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Nonexistence of minimal blow-up solutions of equations $iu_t = -\Delta u - k(x) \mid u \mid^{4/N} u$ in $\mathbb{R}^N$

by

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ABSTRACT. – In this paper, we prove the existence of blow-up solutions of Equation of the form $iu_t = -\Delta u - k(x) \mid u \mid^{4/N} u$ in $\mathbb{R}^N$ under some conditions on $k(x)$. We then consider the problem to find minimal blow-up solutions in L^2.

RÉSUMÉ. – On démontre l'existence de solutions explosives pour des équations de la forme $iu_t = -\Delta u - k(x) \mid u \mid^{4/N} u$ dans $\mathbb{R}^N$, sous certaines conditions sur $k(x)$. On considère ensuite le problème de trouver des solutions singulières minimales dans L^2.

Mots clés : Schrödinger, critique, explosion, minimal, stabilité.

1. INTRODUCTION

In the present paper, we consider the nonhomogeneous nonlinear Schrödinger equation with critical exponent

$$(1.1) \qquad i\,\frac{\partial u}{\partial t} = -\Delta u - k(x) \mid u \mid^{4/N} u$$

and

$$(1.2) \qquad u(0, \cdot) = \phi(\cdot),$$

where Δ is the Laplace operator on $\mathbb{R}^N$, $u : [0, T) \times \mathbb{R}^N \to \mathbb{C}$ and $\phi \in H^1(\mathbb{R}^N)$.

We assume in this paper that k is a given C^1 function such that there are $k_1 > 0$, $k_2 > 0$ and $c > 0$ such that

$$\forall x \in \mathbb{R}^N, \quad k_1 \leq k(x) \leq k_2, \tag{H.1}$$

$$\forall x \in \mathbb{R}^N, \quad |\nabla k(x)| + |x \cdot \nabla k(x)| \leq c, \tag{H.2}$$

$$\text{there is } x_0 \in \mathbb{R}^N, \quad k(x_0) = k_2. \tag{H.3}$$

We say that $u(\cdot)$ is a solution of Eq. (1.1)-(1.2) on [0, T) if $\forall t \in [0, T)$,

$$u(t) = S(t)\phi + i \int_0^t S(t-s)\{k(x)\,|u(s)|^{4/N} u(s)\}\,ds, \tag{1.3}$$

where $S(\cdot)$ is the group with infinitesimal generator $i\Delta$ and, for each t, $u(t)$ denotes the function $x \to u(t, x)$.

It is easy to prove as in the homogeneous case:

$$k(x) \equiv k_0, \tag{1.4}$$

that Eq. (1.1)-(1.2) has a unique solution $u(t)$ in $H^1(\mathbb{R}^N)$ and there exists $T > 0$ such that, $\forall t \in [0, T)$, $u(t) \in H^1(\mathbb{R}^N)$ and either

$$T = +\infty,$$

or

$$T < +\infty \quad \text{and} \quad \lim_{t \to T^-} \|u(t)\|_{H^1} = +\infty,$$

where $\| \quad \|_{H^1}$ is the usual norm on H^1, and H^1 is $H^1(\mathbb{R}^N)$ (*see* Ginibre and Velo [2], Kato [6]).

Furthermore, we have $\forall t \in [0, T)$,

$$\int_{\mathbb{R}^N} |u(t, x)|^2\,dx = \int_{\mathbb{R}^N} |\phi(x)|^2\,dx, \tag{1.5}$$

$$\begin{aligned} E(u(t)) = {} & \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t, x)|^2\,dx \\ & - \frac{1}{\frac{4}{N}+2}\int_{\mathbb{R}^N} k(x)\,|u(t, x)|^{\frac{4}{N}+2}\,dx \\ = {} & E(\phi). \end{aligned} \tag{1.6}$$

In this paper we are interested in the study of singular solutions of Eq. (1.1)-(1.2). In the case where

$$k(x) \not\equiv k_0, \tag{1.7}$$

there are no results available.

Let us first recall some results in the case where $k(x) \equiv k_0$. For such a nonlinearity, there is another identity which is the following.

Let $\phi \in \Sigma = H^1 \cap \{|x|\,\phi \in L^2\}$ then $\forall t < T$, $u(t) \in \Sigma$, and

$$\frac{d}{dt}\int |x|^2\,|u(t,x)|^2\,dx = 4\,\mathrm{Im}\int x\cdot\overline{\nabla u}\,u\,dx, \tag{1.8}$$

and

$$\frac{d^2}{dt^2}\int |x|^2\,|u(t,x)|^2\,dx = 16\,E(\phi). \tag{1.9}$$

From this identity, it follows easily that if

$$E(\phi) < 0 \tag{1.10}$$

then

$$T < +\infty$$

(*see* Zakharov, Sobolev, Synach [15] and Glassey [5]). Moreover blow-up solutions have three important properties.

(i) They are bounded from below in L^2 (Weinstein [18]). That is, let Q_{k_0} be the unique radially symetric solution of

$$\Delta u + k_0\,|u|^{\frac{4}{N}}\,u = u \tag{1.11}$$

(*see* for existence Strauss, Berestycki, Lions, Peletier [1], [16], and for uniqueness Kwong [7]). If $u(t)$ is a blow-up solution then

$$\|\phi\|_{L^2} \geq \|Q_{k_0}\|_{L^2}.$$

(ii) The set of minimal blow-up solutions is known (Merle [10], [11]).

Let $u(t)$ be a blow-up solution with minimal mass in L^2, ($\|\phi\|_{L^2} = \|Q_{k_0}\|_{L^2}$). There are then constants $\theta \in S^1$, $\omega > 0$, $x_0 \in \mathbb{R}^N$, $x_1 \in \mathbb{R}^N$,

$T > 0$ such that

$$u(t, x) = \left(\frac{\omega}{T-t}\right)^{\frac{N}{2}} \exp\left\{i\left(\theta + \frac{|x - x_1|^2}{4(-T+t)} - \frac{\omega^2}{(-T+t)}\right)\right\} \times Q_{k_0}\left(\frac{\omega(x - x_1)}{T-t} - \omega x_0\right). \tag{1.12}$$

In [3], [4], we point out the importance of such solutions as limits of "stable" (from the numerical point of view) blow-up solutions for more complex equations which have (1.1) as a limit case (*see* Landam, Papanicolaou, C. and P. L. Sulem, Wang for numerical simulations [8], [14]).

(iii) At the blow-up time, there is a concentration phenomenon (Merle, Tsutsumi [12], Weinstein [19], Merle [9], Proposition A.3 in [4]). Indeed, let $u(t)$ be a blow-up solution of Eq. (1.1) and T its blow-up time. There is then $x(t)$ for $t > T$ such that

$$\forall R > 0, \quad \liminf_{t \to T} \| u(t) \|^2_{L^2(B(x(t), R))} \geq \| Q_{k_0} \|^2_{L^2},$$

where $B(x, R)$ is the ball of radius R and center x.

We first have the following result about existence of blow-up solutions.

THEOREM 1 (Existence and lower L^2-bound of blow-up solutions, concentration at the blow-up time). – (i) *Lower L^2-bound: Assume that k satisfies* (H.1)-(H.2). *Let $\phi \in H^1$ be such that*

$$\| \phi \|_{L^2} < \| Q_{k_2} \|_{L^2}.$$

Then $u(t)$ is globally defined in time.

(ii) *Existence of blow-up solutions: Let k satisfy* (H.1)-(H.3). *Assume in addition that k satisfies* (H.4) *or* (H.4)$'$ *where*

$$\text{(H.4)} \qquad \textit{There is a } \rho_0 > 0 \textit{ such that} \quad (x - x_0) \cdot \nabla k(x) < 0 \quad \text{for } 0 < |x - x_0| < \rho_0$$

and

$$\text{(H.4)}' \qquad \forall x, \quad (x - x_0) \cdot \nabla k(x) \leq 0,$$

and x_0 is such that $k(x_0) = k_2$. Then there is $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$, there is $\phi_\varepsilon \in H^1$ such that

– $\| \phi_\varepsilon \|_{L^2} = \| Q_{k_2} \|_{L^2} + \varepsilon,$

– *$u_\varepsilon(t)$ blows up in finite time where $u_\varepsilon(t)$ is the solution of Eq.* (1.1) *with initial data ϕ_ε. In addition, $\varepsilon_0 = +\infty$ when k satisfies $(H.4)'$.*

(iii) *Concentration at the blow-up time: Let k satisfy* (H.1)-(H.2), *let $u(t)$ be a blow-up solution of Eq.* (1.1) *and let T be its blow-up time. There is then $x(t)$ for $t < T$ such that*

$$\forall R > 0, \quad \liminf_{t \to T} \| u(t) \|^2_{L^2(B(x(t), R))} \geq \| Q_{k_2} \|^2_{L^2}.$$

Remark. – In part (ii), assumption (H.4) or (H.4)′ can be weaken (*see* section 3) and x_0 can be a local maximum. However, it is still an open problem to show existence of blow-up solutions in the case where there is no local maximum of k.

Let us now consider k satisfying (H.1)-(H.3). The main question is whether there is or not L^2-minimal blow-up solution: Is there a $\phi \in H^1$ such that

– $\| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2}$,

– $u(t)$ blows-up in finite time where $u(t)$ is the solution of (1.1)-(1.2).

These results related to L^2-minimal blow-up solutions have a plysical interest.

– In the case of existence of such a solution, we have a solution which blows up with minimal mass and is in some sense the limit point of numerically stable blow-up solution (*see* [8], [14]).

– In the case of nonexistence of such a solution, we obtain the existence of a space singularity which is in some sense, stable in time with respect to Eq. (1.1). We will call this kind of phenomenon a black hole (*see* Theorem 3).

THEOREM 2 (L^2-minimal blow-up solutions). – *Consider k satisfying* (H.1)-(H.2) *and* (H.5) *where*

(H.5)
$$\begin{gathered} \text{there are } \delta_0 > 0 \text{ and } R_0 > 0 \\ \text{such that for } |x| > R_0, \\ k(x) \leq k_2 - \delta_0 \\ \text{and } M = \{x;\ k(x) = k_2\} \text{ is finite.} \end{gathered}$$

(i) *Characterization: Assume that $\| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2}$ and $u(t)$ blows-up in finite time. There is then $x_0 \in M$ such that*

– *$|u(t, x)|^2 \to \| Q_{k_2} \|^2_{L^2} \delta_{x=x_0}$ in the distribution sense,*

– *$|x - x_0|^2 |u(t, x)|^2 \to 0$ in L^1, as $t \to T$.*

(ii) *Nonexistence result: Assume in addition that for* $x_0 \in M$, *we have the following property:*

(H.6) *there is* ρ_0 *and* $\alpha_0 \in (0, 1)$ *such that*

$$\nabla k(x) \cdot (x - x_0) \leq -|x - x_0|^{1+\alpha_0}, \quad \textit{for } |x - x_0| \leq \rho_0.$$

There is then no blow-up solutions such that

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}.$$

Remark. – In the case where $k(x) \equiv k_2$ globally or $k(x) \equiv k_2$ for x near x_0, we are able to show the existence of minimal blow-up solution. Therefore, the existence of minimal blow-up solutions depends strongly on the form of the function $k(x)$ near the points where k achieves its maximum. However, we do not know exactly the case of limiting behavior near x_0 (where x_0 is such that $k(x_0) = k_2$) of k (between flatness near x_0 and assumption (H.5)) where there is nonexistence of minimal L^2 blow-up solutions.

We can in addition remark that in the elliptic situation in the case where $k(x) \not\equiv k_2$ there is no solution of the equation

$$\Delta v + k(x)|v|^{\frac{4}{N}} v = \omega v$$

where $\omega > 0$ such that

$$\|v\|_{L^2} = \|Q_{k_2}\|_{L^2}.$$

THEOREM 3 (Stability in time of singularity). – *Assume that* x_0 *is such that* $k(x_0) = k_2$ *and* x_0 *is a strict local maximum. Moreover, assume that there is no blow-up solution of Eq.* (1.1)-(1.2) *such that*

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}.$$

Consider now a sequence $\phi_n \in H^1$ *such that*

– $\|\phi_n\|_{L^2}^2 \to \|Q_{k_2}\|_{L^2}^2$,

– $|\phi_n(x)|^2 \rightharpoonup \|Q_{k_2}\|_{L^2} \delta_{x=x_0}$ *in the distribution sense,*

– *there is a* $c > 0$ *such that*

$$E_{\varepsilon_n}(\phi_n) \leq c$$

where $\varepsilon_n \to 0$ *as* $n \to +\infty$, $\varepsilon_n > 0$, $q \in \left(\frac{4}{N} + 1, 1 + \frac{4}{N-2}\right)$, $E_\varepsilon(u) = E(u) + \frac{\varepsilon}{q+1}\int |u|^{q+1}$. *Then* $u_n(t)$, *the solution of equation*

$$iu_t = -\Delta u - k(x)|u|^{\frac{4}{N}} u + \varepsilon_n |u|^{q-1} u, \tag{1.13}$$

$$u(0) = \phi_n, \tag{1.14}$$

is such that

– $u_n(t)$ *is defined for all time,*

– *for all time* $t > 0$,

$$|u_n(t, x)|^2 \rightharpoonup \|Q_{k_2}\|_{L^2}^2\, \delta_{x=x_0} \quad \text{in the distribution sense as } n \to +\infty, \tag{1.15}$$

and

$$\|u_n(t)\|_{L^2} \to \|Q_{k_2}\|_{L^2}, \quad as\ n \to +\infty. \tag{1.16}$$

Remark. – In this case, we say that $\|Q_{k_2}\|_{L^2}^2\, \delta_{x=x_0}$ is a singularity stable in time.

The plan of the paper is the following:

– In section two, we establish some conservation laws for solutions of (1.1) and derive some concentration properties at the blow-up time.

– In section three, we prove some blow-up results.

– Sections four and five are devoted to minimal blow-up solutions.

– Finally, in section six, we study the existence of black holes.

2. CONCENTRATION PROPERTIES OF BLOW-UP SOLUTIONS

In the first subsection, we give various identities satisfied by solutions of Eq. (1.1). We assume that $\phi \in \Sigma = H^1 \cap \{u;\ xu \in L^2\}$.

2.A. Conservation laws

Let us consider $u(t, x)$ solution of Eq. (1.1) and T its blow-up time.

PROPOSITION 2.1. – *We have* $\forall t \in [0, T)$,

(2.1) (i)
$$\int |u(t, x)|^2\, dx = \int |\phi(x)|^2\, dx,$$

(2.2) (ii)
$$E(u(t)) = E(\phi)\ where$$

$$E(u) = \frac{1}{2}\int |\nabla u(x)|^2\, dx - \frac{1}{\frac{4}{N}+2}\int k(x)\,|u(t, x)|^{\frac{4}{N}+2}\, dx,$$

(2.3) (iii) $$\frac{d}{dt}\int |x|^2\,|u(t,x)|^2 dx = 4\,\mathrm{Im}\int \bar{u}\,\nabla u\cdot x,$$

(2.4) $$\frac{d^2}{dt^2}\int |x|^2\,|u(t,x)|^2\,dx$$
$$= 4\left\{4\,E(\phi) + \frac{1}{\frac{2}{N}+1}\int x\cdot\nabla k(x)\,|u(t,x)|^{\frac{4}{N}+2}\,dx\right\}.$$

Proof. – (i) and (ii) follow from direct calculation.

(iii) Let us show that

(2.5) $$\frac{d}{dt}\int \mathrm{Im}\,\bar{u}\,\nabla u\cdot x = 4\,E(\phi)$$
$$+ \frac{1}{\frac{2}{N}+1}\int x\cdot\nabla k(x)\,|u(t,x)|^{\frac{4}{N}+2}\,dx.$$

(2.6) $$\frac{d}{dt}\,\mathrm{Im}\int \bar{u}\,\nabla u\cdot x = \mathrm{Im}\left\{\int x\bar{u}\,\nabla\frac{\partial u}{\partial t} + \int x\,\frac{\partial \bar{u}}{\partial t}\,\nabla u\right\}$$
$$= \mathrm{Im}\left\{2\int x\,\frac{\partial \bar{u}}{\partial t}\,\nabla u - N\int \bar{u}\,\frac{\partial \bar{u}}{\partial t}\right\}.$$

On the one hand,

(2.7) $$N\,\mathrm{Im}\int \bar{u}\,\frac{\partial u}{\partial t} = -N\,\mathrm{Re}\int \bar{u}\,(\Delta u + k(x)\,|u|^{\frac{4}{N}}\,u)$$
$$= -N\int k(x)\,|u|^{\frac{4}{N}+2} + N\int |\nabla u|^2.$$

On the other hand

(2.8) $$2\,\mathrm{Im}\int x\,\frac{\partial \bar{u}}{\partial t}\,\nabla u$$
$$= -2\,\mathrm{Re}\left\{\int x\,\Delta u\,\nabla\bar{u} + \int x k(x)\,|u|^{\frac{4}{N}}\,u\,\nabla\bar{u}\right\}$$

$$= -(N-2)\int |\nabla u|^2 - \frac{1}{\frac{2}{N}+1}\int xk(x)\nabla |u|^{\frac{4}{N}+2}$$

$$= -(N-2)\int |\nabla u|^2 - \frac{N}{\frac{2}{N}+1}\int k(x)\,|u|^{\frac{4}{N}+2}$$

$$+ \frac{1}{\frac{2}{N}+1}\int x\cdot\nabla k\,|u|^{\frac{4}{N}+2};$$

(for more detailed calculations *see* [11], p. 214).

From (2.6)-(2.8), (2.5) follows.

As in the case $k(x) \equiv k_0$, let us derive some consequences of these conservation laws.

COROLLARY 2.2.

(i) $$\frac{d}{dt}\int \vec{x}\,|u(t,x)|^2\,dx = 2\,\mathrm{Im}\int \bar{u}\nabla u,$$

(ii) $$\frac{d^2}{dt^2}\int \vec{x}\,|u(t,x)|^2\,dx = \frac{2}{\frac{2}{N}+1}\int \nabla k\,|u(t,x)|^{\frac{4}{N}+2}\,dx.$$

Proof. – We have for all $x_0 \in \mathbb{R}^N$:

$$\frac{d}{dt}\int |\vec{x}+\vec{x}_0|^2\,|u(t,x)|^2\,dx = 4\,\mathrm{Im}\int \bar{u}\nabla u\cdot(\vec{x}+\vec{x}_0).$$

Therefore,

$$\frac{d}{dt}\left\{|x_0|^2\int |u(t,x)|^2\,dx + \int |x|^2\,|u(t,x)|^2\,dx + 2\vec{x}_0\cdot\int \vec{x}\,|u(t,x)|^2\,dx\right\}$$

$$= 4\,\mathrm{Im}\int \bar{u}\nabla u\cdot x + 4\vec{x}_0\,\mathrm{Im}\int \bar{u}\nabla u,$$

and from Proposition 2.1,

$$\vec{x}_0\cdot 2\frac{d}{dt}\int \vec{x}\,|u(t,x)|^2\,dx = \vec{x}_0\cdot 4\,\mathrm{Im}\int \bar{u}\nabla u. \tag{2.9}$$

(i) follows from the fact that (2.9) is true for all $x_0 \in \mathbb{R}^N$. Proof of part (ii) is similar.

Let us write an energy type identity from Proposition 2.1 derived in the case $k(x) \equiv k_2$ by Anosov and rediscovered by Ginibre and Velo.

COROLLARY 2.3. – *We have*

$$\tilde{E}_t(u(t)) = \tilde{E}_0(u(0)) - \int_0^t \frac{s}{\frac{4}{N}+2} \int x \cdot \nabla k \, |u(s, x)|^{\frac{4}{N}+2} \, dx \, ds$$

$$= \frac{1}{8} \int |x|^2 \, |\phi(x)|^2 \, dx - \int_0^t \frac{s}{\frac{4}{N}+2} \int x \cdot \nabla k \, |u(s, x)|^{\frac{4}{N}+2} \, dx \, ds,$$

where

$$\tilde{E}_t(u) = \frac{1}{2} \int |\nabla v|^2 - \frac{1}{\frac{4}{N}+2} \int k(tx) \, |v|^{\frac{4}{N}+2}$$

with

$$v = |t|^{\frac{N}{2}} e^{-i\frac{|x|^2 t}{4}} u(xt).$$

Proof. – Let $u \in \Sigma$,

$$\tilde{E}_t(u) = \frac{1}{2} \int |t|^N \left| \left(-\frac{ixt}{2} + t\nabla \right) u(xt) \right|^2 - \frac{1}{\frac{4}{N}+2} t^2 \int k(tx) \, |t|^N \, |u(xt)|^{\frac{4}{N}+2} \, dx \tag{2.10}$$

$$= \frac{1}{2} \int \left| \left(-\frac{iy}{2} + t\nabla \right) u(y) \right|^2 dy - \frac{t^2}{\frac{4}{N}+2} \int k(y) \, |u(y)|^{\frac{4}{N}+2} \, dy$$

$$= \frac{1}{2} \left\{ \frac{1}{4} \int |x|^2 \, |u(x)|^2 \, dx - t \, \mathrm{Im} \int x \cdot \nabla u \bar{u} \right\} + t^2 E(u).$$

Let us now consider $\tilde{E}_t(u(t))$

$$\frac{d}{dt}\tilde{E}_t(u(t)) = \frac{1}{2}\left\{\frac{1}{4}\frac{d}{dt}\int |x|^2 |u(t, x)|^2 dx - \text{Im}\int x\cdot\nabla u(t)\,\bar{u}(t)\right\}$$
$$- \frac{t}{2}\frac{d}{dt}\int x\cdot\nabla u(t)\,\bar{u}(t) + 2tE(\phi).$$

From Proposition 2.1, we have

$$\frac{d}{dt}\tilde{E}_t(u(t)) = \frac{1}{2}\left\{\text{Im}\int x\cdot\nabla u(t)\,\bar{u}(t) - \text{Im}\int x\cdot\nabla u(t)\,\bar{u}(t)\right\}$$
$$- \frac{t}{2}\left\{4E(\phi) + \frac{1}{\frac{2}{N}+1}\int x\cdot\nabla k\,|u(t, x)|^{\frac{4}{N}+2} dx\right\}$$
$$+ 2tE(\phi)$$
$$= -\frac{t}{\frac{4}{N}+2}\int x\cdot\nabla k\,|u(t, x)|^{\frac{4}{N}+2} dx,$$

which concludes the proof of Corollary 2.3 and Section 2.A.

2.B. Concentration properties of blow-up solutions of Eq. (1.1)

In this section, we consider a blow-up solution of Eq. (1.1), $u(t)$. Let T be its blow-up time. Assume that

- $0 < k_1 \equiv \inf_{x\in\mathbb{R}^N} k(x) \le k_2 \equiv \sup_{x\in\mathbb{R}^N} k(x) < +\infty,$
- $k \in C^1,$
- $|\nabla k| \le c_0.$

We claim the following

PROPOSITION 2.4. – *There is $x(t) \in \mathbb{R}^N$ such that for all $R > 0$,*

$$\liminf_{t\to T} \| u(t) \|_{L^2(B(x(t), R))} \ge \| Q_{k_2} \|_{L^2},$$

where Q_{k_2} is the unique positive radially symetric solution of

$$v = \Delta v + k_2 |v|^{\frac{4}{N}} v.$$

Remark. – From scaling argument, we have $Q_{k_2} = \frac{1}{k_2^{\frac{N}{4}}} Q$ where Q is the unique radially symetric solution of (II, 1). In particular

$$\| Q_{k_2} \|_{L^2} = \frac{\| Q \|_{L^2}}{k_2^{\frac{N}{4}}}.$$

In fact, we have a slightly more precise result.

PROPOSITION 2.5. – *There is* $x(t) \in \mathbb{R}^N$ *such that for all* $R > 0$,

$$\liminf_{t \to T} \left\{ \frac{\| u(t) \|_{L^2 (B(x(t), R))}}{\| Q_{k(x(t))} \|_{L^2}} \right\} \geq 1.$$

Remark.

$$\| Q_{k(x(t))} \|_{L^2} = \frac{\| Q \|_{L^2}}{[k(x(t))]^{\frac{N}{4}}} \geq \frac{\| Q \|_{L^2}}{k_2^{\frac{N}{4}}}.$$

Proof of Proposition 2.5 follows exactly the proof of Proposition 2.4 and will be omitted (it uses the fact that $\forall R > 0$,

$$\operatorname{Sup}_{|x-y| \leq R'} \left| k\left(\frac{x}{\lambda(t)}\right) - k\left(\frac{y}{\lambda(t)}\right) \right| \leq c_0 \frac{|x-y|}{\lambda(t)} \leq \frac{R c_0}{\lambda(t)} \underset{t \to T}{\longrightarrow} 0,$$

where $\lambda(t) = \| \nabla u(t) \|_{L^2}$).

Sketch of proof of Proposition 2.4. – It is a consequence of similar results in [18], [12], [9], [4]. Indeed, we have

$$\begin{aligned} E_{k_2}(u(t)) &= \frac{1}{2} \int | \nabla u(t, x) |^2 \\ &\quad - \int \frac{k_2}{\frac{4}{N} + 2} | u(t, x) |^{\frac{4}{N}+2} dx \\ &\leq \frac{1}{2} \int | \nabla u(t, x) |^2 \\ &\quad - \int \frac{k(x)}{\frac{4}{N} + 2} | u(t, x) |^{\frac{4}{N}+2} dx \\ &\leq E(u(t)) = E(\phi) \end{aligned} \tag{2.11}$$

and

$$\| u(t) \|_{L^2} = \| \phi \|_{L^2}. \tag{2.12}$$

Let us argue by contradiction. Assume there are $R_0 > 0$, $\delta_0 > 0$ and a sequence $t_n \to T$ such that

$$\operatorname*{Sup}_{x \in \mathbb{R}^N} \left\{ \int_{|x-y|<R_0} |u(t_n, x)|^2 \, dy \right\} \leq \| Q_{k_2} \|_{L^2}^2 - \delta_0.$$

Then from results of [12], [19], [4], we have the existence of constants $c_1 > 0$ and $c_2 > 0$ such that

$$\forall t_n, \quad -c_1 + c_2 \int |\nabla u(t_n, x)|^2 \, dx \leq E_{k_2}(u(t_n)) \tag{2.13}$$

(*see* from example Proposition A.3 in [4]).

From (2.11), we deduce that $\int |\nabla u(t_n, x)|^2 \, dx \leq c$ which contradicts that $t_n \to T$. This concludes the proof of Proposition 2.4 and Theorem 1. (iii).

As a direct consequence of Propostion 2.4 and (2.12), we obtain

COROLLARY 2.6. (Lower bound for blow-up solutions). – *Assume*

$$\| \phi \|_{L^2} < \| Q_{k_2} \|_{L^2} = \frac{\| Q \|_{L^2}}{k_2^{\frac{N}{4}}}.$$

Then the solution $u(t)$ is globally defined in time.

In fact, from the proof of Proposition 2.4, we have a useful corollary (*see* also [19]):

COROLLARY 2.7. – *Let $u_n \in H^1$ be such that $\| u_n \|_{L^2} \to \| Q_{k_2} \|_{L^2}$, $\lambda_n = \| \nabla u_n \|_{L^2} \to +\infty$ as $n \to +\infty$ and $E(u_n) \leq c$ for a $c > 0$. There are sequences $x_n \in \mathbb{R}^N$, $\theta_n \in S^1$ such that*

$$|u_n(x - x_n)|^2 \to \| Q_{k_2} \|_{L^2}^2 \, \delta_{x=x_0},$$

and

$$\lambda_n^{-\frac{N}{2}} e^{i\theta_n} u_n \left(\frac{x - x_n}{\lambda_n} \right) \to Q_{k_2} \quad in \ H^1.$$

3. BLOW-UP THEOREMS FOR SOLUTIONS OF EQ. (1.1)

In the homogeneous case

$$k(x) \equiv k_0; \tag{3.1}$$

blow-up theorems are obtained using the virial identity

$$\frac{d^2}{dt^2}\int |x|^2\,|u(t,x)|^2\,dx = 16\,E(\phi). \tag{3.2}$$

(*see* [5], [15]). If $E(\phi) < 0$, then using the fact

$$\forall t, \quad \int |x|^2 u(t,x)|^2\,dx > 0 \tag{3.3}$$

and (3.2), we obtain a contradiction.

In the case where

$$k(x) \equiv k$$

such an identity is not true anymore (*see* (2.4)) and we have $\forall x_0 \in \mathbb{R}^N$,

$$\frac{d^2}{dt^2}\int |x-x_0|^2\,|u(t,x)|^2\,dx = 16\,E(\phi) + \frac{4}{\frac{2}{N}+1} \times \int (x-x_0)\nabla k\,|u(t,x)|^{\frac{4}{N}+2}\,dx. \tag{3.4}$$

Under some global or local conditions on the sign of

$$(x-x_0).\nabla k(x)$$

we are able to obtain some blow-up theorems for solutions of Eq. (1.1).

THEOREM 3.1 (Global condition on $(x-x_0)\cdot\nabla k(x)$). – *Assume there is $x_0 \in \mathbb{R}^N$ such that*

$$\forall x \in \mathbb{R}^N, \quad (x-x_0)\cdot\nabla k(x) \le 0 \tag{3.5}$$

so that x_0 is global maximum of $k(x)$.

(i) *Let $\phi \in \Sigma$ be such that $E(\phi) < 0$. Then the solution $u(t)$ of Eq.* (1.1) *blows up in finite time.*

For all $\varepsilon > 0$, there is ϕ_ε such that

– $\|\phi_\varepsilon\|_{L^2} = \|Q_{k_2}\|_{L^2} + \varepsilon$,

– *$u_\varepsilon(t)$ blows-up in finite time, where $u_\varepsilon(t)$ is the solution of Eq.* (1.1) *with initial data ϕ_ε.*

THEOREM 3.2 (Local condition on $(x-x_0)\cdot\nabla k(x)$). – *Assume there is $x_0 \in \mathbb{R}^N$ and $\rho_0 > 0$ such that*

$$(x-x_0)\cdot\nabla k(x) < 0, \quad \text{for } 0 < |x-x_0| < \rho_0, \tag{3.6}$$

so that x_0 *is a local strict maximum of* $k(x)$

$$k(x_0) \geq k(x) \quad \text{for } 0 < |x - x_0| < \rho_0.$$

There is ε_0 *such that for all* $0 < \varepsilon < \varepsilon_0$, *there exists* $\phi_\varepsilon \in \Sigma$ *such that*

– $\| \phi_\varepsilon \|_{L^2} = \| Q_{k(x_0)} \|_{L^2} + \varepsilon$,

– $u_\varepsilon(t)$ *blows up in finite time where* $u_\varepsilon(t)$ *is the solution of Eq.* (1.1) *with initial data* ϕ_ε.

Remark. – Theorem 3.2 implies Theorem 3.1 but the proof of Theorem 3.1 is completely elementary. Assumption (3.6) can be weaken and replaced by

$$(3.6)' \qquad \begin{cases} (x - x_0) \cdot \nabla k(x) \leq 0 & \text{for } 0 < |x - x_0| < \rho_0, \\ (x - x_0) \cdot \nabla k(x) < 0 & \text{on } S, \end{cases}$$

where S is a closed hypersurface included in $B(x_0, \rho_0)$ with x_0 in its interior.

In Theorem 3.1 or 3.2, we have to assume that x_0 is a local maximum. An open problem left in this direction is to obtain blow-up theorem in the case where there is no local maximum of k in $\mathbb{R}^N$. For example, consider in $\mathbb{R}$ a function $k(x)$ such that

– $k' < 0$,

– $\lim_{x \to +\infty} k(x) = k_1 > 0$,

– $\lim_{x \to -\infty} k(x) = k_2 > 0$.

Is there a blow-up solution of Eq. (1.1)?

Proof of Theorem 3.1. – The proof is completely elementary.

(i) Let $\phi \in \Sigma$ such that $E(\phi) < 0$. Consider $y(t) = \int |x - x_0|^2 |u(t, x)|^2 dx$ and assume by contradiction that $u(t)$ and $y(t)$ are defined for all time; we have $\forall t > 0$, $y''(t) \leq 16 E(\phi)$. Thus by integration

$$\forall t > 0, \quad y(t) \leq y(0) + t y'(0) + 8 t^2 E(\phi) = z(t).$$

Since $E(\phi) < 0$, $z(t) < 0$ for t large which is contradiction. This concludes the proof of (i).

(ii) (3.5) implies directly that x_0 is a global maximum. Let $k_2 = k(x_0)$. For all $\varepsilon > 0$, consider for $\lambda > 0$, $w_{\varepsilon,\lambda} = (1 + \varepsilon) \frac{1}{\lambda^{\frac{N}{2}}} Q_{k_2} \left(\frac{(x - x_0)}{\lambda} \right)$. $\forall \lambda > 0$,

$$(3.7) \qquad \| w_{\varepsilon,\lambda} \|_{L^2} = (1 + \varepsilon) \| Q_{k_2} \|_{L^2}.$$

In addition,

$$E(w_{\varepsilon,\lambda}) = \frac{1}{2}\int |\nabla\, \omega_{\varepsilon,\lambda}|^2 - \frac{1}{\frac{4}{N}+2}\int k(x)\,|w_{\varepsilon,\lambda}|^{\frac{4}{N}+2}$$

$$= E_{k_2}(w_{\varepsilon,\lambda}) + \frac{1}{\frac{4}{N}+2}\int (k(x_0) - k(x))\,|w_{\varepsilon,\lambda}|^{\frac{4}{N}+2}$$

where $E_{k_2}(w) = \frac{1}{2}\int |\nabla\, w|^2 - \frac{1}{\frac{4}{N}+2}\int k_2\,|w|^{\frac{4}{N}+2}$.

On the one hand, by scaling arguments

$$(3.8)\qquad E_{k_2}(w_{\varepsilon,\lambda}) = (1+\varepsilon)^2\,\frac{1}{\lambda^2}\,E_{k_2}(Q_{k_2}) + ((1+\varepsilon)^2 - (1+\varepsilon)^{\frac{4}{N}+2})\,\frac{1}{\lambda^2}\int Q_{k_2}^{\frac{4}{N}+2}.$$

Since $E_{k_2}(Q_{k_2}) = 0$ (Pohazaev identity),

$$(3.9)\qquad \forall\,\lambda > 0,\quad E_{k_2}(w_{\varepsilon,\lambda}) \le -\frac{c(\varepsilon)}{\lambda^2}\quad \text{where } c(\varepsilon) > 0.$$

Since $\forall\,x$, $Q_{k_2}(x) \le c_0\,e^{-c_1|x|}$ and $|\nabla\,k(x)| \le c_0$, for $\lambda > 1$,

$$(3.10)\qquad \left|\int (k(x_0) - k(x))\,|w_{\varepsilon,\lambda}(x)|^{\frac{4}{N}+2}\right| \le c + \int_{|x-x_0|\le 1} |k(x_0) - k(x)|\,|w_{\varepsilon,\lambda}|^{\frac{4}{N}+2} \le c + c\int_{\mathbb{R}^N} \frac{|x|}{\lambda^{2+N}}\,e^{-c_1\frac{|x|}{\lambda}}\,dx$$
$$c + \frac{c}{\lambda}\int |y|\,e^{-c_1|y|}\,dy \le c\left(1+\frac{1}{\lambda}\right).$$

From (3.9)-(3.10) we derive that for $\lambda \ge \lambda(\varepsilon)$, $E(w_{\varepsilon,\lambda}) < 0$ and for $\varepsilon > 0$, $\phi_\varepsilon = w_{\varepsilon,\lambda(\varepsilon)}$ satisfies the conclusions of Theorem 3.1. This concludes the proof of Theorem 3.1.

Proof of Theorem 3.2. – We remark that we had showed in the proof of Theorem 3.1 (ii) the following lemma.

LEMMA 3.3. – *$\forall\,\varepsilon \in (0,\,1)$, for all $A(\varepsilon) > 0$, there is a $\phi_\varepsilon \in \Sigma$ such that*
– $\|\,\phi_\varepsilon\,\|_{L^2} = \|\,Q_{k(x_0)}\,\|_{L^2} + \varepsilon$,
– $E(\phi_\varepsilon) = -A(\varepsilon)$,
– $\int |\,x\,|^2\,|\,\phi_\varepsilon\,|^2 \leq C$, *(where C is independent of ε and $A(\varepsilon)$)*,
– $\forall\, x \in \mathbb{R}^N,\ \phi_\varepsilon(x) \in \mathbb{R}$,
– $\|\,\nabla\,\phi_\varepsilon\,\|_{L^2} \underset{\varepsilon\to 0}{\longrightarrow} +\infty$ *and* $|\,\phi_\varepsilon(x)\,|^2 \rightharpoonup \|\,Q_{k(x_0)}\,\|_{L^2}^2\,\delta_{x=x_0}$.

Proof. – It follows from the proof of Theorem 3.1 (ii) and direct computations.

We claim now for $A(\varepsilon)$ sufficiently large as $\varepsilon \to 0$, the solution $u_\varepsilon(t)$ associated with ϕ_ε blows up in finite time. We now assume that $A(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} +\infty$. We argue by contradiction. We suppose that $u_\varepsilon(t)$ is globally defined in time. The two key arguments of the proof are

– On one hand, the use of the geometry of $k(x)$ near x_0 to control the evolution of the concentration point;

– On the other hand, the use of local virial identity as in [10], [11]. We prooceed in three steps to obtain a contradiction.

Step 1. – *Concentration properties of $u_\varepsilon(t)$.*

PROPOSITION 3.4 (Concentration in L^2 of $u_\varepsilon(t)$). – *For all $\varepsilon' > 0$, there is ε_0 such that, $\forall\,\varepsilon \in (0,\,\varepsilon_0)$, $\forall\, t \geq 0$,*

$$\left| \int_{|x-x_0|\leq\varepsilon'} |\,u_\varepsilon(t,\,x)\,|^2\,dx - \int_{\mathbb{R}^N} Q_{k(x_0)}^2(x)\,dx \right| < \varepsilon', \tag{3.11}$$

and

$$\int_{|\,x-x_0|\geq\varepsilon'} |\,u_\varepsilon(t,\,x)\,|^2\,dx \leq \varepsilon'. \tag{3.12}$$

Proof of Proposition 3.4. – One uses the fact that x_0 is a strict local maximum and some contraction lemma.

LEMMA 3.5. – *Consider a sequence $t_\varepsilon \in \mathbb{R}$. We then have*

$$\|\,\nabla\,u_\varepsilon(t_\varepsilon)\,\|_{L^2} \underset{\varepsilon\to 0}{\longrightarrow} +\infty. \tag{3.13}$$

Proof. – Indeed, by contradiction, assume there is a $c > 0$ such that for a sequence $\varepsilon_n \to 0$

$$\|\,\nabla\,u_{\varepsilon_n}(t_{\varepsilon_n})\,\|_{L^2} \leq c.$$

Then by Sobolev imbeddings

$$E(\phi_{\varepsilon_n}) = |E(u_{\varepsilon_n}(t_{\varepsilon_n}))| \leq \frac{1}{2} \| \nabla u_{\varepsilon_n}(t_{\varepsilon_n}) \|_{L^2}^2 + c \| u_{\varepsilon_n}(t_{\varepsilon_n}) \|_{L^{\frac{4}{N}+2}}^{\frac{4}{N}+2} \leq c$$

which contradicts the fact

$$|E(\phi_{\varepsilon_n})| = A(\varepsilon_n) \to +\infty \quad \text{as } n \to +\infty.$$

Similarly with Proposition 2.5, we have the following lemma.

LEMMA 3.6. – *Let u_n be such that for constants a, b,*

(3.13) $$\| u_n \|_{L^2} \leq a,$$

(3.14) $$E(u_n) \leq b,$$

(3.15) $$\| \nabla u_n \|_{L^2} \to +\infty \quad \textit{as } n \to +\infty.$$

There is x_n such that for all $R > 0$

$$\liminf_{n \to +\infty} \left\{ \frac{\| u_n \|_{L^2(B(x_n, R))}}{\| Q_{k(x_n)} \|_{L^2}} \right\} \geq 1.$$

Proof. – *See* Corollary 2.7.

Applying Lemma 3.6 with $u_\varepsilon(t)$ $(a = 2 \| Q_{k(x_0)} \|_{L^2}, b = 0)$, we obtain the conclusion.

Indeed, consider $\delta > 0$ such that

(3.16) $$\forall x, \quad \| Q_{k(x)} \|_{L^2}^2 \geq 2\delta.$$

$\left(\text{(3.16) is equivalent to, } \forall x, \ \frac{\| Q \|_{L^2}^2}{k(x)^{\frac{N}{2}}} \geq 2\delta \text{ or equivalently } \delta \leq \frac{\| Q \|_{L^2}^2}{2 k_2^{\frac{N}{2}}} \right).$

Consider, for each $\varepsilon > 0$, $\tilde{T}_\varepsilon$ such that

$$\forall t \in [0, \tilde{T}_\varepsilon), \quad \| u_\varepsilon(t, x) \|^2_{L^2(B(x_0, \frac{\rho_0}{4}))} \geq \| Q_{k(x_0)} \|^2_{L^2} - \delta, \tag{3.17}$$

$$\| u_\varepsilon(\tilde{T}_\varepsilon, x) \|^2_{L^2(B(x_0, \frac{\rho_0}{4}))} = \| Q_{k(x_0)} \|^2_{L^2} - \delta. \tag{3.18}$$

From Lemma 3.3, for ε small enough, $\tilde{T}_\varepsilon > 0$. Let us show that for $\varepsilon < \varepsilon_0$ (where $\varepsilon_0 > 0$)

$$\tilde{T}_\varepsilon = +\infty. \tag{3.19}$$

Indeed, by contradiction, assume that for $\varepsilon_n \to 0$

$$\tilde{T}_{\varepsilon_n} = \tilde{T}_n < +\infty. \tag{3.20}$$

Consider $u_n = u_{\varepsilon_n}(\tilde{T}_n, x)$. u_n satisfies (3.13)-(3.15), therefore from Lemma 3.6, there is x_n

$$\forall R, \quad \liminf_{n \to +\infty} \| u_n \|^2_{L^2(B(x_n, R))} \geq \limsup_{n \to +\infty} \| Q_{k(x_n)} \|^2_{L^2}. \tag{3.21}$$

We chain for n large

$$| x_n - x_0 | \leq \frac{\rho_0}{2}. \tag{3.22}$$

Indeed if not

$$\liminf_{n \to +\infty} \| u_n \|^2_{L^2(B(x_0, \frac{\rho_0}{4}))} \geq \lim_{n \to +\infty} \| u_n \|^2_{L^2(B(x_n, \frac{\rho_0}{4}))}$$
$$\geq \limsup_{n \to +\infty} \| Q_{k(x_0)} \|^2_{L^2} \geq 2\,\delta$$

and from (3.18)

$$\liminf_{n \to +\infty} \| u_n \|^2_{L^2} \geq \| Q_{k(x_n)} \|^2_{L^2} + \delta. \tag{3.23}$$

Since $\| u_n \|_{L^2} = \| u_{\varepsilon_n}(\tilde{T}_{\varepsilon_n}, x) \|_{L^2} = \| \phi_{\varepsilon_n} \|_{L^2} \to \| Q_{k(x_0)} \|_{L^2}$ as $n \to +\infty$, we obtain a contradiction. We then remark that

$$x_n \to x_0 \quad \text{as } n \to +\infty. \tag{3.24}$$

Indeed, we have from (3.21)

$$(3.25)\qquad \begin{aligned}\liminf_{n\to+\infty} \| \phi_{\varepsilon_n} \|_{L^2}^2 &= \liminf_{n\to+\infty} \| u_n \|_{L^2}^2 \\ &\geq \limsup_{n\to+\infty} \| Q_{k(x_n)} \|_{L^2}^2 \\ &\geq \limsup_{n\to+\infty} \frac{\| Q \|_{L^2}^2}{[k(x_n)]^{\frac{N}{2}}} \\ &\geq \| Q_{k(x_0)} \|_{L^2}^2 \limsup_{n\to+\infty} \left[\frac{k(x_0)}{k(x_n)} \right]^{\frac{N}{2}}.\end{aligned}$$

From Lemma 3.3,

$$\| Q_{k(x_0)} \|_{L^2}^2 \geq \| Q_{k(x_0)} \|_{L^2}^2 \limsup_{n\to+\infty} \left[\frac{k(x_0)}{k(x_n)} \right]^{\frac{N}{2}}$$

or

$$\liminf_{n\to+\infty} k(x_n) \geq k(x_0),$$

which is equivalent from (3.6) and (3.22) to

$$x_n \to x_0 \quad \text{as } n \to +\infty.$$

From (3.21)

$$(3.26)\qquad \begin{aligned}\liminf_{n\to+\infty} \| u_{\varepsilon_n}(\tilde{T}_{\varepsilon_n}) \|_{L^2(x_0, \frac{\rho_0}{4})}^2 &\geq \liminf_{n\to+\infty} \| u_{\varepsilon_n}(\tilde{T}_{\varepsilon_n}) \|_{L^2(x_n, \frac{\rho_0}{8})}^2 \\ &\geq \liminf_{n\to+\infty} \| u_n \|_{L^2(x_n, \frac{\rho_0}{8})}^2 \\ &\geq \| Q_{k(x_0)} \|_{L^2}^2\end{aligned}$$

which is a contradiction with (3.18). Therefore there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, $\tilde{T}_\varepsilon = +\infty$.

Let us conclude the proof of Proposition 3.4 by contradiction. We claim that (3.11), (3.12) follow from Lemma 3.3 and the conservation of mass. Assume there is t_{ε_n} and $\varepsilon_n \to 0$, $\varepsilon' > 0$ such that

$$(3.27)\qquad \left| \| Q_{k(x_0)} \|_{L^2}^2 - \int_{|x-x_0|\leq\varepsilon'} |u_n|^2 \right| \geq \varepsilon',$$

where $u_n = u_{\varepsilon_n}(t_{\varepsilon_n}, x)$.

As before, there is x_n such that

$$\text{(3.28)} \qquad \forall R, \quad \left[\liminf_{n \to +\infty} \| u_n \|^2_{L^2 (B(x_0, R)} \left[\frac{k(x_n)}{k(x_0)} \right]^{\frac{N}{2}} \right] \geq \| Q_{k(x_0)} \|^2_{L^2}.$$

We have from (3.17) and (3.19) by the same arguments than before that for n large $| x_n - x_0 | \leq \frac{\rho_0}{2}$ and then $x_n \to x_0$ as $n \to +\infty$.

In particular, from (3.28) as classical arguments,

$$\text{(3.29)} \qquad \forall R, \quad \liminf_{n \to +\infty} \| u_n \|^2_{L^2 (B(x_0, R)} \geq \| Q_{k(x_0)} \|^2_{L^2}.$$

Since

$$\| \phi_{\varepsilon_n} \|_{L^2} = \| u_{\varepsilon_n} \|_{L^2} = \| u_n \|_{L^2} \xrightarrow[n \to +\infty]{} \| Q_{k(x_0)} \|_{L^2},$$

we have

$$\lim_{n \to +\infty} (\| u_n \|^2_{L^2 (B(x_0, R))} - \| Q_{k(x_0)} \|^2_{L^2}) = 0,$$

which is a contradiction with (3.27). Thus Proposition 3.4 is proved.

Remark. – In the case where x_0 is a global maximum, we do not need to prove (3.19).

Step 2. – *Energy estimates outside the concentration point.*

Using local virial identity, we are able to prove the following proposition.

PROPOSITION 3.7. – *There are constants* $0 < B_0 < \frac{\rho_0}{4}$, $c_1 > 0$ *and* $c_2 > 0$ *independent of* ε *such that* $\forall t$,

$$- \left[8 E(\phi_\varepsilon) t^2 + \int_0^t (t-s) \frac{4}{\left(\frac{2}{N} + 1 \right)} \int_{|x - x_0| \leq 2B_0} (x - x_0) \right.$$
$$\left. \times \nabla k(x) | u_\varepsilon (s, x) |^{\frac{4}{N} + 2} dx\, ds \right]$$
$$\geq c_2 \int_0^t (t-s) \int_{|x - x_0| \geq \frac{\rho_0}{2}} | \nabla u_\varepsilon (s, x) |^2 dx\, ds - c_1.$$

Proof of Proposition 3.7. – Let us give some lemmas. As in [10], [11] for $k(x) \equiv k_0$ we have local virial identity.

LEMMA 3.8 (Local virial identity). – *Consider* $\psi \in C^4(\mathbb{R}^N, \mathbb{R})$ *with compact support.*

(i) $$\frac{d}{dt}\int \psi(x)\,|u(t,x)|^2 = 2\,\mathrm{Im}\int \nabla\psi\,\nabla u\bar{u},$$

(ii) $$\frac{d^2}{dt^2}\int \psi(x)\,|u(t,x)|^2 = 2\left\{-\frac{2}{N\left(\frac{2}{N}+1\right)}\int \Delta\psi\, k\,|u|^{\frac{4}{N}+2}\right.$$
$$+2\sum_{i,j}\int \partial_i\partial_j\psi\,\partial_i u\,\overline{\partial_j u} - \frac{1}{2}\int |u|^2\Delta^2\psi$$
$$\left.+\frac{1}{\frac{2}{N}+1}\int \nabla\psi\cdot\nabla k\,|u|^{\frac{4}{N}+2}\,dx\right\}.$$

Proof. – It follows from similar calculation as in [10].

LEMMA 3.9. – *Let* $\rho(x) \in C^1(\mathbb{R}^N, \mathbb{R})$ *such that* $\rho \in L^\infty$ *and* $\nabla\rho \in L^\infty$. *There is a* $c_\rho > 0$ *such that*

$$\forall u \in H^1, \quad \int |u(x)|^{\frac{4}{N}+2}\rho^2(x)\,dx \le c_\rho\left(\int u^2\right)^{\frac{2}{N}}$$
$$\times\left\{\int |\nabla u|^2\rho^2 + \int \nabla\rho^2 u^2\right\}.$$

Proof. – *See* [10] p. 434.

We claim now that applying Lemma 3.8 to a suitable function $\psi(x)$, we obtain Proposition 3.7. Indeed, consider ψ such that

(3.27) $$\psi \in C^4(\mathbb{R}^N, \mathbb{R}) \quad \text{and} \quad \psi(x) = \psi(|x|),$$

(3.28) $$\psi(x) < |x|^2 \quad \text{for } |x| > \beta_0,$$
$$\psi(x) \equiv |x|^2 \quad \text{for } |x| \le \beta_0,$$
$$\frac{1}{2}|x|^2 \le x\cdot\nabla\psi \le 3|x|^2 \quad \text{for } |x| \le 2\beta_0,$$
$$\Delta\psi - 2N \ge 0 \quad \text{for } |x| \le 2\beta_0,$$

(3.29) $$\psi(x) \equiv c \quad \text{for } |x| \ge \frac{\rho_0}{2},$$

there are a constant c_0 and a function g such that for

$$\beta_0 \le |x| \le \frac{\rho_0}{2}, \qquad \forall a \in \mathbb{C}^N \tag{3.30}$$

$$\left(\sum_i |a_i|^2 - \sum_{i,j} \frac{\partial_i \partial_j \psi}{2} a_i \bar{a}_j\right) \ge g(x)\left(\sum_i |a_i|^2\right),$$

where $g(x) \ge c_0$ for $|x| \ge 2\beta_0$ and $g(x) \ge 0$, $\forall x$.

The existence of such a ψ can be proved easily, and the proof is omitted.

We have then by Lemma 3.8 and Lemma 3.3, $\forall \varepsilon$, $\forall t > 0$,

$$\begin{aligned} \int \psi(x - x_0)\,|u_\varepsilon(t, x)|^2\,dx &= \int \psi(x - x_0)\,|\phi_\varepsilon|^2 \\ &+ 2t\,\mathrm{Im} \int \nabla\psi \cdot \nabla\phi_\varepsilon\,\bar{\phi}_\varepsilon \\ &+ 2\int_0^t (t - s)\Bigg\{-\frac{2}{N\left(\frac{2}{N}+1\right)} \int \Delta\psi\, k\,|u_\varepsilon(s)|^{\frac{4}{N}+2} \\ &+ 2\sum_{i,j} \int \partial_i \partial_j \psi\, \partial_i u_\varepsilon(s)\, \overline{\partial_j u_\varepsilon(s)} - \frac{1}{2}\int |u_\varepsilon(s)|^2\, \Delta^2 \psi \\ &+ \frac{1}{\frac{2}{N}+2} \int \nabla k\, \nabla\psi(x - x_0)\,|x_\varepsilon(s)|^{\frac{4}{N}+2}\Bigg\}\,ds. \end{aligned} \tag{3.31}$$

From (3.31), the conservation of mass, Lemma 3.3 and (3.27)-(3.30), we obtain, $\forall \varepsilon$, $\forall t$

$$\begin{aligned} &\Bigg| \int_0^t (t - s)\Bigg\{-\frac{2}{N\left(\frac{2}{N}+1\right)} \int \Delta\psi\, k\,|u_\varepsilon(s)|^{\frac{4}{N}+2} \\ &+ 2\sum_{i,j} \int \partial_i \partial_j \psi\, \partial_i u_\varepsilon(s)\, \overline{\partial_j u_\varepsilon(s)} \\ &+ \frac{1}{\frac{2}{N}+2} \int \nabla k\, \nabla\psi(x - x_0)\,|u_\varepsilon(s)|^{\frac{4}{N}+2}\Bigg\}\,ds\Bigg| \\ &\le c_1 + c_1 t^2. \end{aligned} \tag{3.32}$$

Thus

$$(3.33)\quad \Big| \int_0^t (t-s)\Big\{ 8\,E(\phi_\varepsilon) + \int_{|x-x_0|\geq \beta_0} \frac{-2}{N\left(\frac{2}{N}+1\right)} \times (\Delta\psi - 2\,N)\,k\,|\,u_\varepsilon\,|^{\frac{4}{N}+2} + 2\,(\sum_{i,j} \partial_i\,\partial_j\,\psi\,\partial_i\,\mu_\varepsilon\,\partial_j\,\bar{u}_\varepsilon) - 4\,(\sum_i |\,\partial_i\,u_\varepsilon\,|^2)\Big\}\,dx + \frac{1}{\frac{2}{N}+2}\int \nabla\,k\,\nabla\,\psi\,(x-x_0)\,|\,u_\varepsilon\,|^{\frac{4}{N}+2}\,dx\Big\}\,ds\,\Big| \leq c_1 + c_1\,t^2,$$

or equivalently

$$(3.34)\quad \int_0^t 2\int_{|x-x_0|\geq\beta_0} (\sum_i |\,\partial_i\,u\,|^2) - \frac{1}{2}(\sum_{i,j} \partial_i\,\partial_j\,\psi\,\partial_i\,u\,\overline{\partial_j\,u}) \leq c\left(|\,E(\phi_\varepsilon)\,|\,t^2 + 1 + \int_0^T \int_{\beta_0\leq|x-x_0|\leq\frac{\rho_0}{2}} |\,u\,|^{\frac{4}{N}+2}\,dx\,ds\right) + \left|\int_0^T (T-t)\int_{|\,x-x_0\,|\leq 2\beta_0} \nabla\,k\cdot(x-x_0)\,|\,u\,|^{\frac{4}{N}+2}\,dx\,dt\right|.$$

In addition, from (3.6) and a compactness argument in $\mathbb{R}^N$, we have

$$(3.35)\qquad |\,(x-x_0)\cdot\nabla\,k\,| \geq c_0 > 0 \quad \text{for } \beta_0 \leq |\,x-x_0\,| \leq \frac{\rho_0}{2}\,.$$

Thus (3.34)-(3.35) yield Proposition 3.7.

Step 3. – *Conclusion of the proof.*

From Proposition 2.1, we have $\forall\,\varepsilon$, $\forall\,t>0$,

$$\frac{d}{dt}\int |\,x\,|^2\,|\,u_\varepsilon\,(t,\,x)\,|^2\,dx = 4\,\mathrm{Im}\int \bar{u}_\varepsilon\,\nabla\,u_\varepsilon\,x,$$

$$\frac{d^2}{dt^2}\int |\,x\,|^2\,|\,u_\varepsilon\,(t,\,x)\,|^2\,dx = 4\left\{4\,E(\phi_\varepsilon) + \frac{1}{\frac{2}{N}+1}\int (x-x_0)\,\nabla\,k\,|\,u_\varepsilon\,|^{\frac{4}{N}+2}\right\}.$$

We integrate twice these identities and using Lemma 3.3 we obtain for t,

$$y_\varepsilon(t) = \int |x|^2 |u_\varepsilon(t, x)|^2 \, dx = 8\,E(\phi_\varepsilon)\, t^2 + \int |x|^2 |\phi_\varepsilon|^2 + \int_0^t (t-s) \frac{4}{\frac{2}{4}+1} \times \int (x - x_0)\,\nabla k\, |u_\varepsilon(s)|^{\frac{4}{N}+2} \, dx\, ds \tag{3.36}$$

$$= 8\,E(\phi_\varepsilon)\, t^2 + \int |x|^2 |\phi_\varepsilon|^2 + \int_0^t (t-s) \frac{4}{\frac{2}{N}+1} \times \int_{|x-x_0|\le\rho_0} (x - x_0)\,\nabla k\, |u_\varepsilon(s)|^{\frac{4}{N}+2} \, dx\, ds + \int_0^t (t-s) \frac{4}{\frac{2}{N}+1} \times \int_{|x-x_0|\ge\rho_0} (x - x_0)\,\nabla k\, |u_\varepsilon|^{\frac{4}{N}+2} \, dx\, ds. \tag{3.37}$$

Let us estimate the last term.

LEMMA 3.10. – *There is a constant $c(\varepsilon)$ depending only on ε such that*

(i) $$c(\varepsilon) \to 0 \quad as\ \varepsilon \to 0$$

(ii) $$\forall t, \quad \left| \int_0^t (t-s) \frac{4}{\frac{2}{N}+1} \times \int_{|x-x_0|\ge\rho_0} |x - x_0|\,\nabla k\, |u_\varepsilon|^{\frac{4}{N}+2} \, dx\, ds \right| \le c(\varepsilon) \left| c_1 + 8\,E(\phi_\varepsilon)\, t^2 + \int_0^t (t-s) \frac{4}{\frac{2}{N}+1} \times \int_{|x-x_0|\le\frac{\rho_0}{2}} (x - x_0)\,\nabla k\, |u_\varepsilon|^{\frac{4}{N}+2} \, dx\, ds \right|.$$

Proof. – We have

$$(3.38)\qquad \beta_\varepsilon(t) = \left| \int_0^t (t-s)\frac{4}{\frac{2}{N}+1}\int_{|x-x_0|\geq\rho_0} \times (x-x_0)\nabla k\,|u_\varepsilon|^{\frac{4}{N}+2}\,dx\,ds\right|$$

$$\leq c\int_0^t (t-s)\int_{|x-x_0|\geq\rho_0} |u_\varepsilon|^{\frac{4}{N}+2}\,dx\,ds$$

$$\leq c\int_0^t (t-s)\int \rho^2(x)\,|u_\varepsilon|^{\frac{4}{N}+2}\,dx\,ds,$$

where ρ is a C^∞ function such that

– $0 \leq \rho \leq 1$

– $\rho = 1$ for $|x-x_0| \geq \rho_0$

– $\rho = 0$ for $|x-x_0| \leq \frac{\rho_0}{2}$.

Therefore from Lemma 3.9,

$$(3.39)\qquad \beta_\varepsilon(t) \leq c\int_0^t (t-s)\left\{\left(\int_{|x-x_0|\geq\rho_0} |u_\varepsilon(s,x)|^2\,dx\right)^{\frac{2}{N}} \left(\int |\nabla u_\varepsilon(s,x)|^2\rho^2(x)\,dx + \int \nabla\rho^2(x)\,|u_\varepsilon(s,x)|^2\,dx\right)ds\right.$$

$$\leq c\int_0^t (t-s)\left\{\left(\int_{|x-x_0|\geq\frac{\rho_0}{2}} |u_\varepsilon(s,x)|^2\,dx\right)^{\frac{2}{N}} \left(\int_{|x-x_0|\geq\frac{\rho_0}{2}} |u_\varepsilon(s,x)|^2 + |\nabla u_\varepsilon(s,x)|^2\,dx\right)\right\}ds.$$

From Step 1, we have

$$(3.40)\qquad \beta_\varepsilon(t) \leq c(\varepsilon)\left\{c_1 + t^2 + \int_0^t (t-s) \times \int_{|x-x_0|\geq\frac{\rho_0}{2}} |\nabla u_\varepsilon(s,x)|^2\,dx\,ds\right\}$$

where

$$c(\varepsilon) = \left(\operatorname*{Sup}_{t \in \mathbb{R}} \int_{|x - x_0| \geq \frac{\rho_0}{2}} |u_\varepsilon(t, x)|^2 \right)^{\frac{2}{N}} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

From Proposition 3.7 and (3.40), we conclude the proof of Lemma 3.10.

Let ε_0 be such that

$$c(\varepsilon) \leq \frac{1}{2}, \quad \text{for } \varepsilon \leq \varepsilon_0.$$

For $\varepsilon \leq \varepsilon_0$, $\forall t$,

$$\begin{aligned} y_\varepsilon(t) \leq {} & \frac{1}{8}\, 8\, E(\phi_\varepsilon)\, t^2 + c_1 - \frac{1}{2}\, E(\phi_\varepsilon)\, t^2 \\ & + \frac{3}{2} \int_0^t (t - s)\, \frac{4}{\frac{2}{N} + 1} \\ & \times \int_{|x - x_0| \leq \frac{\rho_0}{2}} (x - x_0)\, \nabla k\, |u_\varepsilon|^{\frac{4}{N} + 2}\, dx\, ds \\ & + \int_0^t (t - s)\, \frac{4}{\frac{2}{N} + 1} \\ & \times \int_{\frac{\rho_0}{2} \leq |x - x_0| \leq \rho_0} (x - x_0)\, \nabla k\, |u_\varepsilon|^{\frac{4}{N} + 2}\, dx\, ds. \end{aligned} \tag{3.41}$$

Since $(x - x_0)\, \nabla k \leq 0$ on $|x - x_0| \leq \rho_0$, we have

$$\forall t, \quad y_\varepsilon(t) \leq c_1 + \frac{1}{2}\, E(\phi_\varepsilon)\, t^2. \tag{3.42}$$

Therefore, from the fact that $y_\varepsilon(1) \geq 0$, we obtain that for a $c > 0$,

$$\forall \varepsilon \leq \varepsilon_0, \quad E(\phi_\varepsilon) \geq -c.$$

This is a contradiction with Lemma 3.3 and the solution $u_\varepsilon(t)$ for $\varepsilon \leq \varepsilon_0$ blows up in finite time. This concludes the proof of the Theorem 3.2 and Section 3.

4. PROPERTIES OF L^2-MINIMAL BLOW-UP SOLUTIONS ($\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$)

In this section, we assume that k is C^1 and

$$0 < k_1 \equiv \inf_{x \in \mathbb{R}^N} k(x) \leq k(x) \leq \operatorname*{Sup}_{x \in \mathbb{R}^N} k(x) \equiv k_2 < +\infty. \tag{4.1}$$

Moreover, we assume compactness and nondegeneracy conditions on $k(x)$, that is

(4.1)′ There are $R_0 > 0$, $c_0 > 0$ and $\delta_0 > 0$ such that for $|x| \geq R_0$,

$$k(x) \leq k_2 - \delta_0, \quad |\nabla k(x)| \leq c_0,$$

and

(4.1)″ there are $x_1, \ldots, x_p$ such that M

$$= \{x;\, k(x) = k_2\} = \{x_1, \ldots, x_p\}.$$

In this section we are interested by qualitative properties satisfied by blow-up solutions such that

$$\| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2} = \frac{\| Q \|_{L^2}}{k_2^{N/4}}. \tag{4.2}$$

We had seen in Section 2 that if

$$\| \phi \|_{L^2} < \| Q_{k_2} \|_{L^2} \tag{4.3}$$

then $u(t)$ is globally defined.

Moreover under some compactness assumptions on $k(x)$ in Section 3, we had seen that for all $\varepsilon > 0$, there is a blow-up solution with initial data ϕ_ε such that

$$\| \phi_\varepsilon \|_{L^2} = \| Q_{k_2} \|_{L^2} + \varepsilon. \tag{4.4}$$

Therefore, if $u(t)$, solution of Eq. (1.1) with initial data ϕ satisfying (4.2), blows-up in finite time $T < +\infty$, then $u(t)$ is a minimal blow-up solution in L^2. Let $u(t)$ be such a solution.

In the case $k(x) \equiv k$, in [10], the following result has been proved: there is $x_0 \in \mathbb{R}^N$ such that

$$|u(t, x)|^2 \rightarrow \| Q_{k_0} \|_{L^2} \delta_{x=x_0} \quad \text{as } t \rightarrow T,$$
$$|x - x_0|^2 |u|^2 \rightarrow 0 \quad \text{in } L^1 \quad \text{as } t \rightarrow T.$$

Using variational arguments we prove the following in the case where $k(x) \not\equiv k$.

PROPOSITION 4.1. – *Assume that* $\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$ *and* $u(t)$ *blows-up in finite time at* $T < +\infty$. *We then have the existence of* x_0 *such that*

– $|u(t, x)|^2 \to \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}$ *in the distribution sence as* $t \to T$,

– $|x - x_0|^2 |u(t, x)|^2 \to 0$ *in* L^1 *as* $t \to T$,

and

$$\nabla k(x_0) = 0, \qquad k(x_0) = k_2.$$

Remark. – It follows from Proposition 4.1 that we do not have ejection of mass in finite time with a minimal mass ($\|Q_{k_2}\|_{L^2}$). That is

$$|u(t, x - x(t))|^2 \to \|Q_{k_2}\|_{L^2} \delta_{x=0} \quad \text{and} \quad |x(t)| \to +\infty \quad \text{as } t \to T.$$

In the case where $k(x)$ does not satisfy (4.1)′ and there is a sequence x_n such that

– $|x_n| \to +\infty$ as $n \to +\infty$,

– $k(x_n) \to k_2$ as $n \to +\infty$,

we still have the existence of $x(t)$ such that

$$|u(t, x + x(t))|^2 \rightharpoonup \|Q_{k_2}\|_{L^2} \delta_{x=0}.$$

But we do not know whether $x(t)$ is bounded or not.

Remark. – For a general initial data ($\|\phi\|_{L^2} > \| \|Q_{k_2}\|_{L^2}$, we don't know whether the concentration point of the solution in L^2 at the blow-up time is a critical point of $k(x)$ or not.

Proof of Proposition 4.1. – We establish the result in three steps. Let us consider $u(t)$ solution of Eq. (1.1) with intial data $\phi \in H^1$ such that

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}, \text{ and } u(t) \text{ blows up at } T < +\infty.$$

Step 1. – *Variational estimates.*

We show that there is $x(t)$ such that

$$|u(t, x - x(t))|^2 \rightharpoonup \|Q_{k_2}\|_{L^2} \delta_{x=0}$$

and

$$\forall \delta > 0, \quad \text{there is a } c_\delta > 0 \text{ such that } \forall t \in [0, T),$$

$$\int_{|x(t) - x| \geq \delta} |\nabla u(t, x)|^2 \, dx \leq c_\delta.$$

Step 2. – *Localization of the concentration point.*

There is $x_0 \in \mathbb{R}^N$ such that $x(t) \to x_0$ as $t \to T$. Moreover $k(x_0) = k_2$ and $\nabla k(x_0) = 0$.

Step 3. – *Control of* $u(t, x)$ *for* x *large and conclusion.*

We then show that

$$|x|\,|u(t, x)| \in L^2 \quad \text{for all } t \in [0, T)$$

and

$$|x - x_0|^2\,|u(t, x)|^2 \to 0 \quad \text{in } L^1 \quad \text{as } t \to T.$$

Step 1. – *Variational estimates: Concentration and compactness outside the concentration point.*

We show that there is $x(t)$ such that

$$|u(t, x + x(t))|^2 \to \|Q_{k_2}\|_{L^2}\,\delta_{x=0} \tag{4.5}$$

and

$$\begin{gathered} \forall\,\delta > 0, \quad \text{there is a } c_\delta > 0 \text{ such that } \forall\,t \in [0, T), \\ \int_{|x(t)-x|\geq\delta} |\nabla u(t, x)|^2\,dx \leq c_\delta. \end{gathered} \tag{4.6}$$

We claim this result as a consequence of the concentration properties (Section 2.B) and a crucial compactness lemma.

LEMMA 4.2 ([10], p. 433). – *Let* $u_n \in H^1(\mathbb{R}^N)$ *and* $R_0 > 0$ *such that for a* c_0, *we have* $\forall\,n$:

– $E_{k_2}(u_n) \leq c_0$,

– $\int |u_n(x)|^2\,dx \leq \int |Q_{k_2}(x)|^2\,dx$,

– $\int |\nabla u_n(x)|^2\,dx \to +\infty$ as $n \to +\infty$,

– $\int_{|x|\geq R_0} |u_n(x)|^2\,dx \leq \varepsilon(N)$,

where $\varepsilon(N) > 0$ *is depending only on* N. *Then there is* $A > 0$ *depending only on* R_0, c_0 *such that*

$$\forall\,n, \quad \int_{|x|\geq 4R_0} |\nabla u_n(x)|^2\,dx \leq A.$$

Proof of (4.5)-(4.6). – Let be $x(t)$ defined in Section 2.B (Proposition 2.4). For all $R > 0$, we have

(4.7)
$$\liminf_{t \to T} \| u(t) \|_{L^2(B(x(t),\, R))} \geq \| Q_{k_2} \|_{L^2}.$$

Let

$$v(t,\, x) = |\, u(t,\, x + x(t))\,|^2.$$

(4.8)
$$\| v(t,\, x) \|_{L^1} = \| u(t,\, x) \|_{L^2}^2 = \| \phi \|_{L^2}^2 = \| Q_{k_2} \|_{L^2}^2,$$

and from (4.7)

(4.9)
$$\forall R > 0,$$
$$\liminf_{t \to T} \int_{|x|<R} v(t,\, x) = \liminf_{t \to T} \int_{|x|<R} |\, u(t,\, x + x(t))\,|^2\, dx \geq \| Q_{k_2} \|_{L^2}^2.$$

Therefore from (4.8)-(4.9)

(4.10)
$$v(t,\, x) \rightharpoonup \| Q_{k_2} \|_{L^2}^2\, \delta_{x=0} \quad \text{as } t \to T,$$

or equivalently

(4.11)
$$|\, u(t,\, x + x(t))\,|^2 \rightharpoonup \| Q_{k_2} \|_{L^2}^2\, \delta_{x=0} \quad \text{as } t \to T.$$

And, $\forall R > 0$

(4.12)
$$\int_{|x|>R} |\, u(t,\, x + x(t))\,|^2\, dx \to 0 \quad \text{as } t \to T.$$

We now claim the following lemma.

LEMMA 4.3. – (i) $\forall t \in [0,\, T)$,

$$-\int (k_2 - k(x))\,|\, u(t,\, x)\,|^{\frac{4}{N}+2}\, dx \leq \left(\frac{4}{N} + 2\right) E(\phi) - E_{k_2}(u(t)) \leq E(\phi).$$

(ii) $\forall \delta > 0$, *there is a* $c_\delta > 0$ *such that* $\forall t \in [0, T)$

$$\int_{|x-x(t)|\geq\delta} |\nabla u(t, x)|^2 \leq c_\delta.$$

Proof. – (i) Indeed $\forall t \in [0, T)$,

$$E(u(t)) = E(\phi).$$

Therefore

$$(4.13) \quad \left\{\frac{1}{2}\int |\nabla u(t, x)|^2 - \frac{1}{\frac{4}{N}+2}\int k_2 |u(t, x)|^{\frac{4}{N}+2} dx\right\} + \frac{1}{\frac{4}{N}+2}\int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N}+2} dx = E(\phi).$$

Since $\| u(t) \|_{L^2} = \| Q_{k_2} \|_{L^2}$, we have

$$(4.14) \quad E_{k_2}(u(t)) = \frac{1}{2}\int |\nabla u(t, x)|^2 - \frac{1}{\frac{4}{N}+2} \times \int k_2 |u(t, x)|^{\frac{4}{N}+2} dx \geq 0$$

and from (4.1),

$$(4.15) \quad \int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N}+2} dx \geq 0.$$

From (4.13)-(4.15), we derive part (i) of the lemma.

(ii) Let $\delta > 0$. From (4.12), there is a $t_\delta < T$ such that

$$\forall t \in [t_\delta, T), \quad \int_{|x|>\frac{\delta}{4}} |u(t, x + x(t))|^2 dx < \varepsilon(N),$$

where $\varepsilon(N)$ is defined in Lemma 4.3.

From Lemma 4.2, we have the existence of $A_\delta > 0$ such that

$$\forall t \in [t_\delta, T), \quad \int_{|x|>\delta} |\nabla u(t, x + x(t))|^2 dx \leq A_\delta.$$

Since $\forall t \in [0, t_\delta]$,

$$\int_{|x|>\delta} |\nabla u(t, x + x(t))|^2 \, dx \le \int |\nabla u(t, x)|^2 \, dx \le c,$$

we have the conclusion. This concludes the proof of Lemma 4.3 and of (4.5)-(4.6).

Step 2. – Localisation of the concentration point.

In this step we use strongly the assumptions (4.1)′-(4.1)″. Since $\int \vec{x} \, |u(t, x)|^2 \, dx$ as $t \to T$ can not be controlled as in the case $k(x) \equiv k_0$, we cannot apply arguments such as in [10].

Let us show that there is x_0 such that

$$x(t) \to x_0 \quad \text{as } t \to T, \tag{4.16}$$

$$k(x_0) = k_2 \quad \text{and} \quad \nabla k(x_0) = 0. \tag{4.17}$$

Proof of (4.16)-(4.17).

LEMMA 4.4. – *There is a constant* $c_0 > 0$ *such that*

$$\forall t \in [0, T), \quad |x(t)| \le c_0.$$

Proof. – Indeed, from Lemma 4.3 and (4.1)′:

$$\forall t, \quad \int (k_2 - k(x)) \, |u(t, x)|^{\frac{4}{N}+2} \, dx \le c, \tag{4.17}$$

and

$$\forall |x| \ge R_0, \quad k_2 - k(x) \ge \delta.$$

Therefore

$$\forall t, \quad \int_{|x| \ge R_0} \delta \, |u(t, x)|^{\frac{4}{N}+2} \, dx \le c,$$

and

$$\forall t, \quad \int_{|x| \ge R_0} |u(t, x)|^{\frac{4}{N}+2} \, dx \le \frac{c}{\delta}. \tag{4.18}$$

Moreover, from (4.11) and Hölder inequality we have

$$\int_{|x-x(t)|\le 1} |u(t, x)|^{\frac{4}{N}+2}\,dx \to +\infty \quad \text{as } t \to T. \tag{4.19}$$

It follows from (4.18)-(4.19) that

$$\limsup_{t\to T} |x(t)| \le R_0 + 1,$$

and the conclusion follows.

LEMMA 4.5. – *There is a x_0 such that*

$$x(t) \to x_0 \quad \textit{as}\ t \to T \quad \textit{and} \quad k(x_0) = k_2.$$

Remark. – It follows directly from $k(x_0) = k_2 = \max_{x\in\mathbb{R}^N} k(x)$ that $\nabla k(x_0) = 0$.

Proof. – (i) We first remark that

$$M(t) = \operatorname*{Min}_{i=1,\dots,p} \{|x(t) - x_i|\} \to 0 \quad \text{as } t \to T,$$

where $x_1, \dots, x_p$ are defined by (4.1)$''$. Indeed, by contradiction, assume that there are $t_n \to T$ as $n \to +\infty$ and $\delta > 0$ such that

$$M(t_n) \ge \delta.$$

Compactness arguments in $\mathbb{R}^N$ yield the existence of $\alpha > 0$ such that

$$\forall n, \quad \forall x \in B\left(x(t_n), \frac{\delta}{2}\right), \quad (k_2 - k(x)) \ge \alpha. \tag{4.20}$$

Therefore from Lemma 4.3,

$$\forall n, \quad \int_{|x-x(t_n)|<\frac{\delta}{2}} (k_2 - k(x))\,|u(t_n, x)|^{\frac{4}{N}+2}\,dx \le c_0$$

and

$$\forall n, \quad \int_{|x-x(t_n)|<\frac{\delta}{2}} |u(t_n, x)|^{\frac{4}{N}+2}\,dx \le c. \tag{4.21}$$

(4.21) contradicts the fact that

$$\int_{|x-x(t_n)|<\frac{\delta}{2}} |u(t_n, x)|^{\frac{4}{N}+2}\,dx \xrightarrow[n\to+\infty]{} +\infty$$

(from (4.11)). Therefore

$$M(t) \to 0 \quad \text{as } t \to T.$$

(ii) Let us show now that there is $i \in \{1, \ldots, p\}$ such that

$$x(t) \to x_i \quad \text{as } t \to T. \tag{4.22}$$

Let $\delta = \dfrac{1}{4} \min\limits_{i \neq j} \{|\,x_i - x_j\,|\} > 0$ and $\psi \in \mathcal{C}^\infty$ such that

$- \psi(x) \equiv 1 \quad \text{for } |\,x\,| < \delta,$

$- 0 \leq \psi(x) \leq 1,$

$- \psi(x) \equiv 0 \quad \text{for } |\,x\,| > 2\,\delta.$

From Part (i) and Lemma 4.3 we have the existence of $c > 0$ such that

$$\forall t \in [0, T), \quad \forall i = 1, \ldots, N, \quad \int_{\delta < |x - x_i| < 2\delta} |\,\nabla u(t, x)\,|^2\,dx \leq c. \tag{4.23}$$

We remark that $\forall i = 1, \ldots, p$, there is e_i such that

$$\int \psi(x - x_i)\,|\,u(t, x)\,|^2\,dx \to e_i \quad \text{as } t \to T. \tag{4.24}$$

Indeed from direct calculations and (4.23),

$$\begin{aligned} &\left| \frac{d}{dt} \int \psi(x - x_i)\,|\,u(t, x)\,|^2\,dx \right| \\ &= \left| 4\,\mathrm{Im} \int \nabla \psi(x - x_i)\,u\,\overline{\nabla u} \right| \\ &= \left| 4\,\mathrm{Im} \int_{\delta < |x - x_i| < 2\delta} \nabla \psi(x - x_i)\,u\,\overline{\nabla u} \right| \\ &\leq c \left(\int_{\delta < |x - x_i| < 2\delta} |\,\nabla u\,|^2 \right)^{\frac{1}{2}} \leq c \end{aligned}$$

and (4.24) follows.

Therefore, from (4.11)-(4.12) and (i), there is $i_0 \in \{1, \ldots, p\}$ such that $e_{i_0} = \|\,Q_{k_2}\,\|_{L^2}$ and

$$x(t) \to x_{i_0} \quad \text{as } t \to T.$$

This concludes the proof of Lemma 4.5 and (4.16)-(4.17).

Step 3. – Control of the solution at infinity and conclusion.

Let us show that $\phi \in \Sigma$, that is

$$|x|\,|\phi(x)| \in L^2, \tag{4.25}$$

and

$$\int |x - x_0|^2\,|u(t, x)|^2\,dx \to 0 \quad \text{as } t \to T. \tag{4.26}$$

The proof will use the same type of argument than in [10].

We remark that from Lemmas 4.3 and 4.5, whe have

$$\forall\,\delta > 0, \quad \text{there is a } c_\delta > 0 \text{ such that } \forall\,t \in [0, T), \quad \int_{|x - x_0| \geq \delta} |\nabla u(t, x)|^2\,dx \leq c_\delta. \tag{4.27}$$

LEMMA 4.6.

(i) $$\int |x|^2\,|\phi(x)|^2 < +\infty.$$

(ii) There is a constant $c > 0$ such that

$$\forall\,t \in [0, T), \quad \int |x - x_0|^2\,|u(t, x)|^2\,dx \leq c.$$

Proof. – Let us argue by contradiction. Suppose $\int |x|^2\,|\phi(x)|^2\,dx = +\infty$.

(i) Let us consider $\psi_A(x) = \tilde{\psi}_A(|x - x_0|)$ where

– $\tilde{\psi}_A(0) = 0$,

– $\tilde{\psi}'_A(r) = 0$ for $r \leq 1$,

– $\tilde{\psi}'_A(r) = r - 1$ for $1 \leq r \leq A$,

– $\tilde{\psi}'_A(r) = 2A - 1 - r$ for $A \leq r \leq 2A - 1$

– $\tilde{\psi}'_A(r) = 0$ for $r \geq 2A - 1$.

By direct calculations, we have for a $c > 0$,

$$\forall x, \quad \forall A \geq 1, \quad | \nabla \psi_A |^2 \leq c \psi_A + c, \tag{4.28}$$

$$1 + \psi_4 (x) \geq \frac{| x - x_0 |^2}{4}, \qquad \forall 1 \leq | x - x_0 | \leq A, \tag{4.29}$$

$$\psi_A (x) \equiv c_A, \quad \text{for } | x - x_0 | \geq 2A - 1. \tag{4.30}$$

Let $Y_A (t) = \int \psi_A (x) | u(t, x) |^2 \, dx$. We have for a $c > 0$,

$$\forall A \geq 1, \quad \forall t, \quad | Y_A' (t) | \leq c \sqrt{Y_A (t) + 1}, \tag{4.31}$$

$$Y_A (0) \to +\infty, \quad \text{as } A \to +\infty, \tag{4.32}$$

$$\forall A, \quad Y_A (t) \to 0 \quad \text{as } t \to T. \tag{4.33}$$

(4.33) follows from (4.11)-(4.12) and (4.30). (4.32) is a consequence of (4.29) and $\int | x |^2 \phi (x) |^2 \, dx = +\infty$. (4.31) can be deduced from (4.28) and (4.27). Indeed,

$$\begin{aligned} Y_A' (t) &= 4 \operatorname{Im} \int \nabla \psi_A \, u \overline{\nabla u} \\ &\leq c \left(\int_{|x - x_0| \geq 1} | \nabla u |^2 \right)^{1/2} \left(\int | \nabla \psi_A |^2 | u |^2 \right)^{1/2} \\ &\leq c \left(\int \psi_A (x) | u |^2 + \int | u |^2 \right)^{1/2} \leq c (Y_A (t) + 1)^{1/2}. \end{aligned}$$

Integrating in time (4.31), we obtain

$$\forall A, \quad \forall t \in [0, T), \quad | \sqrt{Y_A (0) + 1} - \sqrt{Y_A (t) + 1} | \leq c.$$

Letting $t \to T$, we then have $\sqrt{Y_A (0) + 1} \leq c$, which contradicts (4.32). Therefore

$$\int | x |^2 | \phi (x) |^2 \, dx < +\infty.$$

(ii) Considering now

$$\psi(x) = \tilde{\psi}(|x - x_0|)$$

where $\tilde{\psi}(0) = 0$, $\tilde{\psi}'(r) = 0$ for $r \leq 1$, $\tilde{\psi}'(r) = r - 1$ for $r \geq 1$. We obtain

- $|Y'(t)| \leq c\sqrt{Y(t) + 1}$,
- $Y(0) < +\infty$.

Therefore, there is a constant $c > 0$ such that

$$\forall t \in [0, T), \quad Y(t) \leq c,$$

and since $2 + \tilde{\psi}(r) \geq \dfrac{r^2}{4}$,

$$\forall t \in [0, T), \quad \int |x - x_0|^2 |u(t, x)|^2 \, dx \leq c.$$

LEMMA 4.7.

$$\lim_{t \to T} \int |x - x_0|^2 |u(t, x)|^2 \, dx = 0.$$

The proof is the same than the one in [10] (Step 2, p. 442). Let us recall the key parts of the proof. From (4.11)-(4.12), we have $\forall\, A \geq 0$,

$$\lim_{t \to T} \int_{|x - x_0| \leq A} |x - x_0|^2 |u(t, x)|^2 \, dx = 0. \tag{4.34}$$

The conclusion will follow from an uniform integrability property:

(4.35) $\forall \varepsilon > 0$, there is a A_ε such that

$$\forall t \in [0, T), \quad \int_{|x - x_0| \geq A_\varepsilon} |x - x_0|^2 |u(t, x)|^2 \, dx \leq \varepsilon.$$

Proof of (4.35). – Let us consider $\psi \in C^4(\mathbb{R}^N, \mathbb{R})$

- $\psi(x) = \psi(|x - x_0|)$,
- $\psi(x) = 0$ for $|x - x_0| \leq 1$,
- $\dfrac{1}{2}|x|^2 \leq \psi(x) \leq |x|^2$ for $|x - x_0| \geq 2$,
- there is $c > 0$ such that $\forall x$, $\forall r \geq 0$,

$|\nabla \psi(x) \leq c|x - x_0|$ and $|\psi''(r)| + |\psi''(r)| + |\psi''''(r)| \leq c$,

and

$$\psi_A(x) = A^2 \psi\left(\frac{x}{A}\right).$$

Considering $\frac{d}{dt}\int \psi_A(x)|u(t, x)|^2\,dx$, we obtain the existence of $\varepsilon(A) > 0$ such that

$$\sup_{t\in[0,T)} \int \psi_A(x)|u(t, x)|^2\,dx \le \varepsilon(A) \tag{4.36}$$

where $\varepsilon(A) \to 0$ as $A \to +\infty$ (*see* proof below). The fact that $\psi_A(x) \ge \frac{1}{2}|x-x_0|^2$ for $|x-x_0| \ge 2A$ implies (4.35) and the conclusion follows.

Proof of (4.36). – Let us define

$$Y_A(t) = \int \psi_A(x)|u(t, x)|^2\,dx.$$

We have

$$\begin{aligned} |Y_A'(t)| &= \left| 2\,\mathrm{Im}\int \nabla\psi_A(x)\,u\,\overline{\nabla u}\right| \\ &\le 2\left|\int_{|x|\ge A} \nabla\psi_A\,u\,\overline{\nabla u}\right| \\ &\le 2\left(\int_{|x|\ge A} |\nabla\psi_A|^2|u|^2\right)^{1/2} \\ &\quad\times\left(\int_{|x|\ge A} |\nabla u(t, x)|^2\,dx\right)^{1/2}. \end{aligned} \tag{4.37}$$

We can remark that $\forall A \ge 1$,

$$\forall x, \quad |\nabla\psi_A|^2 \le c\psi_A(x) + c.$$

Therefore from Lemma 4.6,

$$\begin{aligned} |Y_A'(t)| &\le c\left(\int_{|x|\ge A} \psi_A(x)|u|^2 + \int_{|x|\ge A}|u|^2\right)^{1/2} \\ &\quad\times\left(\int_{|x|\ge A}|\nabla u|^2\right)^{1/2} \\ &\le c\left(Y_A(t) + \frac{1}{A^2}\right)^{1/2}\left(\int_{|x|\ge A}|\nabla u|^2\right)^{1/2}, \end{aligned} \tag{4.38}$$

or equivalently

$$|Y_A'(t)| \leq Y_A(t) + \frac{1}{A^2} + c \int_{|x|\geq A} |\nabla u|^2. \tag{4.39}$$

Since (4.27),

$$\int_0^T \int_{|x|\geq A} |\nabla u(t, x)|^2 \, dx \, dt \leq c,$$

the convergence dominated theorem yiellds

$$\lim_{A\to+\infty} \int_0^T \int_{|x|\geq A} |\nabla u(t, x)|^2 \, dx \, dt = 0.$$

Therefore by integration of (4.39),

$$\begin{aligned}\lim_{A\to+\infty} \{ \operatorname{Sup}_{t\in[0,T)} Y_A(t)\} &\leq c \left\{ \lim_{A\to+\infty} \int_0^T \int_{|x|\geq A} |\nabla u(t, x)^2 \, dx \, dt \right\} \\ &\quad + c\{ \lim_{A\to+\infty} Y_A(0)\} = 0,\end{aligned}$$

which concludes the proof of (4.36) and of Proposition 4.1.

5. NONEXISTENCE OF L²-MINIMAL BLOW-UP SOLUTIONS

In this section, we discuss nonexistence and existence of L^2-minimal blow-up solutions.

Under some conditions on the function $k(x)$ at infinity, we saw in Section 4 that a blow-up solution such that

$$\| \phi \|_{L^2} = \| Q_{k_2} \|_{L^2} \tag{5.1}$$

concentrates at the blow-up time at a point x_0 such that

$$k(x_0) = k_2, \quad \nabla k(x_0) = 0. \tag{5.2}$$

In subsection 5.1, under some condition on the form of $k(x)$ for x near x_0, we prove that such a solution does not exist. We briefly give the existence of such a solution in subsection 5.2 under some condition of flatness on $k(x)$ for x near x_0.

5.1. Nonexistence of minimal blow-up solution

Let x_0 be such that $k(x_0) = k_2$ (in particular $\nabla k(x_0) = 0$). We assume for a $c_0 > 0$ that

$$\nabla k(x) \cdot (x - x_0) \leq -c_0\,|x - x_0|^{1+\alpha_0} \quad \text{for } x \text{ near } x_0,$$

where $0 < \alpha_0 < 1$. It implies in particular

$$k(x_0) - k(x) \geq c\,|x - x_0|^{1+\alpha_0} \quad \text{for } x \text{ near } x_0; \qquad (5.3)x_0$$

(this condition does not allow $k(x)$ to be C^2 near x_0). We claim the following theorem.

THEOREM 5.1. – *Assume that $k(x)$ satisfies* $(5.3)x_0$. *There is then no blow-up solution such that*

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$$

and

$$|u(t, x)|^2 \rightharpoonup \|Q_{k_2}\|_{L^2}^2\, \delta_{x=x_0} \textit{ in the distribution sense as } t \to T$$

(where T is the blow-up time).

This theorem has the following corollary:

COROLLARY 5.2 (Nonexistence of L^2-minimal blow-up solutions). – *Assume that k satisfies* (4.1), (4.1)′, (4.1)″ *and all x_0 such that $k(x_0) = k_2$ satisfies* $(5.3)x_0$. *There is no blow-up solutions such that*

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}.$$

We remark that the corollary follows directly from Section 4 and Theorem 5.1. Let us prove Theorem 5.1.

Proof of Theorem 5.1. – We argue by contradiction. Assume there is a $\phi \in H^1$ such that

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}, \qquad (5.4)$$

$u(t)$ blows-up in finite time T, and

$$|u(t, x)|^2 \rightharpoonup \|Q_{k_2}\|_{L^2}^2\, \delta_{x=x_0}. \qquad (5.5)$$

A contradiction follows from asymptotic estimates on the solution and energy arguments.

LEMMA 5.3. (Energy estimates). – *We have*

(i) $$E(\phi) \geq E_{k_2}(u(t)) \geq 0,$$

(ii) $$E(\phi) \geq \frac{1}{\frac{4}{N}+2} \int (k_2 - k(x)) \, | u(t, x) |^{\frac{4}{N}+2} \, dx \geq 0,$$

(iii) $$E_{k_2}(u(t)) + \frac{1}{\frac{4}{N}+2} \int (k_2 - k(x)) \, | u(t, x) |^{\frac{4}{N}+2} \, dx \leq E(\phi).$$

Proof. – Parts (i) and (ii) follow from $\| \phi \|_{L^2} = \| u(t) \|_{L^2} \leq \| Q_{k_2} \|_{L^2}$, Part iii), and the definition of k_2. The conservation of the energy yields (iii).

We claim that

(5.6) $$\int (k_2 - k(x)) \, | u(t, x) |^{\frac{4}{N}+2} \, dx \xrightarrow[t \to T]{} +\infty$$

which will be a contradiction with part (ii) of Lemma 5.3.

Proof of (5.6). – From (5.3)x_0, (5.6) is implied by

(5.7) $$\int_{|x-x_0| \leq \rho_0} | x - x_0 |^{1+\alpha_0} \, | u(t, x) |^{\frac{4}{N}+2} \, dx \xrightarrow[t \to T]{} +\infty \quad \text{for a } \rho_0 > 0.$$

LEMMA 5.4. – *We have the existence of* $x(t) \to x_0$, $\theta(t) \in \mathbb{R}^2$, *such that*

(5.8) $$\frac{1}{[\lambda(t)]^{\frac{N}{2}}} e^{i\theta(t)} u\left(t, x(t) + \frac{(x - x(t))}{\lambda(t)}\right) \to Q_{k_2}(x) \quad \text{in } H^1,$$

where $\lambda(t) = \| \nabla u(t) \|_{L^2} \to +\infty$.

Proof. – *See* Corollary 2.7.

Therefore for t near T

(5.9) $$| x_0 - x(t) | < \frac{\rho_0}{2}$$

and

$$\begin{aligned}
&\int_{|x-x_0|\le\rho_0} |x - x_0|^{1+\alpha_0}\, |u(t, x)|^{\frac{4}{N}+2}\, dx \\
&\ge \int_{|x(t)-x|\le\frac{\rho_0}{2}} |(x - x(t) + (x(t) - x_0)|^{1+\alpha_0} \\
&\quad\times |u(t, x - x(t) + x(t))|^{\frac{4}{N}+2}\, dx \\
&\ge \int_{|y|\le\frac{\rho_0}{2}\lambda(t)} \left| \frac{y}{\lambda(t)} + (x(t) - x_0) \right|^{1+\alpha_0} \\
&\quad\times \left| u\left(t, \frac{y}{\lambda(t)} + x(t)\right) \right|^{\frac{4}{N}+2} dx \\
&\ge \lambda(t)^2 \int_{|y|\le 10} \frac{1}{\lambda(t)^{1+\alpha_0}}\, |y + (x(t) - x_0)\,\lambda(t)|^{1+\alpha_0}\, Q_{k_2}^{\frac{4}{N}+2}(y)\, dy \\
&\ge c\left(\int_{|y|\le 10} Q^{\frac{4}{N}+2}(y)\, dy\right) \frac{\lambda(t)^2}{\lambda(t)^{1+\alpha_0}} \\
&\ge c\,\lambda(t)^{1-\alpha_0} \xrightarrow[t\to T]{} +\infty.
\end{aligned}$$

This concludes the proof of (5.6). A contradiction follows and Theorem 5.1 is proved.

5.2. Existence of L^2-blow-up solution and open problems

Using the same method as [9′], that is a fixed point and compactness argument near the solution of the homogeneous Schrödinger equation

$$u(t, x) = \frac{w^{\frac{N}{2}}}{t^{\frac{N}{2}}}\, e^{i\frac{|x-x_0|^2}{4t} - \frac{iw^2}{t}}\, Q_{k_2}\left(\frac{w(x - x_0)}{t}\right),$$

we are able to prove the following proposition.

PROPOSITION 5.4 (Existence L^2-minimal blow-up solution under flatness condition). – *Assume $k(x) \equiv k_2$ for x near x_0. There is then a L^2-minimal blow-up solution $u(t)$ such that*

$$|u(t, x)|^2 \rightharpoonup \|Q_{k_2}\|_{L^2}\, \delta_{x=x_0} \quad as\ t \to T$$

(where T is the blow-up time of $u(t)$).

Remark. – Section 5 leaves open the question of existence and nonexistence of L^2-minimal blow-up solution in the case where k is a C^2 near x_0 and

$$c_1 \le \left| \frac{D^2 k(x, x)}{|x - x_0|^{2+i}} \right| \le c_2$$

for $i = 0, 1, \dots$

In addition, knowing which i (and eventually c_1, c_2) separes the cases of existence and nonexistence is an open question.

6. STABILITY OF SINGULARITY

In this section, we point out the relation between the nonexistence of minimal blow-up solutions and the existence of black holes. We define a black hole as a "space singularity stable in time with respect to initial data". More precisely, assume that there is no minimal blow-up solution $Q(x_0) = k_2$ and x_0 is a strict local maximum. Then the singularity

$$|u|^2 = \| Q_{k_2} \|_{L^2}^2 \, \delta_{x=x_0} \tag{6.1}$$

will be stable in time in some sense. That is,

THEOREM 6.1. – *Consider a sequence of initial data ϕ_n in H^1 such that*

$$\int |\phi_n|^2 \to \| Q_{k_2} \|_{L^2}^2, \quad \text{as } n \to +\infty, \tag{6.2}$$

$$|\phi_n(x)|^2 \rightharpoonup \| Q_{k_2} \|_{L^2}^2 \, \delta_{x=x_0}, \quad \text{as } n \to +\infty \quad \textit{in the distribution sense}, \tag{6.3}$$

$$\text{there is a } c > 0 \text{ such that } E_{\varepsilon_n}(\phi_n) \le c, \tag{6.4}$$

where $E_{\varepsilon_n}(u) = E(u) + \dfrac{\varepsilon_n}{q+1} \displaystyle\int |u|^{q+1}$, $\varepsilon_n > 0$, $\varepsilon_n \xrightarrow[n \to +\infty]{} 0$, *and* $\dfrac{N+2}{N-2} > q > \dfrac{4}{N} + 1$. *Let $u_n(t)$ be the solution of*

$$\begin{cases} iu_t = -\Delta u - k(x) |u|^{\frac{4}{N}} u + \varepsilon_n |u|^{q-1} u \\ u_n(0) = \phi_n. \end{cases} \tag{6.5}$$

$$\text{For all time } t > 0, \tag{6.6}$$

$|u_n(t, x)|^2 \rightharpoonup \| Q_{k_2} \|_{L^2}^2 \, \delta_{x=x_0}$ in the distribution sense as $n \to +\infty$.

Remark. – We have considered $u_n(t)$ solution of equation (6.5) to assure that $u_n(t)$ will be defined for all time. The same conclusions hold for solutions of equation (1.1) ($\varepsilon_n = 0$) on their maximum common time existence interval.

Remark. – In the case of nonexistence of minimal blow-up solution such that

$$E(\phi) \leq a,$$

if we assume $E(\phi_n) \to a$, same conclusion holds.

Remark. – It is an open problem to show that there is no black hole at a mass level different of $\| Q_{k_2} \|_{L^2}^2$. We conjecture there is none.

Proof of Theorem 6.1. – We do it in three steps.

Step 1. – *Reduction.*

We claim using concentration properties that Theorem 6.1 is implied by the following property

$$(6.7) \qquad \forall t, \quad \liminf_{n \to +\infty} \{ \inf_{s \in [0,\, t]} \| \nabla u_n(s) \|_{L^2} \} = +\infty.$$

(6.7) *implies* (6.6). – Indeed, assume (6.7) and let us fix $t > 0$. From Corollary 2.7, there is a $x_n(s)$ such that

$$(6.8) \qquad |u_n(s,\, x - x_n(s))|^2 \rightharpoonup \| Q_{k_2} \|_{L^2}^2\, \delta_{x = x_0}$$

in the distribution sense uniformly in s, that is: $\forall\, \delta_1 > 0$, $\forall\, \delta_2 > 0$, for n large

$$\sup_{s \in [0,\, t]} \int_{|x - x_n(s)| \geq \delta_2} |u_n(s,\, x)|^2\, dx \leq \delta_1.$$

We remark that the energy identity

$$E_{\varepsilon_n}(u_n(t)) = E_{\varepsilon_n}(\phi_n) \leq c$$

implies $E(u_n(t)) \leq c - \dfrac{\varepsilon_n}{q+1} \displaystyle\int |u_n(t)|^{q+1} \leq c.$

Moreover, direct continuity arguments on the solution (with respect to the initial data) show that we can choose for a fixed n, $x_n(\cdot) : [0,\, t] \to \mathbb{R}^N$ continuous with respect to s.

We claim that

$$\lim_{n \to +\infty} \sup_{s \in [0,\, t]} |x_n(s) - x_0| = 0.$$

Indeed, by contradiction, assume there is $\delta > 0$ such that $\forall n$, there is $s_n \in [0, t]$ such that

$$\forall n, \quad | x_n(s_n) - x_0 | \geq \delta. \tag{6.9}$$

We remark from (5.2) that

$$x_n(0) \to x_0 \quad \text{as } n \to +\infty. \tag{6.10}$$

Since $x_n(s)$ is a continuous function of s, there is a sequence $\tau_n \in [0, t]$ such that

$$| x_n(\tau_n) - x_0 | = \delta. \tag{6.11}$$

From the fact that x_0 is a strict local maximum, taking δ small enough, there is $\varepsilon > 0$ such that

$$k(x_n(\tau_n)) \leq k_2 - \varepsilon_0. \tag{6.12}$$

By similar arguments than in the proof of Proposition 2.5, we have in addition

$$\begin{aligned} \liminf_{n \to +\infty} \frac{\| \phi_n \|_{L^2}}{\| Q_{k(x_n(\tau_n))} \|_{L^2}} &\geq \liminf_{n \to +\infty} \frac{\| u_n(\tau_n) \|_{L^2}}{\| Q_{k(x_n(\tau_n))} \|_{L^2}} \\ &\geq \liminf_{n \to +\infty} \frac{\| u_n(\tau_n) \|_{L^2(B(x_n(r_n)))}}{\| Q_{k(x_n(\tau_n))} \|_{L^2}} \geq 1. \end{aligned} \tag{6.13}$$

Going to the limit in (6.13) as $n \to +\infty$, we obtain

$$\begin{aligned} \frac{\| Q \|_{L^2}}{k_2^{\frac{N}{2}}} = \| Q_{k_2} \|_{L^2} &\geq \limsup_{n \to +\infty} \| Q_{k(x_n(\tau_n)} \|_{L^2} \\ &\geq \limsup_{n \to +\infty} \frac{\| Q \|_{L^2}}{[k(x_n(\tau_n)]^{\frac{N}{4}}} \\ &\geq \frac{\| Q \|_{L^2}}{(k_2 - \varepsilon_0)^{\frac{N}{4}}}, \end{aligned} \tag{6.14}$$

which is a contradiction. This concludes the proof of (6.8) and the fact that (6.7) implies (6.6).

Proof of (6.7). – We are now reduced to prove (6.7). Let us argue by contradiction; assume there is a sequence s_n such that

$$(6.15)\qquad |s_n| \le c \quad\text{and}\quad \|\nabla u_n(s_n)\|_{L^2} + \|u_n(s_n)\|_{L^2} \le c.$$

There is then a $\delta_0 > 0$, by Sobolev imbedding such that

$$\int_{|x-x_0|\le\delta_0} |u_n(s_n, x)|^2 \le \frac{1}{2}\|Q_{k_2}\|_{L^2}^2.$$

The fact that x_0 is a strict local maximum implies that taking δ_0 sufficiently small, there is a $\varepsilon_0 > 0$ such that

$$(6.16)\qquad k(x) \le k_2 - \varepsilon_0 \quad \text{for } |x - x_0| = \delta_0.$$

Consider now $t_n \in [0, s_n]$ such that

$$(6.17)\qquad \int_{|x-x_0|\le\delta_0} |u_n(t_n, x)|^2\,dx = \frac{1}{2}\|Q_{k_2}\|_{L^2}^2,$$

$$\text{for } t \in [0, t_n],\quad \int_{|x-x_0|\le\delta_0} |u_n(t, x)|^2\,dx \ge \frac{1}{2}\|Q_{k_2}\|_{L^2}^2.$$

We have then t_n such that for a $c_0 > 0$, $\delta_0 > 0$ and

$$(6.18)\qquad |t_n| \le c_0,$$

$$(6.19)\qquad \|\nabla u_n(t_n)\|_{L^2} \le c_0,$$

$$(6.20)\qquad \int_{|x-x_0|\le\delta_0} |u_n(t_n, x)|^2\,dx = \frac{1}{2}\|Q_{k_2}\|_{L^2}^2.$$

We just have to check (6.19). We argue by contradiction: assume for a subsequence also denoted t_n

$$(6.21)\qquad \|\nabla u_n(t_n)\|_{L^2} \xrightarrow[n\to+\infty]{} +\infty.$$

Then by Corollary 2.7 and Proposition 2.5 (*see* (6.13)), we have

$$(6.22)\qquad |u_n(t_n, x - x_n)|^2 \to \|Q_{k_2}\|_{L^2}^2\,\delta_{x=x_0},$$

and

$$\text{(6.23)}\quad \liminf_{n\to+\infty}\left\{\frac{\|\phi_n\|_{L^2}}{\|Q_{k(x_n)}\|_{L^2}}\right\}\geq \liminf_{n\to+\infty}\left\{\frac{\|u_n(t_n)\|_{L^2(B(x_n,1))}}{\|Q_{k(x_n)}\|_{L^2}}\right\}\geq 1.$$

Since $\|\phi_n\|_{L^2}\to\|Q_{k_2}\|_{L^2}$, we have from (6.17) and (6.22),

$$\text{(6.24)}\qquad x_n\to\hat{x}\qquad |x_0-\hat{x}|=\delta_0.$$

(6.23) implies that

$$\text{(6.25)}\qquad \|Q_{k_2}\|_{L^2}\geq\|Q_{k(\hat{x})}\|_{L^2},$$

that is

$$\frac{\|Q\|_{L^2}}{k_2^{\frac{N}{2}}}\geq\frac{\|Q\|_{L^2}}{[k(\hat{x})]^{\frac{N}{2}}}\qquad\text{or}\qquad k(\hat{x})\geq k_2,$$

which is a contradiction with (6.16) and (6.24). Thus (6.19) is proved. Let us now obtain a contradiction with $u_n(t_n)$.

Step 2. – Compactness of $u_n(t_n)$ in L^2.

LEMMA 6.2. – *There is a $\phi\in H^1$ such that*

$$\text{(6.26)}\qquad u_n(t_n)\to\phi\ \text{in } L^2\quad\text{as } n\to+\infty$$

(*eventually subtracting a subsequence*).

Proof of Lemma 6.2. – From (6.19) and (6.20) and the fact that

$$\text{(6.27)}\qquad \|u(t_n)\|_{L^2}=\|\phi_n\|_{L^2}\xrightarrow[n\to+\infty]{}\|Q_{k_2}\|_{L^2},$$

we have, by standard compactness arguments, (eventually subtracting a subsequence) the existence of $\phi\in H^1$ such that

$$\text{(6.28)}\qquad u_n(t_n)\to\phi\quad\text{in } L^2_{\text{loc}}\quad\text{as } n\to+\infty.$$

In addition,

$$\text{(6.29)}\qquad \|\nabla\phi\|_{H^1}\leq c,$$

$$\text{(6.30)}\qquad \|\phi\|_{L^2}\leq\|Q_{k_2}\|_{L^2}.$$

We claim that in fact

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}. \tag{6.31}$$

Then (6.31) together with (6.27)-(6.28) give that

$$u_n(t_n) \to \phi \quad \text{in } L^2 \quad \text{as } n \to +\infty. \tag{6.32}$$

We show (6.31) by contradiction. We have to avoid in some sense dichotomy. Assume that

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2} - \delta \qquad \text{where} \quad \delta > 0. \tag{6.33}$$

We can remark from (6.37), (6.33) and (6.28) that

$$\frac{1}{2}\|Q_{k_2}\|_{L^2} \leq \|\phi\|_{L^2} \quad \text{or} \quad \delta < \frac{\|Q_{k_2}\|_{L^2}}{2}. \tag{6.34}$$

We then have the existence of R_0 and a sequence $R_n \to +\infty$ such that for n large

$$\|u_n(t_n)\|_{L^2(|x|>R_0)} \geq \|Q_{k_2}\|_{L^2} - \delta - \frac{\delta}{8}; \tag{6.35}$$

and

$$\|u_n(t_n)\|_{L^2(|x|>R_n)} = \delta - \frac{\delta}{8}. \tag{6.36}$$

We consider now ψ such that

$$\psi \in C^\infty, \quad |\psi| \leq 1, \quad \psi \equiv 0 \quad \text{for } |x| \leq \frac{1}{2}, \quad \psi \equiv 1 \quad \text{for } |x| \geq 1.$$

Let us consider t'_n such that

$$\text{for } t \in [t'_n, t_n], \quad \int \psi\left(\frac{x}{R_n}\right)|u_n(t, x)|^2\, dx \geq \frac{\delta}{2}, \tag{6.37}$$

$$\int \psi\left(\frac{x}{R_n}\right)|u_n(t'_n, x)|^2\, dx = \frac{\delta}{2}. \tag{6.38}$$

We have from (6.15), (6.2)-(6.3) that

$$0 < t'_n < t_n \quad \text{and} \quad 0 \leq t_n - t'_n \leq c. \tag{6.39}$$

In addition, we have, for $c > 0$,

$$\forall t \in [t'_n, t_n], \quad \| \nabla u_n (t) \|_{L^2} \leq c. \tag{6.40}$$

Indeed by contradiction Lemma 5.6 (ii) implies that for x_n and $\tau_n \in [t'_n, t_n]$

$$| u_n (\tau_n, x - x_n) |^2 \to \| Q_{k_2} \|_{L^2}^2 \delta_{x=x_0}. \tag{6.41}$$

We have in addition

$$\| u_n \|_{L^2}^2 = \| \phi_n \|_{L^2} \to \| Q_{k_2} \|_{L^2}^2.$$

For n large $| \| u_n \|_{L^2}^2 - \| Q_{k_2} \|_{L^2}^2 | \leq \frac{1}{8} \| Q_{k_2} \|_{L^2}^2$ and from (6.17)

$$\int_{|x-x_0| \leq \delta_0} | u_n (\tau_n, x) |^2 \, dx \geq \frac{1}{2} \| Q_{k_2} \|_{L^2}^2,$$

we obtain using (6.40) that

$$| x - x_0 | \leq 2 \delta_0. \tag{6.42}$$

Then, from (6.41)-(6.42), we obtain for n large

$$\| u_n (\tau_n) \|_{L^2 (|x-x_0| \geq 3\delta_0)} \leq \frac{\delta}{4}$$

or

$$\| u_n (\tau_n) \|_{L^2 (|x| > R_n/2)} \leq \frac{\delta}{4},$$

which is a contadiction with (6.37). Therefore (6.40) is proved.

Let $y_n (s) = \int \psi \left(\frac{x}{R_n} \right) | u_n (t_n - s, x) |^2 \, dx$. We have

$$y_n (0) \geq \int_{|x| > R_n} | u_n (t_n, x) |^2 \, dx \geq \delta - \frac{\delta}{8}, \tag{6.43}$$

$$y_n (t_n - t'_n) = \int \psi \left(\frac{x}{R_n} \right) | u_n (t'_n, x) |^2 \, dx = \frac{\delta}{2}, \tag{6.44}$$

$$\text{for } s \in [0,\, t_n - t'_n],$$

(6.45)
$$|y'_n(s)| \leq \frac{c}{R_n}\left|\int \nabla\psi\left(\frac{x}{R_n}\right)\overline{\nabla u_n}\, u_n\right| \leq \frac{c}{R_n}.$$

Integrating (6.45), we obtain from (6.39)

$$|y_n(t_n - t'_n) - y_n(0)| \leq |t_n - t'_n|\,\frac{c}{R_n} \leq \frac{c}{R_n} \to 0 \quad \text{as } n \to +\infty$$

which is a contradiction with (6.43)-(6.44). This concludes the proofs of (6.31) and of Lemma 6.2.

Remark. – In the case where

– $\varepsilon_n = 0$,

– $\|\phi_n\|_{L^2} \leq \|Q_{k_2}\|_{L^2}$,

– for a $c > 0$, $\int |x|^2 |\phi_n(x)|^2\, dx \leq c$,

there is a simpler proof of (6.31).

Step 3. – Conclusion of the proof.

We have then the existence of $\phi \in H^1$ such that

(6.46)
$$u_n(t_n) \to \phi \quad \text{in } L^2 \quad \text{as } n \to +\infty.$$

Since $\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$ and the fact that there is no minimal blow-up solutions, the solution of Eq. (1.1) with initial data ϕ, $u(t)$ is globally defined for all $t \in \mathbb{R}$ (using conjugation for $t > 0$ and for $t < 0$). Moreover, there is a $c > 0$ such that

$$\text{for } t \in [-c_0,\, 0], \quad \|\nabla u(t)\|_{L^2} \leq c,$$

(where c_0 is defined in (6.18)).

Continuity arguments with respect to the initial data in L^2 implies in fact that

(6.47)
$$u_n(t_n + t) \to u(t) \quad \text{in } C([-c_0,\, 0],\, L^2) \quad \text{as } n \to +\infty.$$

In the case $\varepsilon_n = 0$ it follows from a result of Cazenave and Weissler (Theorem 1.2 of [1′]). In the general case, we can see from Kato [6] that

$$u_n(t_n) \to \phi$$

in standard Cauchy space where continuity with respect initial data is true from (6.19) and $\phi \in H^1$.

Since $|t_n| \leq c_0$, fom (6.45) we have

$$\int |u_n(t_n - t_n) - u(-t_n)|^2 \to 0 \quad \text{as } n \to +\infty$$

or equivalently

$$\int |\phi_n(x) - u(-t_n)|^2 \to 0 \quad \text{as } n \to +\infty. \tag{6.48}$$

From (6.3), (6.48),

$$|u(-t_n, x)|^2 \rightharpoonup \| Q_{k_2} \|_{L^2} \delta_{x=x_0} \quad \text{as } n \to +\infty$$

in the distribution sense which is a contradiction with the fact

$$\| \nabla u(-t_n) \|_{L^2} \leq c.$$

This concludes the proof of (6.7) and of Theorem 6.1.

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