. Define the language

$$x^{-1} L = \{w \in E^* / xw \in L\}.$$

Two words $u, v \in E^*$ are said to be L-equivalent $(u \sim_L v)$ if $u^{-1} L = v^{-1} L$. The relation is clearly an equivalence. For $w_0 \in E^*$ we denote the equivalence class containing w_0 by $[w_0]$.

Consider the automaton

$$M'' = (E^* / \sim_L, [\emptyset], f'')$$

where $\emptyset$ is the empty word in E^* and where

$$f'' : (E^* / \sim_L) \times E \to E^* / \sim_L$$

is defined by

$$f''([w], \varepsilon) = [w \varepsilon].$$

Let us show that M'' is a nontrivial factor of $M = (\Sigma, A_0, f)$ and of $M' = (\Sigma', A'_0, f')$. For $A \in \Sigma$ let $w \in E^*$ be such that $A_0 w = A$. Define $F : \Sigma \to E^* / \sim_L$ by $F(A) = [w]$. This is well defined since if $A = A_0 u$, $u \in E^*$ then for any $x \in E^*$ we have

$$\begin{aligned} wx \in L &\Leftrightarrow A_0 wx \in F \\ &\Leftrightarrow A x \in F \\ &\Leftrightarrow A_0 ux \in F \\ &\Leftrightarrow ux \in L \end{aligned}$$

and therefore $[w] = [u]$.

F is obviously surjective so that $E^* / \sim_L$ is finite. Thus M'' is an automaton and F is a factor map from M to M''. Hence M'' is a factor of M. Similarly M'' is a factor of M'.

Finally, since L is a proper language, $E^* / \sim_L$ contains at least two elements e.g. $[u]$ and $[v]$ where $u \in L$ and $v \notin L$. This shows that M'' is nontrivial, and thus (1) is established.

Q.E.D.

We now give a simple sufficient condition for automata to be adjacent. Let us suppose that the input alphabet E consists of two elements $E = \{\xi, \eta\}$. We agree to say that a state A is pure if either

$$A \xi^{-1} := \{B \in \Sigma / B \xi = A\} = \emptyset$$

or $A \eta^{-1} = \emptyset$. In other words, the incident arrows to A are either all labelled ξ or all labelled η. A state is said to have label ξ (resp. η) if at least one of its incident arrows is ξ (resp. η) (and then all incident arrows of the state are ξ (resp. η)). The automaton M is said to be pure if all its states are pure. (This amounts to saying that the opacity of M vanishes: *see* [1] or [4].)

COROLLARY. – *Nontrivial pure automata are pairwise adjacent provided their initial state has same label.*

Proof. – Let $M = (\Sigma, A_0, f)$ be a pure automaton. Let $F \subset \Sigma$ be the set of states which receive arrows labelled ξ:

$$F = \{A\,\xi / A \in \Sigma\} = \Sigma\xi.$$

The language $L = \{w \in E^* | A_0\, w \in F\}$ reduces to $E^*\,\xi \cup \{\emptyset\}$ or $E^*\,\xi$ according to whether A_0 has label ξ or not. Therefore if M and M' are two pure automata such that the initial state have same label they share the same proper language $E^*\,\xi \cup \{\emptyset\}$ or $E^*\,\xi$. Property (3) in Theorem 1 then shows that M and M' are adjacent.

Q.E.D.

The corollary is easily extended to automata with input alphabet containing more than two elements. Suppose the input alphabet is partitioned in two disjoint proper subsets $E = E_1 \cup E_2$. A state A is (E_1, E_2)-pure if either

$$\{B \in \Sigma / BE_1 = A\} = \emptyset$$

or

$$\{B \in \Sigma / BE_2 = A\} = \emptyset.$$

An automaton is (E_1, E_2)-pure if all its states are (E_1, E_2)-pure. Then it is easily seen that (E_1, E_2)-pure automata are pairwise adjacent provided the initial states have same labels.

4. CONTINUOUS FAMILIES OF AUTOMATA

Let $(M_\alpha, \varphi_\alpha)$ be a family of automata with nonconstant output. The index α is supposed to run through some real interval. The input alphabet E is common to all automata M_α. The output functions φ_α are defined respectively on the set of states Σ_α of each M_α and take their values in the same set Θ. It may be convenient to suppose that $\Theta \subset \mathbb{R}$.

Let us first assume that as α runs through a closed interval I, the automata M_α stay isomorphic one to the other. We say that the family $(M_\alpha, \varphi_\alpha)$ is continuous if the map $\alpha \mapsto \varphi_\alpha$ is continuous *i.e.* for all $A \in \Sigma = \Sigma_\alpha$, $\alpha \mapsto \varphi_\alpha(A)$ is continuous.

We now extend the definition. Suppose that I_1 and I_2 are two contiguous closed intervals. Let $\{\alpha_0\} = I_1 \cap I_2$. Let us assume that $(M_\alpha, \varphi_\alpha)$ is *continuous* as above both on I_1 and on I_2. We say that the family $(M_\alpha, \varphi_\alpha)$ is *continuous* on the union $I_1 \cup I_2$ if furthermore for all input sequences $w \in E^*$

$$(M_{\alpha_0 - 0}, \varphi_{\alpha_0 - 0})\, w = (M_{\alpha_0 + 0}, \varphi_{\alpha_0 + 0})\, w$$

where

$$(M_{\alpha_0 - 0}, \varphi_{\alpha_0 - 0}) \; (\text{resp.}\, (M_{\alpha_0 + 0}, \varphi_{\alpha_0 + 0}))$$

represents the limit of the continuous family $(M_\alpha, \varphi_\alpha)$ as α increases (resp. decreases) to α_0.

The definition extends to any number (finite or infinite) of contiguous closed intervals I_j. In particular $I = \bigcup_j I_j$ may well be an open interval in which case the family $(M_\alpha, \varphi_\alpha)$ is continuous in an open interval.

The important fact should be underlined: if the family $(M_\alpha, \varphi_\alpha)$ is continuous in some interval *I*, then for all $\alpha \in I$, the two automata $M_{\alpha-0}$ and $M_{\alpha+0}$ are adjacent.

5. THE INHOMOGENEOUS ISING CHAIN

Let us consider a finite Ising chain with given interaction coefficients $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{N-1}$ $(\varepsilon_i = \pm 1)$ and with external field $\alpha/2 \in \mathbb{R}$. We introduce boundary interaction coefficients ξ_0 and ξ_N which also take values ± 1. Our problem is to find a configuration $\sigma = (\sigma_i)_{i=0,1,\dots,N}$ with $\sigma_i = \pm 1$ which maximizes the *negative Hamiltonian*

$$H(\sigma) = \sum_{i=0}^{N-1} \varepsilon_i \,\sigma_i \,\sigma_{i+1} + \frac{\alpha}{2} \sum_{i=0}^{N} \sigma_i + \xi_0 \,\sigma_0 + \xi_N \,\sigma_N.$$

σ_i is called *spin* at site i.

We define $\mathcal{G}$ as the set of configurations which maximize *H*:

$$\mathcal{G} := \{\hat{\sigma} \in \{\pm 1\}^{N+1} / H(\hat{\sigma}) = \max_{\sigma} H(\sigma)\}.$$

$\mathcal{G}$ is called the *ground state* for *H*. It depends on $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{N-1}, \xi_0, \xi_N$ and on α.

Let us give two simple examples.

Example 1. – If $\alpha > 4$ then $\mathcal{G}$ consists of one element $\hat{\sigma} = (1, 1, \dots, 1)$. If $\alpha = 4$ then $(1, 1, \dots, 1) \in \mathcal{G}$ but $\mathcal{G}$ can contain more elements. If

$\varepsilon_{i-1} = \varepsilon_i = -1$ then $(1, 1, ..., -1, ..., 1)$ where -1 appears at site i, is also an element of $\mathcal{G}$.

Example 2. – If $\alpha = 0$ then either card $\mathcal{G} = 1$ or card $\mathcal{G} = N+2$. In fact if

$$\xi_0 \, \varepsilon_0 \, \varepsilon_1 ... \varepsilon_{N-1} \, \xi_N = 1$$

then $\mathcal{G}$ contains the only element $\hat{\sigma}$ where

$$\hat{\sigma}_i = \xi_0 \, \varepsilon_0 \, \varepsilon_1 ... \varepsilon_{i-1} \qquad (i = 0, 1, ..., N).$$

If $\xi_0 \, \varepsilon_0 \, \varepsilon_1 ... \varepsilon_{N-1} \, \xi_N = -1$ then $\mathcal{G}$ consists of the $N + 2$ elements $\hat{\sigma}^j$ $(j = 0, 1, ..., N + 1)$ defined by

$$\hat{\sigma}_i^j = \begin{cases} \xi_0 \, \varepsilon_0 ... \varepsilon_{i-1} & \text{if } \; i < j \\ -\xi_0 \, \varepsilon_0 ... \varepsilon_{i-1} & \text{if } \; i \geq j. \end{cases}$$

Our purpose in this section is to find an algorithm to determine $\mathcal{G}$ given $\varepsilon_0, \varepsilon_1, ..., \varepsilon_{N-1}, \xi_0, \xi_N$ and α. We restrict ourselves to the case $0 < \alpha \leq 4$ by reason of symmetry and since as shown in the two previous examples the case $\alpha = 0$ and $\alpha > 4$ are trivial.

For $n = 0, 1, ..., N$ we define the Hamiltonians H_n by

$$H_n(\sigma) = \sum_{i=0}^{n-1} \varepsilon_i \, \sigma_i \, \sigma_{i+1} + \frac{\alpha}{2} \sum_{i=0}^{n} \sigma_i + \xi_0 \, \sigma_0.$$

Note that $H_N \neq H$ since H_N is not affected by the boundary effect at site N. Put

$$\begin{cases} a_n = \max\limits_{\sigma_n = 1} H_n(\sigma) \\ b_n = \max\limits_{\sigma_n = -1} H_n(\sigma) \qquad n = 0, 1, ..., N \\ \delta_n = a_n - b_n. \end{cases}$$

Then

$$\begin{cases} a_0 = \dfrac{\alpha}{2} + \xi_0 \\ b_0 = -\dfrac{\alpha}{2} - \xi_0 \\ \delta_0 = \alpha + 2\,\xi_0. \end{cases}$$

Since

$$H_{n+1} = H_n + \varepsilon_n \, \sigma_n \sigma_{n+1} + \frac{\alpha}{2} \sigma_{n+1} \qquad (n = 0, 1, ..., N - 1)$$

we have

$$a_{n+1} = (\max_{\substack{\sigma_n=1\\ \sigma_{n+1}=1}} H_{n+1}) \vee (\max_{\substack{\sigma_n=-1\\ \sigma_{n+1}=1}} H_{n+1})$$
$$= \left(a_n + \varepsilon_n + \frac{\alpha}{2}\right) \vee \left(b_n - \varepsilon_n + \frac{\alpha}{2}\right)$$

where $a \vee b = \max\{a, b\}$.

Similarly

$$b_{n+1} = \left(a_n - \varepsilon_n - \frac{\alpha}{2}\right) \vee \left(b_n + \varepsilon_n - \frac{\alpha}{2}\right)$$

from where we deduce

$$\delta_{n+1} = a_{n+1} - b_{n+1} = \alpha + \varepsilon_n\,\tau(\delta_n)$$

where

$$\tau(x) = \begin{cases} -2 & \text{if } x < -2 \\ x & \text{if } -2 \leq x \leq 2 \\ 2 & \text{if } x > 2. \end{cases}$$

Thus starting by $\delta_0 = \alpha + 2\,\xi_0$ which is given we can determine δ_n inductively by the relation

$$\delta_{n+1} = \alpha + \varepsilon_n\,\tau(\delta_n) \qquad (n = 0, 1, \dots, N-1).$$

Let

$$\delta'_N := \delta_N + 2\,\xi_N.$$

Since $H = H_N + \xi_N\,\sigma_N$, we have

$$\max_{\sigma} H(\sigma) = (\max_{\sigma_N=1} H(\sigma)) \vee (\max_{\sigma_N=-1} H(\sigma))$$
$$= (a_N + \xi_N) \vee (b_N - \xi_N).$$

This implies that

$$\delta'_N > 0 \quad \Rightarrow \quad a_N + \xi_N > b_N - \xi_N$$

so that $\max_{\sigma} H$ is attained by $\hat{\sigma}$ with $\hat{\sigma}_N = 1$. On the other hand, if $\delta'_N < 0$ then $\max_{\sigma} H$ is attained by $\hat{\sigma}$ with $\hat{\sigma}_N = -1$. Therefore in both cases the ground state $\mathcal{G}$ is determined at site N. If $\delta'_N = 0$ then $\max_{\sigma} H$ is attained by configurations $\hat{\sigma}$ and $\hat{\sigma}'$ with $\hat{\sigma}_N = 1$ and $\hat{\sigma}'_N = -1$. In this case the ground state $\mathcal{G}$ at site N can assume both values ± 1.

We can now proceed backwards to determine $\mathcal{G}$ at sites $N-1, N-2, \ldots, 0$. In fact let

$$\delta'_{N-1} := \delta_{N-1} + 2\,\varepsilon_{N-1}\,\mathrm{sgn}\,(\delta'_N)$$

where if $\delta'_N = 0$ then we can choose arbitrarily $\mathrm{sgn}\,(\delta'_N) = 1$ or -1. Since

$$H = H_{N-1} + \varepsilon_{N-1}\,\sigma_{N-1}\,\sigma_N + \frac{\alpha}{2}\,\sigma_N + \xi_N\,\sigma_N$$

we have

$$\begin{aligned}\max_\sigma H(\sigma) &= \max_{\sigma_N = \mathrm{sgn}\,(\delta'_N)} H(\sigma)\\ &= (\max_{\substack{\sigma_N = \mathrm{sgn}\,(\delta'_N)\\ \sigma_{N-1}=1}} H(\sigma)) \vee (\max_{\substack{\sigma_N = \mathrm{sgn}\,(\delta'_N)\\ \sigma_{N-1}=-1}} H(\sigma))\\ &= \left(a_{N-1} + \varepsilon_{N-1}\,\mathrm{sgn}\,(\delta'_N) + \frac{\alpha}{2}\,\mathrm{sgn}\,(\delta'_N) + \xi_N\,\mathrm{sgn}\,(\delta'_N)\right)\\ &\quad\vee \left(b_{N-1} - \varepsilon_{N-1}\,\mathrm{sgn}\,(\delta'_N) + \frac{\alpha}{2}\,\mathrm{sgn}\,(\delta'_N) + \xi_N\,\mathrm{sgn}\,(\delta'_N)\right).\end{aligned}$$

Hence if $\delta'_{N-1} > 0$ then the first alternative exceeds the second so that $\max_\sigma H$ is attained by $\hat{\sigma}$ with $\hat{\sigma}_{N-1} = 1$ and of course $\hat{\sigma}_N = \mathrm{sgn}\,(\delta'_N)$. On the other hand, if $\delta'_{N-1} < 0$ then $\max_\sigma H$ is attained by $\hat{\sigma}$ with $\hat{\sigma}_{N-1} = -1$ provided that $\hat{\sigma}_N = \mathrm{sgn}\,(\delta'_N)$. If $\delta'_{N-1} = 0$ then $\hat{\sigma}_{N-1} = \pm 1$ and $\hat{\sigma}_N = \mathrm{sgn}\,(\delta'_N)$.

In the same way, define

$$\delta'_n := \delta_n + 2\,\varepsilon_n\,\mathrm{sgn}\,(\delta'_{n+1}), \qquad (n = N-2,\ N-3, \ldots, 0)$$

where as usual $\mathrm{sgn}\,(0) = \pm 1$. We can now state our second theorem which summarizes the above calculations.

THEOREM 2. – *The configuration $\hat{\sigma}$ belongs to the ground state $\mathcal{G}$ if and only if $\hat{\sigma}_n = \mathrm{sgn}\,(\delta'_n)$ $(n = 0, 1, \ldots, N)$ where δ'_n is determined inductively by the equations*

$$\begin{aligned}\delta_0 &= \alpha + 2\,\xi_0\\ \delta_{n+1} &= \alpha + \varepsilon_n\,\tau\,(\delta_n) \qquad (n = 0, 1, \ldots, N-1)\\ \delta'_N &= \delta_N + 2\,\xi_N\\ \delta'_n &= \delta_n + 2\,\varepsilon_n\,\mathrm{sgn}\,(\delta'_{n+1}) \qquad (n = N-1,\ N-2, \ldots, 0).\end{aligned}$$

If $\delta'_{n+1} = 0$ we define $\mathrm{sgn}\,(\delta'_{n+1}) = \hat{\sigma}_{n+1}$.

The sequence (δ'_n) $(0 \leq n \leq N)$ which controls the sign of the spins $\hat{\sigma}_n$ is called the *induced field.*

6. THE INFINITE INHOMOGENEOUS ISING CHAIN

We consider an infinite chain with given interaction constants $(\varepsilon_i)_{i\in\mathbb{Z}}$ and external field $\alpha/2$. The negative Hamiltonian is given by the formal infinite series

$$H(\sigma) = \sum_{i\in\mathbb{Z}} \varepsilon_i\,\sigma_i\,\sigma_{i+1} + \frac{\alpha}{2}\sum_{i\in\mathbb{Z}} \sigma_i.$$

The series diverges since its terms are ± 1 but if two configurations σ and σ' differ in only finitely many places then $H(\sigma) - H(\sigma')$ is a finite sum and its sign allows us to decide which of the two $H(\sigma)$ or $H(\sigma')$ is the largest. We define the ground state $\mathcal{G}$ to be the set of $\hat{\sigma}$ such that for any finitely many changes σ of $\hat{\sigma}$ we have $H(\sigma) \leq H(\hat{\sigma})$.

Let $M \leq N$ be any two positive, negative or vanishing integers. Put

$$H_M^N(\sigma) = \sum_{i=M-1}^{N} \varepsilon_i\,\sigma_i\,\sigma_{i+1} + \frac{\alpha}{2}\sum_{i=M}^{N} \sigma_i.$$

Then

$$\mathcal{G} = \{(\hat{\sigma}_i)_{i\in\mathbb{Z}} / \forall M,\ N\in\mathbb{Z},\ M\leq N,\ H_M^N(\hat{\sigma}) = \max_{\substack{\sigma_{M-1}=\hat{\sigma}_{M-1}\\ \sigma_{N+1}=\hat{\sigma}_{N+1}}} (H_M^N(\sigma))\}.$$

Theorem 2 readily applies to the infinite chain.

COROLLARY. – *$\hat{\sigma}\in\mathcal{G}$ if and only if for all integers $M \leq N$, $\hat{\sigma}_n = \operatorname{sgn}(\delta'_n)$ holds for $n = M,\ M+1, \ldots, N$ where δ'_n is defined by the following equations*:

$$\begin{aligned}
\delta_M &= \alpha + 2\,\varepsilon_{M-1}\,\hat{\sigma}_{M-1}\\
\delta_{n+1} &= \alpha + \varepsilon_n\,\tau(\delta_n) \qquad (n = M,\ M+1, \ldots, N-1)\\
\delta'_N &= \delta_N + 2\,\varepsilon_N\,\hat{\sigma}_{N+1}\\
\delta'_n &= \delta_n + 2\,\varepsilon_n \operatorname{sgn}(\delta'_{n+1}) \qquad (n = N-1,\ N-2, \ldots, M).
\end{aligned}$$

where if $\delta'_{n+1} = 0$ then we define $\operatorname{sgn}(\delta'_{n+1})$ *to be* $\hat{\sigma}_{n+1}$.

7. THE ISING AUTOMATA ([1], [4])

We define an automaton which generates $\delta_1\,\delta_2 \ldots \delta_N$ given the input sequence $\varepsilon_0\,\varepsilon_1 \ldots \varepsilon_{N-1}$. For the moment we fix the external field $\alpha/2$ so that the automaton is indexed by α. We suppose $0 < \alpha \leq 4$.

The set of states Σ of the automaton M_α we are about to define is a subset of $\mathbb{R}$ so that $A \in \Sigma$ will be considered according to the context either as a state or as a real number. Our automaton is an automaton with output: the output function is just the identity map $\Sigma \to \Sigma$. In other words, in this section we identify (M_α, id) with M_α.

The input set is $E = \{-1, +1\}$. For $A \in \mathbb{R}$ and $\varepsilon \in E$ define

$$f(A, \varepsilon) := A\varepsilon := \alpha + \varepsilon\tau(A)$$

where τ is the function defined in paragraph 5. Let Σ be the set of real numbers A such that there exists an input sequence $\varepsilon_1 \varepsilon_2 ... \varepsilon_n$ possibly empty with $A = A_0 \varepsilon_1 \varepsilon_2 ... \varepsilon_n$ where $A_0 = 2 + \alpha$. In fact

$$\Sigma = \{2 + \alpha\} \cup \left\{-2 + i\,\alpha / i = 1, 2, ... \left[\frac{4}{\alpha}\right] + 1\right\}$$
$$\cup \left\{2 - i\,\alpha / i = 0, 1, ..., \left[\frac{4}{\alpha}\right] - 1\right\}.$$

We have thus defined the automaton $M_\alpha = (\sum, A_0, f)$.

From the point of view of isomorphism the M_α's can be classified as follows.

Case 1. $\frac{4}{\alpha} = \mu$ is an integer. Then M_α is isomorphic to the automaton N_μ with $\mu + 1$ states.

$$A_0 = \alpha + 2$$
$$D_i = i\,\alpha - 2,\ 1 \leq i \leq \mu - 1$$
$$D_\mu = 2$$

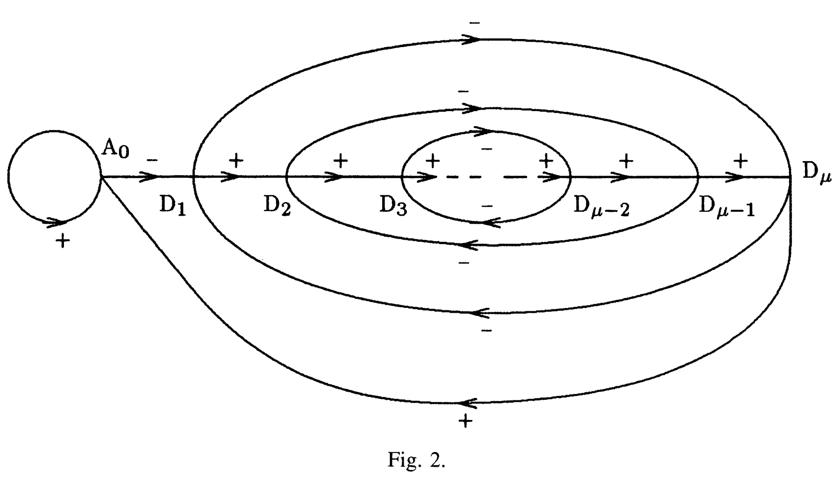

Fig. 2.

A_0 is the initial state and the next state function is given by the labelled arrows.

Case 2. $\mu < \frac{4}{\alpha} < \mu + 1$, $\mu \in \mathbb{N}$. Then M_α is isomorphic to the automaton L_μ with $2(\mu + 1)$ states and with initial state A_0.

$$A_0 = \alpha + 2$$
$$B_i = i\,\alpha - 2$$
$$C_j = 2 - (j - 1)\,\alpha$$

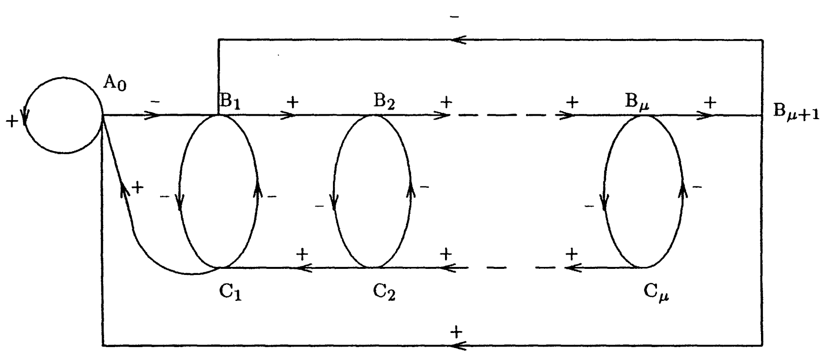

Fig. 3.

Both of these automata appear in [4].

THEOREM 3. – (1) *For* $\mu = 1, 2, \ldots$ $L_\mu = N_\mu \times N_{\mu+1}$

(2) *If* $\mu \neq \nu$ *then* N_μ *and* N_ν *are not adjacent.*

(3) L_μ *and* N_ν *are adjacent if and only if* $\nu = \mu$ *or* $\nu = \mu + 1$.

(4) L_μ *and* L_ν *are adjacent if and only if* $|\mu - \nu| \leq 1$.

(5) *For* $\mu = 2, 3, \ldots$ *the maximal common factor of* L_μ *and* $L_{\mu-1}$ *is* N_μ.

Proof. – * (1) The bijection $F : L_\mu \leftrightarrow N_\mu \times N_{\mu+1}$ is defined as follows:

$$F(A_0) = (A_0, A_0)$$
$$F(B_i) = (D_i, D_i) \qquad (i = 1, 2, \ldots, \mu)$$
$$F(C_i) = (D_{\mu-i+1}, D_{\mu-i+2}) \qquad (i = 1, 2, \ldots, \mu)$$
$$F(B_{\mu+1}) = (A_0, D_{\mu+1}).$$

* (2) Let $\mu > \nu$ and suppose N_μ and N_ν are adjacent. We represent the states of N_ν by A'_0, $D'_1, ..., D'_\nu$ and those of N_μ by A_0, $D_1, ..., D_\mu$. By Theorem 1 there exist nonconstant output functions φ and φ' such that $\varphi(A_0) = \varphi'(A'_0)$ and for all input sequence $\varepsilon_1 \varepsilon_2 ... \varepsilon_n$

$$(N_\mu, \varphi)(\varepsilon_1 \varepsilon_2 ... \varepsilon_n) = (N_\nu, \varphi')(\varepsilon_1 \varepsilon_2 ... \varepsilon_n).$$

Let $a := \varphi(A_0) = \varphi'(A'_0)$. Consider the words $-1\,1^i$, $i = \nu$, $\nu + 1, ..., \mu - 1$ where $1^i = 1...1$ represents 1 repeated i times. Then $\varphi(A_0 - 1\,1^i) = \varphi'(A_0 - 1\,1^i) = \varphi'(A'_0) = a$ so that

$$\varphi(D_{\nu+1}) = \varphi(D_{\nu+2}) = ... = \varphi(D_\mu) = a.$$

Now since $\varphi'(A'_0 - 1 - 1) = \varphi(A_0 - 1 - 1) = \varphi(D_\mu)$ we have $\varphi'(D'_\nu) = a$. We then deduce that

$$\varphi(D_\nu) = \varphi(A_0 - 1\,1^{\nu-1}) = \varphi'(A'_0 - 1\,1^{\nu-1}) = \varphi'(D'_\nu)$$

and thus

$$\varphi(D_\nu) = \varphi(D_{\nu+1}) = ... = \varphi(D_\mu) = a.$$

Since

$$\varphi'(A'_0 - 1\,1 - 1) = \varphi(A_0 - 1\,1 - 1) = \varphi(D_{\mu-1})$$

we have $\varphi'(D'_{\nu-1}) = a$. Then by the same argument $\varphi(D_{\nu-1}) = a$. Repeating the same idea we finally get

$$\varphi(D_1) = \varphi(D_2) = ... = \varphi(D_\mu) = a.$$

Now as $\varphi(A_0) = a$ we conclude that φ is constant and this contradicts our basic assumption.

* (3) Suppose $\mu = \nu$ (resp. $\nu - 1$). According to (1) $L_\mu = N_\nu \times N_{\nu+1}$ (resp. $N_{\nu-1} \times N_\nu$) so that N_ν is indeed adjacent to L_μ.

We now study the converse and assume that L_μ and N_ν are adjacent for $\mu > \nu$. Then by the same argument as in (2) we show that

$$\varphi(B_{\nu+1}) = \varphi(B_{\nu+2}) = ... = \varphi(B_\mu) = \varphi(B_{\mu+1}) = a$$

where $a := \varphi(A_0) = \varphi'(A'_0)$. Keeping the same notations as in (2) and replacing N_μ by L_μ,

$$\begin{aligned}\varphi'(D'_\nu) &= \varphi'(A'_0 - 1\,1^{\mu-1} - 1 - 1)\\ &= \varphi(A_0 - 1\,1^{\mu-1} - 1 - 1) = \varphi(B_\mu)\end{aligned}$$

hence $\varphi'(D'_\nu) = a$. From this we infer $\varphi(B_\nu) = a$, and proceeding in a similar way we prove

$$\varphi'(D'_1) = \varphi'(D'_2) = ... = \varphi'(D'_\nu) = a.$$

Now as $\varphi'(A'_0) = a$ we conclude that φ' is constant and this contradicts our hypothesis.

We now assume that L_μ and N_ν are adjacent for $\mu + 1 < \nu$. Let us show again that this leads to a contradiction. The same argument as in (2) shows that

$$\varphi(D_{\mu+2}) = \varphi(D_{\mu+3}) = \ldots = \varphi(D_\nu) = a$$

where $a := \varphi(A_0) = \varphi'(A'_0)$. The notations and conditions are just like in (2) where N_μ is replaced by N_ν and N_ν by L_μ. Now

$$\begin{aligned}\varphi'(B'_{\mu+1}) &= \varphi'(A'_0 - 1\,1^\mu - 1\,1^\mu)\\ &= \varphi(A_0 - 1\,1^\mu - 1\,1^\mu) = \varphi(D_\nu) = a,\end{aligned}$$

hence $\varphi(D_{\mu+1}) = a$. So now φ is constant on $D_{\mu+1}, D_{\mu+2}, \ldots, D_\nu$. Since

$$\begin{aligned}\varphi'(B'_\mu) &= \varphi'(A'_0 - 1\,1^\mu - 1\,1^{\mu-1})\\ &= \varphi(A_0 - 1\,1^\mu - 1\,1^{\mu-1}) = \varphi(D_\nu) = a\end{aligned}$$

we have $\varphi(D_\mu) = a$. In this way we prove that

$$\varphi(D_1) = \varphi(D_2) = \ldots = \varphi(D_\nu) = a.$$

But $\varphi(A_0) = a$ therefore φ is constant, and this contradicts the assumption.

$*$ (4) The "if" part follows from (1). We prove the converse. Suppose that L_μ and L_ν are adjacent for $\mu > \nu + 1$. By the same argument as in (2), we have

$$\varphi(B_{\nu+2}) = \varphi(B_{\nu+3}) = \ldots = \varphi(B_\mu) = \varphi(B_{\mu+1}) = a.$$

Notations are as in (2) replacing N_μ by L_μ and N_ν by L_ν. Since

$$\begin{aligned}\varphi(C_1) &= \varphi(A_0 - 1\,1^{\nu+1} - 1\,1^{\nu+1})\\ &= \varphi'(A'_0 - 1\,1^{\nu+1} - 1\,1^{\nu+1}) = \varphi'(A'_0) = a\end{aligned}$$

we have

$$\begin{aligned}\varphi'(C'_1) &= \varphi'(A'_0 - 1 - 1)\\ &= \varphi(A_0 - 1 - 1) = \varphi(C_1) = a.\end{aligned}$$

Therefore

$$\begin{aligned}\varphi(B_{\nu+1}) &= \varphi(A_0 - 1\,1^\nu - 1 - 1)\\ &= \varphi'(A'_0 - 1\,1^\nu - 1 - 1) = \varphi'(C'_1) = a.\end{aligned}$$

Also $\varphi'(B'_{\nu+1}) = \varphi(B_{\nu+1}) = a$. Then we can proceed further: since

$$\begin{aligned}\varphi(C_2) &= \varphi(A_0 - 1\,1^{\nu+1} - 1\,1^{\nu})\\ &= \varphi'(A'_0 - 1\,1^{\nu+1} - 1\,1^{\nu}) = \varphi'(B'_{\nu+1}) = a,\end{aligned}$$

we have

$$\begin{aligned}\varphi'(C'_2) &= \varphi'(A'_0 - 1\,1 - 1)\\ &= \varphi(A_0 - 1\,1 - 1) = \varphi(C_2) = a.\end{aligned}$$

Therefore

$$\begin{aligned}\varphi(B_\nu) &= \varphi(A_0 - 1\,1^{\nu} - 1\,1 - 1)\\ &= \varphi'(A'_0 - 1\,1^{\nu} - 1\,1 - 1) = \varphi'(C'_2) = a.\end{aligned}$$

Continuing we finally establish

$$\varphi(B_1) = \varphi(B_2) = \dots = \varphi(B_\mu) = \varphi(B_{\mu+1}) = a.$$

This once again contradicts the assumption according to which φ is nonconstant.

* (5) Let $M'' = (\Sigma'', A''_0, f'')$ be a common factor of $L_\mu = (\Sigma, A_0, f)$ and of $L_{\mu-1} = (\Sigma', A'_0, f')$. Let $\varphi : \Sigma \to \Sigma''$ and $\varphi' : \Sigma' \to \Sigma''$ be the factor maps. Since $A'_0 - 1\,1^{\mu} = A'_0$ we have $A''_0 - 1\,1^{\mu} = A''_0$. Hence

$$\varphi(B_{\mu+1}) = \varphi(A_0 - 1\,1^{\mu}) = A''_0 - 1\,1^{\mu} = A''_0 = \varphi(A_0).$$

By a similar technique we prove

$$\varphi(C_{\mu-i}) = \varphi(B_{i+1}) \qquad \text{for} \quad i = 0, 1, \dots, \mu - 1.$$

Therefore M'' is a function of $L_\mu/\sim$ where $\sim$ stands for the equivalence relation on Σ defined by

$$\begin{aligned}&A_0 \sim B_{\mu+1}\\ &B_{i+1} \sim C_{\mu-i} \qquad (i = 0, 1, \dots, \mu - 1).\end{aligned}$$

On the other hand, by (1), $L_\mu/\sim$ is isomorphic to N_μ. This implies that N_μ is the maximal common factor of L_μ and $L_{\mu-1}$.

Q.E.D.

8. CONTINUITY PROPERTIES OF THE ISING AUTOMATA

A simple consequence of Theorem 3 is that the family of Ising automata (M_α, id) is continuous for $\alpha > 0$. A closer look at the Ising automata enables one to give a quantitative result concerning continuity.

The induced field (δ'_n) is obtained as we have seen, through the sequence (δ_n) which is inductively computed by the relationship

$$\delta_{n+1} = \alpha + \varepsilon_n \, \tau(\delta_n)$$
$$\delta_0 = \alpha + 2\,\xi_0.$$

We underline the dependance of δ_n on the sequence $\varepsilon = (\varepsilon_0, \varepsilon_1 ...)$ and on α by writing $\delta_n = \delta_n^\alpha(\varepsilon)$.

THEOREM 4. – *Let α and α' be two strictly positive numbers. Then for all $n \geq 1$ and all input sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, ...)$*

$$|\delta_n^\alpha(\varepsilon) - \delta_n^{\alpha'}(\varepsilon)| \leq \left(1 + \left[\frac{4}{\alpha''}\right]\right) |\alpha - \alpha'|$$

where $\alpha'' = \min\{\alpha, \alpha'\}$.

Proof. – Let us first assume that α and α' are both in the same interval $]4/(\mu+1),\, 4/\mu]$ for some integer $\mu \geq 1$. Then both automata M_α and $M_{\alpha'}$ have the same number of states. They actually are the same automaton, they only differ by the output function. If A_0, B_i and C_j (resp. A'_0, B'_i, C'_j) are the states of M_α (resp. $M_{\alpha'}$) then

$$\begin{aligned}|\delta_n^\alpha(\varepsilon) - \delta_n^{\alpha'}(\varepsilon)| &\leq \max\{|A_0 - A'_0|, |B_i - B'_i|, |C_j - C'_j|\}\\ &\leq (\mu+1)\,|\alpha - \alpha'| = \left(1 + \left[\frac{4}{\alpha''}\right]\right)|\alpha - \alpha'|\end{aligned}$$

where $\alpha'' = \min\{\alpha, \alpha'\}$. (In this case $\left[\frac{4}{\alpha''}\right] = \left[\frac{4}{\alpha}\right] = \left[\frac{4}{\alpha'}\right]$).

We now suppose that

$$\frac{4}{\mu+1} < \alpha \leq \frac{4}{\mu} \leq \alpha' \leq \frac{4}{\mu - 1}$$

for some integer $\mu \geq 2$. Let $\alpha_0 := 4/\mu$. The automata M_α and $M_{\alpha_0 - 0}$ are isomorphic and contain $2(\mu+1)$ states and similarly $M_{\alpha'}$ and M_{α_0+0} are isomorphic and have $2\,\mu$ states. Therefore since $(M_{\alpha_0-0}, \mathrm{id})\,\varepsilon = (M_{\alpha_0+0}, \mathrm{id})\,\varepsilon$ for all ε

$$\begin{aligned}|\delta_n^\alpha(\varepsilon) - \delta_n^{\alpha'}(\varepsilon)| &\leq |\delta_n^\alpha(\varepsilon) - \delta^{\alpha_0-0}(\varepsilon)| + |\delta_n^{\alpha_0+0}(\varepsilon) - \delta_n^{\alpha'}(\varepsilon)|\\ &\leq (\mu+1)(\alpha_0 - \alpha) + \mu(\alpha' - \alpha_0)\\ &\leq (\mu+1)(\alpha' - \alpha) = \left(1 + \left[\frac{4}{\alpha''}\right]\right)|\alpha' - \alpha|.\end{aligned}$$

The inequality clearly extends to all strictly positive couple α, α'.

Q.E.D.

If one allows α (or α') to vanish then the number of states of M_α explodes to infinity and our analysis breaks down.

The core of M_α or L_μ $\left(\mu = \left[\frac{4}{\alpha}\right]\right)$ contains the states $B_i = i\,\alpha - 2$ $(i = 1, 2, \ldots, \mu + 1)$ and $C_j = 2 - (j - 1)\,\alpha$ $(j = 1, 2, \ldots, \mu)$ with next state function

$$\begin{array}{lll} B_i + 1 = B_{i+1} & C_j + 1 = C_{j-1} & \\ B_i - 1 = C_i & C_j - 1 = B_j & 1 \leq i, j \leq \mu, \end{array}$$

As α tends to 0, all the B_i's tend to -2 and all the C_j's tend to 2. The infinite automaton M_{+0} is therefore isomorphic to the two state automaton known as the Morse automaton which actually describes the Ising chain when the external field vanishes.

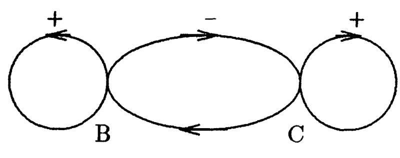

Fig. 4.

So in some sense one can say that continuity is preserved at $\alpha = 0$.

9. THE INFINITE ISING CHAIN REVISITED

Let us consider once again the negative Hamiltonian of an infinite chain with given interactions $(\varepsilon_i)_{i \in \mathbb{Z}}$ and with external field $\alpha/2$, $0 < \alpha \leq 4$:

$$H(\sigma) = \sum_{i \in \mathbb{Z}} \varepsilon_i\, \sigma_i\, \sigma_{i+1} + \frac{\alpha}{2} \sum_{i \in \mathbb{Z}} \sigma_i.$$

We have seen that the system can be analyzed with the help of an automaton with output (M_α, id). An output sequence $(\delta_i)_{i \in \mathbb{Z}}$ is said to *correspond* to the input sequence $(\varepsilon_i)_{i \in \mathbb{Z}}$ if for all $i \in \mathbb{Z}$, $\delta_i\, \varepsilon_i = \delta_{i+1}$. The existence of $(\delta_i)_{i \in \mathbb{Z}}$ is not obvious since there is no "starting point". How can one describe a sequence defined inductively when the initial value is pushed back to $-\infty$? Our first result in this section answers the question.

THEOREM 5. – (1) *There always exists an output sequence* $(\delta_i)_{i \in \mathbb{Z}}$ *corresponding to the input sequence* $(\varepsilon_i)_{i \in \mathbb{Z}}$.

(2) *If* $(\delta_i)_{i\in\mathbb{Z}}$ *corresponds to the input sequence* $(\varepsilon_i)_{i\in\mathbb{Z}}$ *and if*

$$\delta_i > 2$$

holds for infinitely many negative i's, then $(\delta_i)_{i\in\mathbb{Z}}$ *is the unique sequence corresponding to* $(\varepsilon_i)_{i\in\mathbb{Z}}$.

Proof. – $*$ (1) Let λ_n be an arbitrary sequence of states. There exists at least one state, say δ_0 such that $\lambda_n\,\varepsilon_{-n}\,\varepsilon_{-n+1}...\varepsilon_{-1} = \delta_0$ for infinitely many $n \geq 1$. Then choose $\delta_{-1} \in \Sigma$ such that $\delta_{-1}\,\varepsilon_{-1} = \delta_0$ and such that $\lambda_n\,\varepsilon_{-n}\,\varepsilon_{-n+1}...\varepsilon_{-2} = \delta_{-1}$ for infinitely many $n \geq 1$. Choose $\delta_{-2} \in \Sigma$ such that $\delta_{-2}\,\varepsilon_{-2} = \delta_{-1}$ and such that $\lambda_n\,\varepsilon_{-n}\,\varepsilon_{-n+1}...\varepsilon_{-3} = \delta_{-2}$ for infinitely many $n \geq 1$, and so on. In this way we define δ_i for all $i \leq 0$. Now for $i > 0$ put

$$\delta_i := \delta_0\,\varepsilon_0\,\varepsilon_1...\varepsilon_{i-1}.$$

It is then clear that $(\delta_i)_{i\in\mathbb{Z}}$ corresponds to $(\varepsilon_i)_{i\in\mathbb{Z}}$.

It can be observed that the proof of part (1) is valid for all strongly connected automata.

$*$ (2) Let $(\delta_i)_{i\in\mathbb{Z}}$ satisfy the condition stated in the theorem. Since M_α is isomorphic to L_μ or N_μ ($\mu = [4/\alpha]$), it is sufficient to prove that $(\delta_i)_{i\in\mathbb{Z}}$ is the unique sequence corresponding to $(\varepsilon_i)_{i\in\mathbb{Z}}$ in L_μ or N_μ.

We remark that $\delta_i > 2$ implies $\delta_i \in \{B_{\mu+1},\, A_0\}$ in L_μ and $\delta_i = A_0$ in N_μ. Take any input sequence $\eta_1\,\eta_2...\eta_n$ with $n \geq 1$ such that in L_μ

$$B_{\mu+1}\,\eta_1\,\eta_2...\eta_n = B_{\mu+1}.$$

Then for any state A in L_μ it can be shown that

$$A\,\eta_1\,\eta_2...\eta_n \in \{B_{\mu+1},\, A_0\},$$

and hence $A\,\eta_1...\eta_n\,\eta_{n+1}...\eta_m = B_{\mu+1}\,\eta_1...\eta_n\,\eta_{n+1}...\eta_m$ for any $\eta_{n+1},\, \eta_{n+2},...$ and $m > n$. We can also prove that if $A_0\,\eta_1...\eta_m$ satisfies that $A_0\,\eta_1...\eta_k = A_0$ for at least $\mu+1$ number of k's with $1 \leqq k \leqq m$, then $A\,\eta_1...\eta_m = A_0\,\eta_1...\eta_m$ for any state A in L_μ.

Now consider any other sequence $(\delta'_i)_{i\in\mathbb{Z}}$ corresponding to $(\varepsilon_i)_{i\in\mathbb{Z}}$ in L_μ. Then for any $j \in \mathbb{Z}$, there exists $i = j - m$ with $\delta_i > 2$ and m such that $\delta_i\,\varepsilon_i\,\varepsilon_{i+1}...\varepsilon_{i+m-1}$ satisfies either of the above conditions. Then we have $\delta_j = \delta'_j$. This implies uniqueness.

For N_μ, observe that if $A_0 \eta_1 \eta_2 ... \eta_n = A_0$ with $n \geq \mu$, then $A \eta_1 \eta_2 ... \eta_n = A_0$ for any A in N_μ. This leads to the same conclusion.

Q.E.D.

10. THE ENERGY OF THE INFINITE CHAIN

Let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be given such that the output sequence $(\delta_i)_{i \in \mathbb{Z}}$ satisfies

$$\delta_i > 2$$

infinitely many times both for negative and positive i's. Let

$$E_i := (\delta_i - \varepsilon_i) \vee \varepsilon_i - \frac{\alpha}{2}.$$

THEOREM 6. – *Let M_k and N_k be two integer sequences such that $\lim_{k \to \infty} (N_k - M_k) = +\infty$ and such that*

$$E := \lim_{k \to \infty} \frac{1}{N_k - M_k} \sum_{i=M_k}^{N_k - 1} E_i$$

exists. Then for all $\hat{\sigma}$ in the ground state the energy per volume is

$$\lim_{k \to \infty} \frac{1}{N_k - M_k} H_{M_k, N_k} (\hat{\sigma}) = E$$

where

$$H_{M_k, N_k} (\hat{\sigma}) = \sum_{i=M_k}^{N_k - 1} \varepsilon_i \hat{\sigma}_i \hat{\sigma}_{i+1} + \frac{\alpha}{2} \sum_{i=M_k}^{N_k} \hat{\sigma}_i.$$

Proof. – In the same fashion as in paragraph 5 we define

$$a_n = \max_{\sigma_n = 1} \left(\frac{\delta_{M_k} - \alpha}{2} \sigma_{M_k} + H_{M_k, n} (\sigma) \right)$$

$$b_n = \max_{\sigma_n = -1} \left(\frac{\delta_{M_k} - \alpha}{2} \sigma_{M_k} + H_{M_k, n} (\sigma) \right)$$

for $n = M_k, M_k + 1, ..., N_k$. Then $\delta_n = a_n - b_n$. Therefore

$$\max_{\sigma}\left(\frac{\delta_{M_k}-\alpha}{2}\,\sigma_{M_k}+H_{M_k,N_k}(\sigma)\right)=a_{N_k}\vee b_{N_k}$$

$$=\sum_{n=M_k}^{N_k-1}\left(a_{n+1}\vee b_{n+1}-a_n\vee b_n\right)+a_{M_k}\vee b_{M_K}$$

$$=\sum_{n=M_k}^{N_k-1}\left(b_{n+1}+\delta_{n+1}\vee 0-b_n-\delta_n\vee 0\right)+\mathcal{O}(1)$$

$$=\sum_{n=M_k}^{N_k-1}\left(b_{n+1}-b_n\right)+\mathcal{O}(1)$$

$$=\sum_{n=M_k}^{N_k-1}\left(\left(a_n-\varepsilon_n-\frac{\alpha}{2}\right)\vee\left(b_n+\varepsilon_n-\frac{\alpha}{2}\right)-b_n\right)+\mathcal{O}(1)$$

$$=\sum_{n=M_k}^{N_k-1}\left(\delta_n-\varepsilon_n-\frac{\alpha}{2}\right)\vee\left(\varepsilon_n-\frac{\alpha}{2}\right)+\mathcal{O}(1)$$

$$=\sum_{n=M_k}^{N_k-1}E_n+\mathcal{O}(1)$$

On the other hand, since $\hat{\sigma}$ maximizes

$$H_{M_k,N_k}(\sigma)+\varepsilon_{M_k-1}\,\hat{\sigma}_{M_k-1}\,\sigma_{M_k}+\varepsilon_{N_k}\,\hat{\sigma}_{N_k+1}\,\sigma_{N_k}=H_{M_k,N_k}(\sigma)+\mathcal{O}(1)$$

we have

$$H_{M_k,\,N_k}(\hat{\sigma})=\sum_{n=M_k}^{N_k-1}E_n+\mathcal{O}(1)$$

and finally

$$\lim_{k\to\infty}\frac{1}{N_k-M_k}\,H_{M_k,\,N_k}(\hat{\sigma})=E.$$

Q.E.D.

11. THE ENTROPY OF THE INFINITE CHAIN

We assume that $4/\alpha$ is not an integer. Decompose the sequence $(\delta_i)_{i\in\mathbb{Z}}$ into blocks starting either by $A:=\alpha+2$ or by $B:=-2+\left(\left[\frac{4}{\alpha}\right]+1\right)\alpha$

and containing no other A or B. We assume as in the previous paragraph that A or B occurs infinitely many times both for $i < 0$ and $i > 0$. A block is called an i-block if it terminates by $-2 + \left[\frac{4}{\alpha}\right] \alpha$ and contains the symbol 2 exactly i times. Fixing the intergers M and $N\,(M < N)$, let ℓ_i be the number of i-blocks contained in $\delta_M\, \delta_{M+1} ... \delta_{N-1}$. Define the finite sum

$$S_\delta\,(M,\, N) = \sum_{i=1}^{\infty} \ell_i \log\,(i+1).$$

THEOREM 7. – *Let* M_k *and* N_k *be two integer sequences such that* $\lim_{k\to\infty} (N_k - M_k) = +\infty$ *and let*

$$\{\delta_{M_k},\, \delta_{N_k}\} \subset \{A,\, B\}, \qquad (k = 1,\, 2, ...).$$

Assume that

$$S := \lim_{k\to\infty} \frac{1}{N_k - M_k}\, S_\delta\,(M_k,\, N_k)$$

exists. Then the entropy of the ground state $\mathcal{G}$ *defined by*

$$\lim_{k\to\infty} \frac{1}{N_k - M_k} \log\,(\mathrm{card}\, \mathcal{G}_{M_k, N_k})$$

is equal to S. Here

$$\mathcal{G}_{M_k,\, N_k} := \{\hat{\sigma}_{M_k}\, \hat{\sigma}_{M_k+1} ... \hat{\sigma}_{N_k-1} \in \{-1,\, +1\}^{N_k - M_k} / \hat{\sigma} \in \mathcal{G}\}.$$

Proof. – For sufficiently large k, define

$$\delta'_{N_k-1} = \delta_{N_k-1} + 2$$
$$\delta'_n = \delta_n + 2\,\varepsilon_n\, \mathrm{sgn}\,(\delta'_{n+1}), \qquad (n = N_k - 2, ..., M_k).$$

This agrees with our notation in paragraph 6 since

$$\delta_{N_k} > 2 \qquad \text{and} \qquad \varepsilon_{N_k-1} = \hat{\sigma}_{N_k} = 1$$

for all $\hat{\sigma} \in \mathcal{G}$ by Theorem 2 and its corollary.

Then multiplicity occurs if and only if $\delta'_n = 0$. This is equivalent to $\delta_n = 2$ and $\varepsilon_n\, \mathrm{sgn}\,(\delta'_{n+1}) = -1$. But since if $\varepsilon_n = 1$ then $\delta_{n+1} = 2 + \alpha$ and $\mathrm{sgn}\,(\delta'_{n+1}) = 1$, and this is impossible. Hence $\delta'_n = 0$ is equivalent to $\delta_n = 2$, $\varepsilon_n = -1$ and $\mathrm{sgn}\,(\delta'_{n+1}) = 1$. In this case $\delta_{n+1} = -2 + \alpha$.

Consider now any block in $\delta_{M_k}\, \delta_{M_k+1} ... \delta_{N_k-1}$ which finishes by 2. Then it is easy to see that within the block if $\delta_{n+1} = -2 + \alpha$ then $\delta'_{n+1} < 0$. So multiplicity only occurs in the blocks which terminate by $-2 + \left[\frac{4}{\alpha}\right] \alpha$.

Take one of these blocks and consider the last occurrence of $\delta_n = 2$. Then $\varepsilon_n = -1$ and $\delta'_{n+1} > 0$ since the block ends by $-2 + \left[\frac{4}{\alpha}\right] \alpha$. Therefore $\delta'_n = 0$ which leaves us with two possibilities $\mathrm{sgn}\,(\delta'_n) = 1$ or -1.

If we choose $\operatorname{sgn}(\delta'_n) = 1$ then for any former occurrence of $\delta_i = -2+\alpha$ in the block, we have $\delta'_i < 0$ and no other multiplicity. If on the other hand we choose $\operatorname{sgn}(\delta'_n) = -1$, then for the second last occurrence of $\delta_i = 2$ we have another multiplicity. We proceed until the first occurrence of $\delta_i = 2$ and thus the number of multiplicities within the block is exactly the number of occurrences of $\delta_i = 2$ plus one. These choices of $\operatorname{sgn}(\delta'_i)$ when $\delta'_i = 0$ have no influence outside of the block. So finally the number of multiplicities within $\delta_{M_k}\, \delta_{M_k+1} \ldots \delta_{N_k-1}$ is exactly $\exp S_\delta(M_k, N_k)$.

Q.E.D.

Remark. – The above proof uses only the structure of L_μ with $\mu = [4/\alpha]$ so that the entropy stays constant as long as α runs through the open interval $]4/(\mu+1),\, 4/\mu[$. When $4/\alpha$ is an integer there is another formula for the entropy *S*. We decompose again the sequence $(\delta_i)_{i\in\mathbb{Z}}$ into blocks starting at $A = 2+\alpha$ and $B = -2 + ([4/\alpha]+1)\,\alpha$ as before but now $A = B$. A block is called an i-block if it contains the word $(2,\, \alpha - 2)$ i-times and if there exist an even number of places j with $\varepsilon_j = -1$ in the finite sequence $\varepsilon_h, \varepsilon_{h+1}, \ldots, \varepsilon_\ell$. Here h is the largest index such that $\delta_{h-1} = 2$ and $\delta_h = \alpha - 2$, and ℓ is the index of the last element of the block under consideration. Then the same formula holds for *S* and for the entropy of the ground state.

12. TWO EXAMPLES

We illustrate the two previous theorems by giving two examples of an explicit calculation of the energy *E* and of the entropy *S*.

Example 1. – Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a *normal sequence i.e.* for all $n \geq 1$, all words $w \in \{-1, +1\}^n$ appear in the sequence both on the negative side and on the positive side with frequency 2^{-n}.

In Theorems 6 and 7 choose any sequence *M* and *N* with $M \to -\infty$ and $N \to +\infty$. Then *E* and *S* are easily computed:

$$E = 1 + \frac{\alpha}{\mu+2} - \frac{2}{(\mu+1)(\mu+2)}$$

$$S = \frac{1}{2(\mu+1)^2} \sum_{i=1}^{\infty} \left(\frac{\mu}{2(\mu+1)}\right)^i \log(i+1).$$

The formula for *E* stays valid even when $4/\alpha$ is an integer. This is not the case for *S*. Note that *E* is a continuous function of α.

These expressions agree with the results of B. Derrida, J. Vannimenus and Y. Pomeau [2]. In their paper however the ε_i's are i.i.d. random variables $\varepsilon = \pm 1$ with probability 1/2. They are concerned with expected values while we are more interested in the individual $(\varepsilon_i)_{i \in \mathbb{Z}}$.

Example 2. – For a given α such that $0 < \alpha < 4$ and such that $4/\alpha$ is not an integer we can obtain the sequence $(\varepsilon_i)_{i \in \mathbb{Z}}$ which maximizes the entropy S of the ground state. In fact, it suffices to examine periodic sequences $(\varepsilon_i)_{i \in \mathbb{Z}}$ with period

$$\underbrace{-1 - 1 ... - 1}_{2\,k+1}\ \underbrace{1\,1 ... 1}_{\mu} \qquad \left(\mu = \left[\frac{4}{\alpha}\right]\right).$$

Then the unique corresponding output sequence $(\delta_i)_{i \in \mathbb{Z}}$ is periodic with period

$$B_{\mu+1}\ \underbrace{B_1\, C_1 ... B_1\, C_1}_{2\,k}\ B_1\, B_2 ... B_\mu.$$

$(\delta_i)_{i \in \mathbb{Z}}$ only consists of k-blocks and in this case

$$S = \frac{1}{2\,k + 1 + \mu} \log (k + 1).$$

We then can choose k which maximizes S: a simple analysis shows that k can take at most two values. For large μ

$$k \sim \frac{\mu}{2 \log \mu}$$

and as $\alpha \to 0$

$$S \sim \frac{1}{\mu} \log \mu \sim \frac{\alpha}{4} \log \frac{1}{\alpha}.$$

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