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On the validity of Huygens' principle for second order partial differential equations with four independent variables. II. – a sixth necessary condition

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ABSTRACT. – The expansion of Hadamard's necessary and sufficient condition for the validity of Huygens' principle for a second order partial differential equation with four independent variables is taken to fifth order. This expansion takes the form of a covariant Taylor series in normal coordinates, and is extended to a coordinate invariant expansion by imposing invariance of the resulting coefficient under the set of trivial transformations. The requirement that this coefficient vanish for a Huygens' equation provides a sixth necessary condition.

RÉSUMÉ. – On développe au cinquième ordre la condition de validité du principe de Huygens due à Hadamard, s'appliquant à une équation aux dérivées partielles du deuxième ordre. Ce développement prend la forme d'une série de Taylor covariante dans les coordonnées normales, et s'étend aux coordonnées invariantes en imposant l'invariance des coefficients qui en résultent sous l'action d'un ensemble de transformations élémentaires. La condition d'annulation de ces coefficients dans une équation de Huygens fournit une sixième condition nécessaire.

1. INTRODUCTION

We consider second order linear partial differential equations of normal hyperbolic type for an unknown function $u(x^a)$, $a \in \{1, \ldots, n\}$. Such equations can be written in the coordinate invariant form

$$F[u] := g^{ab} u_{;ab} + A^a u_{,a} + C u = 0, \tag{1.1}$$

where g^{ab} is the contravariant metric on a Lorentzian space V_n with signature $(+, -, \cdots, -)$ and ; denotes the covariant derivative with respect to the Levi-Civita connection for this metric. We assume that the coefficients g^{ab}, A^a, C are C^∞ functions and that V_n is a C^∞ manifold. All considerations in this paper are entirely local.

According to Hadamard [14] *Huygens' Principle* is said to hold for Equation (1.1) if and only if for every Cauchy problem the solution at any point $x_0 \in V_n$ depends only on the Cauchy data in an arbitrarily small neighbourhood of the intersection of the past null conoid $C^-(x_0)$ from x_0 with the initial manifold S. Such an equation is called a *Huygens' differential equation.* The problem of determining up to equivalence all Huygens' equations is called *Hadamard's problem.*

We recall that two equations of the form (1.1) are equivalent if and only if one may be transformed into the other by any of the following *trivial transformations* which preserve the Huygens' character of the equation:

(a) coordinate transformations.

(b) multiplication of both sides of wave equation (1.1) by a conformal factor $e^{-2\phi(x)}$, which is equivalent to a conformal transformation of the metric $\tilde{g}_{ab} = e^{-2\phi(x)} g_{ab}$.

(c) replacement of the dependent variable u by $\lambda(x) u$ where $\lambda(x)$ is a nowhere vanishing function.

Hadamard showed that a necessary condition for (1.1) to be a Huygens' equation is that n be even and $\geqq 4$. He also showed that a necessary and sufficient condition for Huygens' principle to be satisfied is the vanishing of the coefficient of the logarithmic term in the fundamental solution.

It was also in [14] that Hadamard conjectured that the only Huygens' equations were the n-dimensional wave equations

$$\frac{\partial^2 u}{\partial x^{02}} - \sum_{\alpha=1}^{n-1} \frac{\partial^2 u}{\partial x^{\alpha 2}} = 0, \tag{1.2}$$

for n even and $\geqq 4$ and those equivalent to them. This is often called *Hadamard's conjecture* in the literature. The conjecture has been proven in the case when V_4 is conformally flat ([4], [15], [18]). However, it has been disproved in general by Stellmacher ([24], [25]) who gave counter examples for all even dimensions $n \geqq 6$ and by Günther [12] who found a family of counter examples in the physically interesting case $n = 4$. These examples arise in the case $A^a = C = 0$, from the metric

$$ds^2 = 2\, dx^0\, dx^3 - a_{\alpha\beta}\, dx^\alpha\, dx^\beta, \tag{1.3}$$

where $\alpha,\ \beta \in \{1,\ 2\}$ and $a_{\alpha\beta}$ is a symmetric positive definite matrix depending only on x^0. The above metric may be interpreted in the context of general relatively as an exact plane wave solution of the Einstein-Maxwell equations.

In light of this counter example, a modification of Hadamard's conjecture was proposed by Carminati and McLanaghan [6], namely that *every Huygens' equation for* $n = 4$ *is equivalent to the ordinary wave equation* (1.2) *or to the wave equation on a plane wave space-time with metric* (1.3). Work by Carminati, Czapor, McLenaghan, Walton and Williams ([5], [6], [7], [8], [21], [22]) based on the conformally invariant Petrov classification, as well as by Günther and Wünsch ([13], [27], [28], [29]), has verified this modified conjecture for wide classes of spacetimes. Nonetheless, the conjecture is still open.

These results are a consequence of necessary conditions for (1.1) to satisfy Huygens' principle obtained by a number of authors ([11], [19], [20], [23], [26]). The first five of these necessary conditions are

$$I \qquad \mathcal{C} := C - \frac{1}{2} A^a{}_{;a} - \frac{1}{4} A_a\, A^a - \frac{1}{6} R = 0, \tag{1.4}$$

$$II \qquad H^k{}_{a;k} = 0, \tag{1.5}$$

$$III \qquad S_{abk;}{}^k - \frac{1}{2} C^k{}_{ab}{}^l\, L_{kl} = -5 \left(H_{ak}\, H_b^k - \frac{1}{4}\, g_{ab}\, H_{kl}\, H^{kl} \right), \tag{1.6}$$

$$IV \qquad TS\, [3\, S_{abk}\, H^k{}_c + C^k{}_{ab}{}^l\, H_{ck;l}] = 0, \tag{1.7}$$

$$\begin{aligned} V \quad TS\, [&3\, C_{kcdl;m}\, C^k{}_{ef}{}^{l}{}_{;}{}^{m} + 8\, C^k{}_{cd}{}^l{}_{;e}\, S_{klf} + 40\, S_{cd}{}^k\, S_{efk} \\ &- 8\, C^k{}_{cd}{}^l\, S_{kle;f} - 24\, C^k{}_{cd}{}^l\, S_{efk;l} + 4\, C^k{}_{cd}{}^l\, C_l{}^m{}_{ek}\, L_{fm} \\ &+ 12\, C^k{}_{cd}{}^l D^m{}_{efl}\, L_{km} + 12\, H_{kc;de}\, H^k{}_f - 16\, H_{kc;d}\, H^k{}_{e;f} \\ &- 84\, H^k{}_c\, C_{cdel}\, H^l{}_f - 18\, H_{kc}\, H^k{}_d\, L_{ef}] = 0. \end{aligned} \tag{1.8}$$

In the above conditions

$$H_{ab} := A_{[a,b]}, \tag{1.9}$$

$$L_{ab} := -R_{ab} + \frac{R}{6} g_{ab}, \tag{1.10}$$

$$C_{abcd} := R_{abcd} - 2\, g_{[a[b}\, L_{b]c]}, \tag{1.11}$$

$$S_{abc} := L_{a[b;c]}, \tag{1.12}$$

where $A_a := g_{ab}\, A^b$, R_{abcd} denotes the Riemann curvature tensor on V_4, $R_{ab} := g^{cd}\, R_{cabd}$ the Ricci tensor, and $R := g^{ab}\, R_{ab}$ the curvature scalar. Our sign conventions are the same as those in [20]. The symbol $TS\,[\cdots]$ denotes the trace-free symmetric part of the enclosed tensor [19]. It should be noted that the conditions *I-V* are necessarily invariant under the trivial transformations. For some of the results mentioned above a further necessary condition *VII*, derived for the self-adjoint equation ($A^a = 0$) by Rinke and Wünsch [23], was required. This condition is too lengthy to be given here (necessary conditions with an odd number of indices vanish identically in the self-adjoint case).

However, recent work [2] indicates that the first five necessary conditions may not be sufficient to prove the conjecture for the non-self-adjoint equations (1.1). Thus, it appears that a continuation of this programme will require the derivation of a sixth necessary condition. The main result of this paper is the derivation of the condition which is given in the following theorem. The condition has been published without proof by Anderson and McLenaghan [3].

THEOREM. – *A necessary condition for any equation* (1.1) *for* $n = 4$ *to satisfy Huygens' principle is that the coefficients satisfy*

$$\begin{aligned} VI \quad TS\,[& 36\, C^k{}_{ab}{}^l\, C_{lcdm;k}\, H^m{}_e - 6\, C^k{}_{ab}{}^l{}_{;c}\, C_{lde}{}^m\, H_{km} \\ & - 138\, S_{ab}{}^k\, C_{kcdl}\, H^l{}_e + 6\, S_{abk}\, H^k{}_{c;de} + 6\, C^k{}_{ab}{}^l{}_{;c}\, H_{kd;le} \\ & - 24\, S_{abk;c}\, H^k{}_{d;e} + 12\, C^k{}_{ab}{}^l\, L_{kc}\, H_{ld;e} \\ & - 9\, C^k{}_{ab}{}^l{}_{;c}\, L_{kd}\, H_{le} - 9\, S_{abk}\, L_{cd}\, H^k{}_e] = 0. \end{aligned} \tag{1.13}$$

The derivation of the condition is based on Hadamard's necessary and sufficient condition extended to C^∞ equations and is an extension of the method employed by one of us [20] to derive condition V. This method is based on the Taylor expansion of the diffusion kernel in normal coordinates about some arbitrary fixed point x_0, with the use of an appropriate choice of the trivial transformations to simplify the calculations.

The plan of the remainder of the paper is as follows: the Hadamard necessary and sufficient condition is given is Section 2. The choice of the trivial transformations is outlined in Section 3. The invariant Taylor expansion of the diffusion kernel to fifth order is obtained in Section 4. The Theorem is proven in Sections 5 and 6. The conclusion is given in Section 7.

2. THE NECESSARY AND SUFFICIENT CONDITION

In the modern version of Hadamard's theory for the equation (1.1), given in the book of Friedlander [10], the fundamental solution is replaced by the scalar distributions $E^{\pm}_{x0}(x)$, where $x_0 \in V_4$ is fixed and x is a variable point in a simply convex set $\Omega \subset V_4$ containing x^0. These distributions satisfy the equation

$$G\,[E^{\pm}_{x0}(x)] = \delta_{x0}(x), \tag{2.1}$$

where

$$G\,[u] := g^{ab}\,u_{;ab} - (A^a\,u)_{;a} + C\,u, \tag{2.2}$$

is the adjoint of $F\,[u]$ and $\delta_{x0}(x)$ is the Dirac delta distribution. For $n = 4$, they decompose as

$$E^{\pm}_{x0}(x) = V(x_0,\, x)\,\delta^{\pm}(\Gamma(x_0,\, x)) + \mathcal{V}^{\pm}(x_0,\, x)\,\Delta^{\pm}(x_0,\, x), \tag{2.3}$$

where

$$V(x_0,\, x) = \frac{1}{2\pi}\exp\left\{-\frac{1}{4}\int_0^{s(x)}(g^{ab}\,\Gamma_{;ab} - 8 - A^a\,\Gamma_{,a})\frac{dt}{t}\right\}, \tag{2.4}$$

$$\delta^{\pm}(\Gamma(x_0,\, x)) = \begin{cases}\delta(\Gamma(x_0,\, x)), & x \in C^{\pm}(x_0),\\ 0, & x \in C^{\mp}(x_0),\end{cases} \tag{2.5}$$

$$\Delta^{\pm}(x_0,\, x) = \begin{cases}1, & x \notin D^{\pm}(x_0),\\ 0, & x \in D^{\pm}(x_0),\end{cases} \tag{2.6}$$

and the function $\mathcal{V}^{\pm}(x_0,\, x)$ is defined by the equations

$$G\,[\mathcal{V}^{\pm}] = 0 \qquad \text{when} \quad x \in D^{\pm}(x_0), \tag{2.7}$$

$$\mathcal{V}^{\pm}(x_0,\, x) = \frac{V(x_0,\, x)}{s}\int_0^{s(x)}\frac{G\,[V]}{V}\,dt \qquad \text{when} \quad x \in C^{\pm}(x_0). \tag{2.8}$$

The symbol $C^+(x_0)\,(C^-(x_0))$ denotes the future (past) light cone or characteristic surface of the point x_0 and $D^+(x_0)\,(D^-(x_0))$ its interior. $\Gamma(x_0, x)$ is the square of the geodesic distance x_0 to x and s is an affine parameter.

If S is a non-compact space-like 3-manifold in Ω, then a weak solution of Cauchy's problem for (1.1) in the future of S is given by

$$\begin{aligned} u(x_0) = \int_{S_3} \star[(u\, V_{,a} - V\, u_{,a} + u\, \mathcal{V}^-\, \Gamma_{,a} - u\, V A_a)\, \delta^-(\Gamma(x_0, x)) \\ + u\, V\, \Gamma_{,a}\, \delta'^-(\Gamma(x_0, x)) \\ + (u\, \mathcal{V}^-_{,a} - \mathcal{V}^- u_{,a} - u\, \mathcal{V}^- A_a)\, \Delta^-(x_0, x)]\, dx^a, \end{aligned} \tag{2.9}$$

where $\star$ is the Hodge star operator. An analogous solution for points in the past of S may be obtained by replacing $\mathcal{V}^-$ with $\mathcal{V}^+$ and similarly for δ and Δ. These solutions will also be classical solutions of equation (1.1) if the Cauchy data is C^∞.

It can be shown that Huygens' principle will be satisfied by (1.1) for both the advanced and retarded Cauchy problems if and only if

$$\mathcal{V}^\pm(x_0, x) = 0, \qquad \forall x_0, \forall x \in D^\pm(x_0). \tag{2.10}$$

It can further be shown from (2.8) that (2.10) is equivalent to

$$[G[V]] = 0, \tag{2.11}$$

where $[\cdots]$ denotes the restriction of the enclosed function to $C(x_0) := C^+(x_0) \cap C^-(x_0)$. The function $[G[V]]$ is called the *diffusion kernel.* This is the form of the necessary and sufficient condition which we will use to derive the sixth necessary condition.

3. THE TRIVIAL TRANSFORMATIONS

It is not possible to calculate $G[V]$ directly in most cases. It will be our approach, therefore, to derive covariant expressions from the necessary and sufficient condition (2.11) which will be necessary but not sufficient conditions for (1.1) to be a Huygens' equation.

In [20] it was proven that (2.11) is invariant under the trivial transformations (*a*), (*b*) and (*c*) listed in Section 1. This implies that the Huygens' nature of (1.1) is invariant under these transformations. By examining the behaviour of the coefficients of equation (1.1) under these transformations, choices were determined which simplify the derivation of necessary conditions for the transformations (*b*) and the transformation that Hadamard [15] called (*bc*), which is defined as:

(*bc*) replacement of the dependent variable u by $\lambda(x)\, u$ and simultaneous multiplication of both sides of wave equation (1.1) by λ^{-1}, where $\lambda(x)$ is nowhere vanishing.

The metric tensor is invariant under transformation (*bc*). The notation $\tilde{a}$ shall be used to denote the quantity obtained by applying transformation (*b*) to a quantity *a*, while $\bar{a}$ shall denote the effect of both (*b*) and (*bc*) on *a*.

First, let us examine the transformations of some quantities that we will be using extensively. We begin with the effect of (*b*) on the covariant and contravariant metric tensors

$$\tilde{g}^{ab} = e^{-2\phi}\, g^{ab} \Rightarrow \tilde{g}_{ab} = e^{2\phi}\, g_{ab}, \tag{3.1}$$

which induces on the Christoffel symbols (of the second kind) the transformation

$$\left\{ \begin{matrix} \tilde{a} \\ bc \end{matrix} \right\} = \left\{ \begin{matrix} a \\ bc \end{matrix} \right\} + 2\, \delta^a_{(b} \phi_{c)} - g_{bc}\, \phi^a, \tag{3.2}$$

where

$$\phi_a := \phi_{,a} \qquad \text{and} \qquad \phi^a := g^{ab}\, \phi_b. \tag{3.3}$$

Using the above transformations, we may now apply transformations (*b*) and (*bc*) to equation (1.1) obtaining

$$\bar{F}[u] := \tilde{g}^{ab}\, u_{;ab} + \bar{A}^a\, u_{,a} + \bar{C}\, u = \lambda^{-1}\, e^{-2\phi}\, F[\lambda\, u], \tag{3.4}$$

where

$$\bar{A}_a := A_a + 2\, (\log\, \lambda)_{,a} - (n-2)\, \phi_{,a}, \tag{3.5}$$

$$\bar{C} := e^{-2\phi}\, (C + \lambda^{-1}\, \Box\, \lambda + A^a\, (\log\, \lambda)_{,a}). \tag{3.6}$$

From (3.1)-(3.6) it is straightforward to derive the transformation properties of the quantities defined in (1.9)-(1.11). We find

$$\tilde{L}_{ab} = L_{ab} - 2\, \phi_{a;b} + 2\, \phi_a\, \phi_b - g_{ab}\, \phi^k\, \phi_k, \tag{3.7}$$

$$\tilde{C}^a{}_{bcd} = C^a{}_{bcd}, \tag{3.8}$$

$$\tilde{H}_{ab} = H_{ab}, \tag{3.9}$$

$$\bar{\mathcal{C}} = e^{-z\phi}\, \mathcal{C}, \tag{3.10}$$

$$\tilde{S}_{abc} = S_{abc} - \phi_k \, C^k{}_{abc}. \tag{3.11}$$

We first specify the choice of the gauge parameter λ which determines the transformation (*bc*). It may be shown [20] that $V(x_0, x)$, which may be rewritten in the form

$$V = \frac{1}{2\pi\sqrt{\varrho}} \exp\left\{\frac{1}{4}\int_0^s A^a \, \Gamma_{,a} \frac{dt}{t}\right\}, \tag{3.12}$$

where ϱ is defined to be

$$\varrho := 16\sqrt{g(x)\, g(x_0)} \left[\det \frac{\partial^2 \Gamma}{\partial x^a \, \partial x_0^b}\right]^{-1}, \tag{3.13}$$

and $g(x) = \det(g_{ab}(x))$, transforms as

$$\bar{V} = \lambda_0^{-1} \lambda \, V. \tag{3.14}$$

Thus, following Hadamard [15] and McLenaghan [20], we choose for the given point x_0

$$\lambda(x) = \exp\left\{-\frac{1}{4}\int_0^{s(x)} A^a \, \Gamma_{,a} \frac{dt}{t}\right\}. \tag{3.15}$$

It follows from (3.14) and (3.12) that we have

$$\bar{V} = \frac{1}{2\pi\sqrt{\varrho}}. \tag{3.16}$$

By (3.12) and equivalent statement of (3.16) is

$$\int_0^{s(x)} \bar{A}^a \, \Gamma_{,a} \frac{dt}{t} = 0 \Rightarrow \bar{A}^a \, \Gamma_{,a} = 0. \tag{3.17}$$

We now consider the conformal transformation (*b*). Following Günther [11] we can choose the derivatives of ϕ at the point x_0 such that

$$\overset{\circ}{\tilde{L}}_{ab} = 0, \tag{3.18}$$

$$\overset{\circ}{\tilde{L}}_{(ab;c)} = 0, \tag{3.19}$$

$$\overset{\circ}{\tilde{L}}_{(ab;cd)} = 0, \tag{3.20}$$

$$\vdots$$

where $\overset{\circ}{\tilde{L}}_{ab} = \tilde{L}_{ab}(x_0)$ and so on. Various consequences of equations (3.18)-(3.20) used in the sequel are compiled in Appendix A. For the remainder of this paper we will assume that our specified choices of the transformations (*b*) and (*bc*) have been made, and we will drop the corresponding tildes and bars from the transformed quantities.

Finally, we specify the choice of coordinates (transformation (*a*)), in which we carry out the expansion of (2.11). Following [20] we choose a system of normal coordinates (x^a) about the point x_0. These coordinates are admissible in the convex set Ω and are defined by the condition

$$g_{ab}\, x^a \overset{\star}{=} \overset{\circ}{g}_{ab}\, x^a. \tag{3.21}$$

In normal coordinates, V takes the simple form

$$V \overset{\star}{=} \frac{1}{2\pi} \left(\frac{\overset{\circ}{g}}{g} \right)^{\frac{1}{4}}, \tag{3.22}$$

where $\overset{\star}{=}$ signifies equality only in a system of normal coordinates. It is then straightforward to show that

$$\begin{aligned} \sigma &:= -4\, \frac{G[V]}{V} \\ &\overset{\star}{=} \gamma + 4\, A^a{}_{,a} + 4\, A^a\, g^{bc}\, g_{bc,a} - 4\, C, \end{aligned} \tag{3.23}$$

where

$$\gamma :\overset{\star}{=} \left[\frac{1}{4}\, g^{ab}\, g_{ab,c}\, g^{cd}\, g^{ef}\, g_{ef,d} + \left(g^{ab}\, g^{cd}\, g_{cd,a} \right)_{,b} \right]. \tag{3.24}$$

Since $V(x_0, x) \neq 0$, $\forall\, x \in \Omega$, it follows that (2.11) is equivalent to

$$[\sigma] = 0, \tag{3.25}$$

This is the form of the necessary and sufficient condition used in our derivation. In order to obtain a Taylor expansion of σ, we will need Taylor expansions of g_{ab}, g^{ab}, A^a, and C. These will be developed in the next section.

4. THE NECESSARY CONDITIONS

Necessary conditions may be obtained from the necessary and sufficient condition (3.25) by performing a Taylor expansion of the quantity σ [20]. We write

$$\sigma = \overset{\circ}{\sigma} + \overset{\circ}{\sigma}_{,a} x^a + \frac{1}{2} \overset{\circ}{\sigma}_{,ab} x^{ab} + \cdots = 0 \qquad \text{on} \quad C(x_0), \tag{4.1}$$

where we have introduced the notation $x^{ab} := x^a x^b$ and so on. It may be shown [1] that if $x^a = k^a s$ are normal coordinates and A is a scalar function in a normal neighbourhood Ω, then

$$A_{;a_1 \cdots a_n} x^{a_1} \cdots x^{a_n} = A_{,a_1 \cdots a_n} x^{a_1} \cdots x^{a_n}, \tag{4.2}$$

for all $n \in \mathbf{Z}^+$. Thus, we may rewrite equation (4.1) as

$$\sigma \overset{\star}{=} \overset{\circ}{\sigma} + \overset{\circ}{\sigma}_{;a} x^a + \overset{\circ}{\sigma}_{;ab} x^{ab} + \cdots = 0 \qquad \text{on} \quad C(x_0), \tag{4.3}$$

which implies

$$\left.\begin{aligned} \sigma &\overset{\star}{=} 0 \\ \sigma_{;a} k^a &\overset{\star}{=} 0 \\ \sigma_{;ab} k^a k^b &\overset{\star}{=} 0 \\ &\vdots \end{aligned}\right\} \quad \mathrm{mod}\, g, \tag{4.4}$$

or equivalently [19]

$$\left\{\begin{aligned} \overset{\circ}{\sigma} &\overset{\star}{=} 0 \\ \overset{\circ}{\sigma}_{;a} &\overset{\star}{=} 0 \\ \overset{\circ}{\sigma}_{;(ab)} &\overset{\star}{=} 0 \\ &\vdots \end{aligned}\right\} \quad \mathrm{mod}\, g, \tag{4.5}$$

where "mod g" signifies that all terms involving the metric tensor $\overset{\circ}{g}_{ab}$ have been removed.

In order to obtain a form of the necessary conditions involving the coefficients of (1.1) we use the form of σ given in equation (3.23). It will be necessary, therefore, to find the Taylor expansion of the right hand side of equation (3.23). Further, since we would like to find conditions which are independent of our choice of normal coordinates, it will be advantageous to express the coefficients of this Taylor series in covariant form. Herglotz [16], Günther [11], and one of us ([19], [20]) have developed, methods which enable one to obtain the required Taylor expansions g_{ab}, g^{ab}, A_a and C. We will not repeat their calculations here but rather states the results:

$$\begin{aligned} g_{ab} \overset{\star}{=} \overset{\circ}{g}_{ab} &+ \frac{1}{3}\, \overset{\circ}{R}_{aijb}\, x^{ij} + \frac{1}{6}\, \overset{\circ}{R}_{aijb;k}\, x^{ijk} \\ &+ \frac{1}{20} \left(\overset{\circ}{R}_{aijb;kl} + \frac{8}{9}\, \overset{\circ}{R}_{aijr}\, \overset{\circ}{R}{}^{r}{}_{klb} \right) x^{ijkl} \\ &+ \frac{1}{90} \left(\overset{\circ}{R}_{aijb;klm} + 2\,(\overset{\circ}{R}_{aijr}\, \overset{\circ}{R}{}^{r}{}_{klb;m} \right. \\ &+ \left. \overset{\circ}{R}_{aijr;k}\, \overset{\circ}{R}{}^{r}{}_{lmb}) \right) x^{ijklm} \\ &+ \frac{1}{504} \left(\overset{\circ}{R}_{aijb;klmn} + \frac{17}{5}\, (\overset{\circ}{R}_{aijr}\, \overset{\circ}{R}{}^{r}{}_{klb;mn} \right. \\ &+ \overset{\circ}{R}_{aijr;kl}\, \overset{\circ}{R}{}^{r}{}_{mnb}) \\ &+ \frac{8}{5}\, \overset{\circ}{R}_{aijr}\, \overset{\circ}{R}{}^{r}{}_{kls}\, \overset{\circ}{R}{}^{s}{}_{mnb} \\ &+ \left. \frac{11}{2}\, \overset{\circ}{R}_{aijr;kl}\, \overset{\circ}{R}{}^{r}{}_{lmb;n} \right) x^{ijklmn} + \ldots, \end{aligned} \tag{4.6}$$

$$\begin{aligned} g^{ab} \overset{\star}{=} \overset{\circ}{g}{}^{ab} &- \frac{1}{3}\, \overset{\circ}{R}{}^{a}{}_{ij}{}^{b}\, x^{ij} - \frac{1}{6}\, \overset{\circ}{R}{}^{a}{}_{ij}{}^{b}{}_{;k}\, x^{ijk} \\ &+ \frac{1}{20} \left(\overset{\circ}{R}{}^{a}{}_{ij}{}^{b}{}_{;kl} - \frac{4}{3}\, \overset{\circ}{R}{}^{a}{}_{ijr}\, \overset{\circ}{R}{}^{r}{}_{kl}{}^{b} \right) x^{ijkl} + \cdots, \end{aligned} \tag{4.7}$$

$$\begin{aligned} A_a \overset{\star}{=} \overset{\circ}{H}_{ai}\, x^i &+ \frac{2}{3}\, \overset{\circ}{H}_{ai;j}\, x^{ij} + \frac{1}{4} \left(\overset{\circ}{H}_{ai;jk} + \frac{1}{3}\, \overset{\circ}{R}_{aij}{}^{r}\, \overset{\circ}{H}_{rk} \right) x^{ijk} \\ &+ \frac{1}{15} \left(\overset{\circ}{H}_{ai;jkl} + \overset{\circ}{R}_{aij}{}^{r}\, \overset{\circ}{H}_{rk;l} + \frac{1}{2}\, \overset{\circ}{R}_{aij}{}^{r}{}_{;k}\, \overset{\circ}{H}_{rl} \right) x^{ijkl} \\ &+ \frac{1}{72} \left(\overset{\circ}{H}_{ai;jklm} + 2\, \overset{\circ}{R}_{aij}{}^{r}\, \overset{\circ}{H}_{rk;lm} + 2\, \overset{\circ}{R}_{aij}{}^{r}{}_{;k}\, \overset{\circ}{H}_{rl;m} \right. \\ &+ \left. \left(\frac{3}{5}\, \overset{\circ}{R}_{aij}{}^{s}{}_{;kl} + \frac{1}{5}\, \overset{\circ}{R}_{aij}{}^{r}\, \overset{\circ}{R}_{rkl}{}^{s} \right) \overset{\circ}{H}_{sm} \right) x^{ijklm} + \cdots, \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
A^a \overset{\star}{=} & \overset{\circ}{H}{}^a{}_i\, x^i + \frac{2}{3}\, \overset{\circ}{H}{}^a{}_{i;j}\, x^{ij} \\
& + \left(\frac{1}{4}\left(\overset{\circ}{H}{}^a{}_{i;jk} + \frac{1}{3}\, \overset{\circ}{R}{}^a{}_{ij}{}^r\, \overset{\circ}{H}_{ri}\right) - \frac{1}{3}\, \overset{\circ}{H}_{ri}\, \overset{\circ}{R}{}^a{}_{jk}{}^r\right) x^{ijk} \\
& + \left(\frac{1}{15}\left(\overset{\circ}{H}{}^a{}_{i;jkl} + \overset{\circ}{R}{}^a{}_{ij}{}^r\, \overset{\circ}{H}{}_{rk;l} + \frac{1}{2}\, \overset{\circ}{R}{}^a{}_{ij}{}^r{}_{;k}\, \overset{\circ}{H}_{rl}\right)\right. \\
& \left. - \frac{2}{9}\, \overset{\circ}{R}{}^a{}_{ij}{}^r{}_{rk;l} - \frac{1}{6}\, \overset{\circ}{R}{}^a{}_{ij}{}^r{}_{;k}\, \overset{\circ}{H}_{rl}\right) x^{ijkl} \\
& + \left(\frac{1}{72}\left(\overset{\circ}{H}{}^a{}_{i;jklm} + 2\, \overset{\circ}{R}{}^a{}_{ij}{}^r\, \overset{\circ}{H}_{rk;lm} + 2\, \overset{\circ}{R}{}^a{}_{ij}{}^r{}_{;k}\, \overset{\circ}{H}_{rl;m}\right.\right. \\
& \left. + \frac{3}{5}\, \overset{\circ}{R}{}^a{}_{ij}{}^r{}_{;kl}\, \overset{\circ}{H}_{rm} + \frac{1}{5}\, \overset{\circ}{R}{}^a{}_{ij}{}^r\, \overset{\circ}{R}{}_{rkl}{}^s\, \overset{\circ}{H}_{sm}\right) \\
& - \frac{1}{12}\, \overset{\circ}{R}{}^a{}_{ij}{}^r \left(\overset{\circ}{H}_{rk;lm} + \frac{1}{2}\, \overset{\circ}{R}{}_{rkl}{}^s\, \overset{\circ}{H}_{sm}\right) \\
& - \frac{1}{9}\, \overset{\circ}{R}{}^a{}_{ij}{}^r{}_{;k}\, \overset{\circ}{H}_{rl;m} \\
& \left. - \frac{1}{20}\left(\overset{\circ}{R}{}^a{}_{ij}{}^s{}_{;kl} - \frac{4}{3}\, \overset{\circ}{R}{}^a{}_{ij}{}^r\, \overset{\circ}{R}{}_{rkl}{}^s\right) \overset{\circ}{H}_{sm}\right) x^{ijklm} + \cdots, \qquad (4.9)
\end{aligned}
$$

$$
C = \overset{\circ}{C} + \overset{\circ}{C}_{,a}\, x^a + \frac{1}{2!}\, \overset{\circ}{C}_{,ab}\, x^{ab} + \ldots \qquad (4.10)
$$

For the purposes of this paper it is necessary to carry these expansions to higher order than was necessary in [20]. This is accomplished using the algorithms in [20] with the aid of the symbolic computation package MAPLE.

The results are:

$$
\begin{aligned}
g_{ab}\,[7] \overset{\star}{=} & \frac{1}{5040}\left(\frac{3}{2}\, \overset{\circ}{R}_{aijb;klmnp} + \frac{23}{3}\, \overset{\circ}{R}_{aijr}\, \overset{\circ}{R}{}^r{}_{klb;mnp}\right. \\
& + \frac{23}{3}\, \overset{\circ}{R}_{aijr;klm}\, \overset{\circ}{R}{}^r{}_{npb} + \frac{33}{2}\, \overset{\circ}{R}_{aijr;k}\, \overset{\circ}{R}{}^r{}_{lmb;np} \\
& + \frac{33}{2}\, \overset{\circ}{R}_{aijr;kl}\, \overset{\circ}{R}{}^r{}_{mnb;p} + \frac{41}{6}\, \overset{\circ}{R}_{aijr}\, \overset{\circ}{R}{}^r{}_{kls}\, \overset{\circ}{R}{}^s{}_{mnb;p} \\
& + \frac{31}{3}\, \overset{\circ}{R}_{aijs}\, \overset{\circ}{R}{}^r{}_{kls;m}\, \overset{\circ}{R}{}^s{}_{npb} \\
& \left. + \frac{41}{6}\, \overset{\circ}{R}_{aijs;k}\, \overset{\circ}{R}{}^r{}_{lms}\, \overset{\circ}{R}{}^s{}_{npb}\right) x^{ijklmnp}, \qquad (4.11)
\end{aligned}
$$

$$g^{ab}\,[5] \overset{\star}{=} -\frac{1}{120}\left(4\,\overset{\circ}{R}{}^{a}{}_{ijr}\,\overset{\circ}{R}{}^{r}{}_{kl}{}^{b}{}_{;m} + 4\,\overset{\circ}{R}{}^{a}{}_{ijr;k}\,\overset{\circ}{R}{}^{r}{}_{lm}{}^{b} - \frac{4}{3}\,\overset{\circ}{R}{}^{a}{}_{ij}{}^{b}{}_{;klm}\right) x^{ijklm}, \tag{4.12}$$

$$\begin{aligned} A_a\,[6] \overset{\star}{=} -\frac{1}{420}\Bigg(&\overset{\circ}{H}_{ai;jklmn} + \frac{10}{3}\,\overset{\circ}{R}_{aij}{}^{r}\,\overset{\circ}{H}_{rk;lmn} \\ &+ 5\,\overset{\circ}{R}_{aij}{}^{r}{}_{;k}\,\overset{\circ}{H}_{rl;mn} \\ &+ 3\,\overset{\circ}{R}_{aij}{}^{r}{}_{;kl}\,\overset{\circ}{H}_{rm;n} + \overset{\circ}{R}_{aij}{}^{r}\,\overset{\circ}{R}_{rkl}{}^{s}\,\overset{\circ}{H}_{sm;n} \\ &+ \frac{2}{3}\,\overset{\circ}{R}_{aij}{}^{r}\,\overset{\circ}{R}_{rkl}{}^{s}{}_{;m}\,\overset{\circ}{H}_{sn} + \frac{1}{3}\,\overset{\circ}{R}_{aij}{}^{r}{}_{;k}\,\overset{\circ}{R}_{rlm}{}^{s}\,\overset{\circ}{H}_{sn} \\ &+ \frac{2}{3}\,\overset{\circ}{R}_{aij}{}^{r}{}_{;klm}\,\overset{\circ}{H}_{rn}\Bigg) x^{ijklmn}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} A^a\,[6] \overset{\star}{=} \Bigg(&\frac{1}{420}\,\overset{\circ}{H}{}^{a}{}_{i;jklmn} - \frac{1}{70} + 5\,\overset{\circ}{R}{}^{a}{}_{ij}{}^{r}\,\overset{\circ}{H}_{rk;lmn} \\ &- \frac{5}{168}\,\overset{\circ}{R}{}^{a}{}_{ij}{}^{r}{}_{;k}\,\overset{\circ}{H}_{rl;mn} \\ &- \frac{11}{420}\,\overset{\circ}{R}{}^{a}{}_{ij}{}^{r}{}_{;kl}\,\overset{\circ}{H}_{rm;n} \\ &+ \frac{31}{1260}\,\overset{\circ}{R}{}^{a}{}_{ij}{}^{r}\,\overset{\circ}{R}_{rkl}{}^{s}\,\overset{\circ}{H}_{sm;n} \\ &+ \frac{1}{42}\,\overset{\circ}{R}{}^{a}{}_{ij}{}^{r}\,\overset{\circ}{R}_{rkl}{}^{s}{}_{;m}\,\overset{\circ}{H}_{sn} \\ &+ \frac{17}{840}\,\overset{\circ}{R}{}^{a}{}_{ij}{}^{r}{}_{;k}\,\overset{\circ}{R}_{rlm}{}^{s}\,\overset{\circ}{H}_{sn} \\ &- \frac{1}{105}\,\overset{\circ}{R}{}^{a}{}_{ij}{}^{r}{}_{;klm}\,\overset{\circ}{H}_{rn}\Bigg) x^{ijklmn}, \end{aligned} \tag{4.14}$$

where the [ ]'s give the order of the expansion in x.

We are now able to obtain the required expansion for σ by substituting the expressions for g^{ab}, g_{ab}, A^a, A_a and C given by (4.6) - (4.14) into (3.23) and (3.24). Again, this is accomplished using MAPLE. The derivation of the first five necessary conditions (1.4) - (1.8) is carried out explicitly in [20]. We will concentrate here on the derivation of the sixth necessary condition in sufficient detail to allow the reader to follow the procedure.

5. THE SIXTH NECESSARY CONDITION

5.1. $\gamma - \frac{2}{3}R$ to fifth order

After gathering together like terms we have the fifth order contribution

$$\begin{aligned}\gamma[5] \overset{\star}{=} \Bigg(&\frac{1}{80}\overset{\circ}{R}_{(nm;abcde)}\overset{\circ}{g}{}^{nm} - \frac{1}{36}\overset{\circ}{R}{}^{k}{}_{(mn}{}^{l}\overset{\circ}{R}_{|k|ab|l|;cde)}\overset{\circ}{g}{}^{mn}\\
&-\frac{2}{27}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\overset{\circ}{R}_{(kl;cde)} - \frac{3}{40}\overset{\circ}{R}{}^{k}{}_{(mn}{}^{l}{}_{;a}\overset{\circ}{R}_{|k|bc|l|;de)}\overset{\circ}{g}{}^{mn}\\
&-\frac{1}{10}\overset{\circ}{R}{}^{k}{}_{(ab}{}^{l}{}_{;k)}\overset{\circ}{R}_{(lc;de)} - \frac{1}{10}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\overset{\circ}{R}_{(kl;de)}\\
&-\frac{1}{20}\overset{\circ}{R}_{(kl;a)}\overset{\circ}{R}{}^{k}{}_{bc}{}^{l}{}_{;de} - \frac{1}{10}\overset{\circ}{R}_{(ka;b)}\overset{\circ}{R}{}^{k}{}_{(lc}{}^{l}{}_{;de)}\\
&+\frac{2}{45}\overset{\circ}{R}{}^{k}{}_{(st|l|}\overset{\circ}{R}{}^{l}{}_{ab|m|}\overset{\circ}{R}{}^{m}{}_{cd|k|;e)}\overset{\circ}{g}{}^{st}\\
&+\frac{1}{27}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\overset{\circ}{R}{}^{m}{}_{(kl|n|}\overset{\circ}{R}{}^{n}{}_{cd|m|;e)}\\
&+\frac{2}{45}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\overset{\circ}{R}_{|km|n|l|}\overset{\circ}{R}{}^{m}{}_{cd}{}^{n}{}_{;e)}\\
&+\frac{2}{9}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\overset{\circ}{R}_{l(km}{}^{m}\overset{\circ}{R}_{cd;e)}\\
&+\frac{1}{90}\overset{\circ}{R}{}^{km}{}_{a}{}^{l}\overset{\circ}{R}_{kbcl}\overset{\circ}{R}_{(md;e)}\Bigg)x^{abcde}. \end{aligned} \tag{5.1}$$

We begin by expanding the symmetrisations of the $\gamma - \frac{2}{3}R$ term in σ. This process is simplified by breaking $\left(\gamma - \frac{2}{3}R\right)$ [5] into parts. Consider first

$$\begin{aligned}\alpha := &\frac{1}{80}\overset{\circ}{R}_{(mn;abcde)}g^{mn}x^{abcde} - \frac{2}{3}\cdot\frac{1}{120}\overset{\circ}{R}_{,abcde}x^{abcde}\\
= &\frac{1}{80}\overset{\circ}{R}_{(mn;abcde)}g^{mn}x^{abcde} - \frac{1}{180}\overset{\circ}{R}_{;abcde}x^{abcde}\\
= &\frac{1}{1680}\Bigg(-\frac{25}{3}\overset{\circ}{R}_{,abcde}\\
&+2\overset{\circ}{R}{}^{k}{}_{a;kbcde} + 2\overset{\circ}{R}{}^{k}{}_{a;bkcde} + 2\overset{\circ}{R}{}^{k}{}_{a;bckde}\\
&+2\overset{\circ}{R}{}^{k}{}_{a;bcdke} + 2\overset{\circ}{R}{}^{k}{}_{a;bcdek} + \overset{\circ}{R}_{ab;}{}^{k}{}_{ckde}\end{aligned}$$

$$+ \overset{\circ}{R}_{ab;}{}^{k}{}_{ckde} + \overset{\circ}{R}_{ab;}{}^{k}{}_{cdke} + \overset{\circ}{R}_{ab;}{}^{k}{}_{cdek} + \overset{\circ}{R}_{ab;c}{}^{k}{}_{kde} + \overset{\circ}{R}_{ab;c}{}^{k}{}_{dke}$$

$$+ \overset{\circ}{R}_{ab;c}{}^{k}{}_{dek} + \overset{\circ}{R}_{ab;cd}{}^{k}{}_{ke} + \overset{\circ}{R}_{ab;cd}{}^{k}{}_{ek} + \overset{\circ}{R}_{ab;cde}{}^{k}{}_{k}\Big) x^{abcde}. \tag{5.2}$$

By the Ricci identity (B.1), our special conformal gauge (A.1)–(A.14), and the contracted Bianchi identity (B.6) we have the following:

$$\overset{\circ}{R}_{ab;cde}{}^{k}{}_{k}\, x^{abcde}$$

$$\overset{\star}{=} (\overset{\circ}{R}_{ab;cd}{}^{k}{}_{ek} + 3\ \overset{\circ}{R}_{ab;}{}^{l}\ \overset{\circ}{R}_{lc;de} + 2\ \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}_{lc;de}{}^{k}$$

$$+ 3\ \overset{\circ}{R}_{ab;}{}^{l}\ \overset{\circ}{R}_{cd;le} + \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}_{cd;le}{}^{k} + \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}_{cd;el}{}^{k})\, x^{abcde}, \tag{5.3}$$

$$\overset{\circ}{R}_{ab;cd}{}^{k}{}_{ek}\, x^{abcde}$$

$$= (\overset{\circ}{R}_{ab;cd}{}^{k}{}_{ke} + 2\ \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}_{lc;de}{}^{k}$$

$$+ \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}_{cd;le}{}^{k} + \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}_{cd;el}{}^{k})\, x^{abcde}, \tag{5.4}$$

$$\overset{\circ}{R}_{ab;cd}{}^{k}{}_{ke}\, x^{abcde}$$

$$\overset{\star}{=} \Big(\overset{\circ}{R}_{ab;c}{}^{k}{}_{dke} + \frac{3}{2}\ \overset{\circ}{R}_{ab;}{}^{l}\ \overset{\circ}{R}_{cd;le}$$

$$+ 3\ \overset{\circ}{R}_{ab;}{}^{l}\ \overset{\circ}{R}_{lc;de} + 2\ \overset{\circ}{R}{}^{l}{}_{abk;c}\ \overset{\circ}{R}_{ld;e}{}^{k}$$

$$+ 2\ \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}_{lc;d}{}^{k}{}_{e} + \overset{\circ}{R}{}^{l}{}_{abk;c}\ \overset{\circ}{R}_{de;l}{}^{k}$$

$$+ \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}_{cd;l}{}^{k}{}_{e}\Big)\, x^{abcde}, \tag{5.5}$$

$$\overset{\circ}{R}_{ab;c}{}^{k}{}_{dek}\, x^{abcde}$$

$$= (\overset{\circ}{R}_{ab;c}{}^{k}{}_{dek} + 2\ \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}_{lc;d}{}^{k}{}_{e}$$

$$+ \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}_{cd;l}{}^{k}{}_{e} + \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}_{cd;el}{}^{k})\, x^{abcde}, \tag{5.6}$$

$$
\begin{aligned}
&\overset{\circ}{R}{}_{ab;c}{}^{k}{}_{dke}\, x^{abcde} \\
&\overset{\star}{=} \Big(\overset{\circ}{R}{}_{ab;c}{}^{k}{}_{kde} + 2\ \overset{\circ}{R}{}^{l}{}_{abk;c}\ \overset{\circ}{R}{}_{ld;e}{}^{k} + 2\ \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}{}_{lc;d}{}^{k}{}_{e} \\
&\quad + \overset{\circ}{R}{}^{l}{}_{abk;c}\ \overset{\circ}{R}{}_{de;l}{}^{k} + \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}{}_{cd;l}{}^{k}{}_{e} \\
&\quad + \frac{1}{2}\ \overset{\circ}{R}{}_{ab;}{}^{l}\ \overset{\circ}{R}{}_{cd;le} \Big)\, x^{abcde},
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
&\overset{\circ}{R}{}_{ab;c}{}^{k}{}_{kde}\, x^{abcde} \\
&\overset{\star}{=} (\overset{\circ}{R}{}_{ab;}{}^{k}{}_{ckde} - 2\ \overset{\circ}{R}{}_{ab;}{}^{l}\ \overset{\circ}{R}{}_{cd;le} + 5\ \overset{\circ}{R}{}_{ab;}{}^{l}\ \overset{\circ}{R}{}_{lc;de} \\
&\quad - \overset{\circ}{R}{}^{l}{}_{abk;cd}\ \overset{\circ}{R}{}_{l}{}^{k};e + 4\ \overset{\circ}{R}{}^{l}{}_{abk;c}\ \overset{\circ}{R}{}_{ld;}{}^{k}{}_{e} \\
&\quad + 2\ \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}{}_{lc;}{}^{k}{}_{de})\, x^{abcde},
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
&\overset{\circ}{R}{}_{ab;}{}^{k}{}_{cdek}\, x^{abcde} \\
&\overset{\star}{=} (\overset{\circ}{R}{}_{ab;}{}^{k}{}_{cdke} + 2\ \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}{}_{lc;}{}^{k}{}_{de} \\
&\quad + \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}{}_{cd;}{}^{k}{}_{le} + \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}{}_{cd;}{}^{k}{}_{el})\, x^{abcde},
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
&\overset{\circ}{R}{}_{ab;}{}^{k}{}_{cdke}\, x^{abcde} \\
&\overset{\star}{=} \Big(\overset{\circ}{R}{}_{ab;}{}^{k}{}_{ckde} + 2\ \overset{\circ}{R}{}^{l}{}_{abk;c}\ \overset{\circ}{R}{}_{lc;}{}^{k}{}_{e} + 2\ \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}{}_{lc;}{}^{k}{}_{de} \\
&\quad + \frac{1}{2}\ \overset{\circ}{R}{}_{ab;}{}^{l}\ \overset{\circ}{R}{}_{cd;}{}^{l}{}_{e} + \overset{\circ}{R}{}^{l}{}_{abk;c}\ \overset{\circ}{R}{}_{de;}{}^{k}{}_{l} \\
&\quad + \overset{\circ}{R}{}^{l}{}_{abk}\ \overset{\circ}{R}{}_{cd;}{}^{k}{}_{le} \Big)\, x^{abcde},
\end{aligned}
\tag{5.10}
$$

$$\overset{\circ}{R}_{ab;}{}^{k}{}_{ckde}\, x^{abcde}$$

$$\stackrel{\star}{=} (\overset{\circ}{R}_{ab;}{}^{k}{}_{kcde} - \overset{\circ}{R}{}^{l}{}_{abk;cd}\, \overset{\circ}{R}{}^{k}{}_{l;e} + 4\, \overset{\circ}{R}{}^{l}{}_{abk;c}\, \overset{\circ}{R}{}^{k}{}_{d;le}$$

$$+ 2\, \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}_{lc;}{}^{k}{}_{de} + \overset{\circ}{R}_{ab;}{}^{l}\, \overset{\circ}{R}_{cd;le} - \overset{\circ}{R}_{ab;}{}^{l}\, \overset{\circ}{R}_{cl;de})\, x^{abcde}. \quad (5.11)$$

We may combine (5.3) - (5.11) to obtain

$$\beta := 10\, \overset{\circ}{R}_{ab;(cdemn)}\, \overset{\circ}{g}{}^{mn}\, x^{abcde}$$

$$\stackrel{\star}{=} (33\, \overset{\circ}{R}_{ab;}{}^{l}\, \overset{\circ}{R}_{lc;de} + 8\, \overset{\circ}{R}_{ab;}{}^{l}\, \overset{\circ}{R}_{cd;le}$$

$$+ 36\, \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}_{lc;}{}^{k}{}_{de}$$

$$+ 18\, \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}_{lc;d}{}^{k}{}_{e} + 6\, \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}_{lc;de}{}^{k}$$

$$+ 12\, \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}_{cd;l}{}^{k}{}_{e}$$

$$+ 4\, \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}_{cd;le}{}^{k} + 4\, \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}_{cd;el}{}^{k}$$

$$+ 80\, \overset{\circ}{R}{}^{l}{}_{abk;c}\, \overset{\circ}{R}_{ld;}{}^{k}{}_{e}$$

$$+ 10\, \overset{\circ}{R}{}^{l}{}_{abk;c}\, \overset{\circ}{R}_{de;l}{}^{k} - 15\, \overset{\circ}{R}{}^{l}{}_{abk;cd}\, \overset{\circ}{R}_{l}{}^{k}{}_{;e}$$

$$+ 10\, \overset{\circ}{R}_{ab;}{}^{k}{}_{kcde})\, x^{abcde}. \quad (5.12)$$

Next, consider

$$\delta := 5\, \overset{\circ}{R}{}^{k}{}_{(k;abcde)}\, x^{abcde}. \quad (5.13)$$

Again, by the Ricci identity (B.1), our conformal gauge (A.1) - (A.14), and the contracted Bianchi identity (B.6) we have the following:

$$\overset{\circ}{R}{}^{k}{}_{a;bcdek}\, x^{abcde}$$

$$= (\overset{\circ}{R}{}^{k}{}_{a;bcdke} + \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{l;cde}$$

$$+ \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{c;lde} + \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{c;dle} + \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{c;del})\, x^{abcde}, \quad (5.14)$$

$$\overset{\circ}{R}{}^{k}{}_{a;bcdke}\, x^{abcde}$$
$$\overset{\star}{=} \Big(\overset{\circ}{R}{}^{k}{}_{a;bckde} + \frac{1}{2}\, \overset{\circ}{R}{}_{ab;}{}^{l}\, \overset{\circ}{R}{}_{lc;de}$$
$$+ \overset{\circ}{R}{}^{l}{}_{abk;c}\, \overset{\circ}{R}{}^{k}{}_{l;de} + \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{l;cde}$$
$$+ 2\, \overset{\circ}{R}{}^{l}{}_{abk;c}\, \overset{\circ}{R}{}^{k}{}_{d;le} + \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{c;lde}$$
$$+ \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{c;dle} \Big)\, x^{abcde}, \qquad (5.15)$$

$$\overset{\circ}{R}{}^{k}{}_{a;bckde}\, x^{abcde}$$
$$\overset{\star}{=} \Big(\overset{\circ}{R}{}^{k}{}_{a;bkcde} + \frac{3}{2}\, \overset{\circ}{R}{}_{ab;}{}^{l}\, \overset{\circ}{R}{}_{lc;de}$$
$$+ \frac{1}{2}\, \overset{\circ}{R}{}^{l}{}_{abk;cd}\, \overset{\circ}{R}{}^{k}{}_{l;e} + 2\, \overset{\circ}{R}{}^{l}{}_{abk;c}\, \overset{\circ}{R}{}^{k}{}_{l;de}$$
$$+ \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{l;cde} + 2\, \overset{\circ}{R}{}^{l}{}_{abk;c}\, \overset{\circ}{R}{}^{k}{}_{d;le}$$
$$+ \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{c;lde} \Big)\, x^{abcde}, \qquad (5.16)$$

$$\overset{\circ}{R}{}^{k}{}_{a;bkcde}\, x^{abcde}$$
$$\overset{\star}{=} \Big(\frac{1}{2}\, \overset{\circ}{R}{}_{;abcde} + 3\, \overset{\circ}{R}{}_{ab;}{}^{l}\, \overset{\circ}{R}{}_{la;bc}$$
$$+ 3\, \overset{\circ}{R}{}^{l}{}_{abk;cd}\, \overset{\circ}{R}{}^{k}{}_{l;e} + 3\, \overset{\circ}{R}{}^{l}{}_{abk;c}\, \overset{\circ}{R}{}^{k}{}_{l;de}$$
$$+ \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{l;cde} \Big)\, x^{abcde}. \qquad (5.17)$$

Thus, putting (5.14) - (5.17) into (5.13) we have

$$\delta \overset{\star}{=} (20\, \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{l;cde} + 12\, \overset{\circ}{R}{}^{l}{}_{abk}\, \overset{\circ}{R}{}^{k}{}_{c;lde}$$

$$+ 6 \overset{\circ}{R}{}^{l}{}_{abk} \overset{\circ}{R}{}^{k}{}_{c;dle} + 2 \overset{\circ}{R}{}^{l}{}_{abk} \overset{\circ}{R}{}^{k}{}_{c;del}$$

$$+ 35 \overset{\circ}{R}{}_{ab;}{}^{l} \overset{\circ}{R}{}_{lc;de} + 40 \overset{\circ}{R}{}^{l}{}_{abk;c} \overset{\circ}{R}{}^{k}{}_{l;de}$$

$$+ 20 \overset{\circ}{R}{}^{l}{}_{abk;c} \overset{\circ}{R}{}^{k}{}_{d;le} + 27 \overset{\circ}{R}{}^{l}{}_{abk;cd} \overset{\circ}{R}{}^{k}{}_{l;e}$$

$$+ 5 \overset{\circ}{R}{}_{;abcde}) \, x^{abcde}. \tag{5.18}$$

Combining equations (5.2), (5.12), and (5.18), and applying the third condition (1.6) we obtain for α

$$\alpha = \frac{1}{1680} \left(\beta + \delta - \frac{25}{3} \overset{\circ}{R}{}_{;abcde} \right) x^{abcde}$$

$$\overset{\star}{\equiv} \frac{1}{1680} \left(68 \overset{\circ}{R}{}_{ab;}{}^{l} \overset{\circ}{R}{}_{lc;de} + 8 \overset{\circ}{R}{}_{ab;}{}^{l} \overset{\circ}{R}{}_{cd;le} \right.$$

$$+40 \overset{\circ}{R}{}^{l}{}_{ab}{}^{k} \overset{\circ}{R}{}_{lk;cde}$$

$$+48 \overset{\circ}{R}{}^{l}{}_{ab}{}^{k} \overset{\circ}{R}{}_{lc;kde} + 24 \overset{\circ}{R}{}^{l}{}_{ab}{}^{k} \overset{\circ}{R}{}_{lc;dke}$$

$$+8 \overset{\circ}{R}{}^{l}{}_{ab}{}^{k} \overset{\circ}{R}{}_{lc;dek}$$

$$+12 \overset{\circ}{R}{}^{l}{}_{ab}{}^{k} \overset{\circ}{R}{}_{cd;lke} + 4 \overset{\circ}{R}{}^{l}{}_{ab}{}^{k} \overset{\circ}{R}{}_{cd;lek}$$

$$+4 \overset{\circ}{R}{}^{l}{}_{ab}{}^{k} \overset{\circ}{R}{}_{cd;elk}$$

$$+100 \overset{\circ}{R}{}^{l}{}_{ab}{}^{k}{}_{;c} \overset{\circ}{R}{}_{lk;de} + 100 \overset{\circ}{R}{}^{l}{}_{ab}{}^{k}{}_{;c} \overset{\circ}{R}{}_{ld;ke}$$

$$+10 \overset{\circ}{R}{}^{l}{}_{ab}{}^{k}{}_{;c} \overset{\circ}{R}{}_{de;lk} + 72 \overset{\circ}{R}{}^{l}{}_{ab}{}^{k}{}_{;cd} \overset{\circ}{R}{}_{lk;e}$$

$$\left. +200 \overset{\circ}{H}{}_{la;bcd} \overset{\circ}{H}{}^{l}{}_{e} + 600 \overset{\circ}{H}{}_{la;bc} \overset{\circ}{H}{}^{l}{}_{d;e} \right) x^{abcde} \mod g. \tag{5.19}$$

Once again, by the Ricci identity (B.1), our conformal gauge (A.1)-(A.14), and the contracted Bianchi identity (B.6) we have the following:

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{kc;del} \, x^{abcde}$$

$$\overset{\star}{=} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \left(\overset{\circ}{R}{}_{kc;dle} - \frac{1}{2} \overset{\circ}{R}{}^{m}{}_{kcl} \overset{\circ}{R}{}_{de;m} \right.$$

$$+\frac{1}{2}\ \overset{\circ}{R}{}^{m}{}_{cdl}\ \overset{\circ}{R}{}_{km;e}\Big)\, x^{abcde}, \tag{5.20}$$

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{kc;dle}\, x^{abcde}$$

$$\overset{\star}{=}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\Big(\overset{\circ}{R}{}_{kc;lde}-\frac{1}{2}\ \overset{\circ}{R}{}^{m}{}_{kcl}\ \overset{\circ}{R}{}_{de;m}$$

$$+\overset{\circ}{R}{}^{m}{}_{cdl}\ \overset{\circ}{R}{}_{km;e}\Big)\, x^{abcde}, \tag{5.21}$$

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{cd;ekl}\, x^{abcde}$$

$$\overset{\star}{=}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,(\overset{\circ}{R}{}_{cd;kel}-\overset{\circ}{R}{}^{m}{}_{cdl}\ \overset{\circ}{R}{}_{km;e})\, x^{abcde}, \tag{5.22}$$

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{cd;kel}\, x^{abcde}$$

$$\overset{\star}{=}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,(\overset{\circ}{R}{}_{cd;kle}+\overset{\circ}{R}{}^{m}{}_{kcl}\ \overset{\circ}{R}{}_{de;m}-\overset{\circ}{R}{}^{m}{}_{cdl}\ \overset{\circ}{R}{}_{km;e})\, x^{abcde}. \tag{5.23}$$

Finally, putting (5.20) - (5.23) into (5.19) we obtain for α

$$\alpha \overset{\star}{\equiv} \frac{1}{1680}\,(68\ \overset{\circ}{R}{}_{ab;}{}^{l}\ \overset{\circ}{R}{}_{lc;de}+8\ \overset{\circ}{R}{}_{ab;}{}^{l}\ \overset{\circ}{R}{}_{cd;le}$$

$$+40\ \overset{\circ}{R}{}^{l}{}_{ab}{}^{k}\ \overset{\circ}{R}{}_{lk;cde}$$

$$+80\ \overset{\circ}{R}{}^{l}{}_{ab}{}^{k}\ \overset{\circ}{R}{}_{lc;kde}+20\ \overset{\circ}{R}{}^{l}{}_{ab}{}^{k}\ \overset{\circ}{R}{}_{cd;lke}$$

$$+100\ \overset{\circ}{R}{}^{l}{}_{ab}{}^{k}{}_{;c}\ \overset{\circ}{R}{}_{lk;de}$$

$$+100\ \overset{\circ}{R}{}^{l}{}_{ab}{}^{k}{}_{;c}\ \overset{\circ}{R}{}_{ld;ke}+10\ \overset{\circ}{R}{}^{l}{}_{ab}{}^{k}{}_{;c}\ \overset{\circ}{R}{}_{de;lk}$$

$$+72\ \overset{\circ}{R}{}^{l}{}_{ab}{}^{k}{}_{;cd}\ \overset{\circ}{R}{}_{lk;e}$$

$$+12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{kmcl}\ \overset{\circ}{R}{}_{de;}{}^{m}+24\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{lcd}{}^{m}\ \overset{\circ}{R}{}_{km;e}$$

$$+200\ \overset{\circ}{H}{}_{la;bcd}\ \overset{\circ}{H}{}^{l}{}_{e}+600\ \overset{\circ}{H}{}_{la;bc}\ \overset{\circ}{H}{}^{l}{}_{d;e})\, x^{abcde} \mod g. \tag{5.24}$$

Next, consider the following terms of $\left(\gamma - \frac{2}{3}R\right)$ [5]

$$\varepsilon := \left(- \frac{1}{36} \overset{\circ}{R}{}^{k}{}_{(mn}{}^{l} \overset{\circ}{R}{}_{|k|ab|l|;cde)} \overset{\circ}{g}{}^{mn} - \frac{2}{27} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{(kl;cde)} \right) x^{abcde}. \tag{5.25}$$

First, we notice that our gauge (A.1) - (A.14), requires

$$\begin{aligned} 0 &\overset{\star}{=} \overset{\circ}{L}{}_{(kl;cde)} \\ &= \overset{\circ}{R}{}_{(kl;cde)} - \frac{1}{6} \overset{\circ}{g}{}_{(kl} \overset{\circ}{R}{}_{;cde)}, \end{aligned} \tag{5.26}$$

which implies

$$\begin{aligned} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{(kl;cde)} x^{abcde} &\overset{\star}{=} \frac{1}{60} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} (\overset{\circ}{g}{}_{kl} \overset{\circ}{R}{}_{;cde} + 6 \overset{\circ}{g}{}_{kc} \overset{\circ}{R}{}_{;lde} \\ &\quad + 3 \overset{\circ}{g}{}_{cd} \overset{\circ}{R}{}_{;kle}) x^{abcde} \\ &\overset{\star}{\equiv} 0 \qquad \mathrm{mod}\, g. \end{aligned} \tag{5.27}$$

Thus, (5.25) becomes

$$\begin{aligned} \varepsilon \overset{\star}{\equiv} \frac{1}{756} (&-2 \overset{\circ}{R}{}^{km}{}_{a}{}^{l} \overset{\circ}{R}{}_{kmbl;cde} - 2 \overset{\circ}{R}{}^{km}{}_{a}{}^{l} \overset{\circ}{R}{}_{kbml;cde} \\ &- 2 \overset{\circ}{R}{}^{km}{}_{a}{}^{l} \overset{\circ}{R}{}_{kbcl;mde} - 2 \overset{\circ}{R}{}^{km}{}_{a}{}^{l} \overset{\circ}{R}{}_{kbcl;dme} \\ &- 2 \overset{\circ}{R}{}^{km}{}_{a}{}^{l} \overset{\circ}{R}{}_{kbcl;dem} - \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{kl;cde} \\ &- 2 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{k}{}^{m}{}_{cl;mde} - 2 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{k}{}^{m}{}_{cl;dme} \\ &- 2 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{k}{}^{m}{}_{cl;dem} - \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{kcdl;}{}^{m}{}_{me} \\ &- \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{kcdl;}{}^{m}{}_{em} \\ &- 2 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{kcdl;e}{}^{m}{}_{m}) x^{abcde} \qquad \mathrm{mod}\, g. \end{aligned} \tag{5.28}$$

By the Ricci and cyclic identities (B.1) and (B.2) we have

$$\begin{aligned}
&\overset{\circ}{R}{}^{km}{}_{a}{}^{l}\ \overset{\circ}{R}{}_{kbcl;dem}\, x^{abcde}\\
&\overset{\star}{=}\overset{\circ}{R}{}^{km}{}_{a}{}^{l}\,(\overset{\circ}{R}{}_{kbcl;dme}+\overset{\circ}{R}{}^{n}{}_{kem}\ \overset{\circ}{R}{}_{nbcl;d}+\overset{\circ}{R}{}^{n}{}_{bem}\ \overset{\circ}{R}{}_{kncl;d}\\
&\quad+\overset{\circ}{R}{}^{n}{}_{cem}\ \overset{\circ}{R}{}_{kbnl;d}+\overset{\circ}{R}{}^{n}{}_{lem}\ \overset{\circ}{R}{}_{kbcn;d}\\
&\quad+\overset{\circ}{R}{}^{n}{}_{dem}\ \overset{\circ}{R}{}_{kbcl;n})\, x^{abcde}\\
&\overset{\star}{=}\overset{\circ}{R}{}^{km}{}_{a}{}^{l}\,(\overset{\circ}{R}{}_{kbcl;dme}+\overset{\circ}{R}{}^{n}{}_{kem}\ \overset{\circ}{R}{}_{nbcl;d}\\
&\quad+\overset{\circ}{R}{}^{n}{}_{bem}\ \overset{\circ}{R}{}_{kncl;d}-\overset{\circ}{R}{}^{n}{}_{cem}\ \overset{\circ}{R}{}_{nkbl;d}+\overset{\circ}{R}{}^{n}{}_{cem}\ \overset{\circ}{R}{}_{nbkl;d}\\
&\quad-\overset{\circ}{R}{}^{n}{}_{eml}\ \overset{\circ}{R}{}_{kbcn;d}+\overset{\circ}{R}{}^{n}{}_{mel}\ \overset{\circ}{R}{}_{kbcn;d}\\
&\quad+\overset{\circ}{R}{}^{n}{}_{dem}\ \overset{\circ}{R}{}_{kbcl;n})\, x^{abcde}. \qquad (5.29)
\end{aligned}$$

Similarly, making use also of the conformal gauge (A.1) - (A.14), and the contracted Bianchi identity (B.6) we have

$$\begin{aligned}
&\overset{\circ}{R}{}^{km}{}_{a}{}^{l}\ \overset{\circ}{R}{}_{kbcl;dme}\, x^{abcde}\\
&\overset{\star}{=}\overset{\circ}{R}{}^{km}{}_{a}{}^{l}\,(\overset{\circ}{R}{}_{kbcl;mde}+\overset{\circ}{R}{}^{n}{}_{kdm;e}\ \overset{\circ}{R}{}_{nbcl}\\
&\quad+\overset{\circ}{R}{}^{n}{}_{kdm}\ \overset{\circ}{R}{}_{nbcl;e}+\overset{\circ}{R}{}^{n}{}_{bdm;e}\ \overset{\circ}{R}{}_{kncl}+\overset{\circ}{R}{}^{n}{}_{bdm}\ \overset{\circ}{R}{}_{kncl;e}\\
&\quad+\overset{\circ}{R}{}^{n}{}_{cdm;e}\ \overset{\circ}{R}{}_{nbkl}+\overset{\circ}{R}{}^{n}{}_{cdm;e}\ \overset{\circ}{R}{}_{nbkl}+\overset{\circ}{R}{}^{n}{}_{cdm}\ \overset{\circ}{R}{}_{nkbl;e}\\
&\quad+\overset{\circ}{R}{}^{n}{}_{cdm}\ \overset{\circ}{R}{}_{nbkl;e}+\overset{\circ}{R}{}^{n}{}_{dml;e}\ \overset{\circ}{R}{}_{kbcn}+\overset{\circ}{R}{}^{n}{}_{mdl;e}\ \overset{\circ}{R}{}_{kbcn}\\
&\quad+\overset{\circ}{R}{}^{n}{}_{dml}\ \overset{\circ}{R}{}_{kbcn;e}+\overset{\circ}{R}{}^{n}{}_{mdl}\ \overset{\circ}{R}{}_{kbcn;e})\, x^{abcde}, \qquad (5.30)
\end{aligned}$$

$$\begin{aligned}
&\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{kmcl;de}{}^{m}\, x^{abcde}\\
&\overset{\star}{=}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,(\overset{\circ}{R}{}_{kmcl;d}{}^{m}{}_{e}+\overset{\circ}{R}{}^{n}{}_{ke}{}^{m}\ \overset{\circ}{R}{}_{nmcl;d}
\end{aligned}$$

$$+ \overset{\circ}{R}{}^{n}{}_{ce}{}^{m} \ \overset{\circ}{R}{}_{kmnl;d} + \overset{\circ}{R}{}^{n}{}_{e}{}^{m}{}_{l} \ \overset{\circ}{R}{}_{kmcn;d}$$

$$+ \overset{\circ}{R}{}^{nm}{}_{el} \ \overset{\circ}{R}{}_{kmcn;d} + \overset{\circ}{R}{}^{n}{}_{de}{}^{m} \ \overset{\circ}{R}{}_{kmcl;n}) \, x^{abcde}, \qquad (5.31)$$

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \ \overset{\circ}{R}{}_{kmcl;d}{}^{m}{}_{e} \, x^{abcde}$$

$$\overset{\star}{=} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \Big(\overset{\circ}{R}{}_{kl;cde} - \overset{\circ}{R}{}_{kc;lde} + \overset{\circ}{R}{}^{n}{}_{kd}{}^{m}{}_{;e} \ \overset{\circ}{R}{}_{nmcl}$$

$$+ \overset{\circ}{R}{}^{n}{}_{kd}{}^{m} \ \overset{\circ}{R}{}_{nmcl;e} + \frac{1}{2} \ \overset{\circ}{R}{}_{kncl} \ \overset{\circ}{R}{}_{de}{}^{n}$$

$$+ \overset{\circ}{R}{}^{n}{}_{cd}{}^{m}{}_{;e} \ \overset{\circ}{R}{}_{kmnl} + \overset{\circ}{R}{}^{n}{}_{cd}{}^{m} \ \overset{\circ}{R}{}_{kmnl;e}$$

$$+ \overset{\circ}{R}{}^{n}{}_{ld}{}^{m}{}_{;e} \ \overset{\circ}{R}{}_{kcnm} - \overset{\circ}{R}{}^{n}{}_{ld}{}^{m}{}_{;e} \ \overset{\circ}{R}{}_{kncm}$$

$$+ \overset{\circ}{R}{}^{n}{}_{ld}{}^{m} \ \overset{\circ}{R}{}_{kcnm;e} - \overset{\circ}{R}{}^{n}{}_{ld}{}^{m} \ \overset{\circ}{R}{}_{kncm;e}$$

$$+ \overset{\circ}{R}{}^{nm}{}_{el} \ \overset{\circ}{R}{}_{kmcn;d} + \overset{\circ}{R}{}^{n}{}_{de}{}^{m} \ \overset{\circ}{R}{}_{kmcl;n} \Big) \, x^{abcde}, \qquad (5.32)$$

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \ \overset{\circ}{R}{}_{kcdl;e}{}^{m}{}_{m} \, x^{abcde}$$

$$\overset{\star}{=} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \, (\overset{\circ}{R}{}_{kcdl;}{}^{m}{}_{em} + 2 \ \overset{\circ}{R}{}^{n}{}_{ke}{}^{m} \ \overset{\circ}{R}{}_{ncdl;m}$$

$$+3 \ \overset{\circ}{R}{}_{ce;}{}^{n} \ \overset{\circ}{R}{}_{kndl} + 2 \ \overset{\circ}{R}{}^{n}{}_{ce}{}^{m} \ \overset{\circ}{R}{}_{kndl;m}) \, x^{abcde}, \qquad (5.33)$$

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \ \overset{\circ}{R}{}_{kcdl;}{}^{m}{}_{em} \, x^{abcde}$$

$$\overset{\star}{=} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \, (\overset{\circ}{R}{}_{kcdl;}{}^{m}{}_{me} + 2 \ \overset{\circ}{R}{}^{n}{}_{ke}{}^{m} \ \overset{\circ}{R}{}_{ncdl;m}$$

$$+2 \ \overset{\circ}{R}{}^{n}{}_{ce}{}^{m} \ \overset{\circ}{R}{}_{kndl;m}) \, x^{abcde}. \qquad (5.34)$$

We may combine some of the above terms by noting that, using the cyclic identity (B.2)

$$R^k{}_{ab}{}^l\, R^n{}_{lem}\, R_{kmcn;d}\, x^{abcde} = R^k{}_{ab}{}^l\, (R^n{}_{mel} - R^n{}_{eml})\, R_{kmcn;d}\, x^{abcde}$$
$$= R^k{}_{ab}{}^l\, R^n{}_{lem}\, (R_{kncm;d} - R_{kcnm;d})\, x^{abcde},$$
$$\Rightarrow R^k{}_{ab}{}^l\, R_{lncm;e}\, R^n{}_d{}^m{}_k\, x^{abcde} = R^k{}_{ab}{}^l\, R_{lncm}\, R^n{}_d{}^m{}_{k;e}\, x^{abcde}. \tag{5.35}$$

Combining (5.25)-(5.35) we obtain for ε

$$\begin{aligned}
\varepsilon \overset{\star}{\equiv} \frac{1}{756} (&-2\, \mathring{R}^{km}{}_a{}^l\, \mathring{R}_{kmbl;cde} - 2\, \mathring{R}^{km}{}_a{}^l\, \mathring{R}_{kbml;cde} \\
&- 6\, \mathring{R}^{km}{}_a{}^l\, \mathring{R}_{kbcl;mde} \\
&- 3\, \mathring{R}^k{}_{ab}{}^l\, \mathring{R}_{kcdl;}{}^m{}_{me} - 7\, \mathring{R}^k{}_{ab}{}^l\, \mathring{R}_{kl;cde} + 6\, \mathring{R}^k{}_{ab}{}^l\, \mathring{R}_{kc;lde} \\
&+ 16\, \mathring{R}^k{}_{ab}{}^l{}_{;e}\, \mathring{R}_{lcmn}\, \mathring{R}^n{}_d{}^m{}_k - 6\, \mathring{R}^k{}_{ab}{}^l\, \mathring{R}_{lcmn}\, \mathring{R}^n{}_d{}^m{}_{k;e} \\
&- 14\, \mathring{R}^k{}_{ab}{}^l{}_{;e}\, \mathring{R}_{lmcn}\, \mathring{R}^n{}_d{}^m{}_k - 6\, \mathring{R}^k{}_{ab}{}^l\, \mathring{R}_{lmcn}\, \mathring{R}^n{}_d{}^m{}_{k;e} \\
&- 4\, \mathring{R}^k{}_{ab}{}^l{}_{;e}\, \mathring{R}^m{}_{cd}{}^n\, \mathring{R}_{kmnl} - 6\, \mathring{R}^k{}_{ab}{}^l\, \mathring{R}^m{}_{cd}{}^n\, \mathring{R}_{kmnl;e} \\
&- 5\, \mathring{R}^k{}_{ab}{}^l\, \mathring{R}_{kncl;}\, \mathring{R}_{de;}{}^n - 2\, \mathring{R}^k{}_{ab}{}^l{}_{;n}\, \mathring{R}^m{}_{cd}{}^n\, \mathring{R}_{kmel} \\
&- 8\, \mathring{R}^k{}_{ab}{}^l\, \mathring{R}^m{}_{cd}{}^n\, \mathring{R}_{kmel;n} \\
&- 6\, \mathring{R}^k{}_{ab}{}^l\, \mathring{R}_{lcdn;m}\, \mathring{R}^n{}_{ke}{}^m)\, x^{abcde} \qquad \text{mod } g.
\end{aligned} \tag{5.36}$$

Using the identities (B.12) and (B.14) we may combine the following terms

$$\begin{aligned}
&-2\, \mathring{R}^{kml}{}_a\, \mathring{R}_{kmlb;cde} - 2\, \mathring{R}^{kml}{}_a\, \mathring{R}_{lmkb;cde} \\
&+6\, \mathring{R}^{kml}{}_a\, \mathring{R}_{kbcl;mde} = -6\, \mathring{R}^{kml}{}_a\, \mathring{R}_{kmlb;cde},
\end{aligned} \tag{5.37}$$

and using the Bianchi identity (B.3), the Ricci identity (B.1), the cyclic identity (B.2), the conformal gauge (A.1)-(A.14), and the contracted Bianchi identity (B.6)

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{kcdl;}{}^{m}{}_{me}\, x^{abcde}$$

$$=\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,(\overset{\circ}{R}{}_{kcdm;l}{}^{m}{}_{e}-\overset{\circ}{R}{}_{kclm;d}{}^{m}{}_{e})\, x^{abcde}$$

$$=\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,(\overset{\circ}{R}{}_{kclm;}{}^{m}{}_{de}+[-R^{n}{}_{l}\,R_{ndck}+R^{n}{}_{cl}{}^{m}\,R_{mndk}$$
$$+(R_{ld}{}^{nm}-R_{l}{}^{n}{}_{d}{}^{m})\,R_{mcnk}+R^{n}{}_{kl}{}^{m}R_{mcdn}]_{;e}|_{0}$$
$$-\overset{\circ}{R}{}_{mlck;}{}^{m}{}_{de}-[-R^{n}{}_{d}\,R_{nlck}+R^{n}{}_{ld}{}^{m}\,R_{mnck}$$
$$+R^{n}{}_{cd}{}^{m}\,R_{mlnk}+(R^{nm}{}_{dk}-R^{n}{}_{d}{}^{m}{}_{k})\,R_{mlcn}]_{;e}|_{0})\, x^{abcde}$$
$$\overset{\star}{=}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,(\overset{\circ}{R}{}_{kl;cde}-2\,\overset{\circ}{R}{}_{kc;lde}+\overset{\circ}{R}{}_{cd;kle}$$
$$-2\,\overset{\circ}{R}{}_{lcd}{}^{n}\,\overset{\circ}{R}{}_{kn;e}+8\,\overset{\circ}{R}{}_{lcm}\,\overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e}$$
$$-4\,\overset{\circ}{R}{}_{lmcn}\,\overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e}+2\,\overset{\circ}{R}{}^{m}{}_{cd}{}^{n}{}_{;e}\,\overset{\circ}{R}{}_{kmnl}$$
$$+2\,\overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\,\overset{\circ}{R}{}_{kmnl;e})\, x^{abcde}. \tag{5.38}$$

Combining (5.36)-(5.38) one obtains

$$\varepsilon \overset{\star}{\equiv} \frac{1}{756}\,(-6\,\overset{\circ}{R}{}^{kml}{}_{a}\,\overset{\circ}{R}{}_{kmlb;cde}-10\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{R}{}_{kl;cde}$$
$$+12\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{R}{}_{kc;lde}$$
$$-3\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{R}{}_{cd;kle}+16\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e}\,\overset{\circ}{R}{}^{l}{}_{cmn}\,\overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k}$$
$$-30\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{R}{}^{l}{}_{cmn}\,\overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e}-14\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e}\,\overset{\circ}{R}{}^{l}{}_{mcn}\,\overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k}$$
$$+6\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{R}{}^{l}{}_{mcn}\,\overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e}-10\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e}\,\overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\,\overset{\circ}{R}{}_{kmnl}$$
$$-12\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\,\overset{\circ}{R}{}_{kmnl;e}-5\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{R}{}_{kncl}\,\overset{\circ}{R}{}_{de;}{}^{n}$$
$$+6\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{R}{}_{lcd}{}^{n}\,\overset{\circ}{R}{}_{kn;e}-2\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;n}\,\overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\,\overset{\circ}{R}{}_{kmel}$$
$$-8\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\,\overset{\circ}{R}{}_{kmel;n}$$
$$-6\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{R}{}_{lcdn;m}\,\overset{\circ}{R}{}^{n}{}_{ke}{}^{m})\, x^{abcde} \qquad \mathrm{mod}\ g. \tag{5.39}$$

The third group of terms $\left(\gamma - \frac{2}{3}R\right)$ [5] that we wish to consider are

$$\zeta := \left(-\frac{3}{40} \overset{\circ}{R}{}^{k}{}_{(mn}{}^{l}{}_{;a} \overset{\circ}{R}{}_{|k|bc|l|;de)} \overset{\circ}{g}{}^{mn} - \frac{1}{10} \overset{\circ}{R}{}^{k}{}_{(ab}{}^{l}{}_{;k)} \overset{\circ}{R}{}_{(lc;de)} - \frac{1}{10} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{(kl;de)} - \frac{1}{20} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;cd} \overset{\circ}{R}{}_{(kl;e)} - \frac{1}{10} \overset{\circ}{R}{}^{k}{}_{(la}{}^{l}{}_{;bc)} \overset{\circ}{R}{}_{(kd;e)} \right) x^{abcde}. \tag{5.40}$$

Clearly, the last two terms in (5.40) vanish due to the conformal gauge (A.1)-(A.14). Expanding the symmetrisations and making further use of the gauge we obtain

$$\begin{aligned} \zeta \overset{\star}{=} \frac{1}{840} (&-3 \overset{\circ}{R}{}^{kl}{}_{;a} \overset{\circ}{R}{}_{kbcl;de} - 6 \overset{\circ}{R}{}^{km}{}_{a}{}^{l}{}_{;}{}^{m} \overset{\circ}{R}{}_{kbcl;de} \\ &- 6 \overset{\circ}{R}{}^{km}{}_{a}{}^{l}{}_{;b} \overset{\circ}{R}{}_{kmcl;de} - 6 \overset{\circ}{R}{}^{km}{}_{a}{}^{l}{}_{;b} \overset{\circ}{R}{}_{kcml;de} \\ &- 6 \overset{\circ}{R}{}^{km}{}_{a}{}^{l}{}_{;a} \overset{\circ}{R}{}_{kcdl;me} \\ &- 6 \overset{\circ}{R}{}^{km}{}_{a}{}^{l}{}_{;b} \overset{\circ}{R}{}_{kcdl;em} - 6 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m} \overset{\circ}{R}{}_{kmcl;de} \\ &- 3 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m} \overset{\circ}{R}{}_{kcdl;me} \\ &- 3 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m} \overset{\circ}{R}{}_{kcdl;em} - 17 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{kl;de} \\ &- 6 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{k}{}^{m}{}_{dl;me} \\ &- 6 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{k}{}^{m}{}_{dl;em} - 3 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{kdel;m}{}^{m} \\ &- 28 \overset{\circ}{R}{}_{ab;}{}^{l} \overset{\circ}{R}{}_{lc;de} \\ &- 28 \overset{\circ}{R}{}_{ab;}{}^{l} \overset{\circ}{R}{}_{cd;le} - 56 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{kd;le} \\ &- 14 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{de;kl}) x^{abcde}. \end{aligned} \tag{5.41}$$

Once more we use the Ricci identity (B.1), and the cyclic identity (B.2), to obtain

$$\overset{\circ}{R}{}^{km}{}_{a}{}^{l}{}_{;b}\ \overset{\circ}{R}{}_{kcdl;em}\ x^{abcde}$$

$$\overset{\star}{=}\overset{\circ}{R}{}^{km}{}_{a}{}^{l}{}_{;b}\,(\overset{\circ}{R}{}_{kcdl;me}+\overset{\circ}{R}{}^{n}{}_{kem}\ \overset{\circ}{R}{}_{ncdl}$$

$$+\overset{\circ}{R}{}^{n}{}_{cem}\ \overset{\circ}{R}{}_{kndl}-\overset{\circ}{R}{}^{n}{}_{dem}\ \overset{\circ}{R}{}_{nkcl}$$

$$+\overset{\circ}{R}{}^{n}{}_{dem}\ \overset{\circ}{R}{}_{nckl}-\overset{\circ}{R}{}^{n}{}_{eml}\ \overset{\circ}{R}{}_{kcdn}+\overset{\circ}{R}{}^{n}{}_{mel}\ \overset{\circ}{R}{}_{kcdn})\,x^{abcde}, \qquad (5.42)$$

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \overset{\circ}{R}{}_{kcdl;em}\ x^{abcde}$$

$$\overset{\star}{=}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\,(\overset{\circ}{R}{}_{kcdl;me}+2\ \overset{\circ}{R}{}^{n}{}_{kem}\ \overset{\circ}{R}{}_{ncdl}$$

$$+\overset{\circ}{R}{}^{n}{}_{cem}\ \overset{\circ}{R}{}_{kndl})\,x^{abcde} \qquad (5.43)$$

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\ \overset{\circ}{R}{}_{kmdl;e}{}^{m}\ x^{abcde}$$

$$\overset{\star}{=}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\,(\overset{\circ}{R}{}_{kmdl;}{}^{m}{}_{e}+\overset{\circ}{R}{}^{n}{}_{ke}{}^{m}\ \overset{\circ}{R}{}_{nmdl}$$

$$+\overset{\circ}{R}{}^{n}{}_{de}{}^{m}\ \overset{\circ}{R}{}_{kmnl}-\overset{\circ}{R}{}^{n}{}_{e}{}^{m}{}_{l}\ \overset{\circ}{R}{}_{kmdn}$$

$$+\overset{\circ}{R}{}^{nm}{}_{el}\ \overset{\circ}{R}{}_{kmdn})\,x^{abcde}. \qquad (5.44)$$

From (B.12) and (B.14) we have

$$-6\ \overset{\circ}{R}{}^{kml}{}_{a;b}\ \overset{\circ}{R}{}_{kmlc;de}-6\ \overset{\circ}{R}{}^{kml}{}_{a;b}\ \overset{\circ}{R}{}_{lmkc;de}$$

$$-12\ \overset{\circ}{R}{}^{km}{}_{a}{}^{l}{}_{;b}\ \overset{\circ}{R}{}_{kcdl;me}-6\ \overset{\circ}{R}{}^{km}{}_{a}{}^{l}{}_{;bc}\ \overset{\circ}{R}{}_{kdel;m}$$

$$=-18\ \overset{\circ}{R}{}^{kml}{}_{a;b}\ \overset{\circ}{R}{}_{kmlc;de}. \qquad (5.45)$$

Also, from the contracted Bianchi identity (B.6) and the conformal gauge (A.1)-(A.14) we have

$$\overset{\circ}{R}{}_{kmdl;}{}^{m}{}_{e} = \overset{\circ}{R}{}_{kl;de} - \overset{\circ}{R}{}_{kd;le}, \tag{5.46}$$

$$\overset{\circ}{R}{}_{kmal;}{}^{m} \overset{\star}{=} \frac{3}{2} \overset{\circ}{R}{}_{kl;a}. \tag{5.47}$$

Using a similar argument to that for (5.38) we obtain

$$\begin{aligned} &\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e} \overset{\circ}{R}{}_{kcdl;m}{}^{m} x^{abcde} \\ &\quad \overset{\star}{=} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e} (\overset{\circ}{R}{}_{kl;cd} - 2 \overset{\circ}{R}{}_{kc;ld} + \overset{\circ}{R}{}_{cd;kl} + 2 \overset{\circ}{R}{}^{m}{}_{cd}{}^{n} \overset{\circ}{R}{}_{kmnl} \\ &\qquad + 4 \overset{\circ}{R}{}_{lcmn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k} - 2 \overset{\circ}{R}{}_{lmcn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k}) x^{abcde}. \end{aligned} \tag{5.48}$$

Thus, we have for ζ

$$\begin{aligned} \zeta \overset{\star}{=} \frac{1}{840} (&-18 \overset{\circ}{R}{}^{kml}{}_{a;b} \overset{\circ}{R}{}_{kmlc;de} - 32 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{kl;de} \\ &- 38 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{kd;le} - 17 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{de;kl} \\ &- 12 \overset{\circ}{R}{}_{kabl;cd} \overset{\circ}{R}{}^{kl}{}_{;e} - 28 \overset{\circ}{R}{}_{ab;}{}^{l} \overset{\circ}{R}{}_{lc;de} - 28 \overset{\circ}{R}{}_{ab;}{}^{l} \overset{\circ}{R}{}_{cd;le} \\ &- 24 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e} \overset{\circ}{R}{}_{lcmn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k} + 18 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{lcmn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e} \\ &+ 12 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e} \overset{\circ}{R}{}_{lmcn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k} - 18 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{lmcn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e} \\ &- 12 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e} \overset{\circ}{R}{}^{m}{}_{cd}{}^{n} \overset{\circ}{R}{}_{kmnl} - 6 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;m} \overset{\circ}{R}{}^{m}{}_{cd}{}^{n} \overset{\circ}{R}{}_{knel} \\ &- 6 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m} \overset{\circ}{R}{}_{lcd}{}^{n} \overset{\circ}{R}{}_{nkem} - 6 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m} \overset{\circ}{R}{}_{kcdl;me}) x^{abcde}. \end{aligned} \tag{5.49}$$

Finally, we collect the remaining terms of $\left(\gamma - \frac{2}{3}R\right)$ [5] together into the term

$$\begin{aligned} \eta := \Bigg(&\frac{2}{45} \overset{\circ}{R}{}^{k}{}_{(st|l|} \overset{\circ}{R}{}^{l}{}_{ab|m} \overset{\circ}{R}{}^{m}{}_{cd|k|;e)} \overset{\circ}{g}{}^{st} \\ &+ \frac{2}{27} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}^{m}{}_{(kl|n|} \overset{\circ}{R}{}^{n}{}_{cd|m|;e)} \end{aligned}$$

$$+\frac{2}{45}\ \mathring{R}^{k}{}_{(ab}{}^{l}\ \mathring{R}_{(km|n|l|}\ \mathring{R}^{m}{}_{cd}{}^{n}{}_{;e)}$$
$$+\frac{2}{9}\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{l(km}{}^{m}\ \mathring{R}_{cd;e)}$$
$$+\frac{1}{90}\ \mathring{R}^{km}{}_{a}{}^{l}\ \mathring{R}_{kbcl}\ \mathring{R}_{(md;e)}\Big)\ x^{abcde}. \tag{5.50}$$

Again, the last two terms vanish identically due to our choice of gauge, and we may expand the symmetrisation brackets and use the cyclic identity (B.2) to obtain

$$\eta \overset{\star}{=} \frac{1}{945}\,(\mathring{R}^{k}{}_{ab}{}^{l}{}_{;e}\ \mathring{R}_{lcmn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k} + 2\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{lcmn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k;e}$$
$$+ 16\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;e}\ \mathring{R}_{lmcn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k} + 32\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{lmcn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k;e}$$
$$+ 14\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;e}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{kmnl} + 7\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{kmnl;e}$$
$$+ 8\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{lcd}{}^{n}\ \mathring{R}_{kn;e} + 14\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{kmcl}\ \mathring{R}_{de;}{}^{m}$$
$$+ 4\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \mathring{R}_{lcd}{}^{n}\ \mathring{R}_{nmek} + 4\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \mathring{R}_{lcd}{}^{n}\ \mathring{R}_{nemk}$$
$$+ 14\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;m}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{knel}$$
$$+ 14\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{knel;m})\ x^{abcde}. \tag{5.51}$$

Combining (5.1), (5.24), (5.39), (5.49), and (5.51) we have $\left(\gamma - \frac{2}{3}R\right)[5] \overset{\star}{=} (\alpha + \varepsilon + \zeta + \eta)$

$$\overset{\star}{\equiv} \frac{1}{15120}\,(108\ \mathring{R}_{ab;}{}^{k}\ \mathring{R}_{kc;de} -$$
$$- 432\ \mathring{R}_{ab;}{}^{k}\ \mathring{R}_{cd;ke} + 160\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{kl;cde}$$
$$+ 960\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{kc;lde} + 120\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{cd;kle}$$
$$+ 324\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;c}\ \mathring{R}_{kl;de} + 216\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;c}\ \mathring{R}_{kd;le}$$
$$- 216\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;c}\ \mathring{R}_{de;kl} + 432\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;cd}\ \mathring{R}_{kl;e}$$

$$
\begin{aligned}
&- 108\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \mathring{R}_{kcdl;me} - 120\ \mathring{R}^{kml}{}_{a}\ \mathring{R}_{kmlb;cde}\\
&- 324\ \mathring{R}^{kml}{}_{a;b}\ \mathring{R}_{kmlc;de} + 232\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{kncl}\ \mathring{R}_{de;}{}^{n}\\
&+ 464\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{lcd}{}^{n}\ \mathring{R}_{kn;e} - 96\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;e}\ \mathring{R}_{lcmn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k}\\
&- 244\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{lcmn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k;e} + 192\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;e}\ \mathring{R}_{lmcn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k}\\
&+ 308\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{lmcn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k;e} - 192\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;e}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{kmnl}\\
&- 128\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{kmnl;e} - 12\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \mathring{R}_{lcd}{}^{n}\ \mathring{R}_{knem}\\
&+ 76\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;m}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{knel} + 64\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{knel;m}\\
&+ 64\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \mathring{R}_{lcd}{}^{n}\ \mathring{R}_{nmek} + 64\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \mathring{R}_{lcd}{}^{n}\ \mathring{R}_{nemk}\\
&+ 1800\ \mathring{H}_{la;bcd}\ \mathring{H}^{l}{}_{e} + 5400\ \mathring{H}_{la;bc}\ \mathring{H}^{l}{}_{d;e})\, x^{abcde} \quad \mathrm{mod}\ g. \qquad (5.52)
\end{aligned}
$$

Equation (5.52) can be reduced somewhat by comparing it to the covariant derivative of the fifth condition (1.8) at the origin of normal coordinates. To convert the fifth condition to Riemann and Ricci tensors, we will consider one term at a time. During this conversion, repeated use will be made of the conformal gauge (A.1)-(A.14). It will be assumed that all tensors are symmetric with respect to the indices a, b, c, d and e and that $\overset{\star}{\equiv}$ refers to equivalence mod g (with respect to the special gauge). We may begin with

$$
\begin{aligned}
3\,(C^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\, C_{kcdl;m})_{;e}|_0 &= 6\ \mathring{C}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}{}_{e}\ \mathring{C}_{kcdl;m}\\
&\overset{\star}{\equiv} 6\left(\mathring{R}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}{}_{e} + \frac{1}{2}\,(\mathring{g}^{kl}\ \mathring{L}_{ab;}{}^{m}{}_{e} - \delta^{k}_{b}\ \mathring{L}_{a}{}^{l}{}_{;}{}^{m}{}_{e}\right.\\
&\qquad \left. - \delta^{l}_{a}\ \mathring{L}^{k}{}_{b;}{}^{m}{}_{e} + \mathring{g}_{ab}\ \mathring{L}^{kl}{}_{;}{}^{m}{}_{e})\right)\\
&\qquad \times \left(\mathring{R}_{kabl;m} + \frac{1}{2}\,(-\ \mathring{g}_{kl}\ \mathring{R}_{ab;m} + \mathring{g}_{kb}\ \mathring{R}_{al;m}\right.\\
&\qquad \left. + \mathring{g}_{al}\ \mathring{R}_{kb;m} - \mathring{g}_{ab}\ \mathring{R}_{kl;m})\right)\\
&\overset{\star}{\equiv} 6\,(\mathring{R}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}{}_{e}\ \mathring{R}_{kcdl;m} - \frac{1}{2}\ \mathring{R}_{ab;}{}^{m}\ \mathring{R}_{cd;em}). \qquad (5.53)
\end{aligned}
$$

Likewise

$$
\begin{aligned}
&8\,(C^{k}{}_{ab}{}^{l}{}_{;c}\,S_{kld})_{;e}\,|_0\\
&= 8\,(\mathring{C}^{k}{}_{ab}{}^{l}{}_{;cd}\,\mathring{L}_{k[l;e]} + \mathring{C}^{k}{}_{ab}{}^{l}{}_{;c}\,\mathring{L}_{k[l;d]e})\\
&\stackrel{\star}{=} 8\,(\mathring{R}^{k}{}_{ab}{}^{l}{}_{;cd}\\
&\quad + \frac{1}{2}\,(\mathring{g}^{kl}\,\mathring{L}_{ab;cd} - \delta^k_b\,\mathring{L}_a{}^{l}{}_{;cd}\\
&\quad - \delta^l_a\,\mathring{L}^{k}{}_{b;cd} + \mathring{g}_{ab}\,\mathring{L}^{kl}{}_{;cd}))\left(-\frac{3}{4}\right)\mathring{R}_{kl;e}\\
&\quad + \delta^l_a\,\mathring{R}^{k}{}_{b;c} - \mathring{g}_{ab}\,\mathring{R}^{kl}{}_{;c}))\ \ \frac{1}{2}\,(\mathring{L}_{kl;de} - \mathring{L}_{kd;le})\\
&\stackrel{\star}{\equiv} (-6\,\mathring{R}^{k}{}_{ab}{}^{l}{}_{;cd}\,\mathring{R}_{kl;e} - 4\,\mathring{R}^{k}{}_{ab}{}^{l}{}_{;c}\,\mathring{R}_{kl;de} + 4\,\mathring{R}^{k}{}_{ab}{}^{l}{}_{;c}\,\mathring{R}_{kd;le}\\
&\quad - \mathring{R}_{ab;}{}^{k}\,\mathring{R}_{cd;ke} + 4\,\mathring{R}_{ab;}{}^{k}\,\mathring{R}_{kc;de}),
\end{aligned}
\tag{5.54}
$$

$$
\begin{aligned}
-8\,(C^{k}{}_{ab}{}^{l}\,S_{klc;d})_{;e}\,|_0 &= -8\,(\mathring{C}^{k}{}_{ab}{}^{l}{}_{;c}\,\mathring{L}_{k[l;d]e} + \mathring{C}^{k}{}_{ab}{}^{l}\,\mathring{L}_{k[l;c]de}\\
&\stackrel{\star}{=} -8\,(\mathring{R}^{k}{}_{ab}{}^{l}{}_{;c} + \frac{1}{2}\,(-\,\mathring{g}^{kl}\,\mathring{R}_{ab;c} + \delta^k_b\,\mathring{R}_a{}^{l}{}_{;c}\\
&\quad + \delta^l_a\,\mathring{R}^{k}{}_{b;c} - \mathring{g}_{ab}\,\mathring{R}^{kl}{}_{;c}))\,\frac{1}{2}\,(\mathring{L}_{kl;de} - \mathring{L}_{kd;le})\\
&\quad - 4\,\mathring{R}^{k}{}_{ab}{}^{l}\,(\mathring{L}_{kl;cde} - \mathring{L}_{kc;lde})\\
&\stackrel{\star}{\equiv} (4\,\mathring{R}^{k}{}_{ab}{}^{l}{}_{;c}\,\mathring{R}_{kl;de} - 4\,\mathring{R}^{k}{}_{ab}{}^{l}{}_{;c}\,\mathring{R}_{kd;le}\\
&\quad + 4\,\mathring{R}^{k}{}_{ab}{}^{l}\,\mathring{R}_{kl;cde} - 4\,\mathring{R}^{k}{}_{ab}{}^{l}\,\mathring{R}_{kc;lde}\\
&\quad + \mathring{R}_{ab;}{}^{k}\,\mathring{R}_{cd;ke} - \mathring{R}_{ab;}{}^{k}\,\mathring{R}_{kc;de}),
\end{aligned}
\tag{5.55}
$$

$$
\begin{aligned}
&-\,40\,(S_{ab}{}^{k}\,S_{cdk})_{;e}\,|_0 - 80\,\mathring{S}_{ab}{}^{k}\,\mathring{S}_{cdk;e}\\
&\stackrel{\star}{=} -20\,(\mathring{L}_{ab;}{}^{k} - \mathring{L}_a{}^{k}{}_{;b})\,(\mathring{L}_{cd;ke} - \mathring{L}_{ck;de})\\
&\stackrel{\star}{\equiv} -30\,(\mathring{R}_{ab;}{}^{k}\,\mathring{R}_{kc;de} - \mathring{R}_{ab;}{}^{k}\,\mathring{R}_{cd;ke}),
\end{aligned}
\tag{5.56}
$$

$$
\begin{aligned}
&- 24\,(c^{k}{}_{ab}{}^{l}\, S_{cdk;l})_{;e}\,|_0 \\
&\quad = -24\,(\overset{\circ}{C}{}^{k}{}_{ab}{}^{l}{}_{;c}\ \overset{\circ}{L}{}_{d[e;k]l} + \overset{\circ}{C}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{L}{}_{c[d;k]le}) \\
&\quad \overset{\star}{=} -24\,(\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} + \frac{1}{2}\,(-\overset{\circ}{g}{}^{kl}\ \overset{\circ}{R}{}_{ab;c} + \delta^{k}_{b}\ \overset{\circ}{R}{}_{a}{}^{l}{}_{;c} \\
&\qquad + \delta^{l}_{a}\ \overset{\circ}{R}{}^{k}{}_{b;c} - \overset{\circ}{g}{}_{ab}\ \overset{\circ}{R}{}^{kl}{}_{;c}))\ \frac{1}{2}\,(\overset{\circ}{L}{}_{de;kl} - \overset{\circ}{L}{}_{dk;el}) \\
&\qquad - 12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,(\overset{\circ}{L}{}_{cd;kle} - \overset{\circ}{L}{}_{ck;dle}) \\
&\overset{\star}{\equiv} (12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\ \overset{\circ}{R}{}_{de;kl} - 12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\ \overset{\circ}{R}{}_{dk;el} \\
&\quad + 12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{cd;kle} - 12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{ck;lde} \\
&\quad + 6\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{lmck}\ \overset{\circ}{R}{}_{de;}{}^{m} + 12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{lcd}{}^{m}\ \overset{\circ}{R}{}_{km;e} \\
&\quad + 3\ \overset{\circ}{R}{}_{ab;}{}^{k}\ \overset{\circ}{R}{}_{cd;ke} - 3\ \overset{\circ}{R}{}_{ab;}{}^{k}\ \overset{\circ}{R}{}_{kc;de}),
\end{aligned}
\tag{5.57}
$$

$$
4\,(C^{k}{}_{ab}{}^{l}\, C_{lmck}\, L_{d}{}^{m})_{;e}\,|_0 \overset{\star}{=} 2\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{lmck}\ \overset{\circ}{R}{}_{de;}{}^{m},
\tag{5.58}
$$

$$
12\,(C^{k}{}_{ab}{}^{l}\, C_{lcd}{}^{m}\, L_{km})_{;e}\,|_0 \overset{\star}{=} -12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{lcd}{}^{m}\ \overset{\circ}{R}{}_{km;e}.
\tag{5.59}
$$

Thus, the covariant derivative of the fifth condition at the origin of the normal coordinate system is

$$
\begin{aligned}
0 \overset{\star}{\equiv}\ & 24\ \overset{\circ}{R}{}_{ab;}{}^{k}\ \overset{\circ}{R}{}_{cd;ke} - 24\ \overset{\circ}{R}{}_{ab;}{}^{k}\ \overset{\circ}{R}{}_{kc;de} - 4\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{kl;cde} \\
& - 16\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{kc;lde} + 12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{cd;kle} - 12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\ \overset{\circ}{R}{}_{dk;el} \\
& + 12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\ \overset{\circ}{R}{}_{de;kl} - 6\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;cd}\ \overset{\circ}{R}{}_{kl;e} + 6\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \overset{\circ}{R}{}_{kcdl;me} \\
& - 4\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{kmcl}\ \overset{\circ}{R}{}_{de;}{}^{m} - 24\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{lcd}{}^{m}\ \overset{\circ}{R}{}_{km;e} \\
& + 12\ \overset{\circ}{H}{}^{k}{}_{a;bcd}\ \overset{\circ}{H}{}^{k}{}_{e} - 20\ \overset{\circ}{H}{}^{k}{}_{a;bc}\ \overset{\circ}{H}{}^{k}{}_{d;e} \\
& - 168\ \overset{\circ}{H}{}_{ka}\ \overset{\circ}{C}{}^{k}{}_{bc}{}^{l}\ \overset{\circ}{H}{}_{ld;e} - 84\ \overset{\circ}{H}{}_{ka}\ \overset{\circ}{C}{}^{k}{}_{bc}{}^{l}{}_{;d}\ \overset{\circ}{H}{}_{le} \quad \mathrm{mod}\, g.
\end{aligned}
\tag{5.60}
$$

Multiplying (5.60) by $\frac{18}{15120}$ and adding it to (5.52), we obtain for $\left(\gamma - \frac{2}{3}R\right)$ [5]

$$
\begin{aligned}
&\overset{\star}{\equiv} \frac{1}{15120}\,(-324\ \mathring{R}_{ab;}{}^{k}\ \mathring{R}_{kc;de} + 112\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{kl;cde} \\
&+ 672\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{kc;lde} + 336\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{cd;kle} \\
&+ 324\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;c}\ \mathring{R}_{kl;de} + 324\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;cd}\ \mathring{R}_{kl;e} \\
&- 120\ \mathring{R}^{kml}{}_{a}\ \mathring{R}_{kmlb;cde} - 324\ \mathring{R}^{kml}{}_{a;b}\ \mathring{R}_{kmlc;de} \\
&+ 160\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{kncl}\ \mathring{R}_{de;}{}^{n} + 32\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{lcd}{}^{n}\ \mathring{R}_{kn;e} \\
&- 96\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;e}\ \mathring{R}_{lcmn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k} - 244\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{lcmn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k;e} \\
&+ 192\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;e}\ \mathring{R}_{lmcn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k} + 308\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}_{lmcn}\ \mathring{R}^{n}{}_{d}{}^{m}{}_{k;e} \\
&- 192\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;e}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{kmnl} - 128\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{kmnl;e} \\
&+ 76\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;m}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{knel} + 64\ \mathring{R}^{k}{}_{ab}{}^{l}\ \mathring{R}^{m}{}_{cd}{}^{n}\ \mathring{R}_{knel;m} \\
&+ 76\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \mathring{R}_{lcd}{}^{n}\ \mathring{R}_{nmek} + 52\ \mathring{R}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \mathring{R}_{lcd}{}^{n}\ \mathring{R}_{nemk} \\
&+ 2016\ \mathring{H}_{la;bcd}\ \mathring{H}^{l}{}_{e} + 5040\ \mathring{H}_{la;bc}\ \mathring{H}^{l}{}_{d;e} \\
&- 3024\ \mathring{H}_{ka}\ \mathring{C}^{k}{}_{bc}{}^{l}\ \mathring{H}_{id;e} \\
&- 1512\ \mathring{H}_{ka}\ \mathring{C}^{k}{}_{bc}{}^{l}{}_{;d}\ \mathring{H}_{le})\, x^{abcde} \quad \mathrm{mod}\, g. \qquad (5.61)
\end{aligned}
$$

Next, consider (B.25)

$$\mathring{C}^{klm}{}_{a;bc}\ \mathring{C}_{klmd;e}\, x^{abcde} \equiv 0 \quad \mathrm{mod}\, g, \qquad (5.62)$$

$$
\begin{aligned}
&\Rightarrow 0 \overset{\star}{\equiv} \mathring{R}^{klm}{}_{a;bc} + \frac{1}{2}\,[\delta^{k}_{a}\ \mathring{L}^{lm}{}_{;bc} - \mathring{g}^{km}\ \mathring{L}^{l}{}_{a;bc} \\
&- \delta^{l}_{a}\ \mathring{L}^{km}{}_{;bc} + \mathring{g}^{lm}\ \mathring{L}^{k}{}_{a;bc}])
\end{aligned}
$$

$$\times \left(\overset{\circ}{R}{}^{klm}{}_{d;e} + \frac{1}{2} [-\delta^k_d \overset{\circ}{R}{}^{lm}{}_{;e} + \overset{\circ}{g}{}^{km} \overset{\circ}{R}{}^{l}{}_{d;e} + \delta^l_d \overset{\circ}{R}{}^{km}{}_{;e} - \overset{\circ}{g}{}^{lm} \overset{\circ}{R}{}^{k}{}_{d;e}] \right) x^{abcde} \mod g$$

$$\overset{\star}{\equiv} (\overset{\circ}{R}{}^{klm}{}_{a;bc} \overset{\circ}{R}{}_{klmd;e} - \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{kl;de} - \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;cd} \overset{\circ}{R}{}_{kl;e} + \overset{\circ}{R}{}_{ab;}{}^{k} \overset{\circ}{R}{}_{kc;de}) x^{abcde} \mod g. \quad (5.63)$$

However, from (B.23) we have

$$(2 \overset{\circ}{R}{}^{klm}{}_{a;bcd} \overset{\circ}{R}{}_{klme} + 6 \overset{\circ}{R}{}^{klm}{}_{a;bc} \overset{\circ}{R}{}_{klmd;e}) x^{abcde}$$
$$\overset{\star}{\equiv} (6 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;cd} \overset{\circ}{R}{}_{kl;e} + 6 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{kl;de} + 2 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{kl;cde} - 6 \overset{\circ}{R}{}_{ab;}{}^{k} \overset{\circ}{R}{}_{kc;de}) x^{abcde} \mod g. \quad (5.64)$$

Combining (5.63) and (5.64) we have

$$\overset{\circ}{R}{}^{klm}{}_{a} \overset{\circ}{R}{}_{klmb;cde} x^{abcde} \overset{\star}{\equiv} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{kl;cde} x^{abcde} \mod g. \quad (5.65)$$

Furthermore, by the Bianchi identity we have

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m} \overset{\circ}{R}{}_{lcd}{}^{n} \overset{\circ}{R}{}_{nemk} x^{abcde} = (- \overset{\circ}{R}{}^{k}{}_{a}{}^{m}{}_{b;}{}^{l} \overset{\circ}{R}{}_{lcd}{}^{n} \overset{\circ}{R}{}_{nemk} - \overset{\circ}{R}{}^{k}{}_{a}{}^{lm}{}_{;b} \overset{\circ}{R}{}_{lcd}{}^{n} \overset{\circ}{R}{}_{nemk}) x^{abcde}$$
$$= (0 + \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{lcmn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e}) x^{abcde}, \quad (5.66)$$

and

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}^{n}{}_{cd}{}^{m} \overset{\circ}{R}{}_{knel;m} x^{abcde}$$
$$= \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}^{n}{}_{cd}{}^{m} (\overset{\circ}{R}{}_{kmnl;e} + \overset{\circ}{R}{}_{knem;l}) x^{abcde}, \quad (5.67)$$

$$\Rightarrow \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\, \overset{\circ}{R}{}^{n}{}_{cd}{}^{m}\, \overset{\circ}{R}{}_{knel;m}\, x^{abcde}$$
$$= \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\, \overset{\circ}{R}{}^{n}{}_{cd}{}^{m}\, \overset{\circ}{R}{}_{kmnl;e}\, x^{abcde}. \tag{5.68}$$

Using the identities (5.63), (5.65), and (5.68), equation (5.61) becomes

$$\left(\gamma - \frac{2}{3} R\right)[5]$$
$$\overset{\star}{\equiv} \frac{1}{15120}\Big(112\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\, \overset{\circ}{R}{}_{kl;cde} + 672\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\, \overset{\circ}{R}{}_{kc;lde}$$
$$+ 336\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\, \overset{\circ}{R}{}_{cd;kle} + 160\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\, \overset{\circ}{R}{}_{kncl}\, \overset{\circ}{R}{}_{de;}{}^{n}$$
$$+ 32\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\, \overset{\circ}{R}{}_{lcd}{}^{n}\, \overset{\circ}{R}{}_{kn;e} - 96\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e}\, \overset{\circ}{R}{}_{lcmn}\, \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k}$$
$$- 192\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\, \overset{\circ}{R}{}_{lcmn}\, \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e} + 192\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e}\, \overset{\circ}{R}{}_{lmcn}\, \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k}$$
$$+ 308\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\, \overset{\circ}{R}{}_{lmcn}\, \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e} - 192\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e}\, \overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\, \overset{\circ}{R}{}_{kmnl}$$
$$- 96\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\, \overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\, \overset{\circ}{R}{}_{kmnl;e} + 76\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;m}\, \overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\, \overset{\circ}{R}{}_{knel}$$
$$+ 76\, \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\, \overset{\circ}{R}{}_{lcd}{}^{n}\, \overset{\circ}{R}{}_{nmek} + 2016\, \overset{\circ}{H}{}_{la;bcd}\, \overset{\circ}{H}{}^{l}{}_{e}$$
$$+ 5040\, \overset{\circ}{H}{}_{la;bc}\, \overset{\circ}{H}{}^{l}{}_{d;e} - 3024\, \overset{\circ}{H}{}_{ka}\, \overset{\circ}{C}{}^{k}{}_{bc}{}^{l}\, \overset{\circ}{H}{}_{ld;e}$$
$$- 1512\, \overset{\circ}{H}{}_{ka}\, \overset{\circ}{C}{}^{k}{}_{bc}{}^{l}{}_{;d}\, \overset{\circ}{H}{}_{le}\Big)\, x^{abcde} \quad \bmod g. \tag{5.69}$$

We may further simplify by again using our conformal gauge

$$0 \overset{\star}{\equiv} \overset{\circ}{L}{}_{(kl;cde)}\, x^{cde}$$
$$\overset{\star}{\equiv} \frac{1}{10}\Big(\overset{\circ}{L}{}_{(kl);cde} + 2\, \overset{\circ}{L}{}_{(k|c|;l)le} + 2\, \overset{\circ}{L}{}_{(k|c;d|l)e}$$
$$+ 2\, \overset{\circ}{L}{}_{(k|c;de|l)} + \overset{\circ}{L}{}_{cd;(kl)e} + \overset{\circ}{L}{}_{cd;(k|e|l)} + \overset{\circ}{L}{}_{cd;e(kl)}\Big)\, x^{cde}, \tag{5.70}$$

which implies, when we use (5.20)-(5.23)

$$0 \stackrel{\star}{\equiv} (\overset{\circ}{R}_{kl;cde} + 6\ \overset{\circ}{R}_{kc;lde} + 3\ \overset{\circ}{R}_{cd;kle}$$
$$+ 2\ \overset{\circ}{R}_{lcd}{}^{n}\ \overset{\circ}{R}_{kn;e} + \overset{\circ}{R}_{knel}\ \overset{\circ}{R}_{cd;}{}^{n})\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\, x^{abcde} \quad \mathrm{mod}\, g. \quad (5.71)$$

Equation (B.34) implies

$$-\frac{1}{6}\, RC^{k}{}_{(ab}{}^{l}\, C_{|k|cd,l} \equiv C^{k}{}_{(ab}{}^{l}\, C_{|l|cd)}{}^{m}\, L_{km}$$
$$+ C^{k}{}_{(ab}{}^{l}\, C_{|k|c|l|}{}^{m}\, L_{d)m}, \ \mathrm{mod}\, g, \quad (5.72)$$

$$\Rightarrow \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}_{lcd}{}^{m}\ \overset{\circ}{R}_{km;e}\, x^{abcde}$$
$$\stackrel{\star}{\equiv} \frac{1}{2}\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}_{kcml}\ \overset{\circ}{R}_{de;}{}^{m}\, x^{abcde} \quad \mathrm{mod}\, g. \quad (5.73)$$

From the cyclic and Bianchi identities (B.2) and (B.3) and (5.66) we get

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;m}\ \overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\ \overset{\circ}{R}_{knel}\, x^{abcde}$$
$$\stackrel{\star}{=} (\overset{\circ}{R}{}^{k}{}_{a}{}^{l}{}_{m;b} + \overset{\circ}{R}{}^{k}{}_{abm;}{}^{l})\ \overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\ \overset{\circ}{R}_{knel}\, x^{abcde}$$
$$\stackrel{\star}{=} (-\ \overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\,(-\ \overset{\circ}{R}_{knel} + \overset{\circ}{R}_{klen})\ \overset{\circ}{R}{}^{k}{}_{a}{}^{l}{}_{m;b}$$
$$-\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \overset{\circ}{R}_{lcd}{}^{n}\,(-\ \overset{\circ}{R}_{nemk} + \overset{\circ}{R}_{nmek}))\, x^{abcde}$$
$$\stackrel{\star}{=} (\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}_{lmcn}\ \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e} - \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}_{lcmn}\ \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e}$$
$$+ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \overset{\circ}{R}_{lcd}{}^{n}\ \overset{\circ}{R}_{nemk} - \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \overset{\circ}{R}_{lcd}{}^{n}\ \overset{\circ}{R}_{nmek})\, x^{abcde}, \quad (5.74)$$

$$\Rightarrow \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}_{lmcn}\ \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e}\, x^{abcde}$$
$$\stackrel{\star}{=} (\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;m}\ \overset{\circ}{R}{}^{m}{}_{cd}{}^{n}\ \overset{\circ}{R}_{knel} + \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;}{}^{m}\ \overset{\circ}{R}_{lcd}{}^{n}\ \overset{\circ}{R}_{nmek})\, x^{abcde}. \quad (5.75)$$

In view of (5.71), (5.73), and (5.75) (5.69) becomes

$$\left(\gamma - \frac{2}{3} R\right) [5]$$

$$\begin{aligned} &\overset{\star}{\equiv} \frac{1}{15120} (48 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{kncl} \overset{\circ}{R}{}_{de;}{}^{n} + 192 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{lcd}{}^{n} \overset{\circ}{R}{}_{kn;e} - \\ &- 96 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e} \overset{\circ}{R}{}_{lcmn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k} - 192 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{lcmn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e} + \\ &+ 192 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e} \overset{\circ}{R}{}_{lmcn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k} + 384 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{lmcn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k;e} - \\ &- 192 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;e} \overset{\circ}{R}{}^{m}{}_{cd}{}^{n} \overset{\circ}{R}{}_{kmnl} - 96 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}^{m}{}_{cd}{}^{n} \overset{\circ}{R}{}_{kmnl;e} + \\ &+ 2016 \overset{\circ}{H}{}_{la;bcd} \overset{\circ}{H}{}^{l}{}_{e} + 5040 \overset{\circ}{H}{}_{la;bc} \overset{\circ}{H}{}^{l}{}_{d;e} - \\ &- 3024 \overset{\circ}{H}{}_{ka} \overset{\circ}{C}{}^{k}{}_{bc}{}^{l} \overset{\circ}{H}{}_{ld;e} - \\ &- 1512 \overset{\circ}{H}{}_{ka} \overset{\circ}{C}{}^{k}{}_{bc}{}^{l}{}_{;d} \overset{\circ}{H}{}_{le}) x^{abcde} \quad \mathrm{mod}\, g. \end{aligned} \tag{5.76}$$

Finally, we can further reduce (5.76) by using (B.38). The pertinent equations are

$$C^{k}{}_{(ab}{}^{l} C_{|l|c|mn|} C^{n}{}_{d)}{}^{m}{}_{k} \equiv \frac{1}{2} C^{k}{}_{(ab}{}^{l} C_{|lm|c|n|} C^{n}{}_{d)}{}^{m}{}_{k} \quad \mathrm{mod}\, g, \tag{5.77}$$

$$\begin{aligned} &\Rightarrow \\ &(\overset{\circ}{C}{}^{k}{}_{ab}{}^{l}{}_{;e} \overset{\circ}{R}{}_{lcmn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k} \\ &+ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{C}{}_{lcmn;e} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k} + \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{lcmn} \overset{\circ}{C}{}^{n}{}_{d}{}^{m}{}_{k;e}) x^{abcde} \\ &\overset{\star}{\equiv} \left(\frac{1}{2} \overset{\circ}{C}{}^{k}{}_{ab}{}^{l}{}_{;e} \overset{\circ}{R}{}_{lmcn} \overset{\circ}{R}{}^{n}{}_{d}{}^{m}{}_{k} \right. \\ &\left. + \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{lmcn} \overset{\circ}{C}{}^{n}{}_{d}{}^{m}{}_{k;e}\right) x^{abcde} \quad \mathrm{mod}\, g, \end{aligned} \tag{5.78}$$

and

$$C^k{}_{(ab}{}^l C_{|l|c|mn|} C^n{}_{d)}{}^m{}_k \equiv \frac{1}{3} C^k{}_{(ab}{}^l C^m{}_{cd)}{}^n C_{kmnl} \mod g, \tag{5.79}$$

$$\Rightarrow$$

$$\begin{aligned} &(\overset{\circ}{C}{}^k{}_{ab}{}^l{}_{;e} \overset{\circ}{R}{}_{lcmn} \overset{\circ}{R}{}^n{}_d{}^m{}_k \\ &+ \overset{\circ}{R}{}^k{}_{ab}{}^l \overset{\circ}{C}{}_{lcmn;e} \overset{\circ}{R}{}^n{}_d{}^m{}_k + \overset{\circ}{R}{}^k{}_{ab}{}^l \overset{\circ}{R}{}_{lcmn} \overset{\circ}{C}{}^n{}_d{}^m{}_{k;e}) x^{abcde} \\ &\quad \overset{\star}{\equiv} \left(\frac{2}{3} \overset{\circ}{C}{}^k{}_{ab}{}^l{}_{;e} \overset{\circ}{R}{}^m{}_{cd}{}^n \overset{\circ}{R}{}_{kmnl} \right. \\ &\qquad \left. + \frac{1}{3} \overset{\circ}{R}{}^k{}_{ab}{}^l \overset{\circ}{R}{}^m{}_{cd}{}^n \overset{\circ}{C}{}_{kmnl;e}\right) x^{abcde} \mod g, \end{aligned} \tag{5.80}$$

where the covariant derivatives of the Weyl tensor may be expanded as

$$\overset{\circ}{C}{}^k{}_{ab}{}^l{}_{;e} \overset{\star}{\equiv} \overset{\circ}{R}{}^k{}_{ab}{}^l{}_{;e} + \frac{1}{2} (\delta^k_b \overset{\circ}{R}{}_a{}^l{}_{;e} + \delta^l_a \overset{\circ}{R}{}^k{}_{b;e}) \mod g, \tag{5.81}$$

$$\begin{aligned} \overset{\circ}{C}{}_{lcmn;e} \overset{\star}{\equiv} & \overset{\circ}{R}{}_{lcmn;e} + \frac{1}{2} (- \overset{\circ}{g}{}_{ln} \overset{\circ}{R}{}_{cm;e} + \overset{\circ}{g}{}_{lm} \overset{\circ}{R}{}_{cn;e} \\ & + \overset{\circ}{g}{}_{cn} \overset{\circ}{R}{}_{lm;e} - \overset{\circ}{g}{}_{cm} \overset{\circ}{R}{}_{ln;e}) \mod g, \end{aligned} \tag{5.82}$$

$$\begin{aligned} \overset{\circ}{C}{}^n{}_d{}^m{}_{k;e} \overset{\star}{\equiv} & \overset{\circ}{R}{}^n{}_d{}^m{}_{k;e} + \frac{1}{2} (-\delta^n_k \overset{\circ}{R}{}_d{}^m{}_{;e} + \overset{\circ}{g}{}^{nm} \overset{\circ}{R}{}_{dk;e} \\ & + \overset{\circ}{g}{}_{dk} \overset{\circ}{R}{}^{mn}{}_{;e} - \delta^m_d \overset{\circ}{R}{}_k{}^n{}_{;e}) \mod g, \end{aligned} \tag{5.83}$$

$$\begin{aligned} \overset{\circ}{C}{}_{kmnl;e} \overset{\star}{\equiv} & \overset{\circ}{R}{}_{kmnl;e} + \frac{1}{2} (- \overset{\circ}{g}{}_{kl} \overset{\circ}{R}{}_{mn;e} + \overset{\circ}{g}{}_{km} \overset{\circ}{R}{}_{ml;e} \\ & + \overset{\circ}{g}{}_{ml} \overset{\circ}{R}{}_{kn;e} - \overset{\circ}{g}{}_{mn} \overset{\circ}{R}{}_{kl;e}) \mod g. \end{aligned} \tag{5.84}$$

Equations (5.77)-(5.80) together with (5.81)-(5.84) gives us

$$\begin{aligned}0 \overset{\star}{\equiv} & [(C^k{}_{ab}{}^l\, C_{lmcn}\, C^n{}_d{}^m{}_k)_{;e}\,|_0 - 2\,(C^k{}_{ab}{}^l\, C_{lcmn}\, C^n{}_d{}^m{}_k)_{;e}\,|_0 \\ & + (C^k{}_{ab}{}^l\, C^m{}_{cd}{}^n\, C_{kmnl})_{;e}\,|_0]\, x^{abcde} \quad \bmod g \\ \overset{\star}{\equiv} & (\overset{\circ}{R}{}^k{}_{ab}{}^l{}_{;e}\, \overset{\circ}{R}{}_{lcmn}\, \overset{\circ}{R}{}^n{}_d{}^m{}_k + 2\, \overset{\circ}{R}{}^k{}_{ab}{}^l\, \overset{\circ}{R}{}_{lcmn}\, \overset{\circ}{R}{}^n{}_d{}^m{}_{k;e} \\ & - 2\, \overset{\circ}{R}{}^k{}_{ab}{}^l{}_{;e}\, \overset{\circ}{R}{}_{lmcn}\, \overset{\circ}{R}{}^n{}_d{}^m{}_k - 4\, \overset{\circ}{R}{}^k{}_{ab}{}^l\, \overset{\circ}{R}{}_{lmcn}\, \overset{\circ}{R}{}^n{}_d{}^m{}_{k;e} \\ & + 2\, \overset{\circ}{R}{}^k{}_{ab}{}^l{}_{;e}\, \overset{\circ}{R}{}^m{}_{cd}{}^n\, \overset{\circ}{R}{}_{kmnl} + \overset{\circ}{R}{}^k{}_{ab}{}^l\, \overset{\circ}{R}{}^m{}_{cd}{}^n\, \overset{\circ}{R}{}_{kmnl;e} - \\ & - \frac{1}{2}\, \overset{\circ}{R}{}^k{}_{ab}{}^l\, \overset{\circ}{R}{}_{kncl}\, \overset{\circ}{R}{}_{de;}{}^n \\ & + 2\, \overset{\circ}{R}{}^k{}_{ab}{}^l\, \overset{\circ}{R}{}_{lcd}{}^n\, \overset{\circ}{R}{}_{kn;e})\, x^{abcde} \bmod g. \end{aligned} \tag{5.85}$$

So finally, combining (5.76) and (5.85) we have

$$\begin{aligned}\left(\gamma - \frac{2}{3} R\right) [5] \overset{\star}{\equiv} & \frac{1}{15120} (2016\, \overset{\circ}{H}{}_{la;bcd}\, \overset{\circ}{H}{}^l{}_e + 5040\, \overset{\circ}{H}{}_{la;bc}\, \overset{\circ}{H}{}^l{}_{d;e} \\ & - 3024\, \overset{\circ}{H}{}_{ka}\, \overset{\circ}{C}{}^k{}_{bc}{}^l\, \overset{\circ}{H}{}_{ld;e} \\ & - 1512\, \overset{\circ}{H}{}_{ka}\, \overset{\circ}{C}{}^k{}_{bc}{}^l{}_{;d}\, \overset{\circ}{H}{}_{le})\, x^{abcde} \bmod g. \end{aligned} \tag{5.86}$$

5.2. $A^a{}_{,a}$ and $A_a A^a$ to fifth order

The remaining terms to be calculated in σ [5] are the terms involving the Maxwell vector A_a. Using (4.13) and (4.14) we find

$$\begin{aligned}A^k A_k\, [5] = & (g_{kl}\, A^k\, A^l)\, [5] \\ \overset{\star}{\equiv} & \left(\frac{1}{3}\, \overset{\circ}{H}{}_{ka;bc}\, \overset{\circ}{H}{}^k{}_{d;e} + \frac{2}{15}\, \overset{\circ}{H}{}_{ka;bcd}\, \overset{\circ}{H}{}^k{}_e \right. \\ & - \frac{1}{5}\, \overset{\circ}{H}{}_{ka}\, \overset{\circ}{R}{}^k{}_{bc}{}^l\, \overset{\circ}{H}{}_{id;e} \\ & \left. - \frac{1}{10}\, \overset{\circ}{H}{}_{ka}\, \overset{\circ}{R}{}^k{}_{bc}{}^l{}_{;d}\, \overset{\circ}{H}{}_{le} \right) x^{abcde}. \end{aligned} \tag{5.87}$$

Recalling (A.1)-(A.14) and (5.81), and that H_{ab} is anti-symmetric, we may rewrite (5.87) as

$$A^k A_k [5] \stackrel{\star}{\equiv} \Big(\frac{1}{3} \overset{\circ}{H}{}_{ka;bc} \overset{\circ}{H}{}^k{}_{d;e} + \frac{2}{15} \overset{\circ}{H}{}_{ka;bcd} \overset{\circ}{H}{}^k{}_e - \frac{1}{5} \overset{\circ}{H}{}_{ka} \overset{\circ}{C}{}^k{}_{bc}{}^l \overset{\circ}{H}{}_{id;e} - \frac{1}{10} \overset{\circ}{H}{}_{ka} \overset{\circ}{C}{}^k{}_{bc}{}^l{}_{;d} \overset{\circ}{H}{}_{le} \Big) x^{abcde} \bmod g. \tag{5.88}$$

The expansion of the $A^a{}_{,a}$ term is a bit more involved. For example, the first term on the right hand side of (4.14) will give rise, when differentiated with respect to the free index, to a term of the form $\overset{\circ}{H}{}^k{}_{(k;abcde)}$. The expansion of such terms may be carried out using the Ricci identity (B.1) is a manner similar to that used in the previous section. The pertinent expansions are

$$\overset{\circ}{H}{}^k{}_{a;bcdek}\, x^{abcde} \stackrel{\star}{=} (\overset{\circ}{H}{}^k{}_{a;bcdke} + \overset{\circ}{R}{}^l{}_{abk} \overset{\circ}{H}{}^k{}_{c;lde} + \overset{\circ}{R}{}^l{}_{abk} \overset{\circ}{H}{}^k{}_{c;dle} + \overset{\circ}{R}{}^l{}_{abk} \overset{\circ}{H}{}^k{}_{c;del})\, x^{abcde}, \tag{5.89}$$

$$\overset{\circ}{H}{}^k{}_{a;bcdke}\, x^{abcde} \stackrel{\star}{=} \Big(\overset{\circ}{H}{}^k{}_{a;bckde} + \frac{1}{2} \overset{\circ}{R}{}_{ab;k} \overset{\circ}{H}{}^k{}_{c;de} + \overset{\circ}{R}{}^l{}_{abk;c} \overset{\circ}{H}{}^k{}_{d;le} + \overset{\circ}{R}{}^l{}_{abk} \overset{\circ}{H}{}^k{}_{c;lde} + \overset{\circ}{R}{}^l{}_{abk;c} \overset{\circ}{H}{}^k{}_{d;le} + \overset{\circ}{R}{}^l{}_{abk} \overset{\circ}{H}{}^k{}_{c;dle} \Big) x^{abcde}, \tag{5.90}$$

$$\overset{\circ}{H}{}^k{}_{a;bckde}\, x^{abcde} \stackrel{\star}{=} (\overset{\circ}{H}{}^k{}_{a;bkcde} - \overset{\circ}{R}{}_{ka;bc} \overset{\circ}{H}{}^k{}_{d;e} + \overset{\circ}{R}{}_{ab;k} \overset{\circ}{H}{}^k{}_{c;de} + \overset{\circ}{R}{}^l{}_{abk;cd} \overset{\circ}{H}{}^k{}_{e;l} + 2\, \overset{\circ}{R}{}^l{}_{abk;c} \overset{\circ}{H}{}^k{}_{d;le} + \overset{\circ}{R}{}^l{}_{abk} \overset{\circ}{H}{}^k{}_{c;lde})\, x^{abcde}, \tag{5.91}$$

$$\overset{\circ}{H}{}^{k}{}_{a;bkcde}\,x^{abcde} \overset{\star}{=} \left(\overset{\circ}{R}{}_{ka;bcd}\,\overset{\circ}{H}{}^{k}{}_{e} - 3\,\overset{\circ}{R}{}_{ka;bc}\,\overset{\circ}{H}{}^{k}{}_{d;e} + \frac{3}{2}\,\overset{\circ}{R}{}_{ab;k}\,\overset{\circ}{H}{}^{k}{}_{c;de}\right) x^{abcde}. \tag{5.92}$$

Taking the partial derivative of (4.14) and using (5.89)-(5.92) we have

$$\begin{aligned} A^{k}{}_{,k}[5] \overset{\star}{=} \frac{1}{2520}\,(&-18\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{H}{}_{kc;dle} - 30\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{H}{}_{kc;del} - 90\,\overset{\circ}{R}{}_{ab;k}\,\overset{\circ}{H}{}^{k}{}_{c;de} \\ &- 27\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\,\overset{\circ}{H}{}_{kd;le} - 63\,\overset{\circ}{R}{}^{k}{}_{abl;c}\,\overset{\circ}{H}{}_{kd;el} \\ &+ 24\,\overset{\circ}{R}{}^{k}{}_{a;bc}\,\overset{\circ}{H}{}_{kd;e} - 48\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;cd}\,\overset{\circ}{H}{}_{ke;l} \\ &+ 62\,\overset{\circ}{R}{}^{k}{}_{abl}\,\overset{\circ}{R}{}_{lckm}\,\overset{\circ}{H}{}^{m}{}_{d;e} + 62\,\overset{\circ}{R}{}^{k}{}_{abl}\,\overset{\circ}{R}{}_{lcdm}\,\overset{\circ}{H}{}_{ke;}{}^{m} \\ &+ 60\,\overset{\circ}{R}{}^{k}{}_{abl}\,\overset{\circ}{R}{}_{lckm;d}\,\overset{\circ}{H}{}^{m}{}_{e} + 60\,\overset{\circ}{R}{}^{k}{}_{abl}\,\overset{\circ}{R}{}_{lcdm;k}\,\overset{\circ}{H}{}^{m}{}_{e} \\ &+ 102\,\overset{\circ}{R}{}_{ab;}{}^{k}\,\overset{\circ}{R}{}_{kcdl}\,\overset{\circ}{H}{}^{l}{}_{e} + 51\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\,\overset{\circ}{R}{}_{ldkm}\,\overset{\circ}{H}{}^{m}{}_{e} \\ &+ 9\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\,\overset{\circ}{R}{}_{lde}{}^{m}\,\overset{\circ}{H}{}_{km} - 24\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;kcd}\,\overset{\circ}{H}{}_{le} \\ &- 24\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;ckd}\,\overset{\circ}{H}{}_{le} - 24\,\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;cdk}\,\overset{\circ}{H}{}_{le})\,x^{abcde}, \end{aligned} \tag{5.93}$$

where we have once again taken advantage of the fact that due to our gauge (B.6) we have $\overset{\circ}{R}{}_{k(a;b)} \overset{\star}{=} \frac{1}{2}\,\overset{\circ}{R}{}_{ab;k}$. We may now make further use of the Ricci identity (B.1) to obtain further simplification of (5.93) as follows

$$\begin{aligned} &\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{H}{}_{kc;del}\,x^{abcde} \\ &\quad\overset{\star}{=} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,(\overset{\circ}{H}{}_{kc;dle} + \overset{\circ}{R}{}^{m}{}_{kel}\,\overset{\circ}{H}{}_{mc;d} + \overset{\circ}{R}{}^{m}{}_{del}\,\overset{\circ}{H}{}_{kc;m})\,x^{abcde}, \end{aligned} \tag{5.94}$$

$$\begin{aligned} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,\overset{\circ}{H}{}_{kc;dle}\,x^{abcde} \overset{\star}{=} \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\,(&\overset{\circ}{H}{}_{kc;lde} + \overset{\circ}{R}{}^{m}{}_{kdl;e}\,\overset{\circ}{H}{}_{mc} \\ &+ \overset{\circ}{R}{}^{m}{}_{kel}\,\overset{\circ}{H}{}_{mc;d} + \overset{\circ}{R}{}^{m}{}_{cdl;e}\,\overset{\circ}{H}{}_{km})\,x^{abcde}, \end{aligned} \tag{5.95}$$

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\ \overset{\circ}{H}{}_{kd;el}\, x^{abcde}$$
$$\stackrel{\star}{=}\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}(\overset{\circ}{H}{}_{kd;le}+\overset{\circ}{R}{}^{m}{}_{kel}\ \overset{\circ}{H}{}_{md}+\overset{\circ}{R}{}^{m}{}_{del}\ \overset{\circ}{H}{}_{km})\, x^{abcde}, \quad (5.96)$$

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;cdk}\ \overset{\circ}{H}{}_{le}\, x^{abcde}$$
$$\stackrel{\star}{=}\overset{\circ}{H}{}_{le}\ (\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;ckd}+\overset{\circ}{R}{}^{m}{}_{abk}\ \overset{\circ}{R}{}^{k}{}_{cm}{}^{l}{}_{;d}$$
$$+\overset{\circ}{R}{}^{ml}{}_{ak}\ \overset{\circ}{R}{}^{k}{}_{bcm;d}+\overset{\circ}{R}{}^{m}{}_{abk}\ \overset{\circ}{R}{}^{k}{}_{cd}{}^{l}{}_{;m})\, x^{abcde}, \quad (5.97)$$

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;ckd}\ \overset{\circ}{H}{}_{le;}\, x^{abcde}$$
$$\stackrel{\star}{=}\overset{\circ}{H}{}_{le}\left(\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;kcd}+\frac{1}{2}\ \overset{\circ}{R}{}_{ab;m}\ \overset{\circ}{R}{}^{m}{}_{cd}{}^{l}\right)x^{abcde}. \quad (5.98)$$

Equations (5.94)-(5.98) and (B.6) combine with (5.93) to give

$$A^{k}{}_{,k}\,[5] \stackrel{\star}{=} \frac{1}{2520}\,(-48\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{H}{}_{kc;lde} - 90\ \overset{\circ}{R}{}_{ab;k}\ \overset{\circ}{H}{}^{k}{}_{c;de}$$
$$-\,90\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\ \overset{\circ}{H}{}_{kd;le} + 240\ \overset{\circ}{R}{}^{k}{}_{a;bc}\ \overset{\circ}{H}{}_{kd;e}$$
$$-\,48\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;cd}\ \overset{\circ}{H}{}_{ke;l} + 72\ \overset{\circ}{R}{}^{k}{}_{a;bcd}\ \overset{\circ}{H}{}_{ke}$$
$$-\,72\ \overset{\circ}{R}{}_{ab;}{}^{k}{}_{cd}\ \overset{\circ}{H}{}_{ke} + 16\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{lckm}\ \overset{\circ}{H}{}^{m}{}_{d;e}$$
$$+\,32\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{lcdm}\ \overset{\circ}{H}{}_{ke;}{}^{m} + 12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{lckm;d}\ \overset{\circ}{H}{}^{m}{}_{e}$$
$$+\,36\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}\ \overset{\circ}{R}{}_{lcdm;k}\ \overset{\circ}{H}{}^{m}{}_{e} + 78\ \overset{\circ}{R}{}_{ab;}{}^{k}\ \overset{\circ}{R}{}_{kcdl}\ \overset{\circ}{H}{}^{l}{}_{e}$$
$$-\,12\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\ \overset{\circ}{R}{}_{ldkm}\ \overset{\circ}{H}{}^{m}{}_{e}$$
$$-\,6\ \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c}\ \overset{\circ}{R}{}_{lde}{}^{m}\ \overset{\circ}{H}{}_{km})\, x^{abcde}. \quad (5.99)$$

Equations (5.99) may be further simplified by considering the second covariant derivative of the fourth necessary condition (1.7)

$$
\begin{aligned}
0 &\equiv (3\, S_{abk}\, H^k{}_c - C^k{}_{ab}{}^l\, H_{kc;l})_{;de}\,|_0\, x^{abcde} \quad \bmod g\\
&\equiv (3\, \overset{\circ}{S}_{abk;cd}\, \overset{\circ}{H}{}^k{}_e + 6\, \overset{\circ}{S}_{abk;c}\, \overset{\circ}{H}{}^k{}_{d;e} + 3\, \overset{\circ}{S}_{abk}\, \overset{\circ}{H}{}^k{}_{c;de}\\
&\quad - \overset{\circ}{C}{}^k{}_{ab}{}^l{}_{;cd}\, \overset{\circ}{H}_{ke;l} - 2\, \overset{\circ}{C}{}^k{}_{ab}{}^l{}_{;c}\, \overset{\circ}{H}_{kd;le} - \overset{\circ}{C}{}^k{}_{ab}{}^l\, \overset{\circ}{H}_{kc;lde})\, x^{abcde} \quad \bmod g\\
&\overset{\star}{=} \left(-\frac{3}{2}\, \overset{\circ}{R}_{ab;kcd}\, \overset{\circ}{H}{}^k{}_e + \frac{3}{2}\, \overset{\circ}{R}_{ak;bcd}\, \overset{\circ}{H}{}^k{}_e \right.\\
&\quad + 6\, \overset{\circ}{R}_{ab;kc}\, \overset{\circ}{H}{}^k{}_{d;e} - \frac{9}{4}\, \overset{\circ}{R}_{ab;k}\, \overset{\circ}{H}{}^k{}_{c;de}\\
&\quad - \left(\overset{\circ}{R}{}^k{}_{ab}{}^l{}_{;cd} + \frac{1}{2}\, \delta^l_a\, \overset{\circ}{R}{}^k{}_{b;cd} \right) \overset{\circ}{H}_{ke;l}\\
&\quad - 2 \left(\overset{\circ}{R}{}^k{}_{ab}{}^l{}_{;c} - \frac{1}{4}\, \delta^l_a\, \overset{\circ}{R}_{bc;}{}^k \right) \overset{\circ}{H}_{kd;le}\\
&\quad \left. - \overset{\circ}{R}{}^k{}_{ab}{}^l\, \overset{\circ}{H}_{kc;lde} \right) x^{abcde} \quad \bmod g,
\end{aligned} \tag{5.100}
$$

$$
\begin{aligned}
\Rightarrow 0 \overset{\star}{=} (&-6\, \overset{\circ}{R}_{ab;kcd}\, \overset{\circ}{H}{}^k{}_e + 6\, \overset{\circ}{R}_{ka;bcd}\, \overset{\circ}{H}{}^k{}_e\\
&+ 22\, \overset{\circ}{R}_{ka;bc}\, \overset{\circ}{H}{}^k{}_{d;e} - 7\, \overset{\circ}{R}_{ab;k}\, \overset{\circ}{H}{}^k{}_{c;de}\\
&- 4\, \overset{\circ}{R}{}^k{}_{ab}{}^l{}_{;cd}\, \overset{\circ}{H}_{ke;l} - 8\, \overset{\circ}{R}{}^k{}_{ab}{}^l{}_{;c}\, \overset{\circ}{H}_{kd;le}\\
&- 4\, \overset{\circ}{R}{}^k{}_{ab}{}^l\, \overset{\circ}{H}_{kc;lde})\, x^{abcde} \quad \bmod g.
\end{aligned} \tag{5.101}
$$

Equations (5.99) and (5.101) yield

$$
\begin{aligned}
A^k{}_{,k}\,[5] \overset{\star}{\equiv} \frac{1}{2520} (&-6\, \overset{\circ}{R}_{ab;k}\, \overset{\circ}{H}{}^k{}_{c;de} + 6\, \overset{\circ}{R}{}^k{}_{ab}{}^l{}_{;c}\, \overset{\circ}{H}_{kd;le}\\
&- 24\, \overset{\circ}{R}{}^k{}_{a;bc}\, \overset{\circ}{H}_{kd;e} + 16\, \overset{\circ}{R}{}^k{}_{ab}{}^l\, \overset{\circ}{R}_{kcml}\, \overset{\circ}{H}{}^m{}_{d;e}\\
&+ 32\, \overset{\circ}{R}{}^k{}_{ab}{}^l\, \overset{\circ}{R}_{lcdm}\, \overset{\circ}{H}_{ke;}{}^m + 12\, \overset{\circ}{R}{}^k{}_{ab}{}^l\, \overset{\circ}{R}_{kmcl;d}\, \overset{\circ}{H}{}^m{}_e
\end{aligned}
$$

$$- 12 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{ldkm} \overset{\circ}{H}{}^{m}{}_{e} + 36 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l} \overset{\circ}{R}{}_{lcdm;k} \overset{\circ}{H}{}^{m}{}_{e}$$

$$+ 78 \overset{\circ}{R}{}_{ab;}{}^{k} \overset{\circ}{R}{}_{kcdl} \overset{\circ}{H}{}^{l}{}_{e}$$

$$- 6 \overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{R}{}_{lde}{}^{m} \overset{\circ}{H}{}_{km}) x^{abcde} \mod g. \qquad (5.102)$$

To proceed, it will be convenient to convert (5.102) from an expression in the Riemann tensor R_{abcd} and the Ricci tensor R_{ab} and their covariant derivatives to one involving the Weyl tensor C_{abcd} and the tensor S_{abc} and their covariant derivatives. Using (1.10)-(1.12) and (A.1)-(A.14) we can derive

$$\overset{\circ}{R}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\star}{\equiv} \overset{\circ}{C}{}^{k}{}_{ab}{}^{l}{}_{;c} - \frac{1}{2}\left(- \overset{\circ}{g}{}^{kl} \overset{\circ}{R}{}_{ab;c} - \frac{1}{2}\delta^{k}_{b} \overset{\circ}{R}{}_{ac;}{}^{l} - \frac{1}{2}\delta^{l}_{a} \overset{\circ}{R}{}_{bc;}{}^{k}\right) \mod g, \qquad (5.103)$$

$$\overset{\circ}{R}{}_{kmcl;d} \overset{\star}{\equiv} \overset{\circ}{C}{}_{kmcl;d} - \frac{1}{2}\left(- \overset{\circ}{g}{}_{kl} \overset{\circ}{R}{}_{mc;d} + \overset{\circ}{g}{}_{kc} \overset{\circ}{R}{}_{ml;d} - \frac{1}{2} \overset{\circ}{g}{}_{ml} \overset{\circ}{R}{}_{cd;k} - \overset{\circ}{g}{}_{mc} \overset{\circ}{R}{}_{kl;d}\right) \mod g, \quad (5.104)$$

$$\overset{\circ}{R}{}_{lcd}{}^{m}{}_{;k} \overset{\star}{\equiv} \overset{\circ}{C}{}_{lcd}{}^{m}{}_{;k} - \frac{1}{2}\left(-\delta_{l}{}^{m} \overset{\circ}{R}{}_{cd;k} + \overset{\circ}{g}{}_{ld} \overset{\circ}{R}{}_{c}{}^{m}{}_{;k} - \delta_{c}{}^{m} \overset{\circ}{R}{}_{ld;k}\right) \mod g, \qquad (5.105)$$

$$\overset{\circ}{C}{}^{k}{}_{(a;bc)} \overset{\star}{=} - \overset{\circ}{L}{}^{k}{}_{(a;bc)} + \frac{1}{6}\delta^{k}_{(a} \overset{\circ}{R}{}_{;bc)}$$

$$\overset{\star}{=} - \overset{\circ}{S}{}_{ab}{}^{k}{}_{;c} + \frac{1}{6}\delta^{k}_{(a} \overset{\circ}{R}{}_{;bc)}, \qquad (5.106)$$

$$\overset{\circ}{R}_{k(a;b)} \overset{\star}{=} - \overset{\circ}{L}_{k(a;b)}$$
$$\overset{\star}{=} - \overset{\circ}{S}_{abk}. \tag{5.107}$$

Thus, using (A.1)-(A.14) and (5.103)-(5.107), we can write (5.102) as

$$\begin{aligned} A^k{}_{,k}[5] \overset{\star}{\equiv} \frac{1}{2520} (6\, & \overset{\circ}{S}_{abk} \overset{\circ}{H}{}^k{}_{c;de} + 6\, \overset{\circ}{C}{}^k{}_{ab}{}^l{}_{;c} \overset{\circ}{H}{}_{kd;le} \\ & - 24\, \overset{\circ}{S}_{abk;c} \overset{\circ}{H}{}^k{}_{d;e} + 16\, \overset{\circ}{C}{}^k{}_{ab}{}^l \overset{\circ}{C}{}_{kcml} \overset{\circ}{H}{}^m{}_{d;e} \\ & + 32\, \overset{\circ}{C}{}^k{}_{ab}{}^l \overset{\circ}{C}{}_{lcdm} \overset{\circ}{H}{}_{ke;}{}^m + 12\, \overset{\circ}{C}{}^k{}_{ab}{}^l \overset{\circ}{C}{}_{kmcl;d} \overset{\circ}{H}{}^m{}_e \\ & - 12\, \overset{\circ}{C}{}^k{}_{ab}{}^l{}_{;c} \overset{\circ}{C}{}_{ldkm} \overset{\circ}{H}{}^m{}_e + 36\, \overset{\circ}{C}{}^k{}_{ab}{}^l \overset{\circ}{C}{}_{lcdm;k} \overset{\circ}{H}{}^m{}_e \\ & - 6\, \overset{\circ}{C}{}^k{}_{ab}{}^l{}_{;c} \overset{\circ}{C}{}_{lde}{}^m \overset{\circ}{H}{}_{km} \\ & - 136\, \overset{\circ}{S}_{ab}{}^k \overset{\circ}{C}{}_{kcdl} \overset{\circ}{H}{}^l{}_e)\, x^{abcde} \quad \bmod g. \end{aligned} \tag{5.108}$$

It is not immediately obvious that the fourth and fifth terms and the sixth and seventh terms on the right hand side of (5.108) are identically zero. That this is, however, the case may be shown following the method of McLenaghan [20] by using the identities of Lovelock [17]. The essence of these identities is than in an n-dimensional space, the generalised Kronecker delta with $n+1$ indices, defined by

$$\delta^{a_1\, a_2 \ldots a_{n+1}}_{b_1\, b_2 \ldots b_{n+1}} := n! \delta^{[a_1}_{b_1} \delta^{a_2}_{b_2} \ldots \delta^{a_{n+1}]}_{b_{n+1}}, \tag{5.109}$$

is identically zero. Since we are working in the case $n = 4$, we have

$$\begin{aligned} 0 \equiv \delta^{klmnf}_{pqrsd} & \overset{\circ}{C}{}_k{}^r{}_{am} \overset{\circ}{C}{}_n{}^s{}_b{}^p \overset{\circ}{H}{}_{le;}{}^q \overset{\circ}{g}{}_{fc}\, x^{abcde} \\ = (-2\, & \overset{\circ}{C}{}^k{}_{ab}{}^l \overset{\circ}{C}{}_{lcd}{}^m \overset{\circ}{H}{}_{ke;m} \\ & - \overset{\circ}{C}{}^k{}_{ab}{}^l \overset{\circ}{C}{}_{kmcl} \overset{\circ}{H}{}^m{}_{d;e})\, x^{abcde}, \end{aligned} \tag{5.110}$$

and

$$0 \equiv \delta^{klmnf}_{pqrsd} \overset{\circ}{C}{}_{km}{}^{pr}{}_{;a} \overset{\circ}{C}{}^{qs}{}_{bn} \overset{\circ}{H}{}_{le} \overset{\circ}{g}{}_{fc} x^{abcde}$$
$$= (2 \overset{\circ}{C}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{C}{}_{kmdl} \overset{\circ}{H}{}^{m}{}_{e}$$
$$- 2 \overset{\circ}{C}{}^{k}{}_{ab}{}^{l} \overset{\circ}{C}{}_{kmcl;d} \overset{\circ}{H}{}^{m}{}_{e}) x^{abcde}, \tag{5.111}$$

Therefore, we have from (5.108), (5.110), and (5.111)

$$A^{k}{}_{,k}[5] \overset{\star}{\equiv} \frac{1}{2520} (6 \overset{\circ}{S}{}_{abk} \overset{\circ}{H}{}^{k}{}_{c;de} + 6 \overset{\circ}{C}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{H}{}_{kd;le}$$
$$- 24 \overset{\circ}{S}{}_{abk;c} \overset{\circ}{H}{}^{k}{}_{d;e} + 36 \overset{\circ}{C}{}^{k}{}_{ab}{}^{l} \overset{\circ}{C}{}_{lcdm;k} \overset{\circ}{H}{}^{m}{}_{e}$$
$$- 6 \overset{\circ}{C}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{C}{}_{lde}{}^{m} \overset{\circ}{H}{}_{km}$$
$$- 136 \overset{\circ}{S}{}_{ab}{}^{k} \overset{\circ}{C}{}_{kcdl} \overset{\circ}{H}{}^{l}{}_{e}) x^{abcde} \mod g. \tag{5.112}$$

6. CONFORMAL INVARIANCE OF THE SIXTH NECESSARY CONDITION

We may, at last, now construct $\sigma[5]$. First, we note that the first condition, (1.4), allows one to rewrite (1.10) as

$$\sigma \overset{\star}{=} \gamma - \frac{2}{3} R + 2 A^{a}{}_{,a} - A^{a} A^{a}. \tag{6.1}$$

Substituting (5.86), (5.88), and (5.112) into (6.1), we observe that to fifth order only the $A^{k}{}_{,k}$ term contributes, so that in our special conformal gauge, the sixth necessary condition is

$$0 \overset{\star}{\equiv} (3 \overset{\circ}{S}{}_{abk} \overset{\circ}{H}{}^{k}{}_{c;de} + 3 \overset{\circ}{C}{}^{k}{}_{ab;}{}^{l}{}_{;c} \overset{\circ}{H}{}_{kd;le}$$
$$- 12 \overset{\circ}{S}{}_{abk;c} \overset{\circ}{H}{}^{k}{}_{d;e} + 18 \overset{\circ}{C}{}^{k}{}_{ab}{}^{l} \overset{\circ}{C}{}_{lcdm;k} \overset{\circ}{H}{}^{m}{}_{e}$$
$$- 3 \overset{\circ}{C}{}^{k}{}_{ab}{}^{l}{}_{;c} \overset{\circ}{C}{}_{lde}{}^{m} \overset{\circ}{H}{}_{km}$$
$$- 69 \overset{\circ}{S}{}_{ab}{}^{k} \overset{\circ}{C}{}_{kcdl} \overset{\circ}{H}{}^{l}{}_{e}) x^{abcde} \mod g. \tag{6.2}$$

It remains to find the form of the sixth condition for an arbitrary choice of trivial transformation. Following [20] this can be done by finding a conformally invariant expression that reduces to (6.2) in our conformal gauge. The first step in finding such an expression is to observe the behaviour of the right and side of (6.2) under a general conformal transformation. This will be done on a term basis with the aid of (3.1), (3.6), (3.9), (3.2), and (3.11). For the remainder of this section we will assume that all expressions are symmetric in the tensor indices a, b, c, d and e and that $\equiv$ denotes equivalence mod g.

Consider first

$$\tilde{C}^k{}_{ab}{}^l{}_{\tilde{;}c}\, \tilde{C}_{lde}{}^m\, \bar{H}_{km} \equiv e^{-2\phi}\,(C^k{}_{ab}{}^l{}_{;c}\, C_{lde}{}^m\, H_{km} - C^k{}_{ab}{}^l\, C_{lcd}{}^m\, H_{me}\, \phi_k), \tag{6.3}$$

$$\begin{aligned}\tilde{C}^k{}_{ab}{}^l\, \tilde{C}_{lcd}{}^m{}_{\tilde{;}k}\, \bar{H}_{me} &\equiv e^{-2\phi}\, C^k{}_{ab}{}^l\, (C_{lcdm;k} - 2\,\phi_k\, C_{lcdm}\\ &\qquad + 2\,\phi_{[l}\, C_{c]kdm} + 2\, C_{lck[d}\, \phi_{m]}\\ &\qquad + 2\, g_{k[l}\, C^n{}_{c]dm}\, \phi_n\\ &\qquad + 2\, C_{lc[d}{}^n\, g_{m]k}\, \phi_n)\, H^m{}_e\\ &\equiv e^{-2\phi}\,(C^k{}_{ab}{}^l\, C_{lcdm;k}\, H^m{}_e\\ &\qquad - 2\,\phi_k\, C^k{}_{ab}{}^l\, C_{lcdm}\, H^m{}_e\\ &\qquad - \phi_m\, C^k{}_{ab}{}^l\, C_{kcdl}\, H^m{}_e\\ &\qquad + \phi_e\, C^k{}_{ab}{}^l\, C_{kmcl}\, H^m{}_d),\end{aligned} \tag{6.4}$$

$$\tilde{S}_{abk}\, \tilde{C}^k{}_{cd}{}^l\, \bar{H}_{le} \equiv e^{-2\phi}\,(S_{abk} C^k{}_{cd}{}^l\, H_{le} - \phi_k\, C^k{}_{ab}{}^l\, C_{lcd}{}^m\, H_{me}), \tag{6.5}$$

$$\begin{aligned}\tilde{S}_{abk}\, \bar{H}^k{}_{c\tilde{;}de} &\equiv e^{-2\phi}\,(S_{abk} - \phi_l\, C^l{}_{abk})\,(H^k{}_{c;d} - 3\,\phi_d\, H^k{}_c)_{\tilde{;}e}\\ &\equiv e^{-2\phi}\,(S_{abk} - \phi_l\, C^l{}_{abk})\,(H^k{}_{c;de} - 3\,\phi_{d;e}\, H^k{}_c\\ &\qquad - 8\,\phi_e\, H^k{}_{c;d} + 15\,\phi_d\,\phi_e\, H^k{}_c)\\ &\equiv e^{-2\phi}\,(S_{abk}\, H^k{}_{c;de} - \phi_l\, C^l{}_{abk}\, H^k{}_{c;de}\\ &\qquad - 3\,\phi_{d;e}\, S_{abk}\, H^k{}_c + 3\,\phi_{d;e}\,\phi_l\, C^l{}_{abk}\, H^k{}_c\end{aligned}$$

$$
\begin{aligned}
&- 8\, \phi_e\, S_{abk}\, H^k{}_{c;d} + 8\, \phi_e\, \phi_l\, C^l{}_{abk}\, H^k{}_{c;d} \\
&+ 15\, \phi_d\, \phi_e\, S_{abk}\, H^k{}_c \\
&- 15\, \phi_d\, \phi_e\, \phi_l\, C^l{}_{abk}\, H^k{}_c),
\end{aligned} \tag{6.6}
$$

$$
\begin{aligned}
\tilde{S}_{abk\tilde{;}c}\, \bar{H}^k{}_{\tilde{;}de} \equiv\; & e^{-2\phi}\, (S_{abk;c} - \phi_{l;c}\, C^l\, \phi_l\, C^l{}_{abk;c} \\
& - 5\, \phi_c\, S_{abk} + 5\, \phi_c\, \phi_l\, C^l{}_{abk} \\
& + g_{kc}\, \phi^m\, S_{abm} - g_{kc}\, \phi_l\, \phi^m\, C^l{}_{abm}) \\
& \times (H^k{}_{d;e} - 3\, \phi_e\, H^k{}_d + \delta^k_e\, \phi^n\, H_{nd}) \\
\equiv\; & e^{-2\phi}\, (S_{abk;c}\, H^k{}_{d;e} - 3\, \phi_e\, S_{abk;c}\, H^k{}_d \\
& - \phi_{l;e}\, C^l{}_{abk}\, H^k{}_{c;d} + 3\, \phi_d\, \phi_{l;e}\, C^l{}_{abk}\, H^k{}_c \\
& - \phi_l\, C^l{}_{abk;c}\, H^k{}_{d;e} + 3\, \phi_e\, \phi_l\, C^l{}_{abk;c}\, H^k{}_d \\
& - 5\, \phi_e\, S_{abk}\, H^k{}_{c;d} + 15\, \phi_d\, \phi_e\, S_{abk}\, H^k{}_c \\
& + 5\, \phi_e\, \phi_l\, C^l{}_{abk}\, H^k{}_{c;d} \\
& - 15\, \phi_d\, \phi_e\, \phi_l\, C^l{}_{abk}\, H^k{}_c),
\end{aligned} \tag{6.7}
$$

$$
\begin{aligned}
\tilde{C}^k{}_{ab}{}^l{}_{\tilde{;}c}\, \bar{H}_{kd\tilde{;}le} \equiv\; & e^{-2\phi}\, (C^k{}_{ab}{}^l{}_{;c} - 4\, \phi_c\, C^k{}_{ab}{}^l \\
& + \delta^k_c\, C^m{}_{ab}{}^l\, \phi_m + \delta^l_c\, C^k{}_{ab}{}^m\, \phi_m) \\
& \times (H_{kd;l} - 3\, \phi_l\, H_{kd})_{\tilde{;}e} \\
\equiv\; & e^{-2\phi}\, (C^k{}_{ab}{}^l{}_{;c} - 4\, \phi_c\, C^k{}_{ab}{}^l \\
& + \delta^k_c\, C^m{}_{ab}{}^l\, \phi_m + \delta^l_c\, C^k{}_{ab}{}^m\, \phi_m) \\
& \times (H_{kd;le} - 3\, \phi_{l;e}\, H_{kd} \\
& - 4\, \phi_l\, H_{kd;e} - 4\, \phi_l\, H_{kd;e} + 15\, \phi_l\, \phi_e\, H_{kd}) \\
\equiv\; & e^{-2\phi}\, (C^k{}_{ab}{}^l{}_{;c}\, H_{kd;le} - 3\, \phi_{l;e}\, C^k{}_{ab}{}^l{}_{;c}\, H_{kd} \\
& - 4\, \phi_l\, C^k{}_{ab}{}^l{}_{;c}\, H_{kd;e} - 4\, \phi_l\, C^k{}_{ab}{}^l{}_{;c}\, H_{kd;e} \\
& + 15\, \phi_l\, \phi_e\, C^k{}_{ab}{}^l{}_{;c}\, H_{kd} - 4\, \phi_e\, C^k{}_{ab}{}^l\, H_{kc;ld} \\
& + 12\, \phi_{l;d}\, \phi_e\, C^k{}_{ab}{}^l\, H_{kc} + 8\, \phi_l\, \phi_e\, C^k{}_{ab}{}^l\, H_{kc;d} \\
& + 16\, \phi_d\, \phi_e\, C^k{}_{ab}{}^l\, H_{kc;l}\, 45\, \phi_l\, \phi_d\, \phi_e\, C^k{}_{ab}{}^l\, H_{kc} \\
& + C^k{}_{ab}{}^l\, \phi_l\, H_{kc;de} - 3\, \phi_{d;e}\, C^k{}_{ab}{}^l\, \phi_l\, H_{kd}).
\end{aligned} \tag{6.8}
$$

Although again not immediately obvious, it can be shown that the last three terms on the right and side of (6.2) are conformally invariant with the help of a further identity generated by the Lovelock method discussed in the previous section (Section 5.2). This time we begin with

$$\begin{aligned} 0 &\equiv \delta^{klmnf}_{pqrsd} \overset{\circ}{C}{}_{k}{}^{r}{}_{am} \overset{\circ}{C}{}_{n}{}^{s}{}_{b}{}^{p} \phi^{q} \overset{\circ}{H}{}_{le} \overset{\circ}{g}{}_{fc} \\ &= \overset{\circ}{C}{}^{k}{}_{ab}{}^{l} \overset{\circ}{C}{}_{kmcl} \overset{\circ}{H}{}^{m}{}_{d} \phi_{e} - 2 \overset{\circ}{C}{}^{k}{}_{ab}{}^{l} \overset{\circ}{C}{}_{lcd}{}^{m} \overset{\circ}{H}{}_{me} \phi_{k} \\ &+ \overset{\circ}{C}{}^{k}{}_{ab}{}^{l} \overset{\circ}{C}{}_{kcdl} \overset{\circ}{H}{}_{me} \phi^{m}. \end{aligned} \tag{6.9}$$

Thus we have, using (6.3)-(6.5) and (6.9),

$$\begin{aligned} &18 \tilde{C}^{k}{}_{ab}{}^{l} \tilde{C}_{lcdm\tilde{;}k} \bar{H}^{m}{}_{e} - 3 \tilde{C}^{k}{}_{ab}{}^{l}{}_{\tilde{;}c} \tilde{C}_{lde}{}^{m} \bar{H}_{km} - 69 \tilde{S}_{ab}{}^{k} \tilde{C}_{kcdl} \bar{H}^{l}{}_{e} \\ &\equiv e^{-2\phi} (18 C^{k}{}_{ab}{}^{l} \overset{\circ}{C}{}_{lcdm;k} \overset{\circ}{H}{}^{m}{}_{e} - 3 \overset{\circ}{C}{}^{k}{}_{ab}{}^{l}{}_{;c} C_{lde}{}^{m} H_{km} \\ &- 69 S_{ab}{}^{k} C_{kcdl} H^{l}{}_{e}). \end{aligned} \tag{6.10}$$

Cursory inspection of the conformal transformations of the remaining three terms from the right hand side of (6.2) leads quickly to the conclusion that these terms are not, by themselves, conformally invariant. In particular, terms involving the second invariant derivatives of the conformal factor ϕ (eg. $\phi_{l;e}$) cannot be made to vanish using a Lovelock type of identity. It seems, then, that one or more terms which vanish in the special gauge used to obtain (6.2) will need to be added to obtain the sixth condition for a general choice of trivial transformations.

Following [20], we notice that the transformation (3.7) for the tensor L_{ab} has a term containing the second covariant derivative of ϕ. Further, we note that L_{ab} vanishes identically in the special conformal gauge (A.1)(A.14). One is then naturally led to calculate the conformal transformations of the following expressions:

$$\begin{aligned} \tilde{C}^{k}{}_{ab}{}^{l} \tilde{L}_{kc} \bar{H}_{ld\tilde{;}e} \equiv e^{-2\phi} C^{k}{}_{ab}{}^{l} (L_{kc} &- 2\phi_{k;c} \\ &+ 2\phi_{k}\phi_{c} - g_{kc}\phi_{m}\phi^{m}) \\ &\times (H_{ld;e} - 2\delta^{n}_{(l}\phi_{e} H_{nd} \\ &+ g_{le}\phi^{n} H_{nd} - 2\delta^{n}_{d}\phi_{e} H_{ln}) \end{aligned}$$

$$\equiv e^{-2\phi}\,(C^k{}_{ab}{}^l L_{kc}\,H_{ld;e} - 3\,\phi_e\,C^k{}_{ab}{}^l\,L_{kc}\,H_{ld} - 2\,\phi_{k;e}\,C^k{}_{ab}{}^l\,H_{lc;d} + 6\,\phi_{k;d}\,\phi_e\,C^k{}_{ab}{}^l\,H_{lc} + 2\,\phi_k\,\phi_e\,C^k{}_{ab}{}^l\,H_{lc;d} - 6\,\phi_k\,\phi_d\,\phi_e\,C^k{}_{ab}{}^l\,H_{lc}), \tag{6.11}$$

$$\tilde{C}^k{}_{ab}{}^l{}_{\tilde{;}c}\,\tilde{L}_{kd}\,\bar{H}_{le} \equiv e^{-2\phi}\,H_{le}\,(C^k{}_{ab}{}^l{}_{;c} - 4\,\phi_c\,C^k{}_{ab}{}^l + \delta^k_c\,C^m{}_{ab}{}^l\,\phi_m + \delta^l_c\,C^k{}_{ab}{}^m\,\phi_m) \times (L_{kd} - 2\,\phi_{k;d} + 2\,\phi_k\,\phi_d - g_{kd}\,\phi_m\,\phi^m)$$
$$\equiv e^{-2\phi}\,(C^k{}_{ab}{}^l{}_{;c} L_{kc}\,H_{le} - 4\,\phi_e\,C^k{}_{ab}{}^l\,L_{kc}\,H_{ld} + \phi_k\,C^k{}_{ab}{}^l\,L_{cd}\,H_{le} - 2\,\phi_{k;e}\,C^k{}_{ab}{}^l{}_{;c}\,H_{ld} + 8\,\phi_{k;d}\,\phi_e\,C^k{}_{ab}{}^l\,H_{lc} - 2\,\phi_{d;e}\,\phi_k\,C^k{}_{ab}{}^l\,H_{lc} + 2\,\phi_k\,\phi_e\,C^k{}_{ab}{}^l{}_{;c}\,H_{ld} - 8\,\phi_k\,\phi_d\,\phi_e\,C^k{}_{ab}{}^l\,H_{lc}), \tag{6.12}$$

$$\tilde{S}_{abk}\,\tilde{L}_{cd}\,\bar{H}^k{}_e \equiv e^{-2\phi}\,H^k{}_e\,(S_{abk} - \phi_l\,C^l{}_{abk}) \times (L_{cd} - 2\,\phi_{c;d} + 2\,\phi_c\,\phi_d - g_{cd}\,\phi_m\,\phi^m$$
$$\equiv e^{-2\phi}\,(S_{abk}\,L_{cd}\,H^k{}_e - 2\,\phi_{d;e}\,S_{abk}\,H^k{}_c + 2\,\phi_d\,\phi_e\,S_{abk}\,H^k{}_c - \phi_l\,C^l{}_{abk}\,L_{cd}\,H^k{}_e + 2\,\phi_l\,\phi_{d;e}\,C^l{}_{abk}\,H^k{}_c - 2\,\phi_l\,\phi_d\,\phi_e\,C^l{}_{abk}\,H^k{}_c). \tag{6.13}$$

Combining (6.6)-(6.8) and (6.11)-(6.13) we therefore conclude that

$$(6\,\tilde{S}_{abk}\,\bar{H}^k{}_{c\tilde{;}de} + 6\,\tilde{C}^k{}_{ab}{}^l{}_{\tilde{;}c}\,\bar{H}_{kd\tilde{;}le} - 24\,\tilde{S}_{abk\tilde{;}c}\,\bar{H}^k{}_{d\tilde{;}e} + 12\,\tilde{C}^k{}_{ab}{}^l\,\tilde{L}_{kc}\,\bar{H}_{ld\tilde{;}e} - 9\,\tilde{C}^k{}_{ab}{}^l{}_{\tilde{;}c}\,\tilde{L}_{kd}\,\bar{H}_{le} - 9\,\tilde{S}_{abk}\,\tilde{L}_{cd}\,\bar{H}^k{}_e)$$

$$\begin{aligned}\equiv e^{-2\phi}\,(6\,S_{abk}\,H^k{}_{c;de} + 6\,C^k{}_{ab}{}^l{}_{;c}\,H_{kd;le} - 24\,S_{abk;c}\,H^k{}_{d;e}\\ + 12\,C^k{}_{ab}{}^l\,L_{kc}\,H_{ld;e} - 9\,C^k{}_{ab}{}^l{}_{;c}\,L_{kd}\,H_{le} - 9\,S_{abk}\,L_{cd}\,H^k{}_e\\ + 24\,\phi_e\,(3\,S_{abk}\,H^k{}_c - C^k{}_{ab}{}^l\,H_{kc;l})_{;d}\\ + 24\,\phi_d\,\phi_e\,(3\,S_{abk}\,H^k{}_c - C^k{}_{ab}{}^l\,H_{kc;l})).\end{aligned} \tag{6.14}$$

We note that the last two groups of terms on the right hand side of (6.14), which involve the derivative of the conformal factor ϕ, vanish identically $(\text{mod}\,\text{g})$ by the fourth necessary condition, (1.7). Thus, we have at last the conformally invariant form of the sixth necessary condition, which is, in the notation of [20]

$$\begin{aligned}0 = TS\,(36\,C^k{}_{ab}{}^l\,C_{lcdm;k}\,H^m{}_e\\ - 6\,C^k{}_{ab}{}^l{}_{;c}\,C_{lde}{}^m\,H_{km} - 138\,S_{ab}{}^k\,C_{kcdl}\,H^l{}_e\\ + 6\,S_{abk}\,H^k{}_{c;de} + 6\,C^k{}_{ab}{}^l{}_{;c}\,H_{kd;le} - 24\,S_{abk;c}\,H^k{}_{d;e}\\ + 12\,C^k{}_{ab}{}^l\,L_{kc}\,H_{ld;e} - 9\,C^k{}_{ab}{}^l{}_{;c}\,L_{kd}\,H_{le} - 9\,S_{abk}\,L_{cd}\,H^k{}_e).\end{aligned} \tag{6.15}$$

Notice that, for the self adjoint case $H_{ab} = 0$, this condition is vascuous $(0 = 0)$ as expected.

7. CONCLUSION

The modified version of Hadamard's conjecture on Huygen's principle is still an open question. Considerable progress has been made following the programme outlined by Carminati and McLenaghan in answering the conjecture for self-adjoint scalar wave equations. However, there has as yet been little progress on the question of non-self-adjoint equations, although work continues. In particular, it is found in [2] that a further necessary condition is required in order to proceed. This sixth necessary condition has been derived here.

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APPENDIX

A. SOME CONSEQUENCES OF THE CONFORMAL GAUGE

Our choice of conformal gauge, which is expressed in (3.18)-(3.20), has various consequences which simplify the derivation of the necessary conditions. There are derived in [20]. We begin by contracting (3.18) and recalling the definition of L_{ab} from (1.10) to obtain

$$\overset{\circ}{R} = 0, \tag{A.1}$$

$$\Rightarrow \quad \overset{\circ}{R}_{ab} = 0, \tag{A.2}$$

$$\Rightarrow \quad \overset{\circ}{C}_{abcd} = \overset{\circ}{R}_{abcd}. \tag{A.3}$$

Also, using the Ricci identity (B.1) we get

$$\overset{\circ}{R}_{ab;cd} = \overset{\circ}{R}_{ab;dc}, \tag{A.4}$$

which, with (1.10) and the fact that $R_{;cd} = R_{;dc}$ implies

$$\overset{\circ}{L}_{ab;cd} = \overset{\circ}{L}_{ab;dc}. \tag{A.5}$$

Equation (3.19) can be expanded to yield

$$\overset{\circ}{R}_{,c} + 2 \overset{\circ}{R}_{c}{}^{a}{}_{;a} - \frac{1}{2} \overset{\circ}{R}_{,c} = 0, \tag{A.6}$$

while by contracting the Bianchi identity (B.3) we get

$$2 R_{c}{}^{a}{}_{;a} - R_{,c} = 0. \tag{A.7}$$

Equations (A.6) and (A.7) imply

$$\overset{\circ}{R}_{,c} = 0, \tag{A.8}$$

$$\Rightarrow \quad \overset{\circ}{R}_{ab;c} = - \overset{\circ}{R}_{ab;c}. \tag{A.9}$$

If one substitutes from (1.10) into (3.20) and contracts on indices b and c, one obtains

$$-\Box \overset{\circ}{R}_{;ad} + \frac{1}{6} g_{ad} \Box \overset{\circ}{R} - \frac{5}{3} \overset{\circ}{R}_{;ad} = 0, \tag{A.10}$$

where the covariant derivative of (A.7) has been used. A further contraction on (A.10) yields

$$\Box \overset{\circ}{R} = 0, \tag{A.11}$$

$$\Rightarrow \Box \overset{\circ}{R}_{ab} = -\frac{5}{3} \overset{\circ}{R}_{;ab}. \tag{A.12}$$

Lastly, we have, considering (1.12), (A.9), and (3.19)

$$\overset{\circ}{S}_{(ab)c} = \frac{3}{4} L_{ab;c}, \tag{A.13}$$

and, considering (3.20)

$$L_{ab;cd} = \frac{5}{3} \overset{\circ}{S}_{(ab)(c;d)} - \frac{1}{3} S_{(cd)(a;b)}. \tag{A.14}$$

B. SOME USEFUL IDENTIFIES

A number of rather specialised identities are necessary for the derivations in this paper, especially in Section 5. We begin by recalling some elementary identities from differential geometry which are used frequently in this paper. The first is the Ricci identity. Given any tensor of the form X_{ab} on a manifold with metric g_{ab} and corresponding curvature R_{abcd}, the second covariant derivatives of X_{ab} obey the Ricci identity

$$X_{ab;[cd]} = \frac{1}{2} (R^k{}_{acd} X_{kb} + R^k{}_{bcd} X_{ak}). \tag{B.1}$$

This is generalised to higher rank tensors in the obvious way. Two identities obeyed by the Riemann curvature tensor which we will need are the cyclic identity

$$R_{a[bcd]} = 0, \tag{B.2}$$

and the Bianchi identity

$$R_{ab[cd;e]} = 0. \tag{B.3}$$

From these we may derive a number of other identities. To begin with, consider the Bianchi identity in the form

$$R_{abcd;e} + R_{abde;e} + R_{abec;d} = 0. \tag{B.4}$$

Contracting on the indices a and e in (B.4) we get

$$R^k{}_{bcd;k} + R_{bd;c} - R_{bc;d} = 0, \tag{B.5}$$

or, gathering the two Ricci tensor terms to the other side

$$R^k{}_{abc;k} = 2\,R_{a[b;c]}, \tag{B.6}$$

which is our first identity. Our second follows simply by expressing the first in the quantities defined in (1.11) and (1.12), and is

$$C^k{}_{abc;k} = -S_{abc}. \tag{B.7}$$

The third identity also follows immediately from the first by contracting the indices b and d on (B.5), which produces

$$-R^k{}_{c;k} + R^{,c} - R^k{}_{c;k} = 0, \tag{B.8}$$

or, collecting like terms

$$R_{ak;}{}^k = \frac{1}{2}\,R_{,a}. \tag{B.9}$$

The fourth identity is derived from the cyclic identity (B.2). Consider the expression $R_{klma;a_1\ldots a_p}\,R^{mlk}{}_{b;b_1\ldots b_q}$. Applying the cyclic identity to one of the Riemann tensors one has

$$\begin{aligned} &R_{klma;a_1\ldots a_p}\,R^{mlk}{}_{b;b_1\ldots b_q} \\ &\quad = R^{klm}{}_{b;b_1\ldots b_q}\left(-R_{lkma;a_1\ldots a_p} - R_{kmla;a_1\ldots a_p}\right) \end{aligned} \tag{B.10}$$

$$= R_{klma;a_1\ldots a_p}\,R^{klm}{}_{b;b_1\ldots b_q} - R_{klma;a_1\ldots a_p}\,R^{klm}{}_{b;b_1\ldots b_q}, \tag{B.11}$$

which implies, when one gathers like terms,

$$R_{klma;a_1\ldots a_p}\,R^{mlk}{}_{b;b_1\ldots b_q} = \frac{1}{2}\,R_{klma;a_1\ldots a_p}\,R^{klm}{}_{b;b_1\ldots b_q}. \tag{B.12}$$

The fifth identity follows in a similar way from the Bianchi identity. Consider this time the expression $R_{kaml;a_1\ldots a_p}\,R^k{}_{bc}{}^l{}_;{}^m{}_{c_1\ldots c_q}$. We expand one of the derivatives of the Riemann tensor with the Bianchi identity

$$\begin{aligned} &R_{kaml;a_1\ldots a_p}\,R^k{}_{bc}{}^l{}_;{}^m{}_{c_1\ldots c_q} \\ &\quad = R_{mlka;a_1\ldots a_p}\left(-R^k{}_b{}^{lm}{}_{;cc_1\ldots c_q} - R^k{}_b{}^m{}_{c;}{}^l{}_{c_1\ldots c_q}\right) \\ &\quad = R_{klma;a_1\ldots a_p}\,R^{klm}{}_{b;cc_1\ldots b_q} - R_{kaml;a_1\ldots a_p}\,R^k{}_{bc}{}^l{}_;{}^m{}_{c_1\ldots c_q}, \end{aligned} \tag{B.13}$$

$$\Rightarrow R_{kaml;a_1\ldots a_p}\, R^k{}_{bc}{}^l{}_;{}^m{}_{c_1\ldots c_q} = \frac{1}{2} R_{klma;a_1\ldots a_p}\, R^{klm}{}_{\cdot b;cc_1\ldots b_q}. \qquad (B.14)$$

The sixth identity follows from considering the expression $C_{klma}\, C^{klm}{}_b$. Converting this expression to spinorial form via the correspondence

$$C_{abcd} \leftrightarrow \Psi_{ABCD}\, \varepsilon_{A'B'}\, \varepsilon_{D'C'} + \overline{\Psi}_{A'B'C'D'}\, \varepsilon_{AB}\, \varepsilon_{DC}, \qquad (B.15)$$

we get

$$\begin{aligned} C_{klma}\, C^{klm}{}_b \leftrightarrow & (\Psi_{KLMA}\, \varepsilon_{K'L'}\, \varepsilon_{M'A'} + \overline{\Psi}_{K'L'M'A'}\, \varepsilon_{KL}\, \varepsilon_{MA}) \\ & (\Psi^{KLM}{}_B\, \varepsilon^{K'L'}\, \varepsilon^{M'}{}_{'B} + \overline{\Psi}^{K'L'M'}{}_{'B}\, \varepsilon^{KL}\, \varepsilon^M{}_B) \end{aligned} \qquad (B.16)$$

$$= 4\, \Psi^{KLM}{}_A\, \Psi_{KLMB}\, \varepsilon_{A'B'} + \text{c.c.}, \qquad (B.17)$$

where c.c. means the complex conjugate of the preceding terms. But, observe that

$$\Psi^{KLM}{}_A\, \Psi_{KLMB} = -\Psi_{KLMA}\, \Psi^{KLM}{}_B. \qquad (B.18)$$

Recalling that any antisymmetric two spinor is proportional to the metric ε_{AB},

$$2\, \xi_{[AB]} = \varepsilon_{AB}\, \xi_C{}^C, \qquad (B.19)$$

we see that (B.18) implies

$$C_{klma}\, C^{klm}{}_b \leftrightarrow \Psi^{KLMN}\, \Psi_{KLMN}\, \varepsilon_{AB}\, \varepsilon_{A'B'} + \text{c.c.} \qquad (B.20)$$

On the other hand, if we consider the expression $g_{ab}\, C_{klmn}\, C^{klmn}$ in spinorial form we get

$$\begin{aligned} & g_{ab}\, C_{klmn}\, C^{klmn} \\ & \leftrightarrow (\varepsilon_{AB}\, \varepsilon_{A'B'})\, (\Psi^{KLMN}\, \varepsilon^{K'L'}\, \varepsilon^{M'N'} + \overline{\Psi}^{K'L'M'N'}\, \varepsilon^{KL}\, \varepsilon^{MN}) \\ & \quad (\Psi_{KLMN}\, \varepsilon_{K'L'}\, \varepsilon_{M'N'} + \overline{\Psi}_{K'L'M'N'}\, \varepsilon_{KL}\, \varepsilon_{MN}) \\ & = 4\, \Psi^{KLMN}\, \Psi_{KLMN}\, \varepsilon_{AB}\, \varepsilon_{A'B'} + \text{c.c.} \end{aligned} \qquad (B.21)$$

Comparing (B.20) and (B.21) we have

$$C_{klma}\, C^{klm}{}_b = \frac{1}{4}\, g_{ab}\, C_{klmn}\, C^{klmn}. \tag{B.22}$$

Now, we write (B.22) in terms of the Riemann tensor and its contractions using (1.10)-(1.12) to obtain

$$R_{klma}\, R^{klm}{}_b \equiv 2\, R_{kabl}\, R^{kl} + 2\, R_{ka}\, R^k{}_b - RR_{ab} \quad \operatorname{mod} g, \tag{B.23}$$

which is the sixth identity.

Next, we wish to prove an identity given in [19], specifically that the trace free symmetric part of $C_{klma;b}\, C^{klm}{}_{c;d}$ vanishes identically. We begin by again converting to spinors, i.e.

$$\begin{aligned} &TS\,(C_{klma;b}\, C^{klm}{}_{c;d}) \\ &\leftrightarrow S\,[(\Psi_{KLMA;BB'}\, \varepsilon_{K'L'}\, \varepsilon_{M'A'} + \overline{\Psi}_{K'L'M'A';BB'}\, \varepsilon_{KL}\, \varepsilon_{MA}) \\ &\quad (\Psi^{KLM}{}_{C;DD'}\, \varepsilon^{K'L'}\, \varepsilon^{M'\prime}{}_C + \overline{\Psi}^{K'L'M'}{}_{C';DD'}\, \varepsilon^{KL}\, \varepsilon^M{}_C)] \\ &= S\,[\Psi_{KLMA;BB'}\, \Psi^{KLM}{}_{C;DD'}\, \varepsilon_{C'A'} + \text{c.c.}] \\ &= 0, \end{aligned} \tag{B.24}$$

where S denotes the symmetric (in A, B, C, D and A', B', C', D') parts, which implies that

$$TS\,(C_{klma;b}\, C^{klm}{}_{c;d}) = 0, \tag{B.25}$$

which is the seventh identity we require.

The eighth identity we will need is due to Rinke and Wünsch [23]. We begin by considering the quantity $C^k{}_{abu}\, C_{kcdv}$ which we are assuming is symmetrized on the indices a, b, c, d. Converting this quantity to spinors we have

$$\begin{aligned} C^k{}_{abu}\, C_{kcdv} &\leftrightarrow (\Psi^K{}_{ABU}\, \varepsilon^{K'}{}_{A'}\, \varepsilon_{B'U'} + \overline{\Psi}^{K'}{}_{A'B'U'}\, \varepsilon^K{}_A\, \varepsilon_{BU} \\ &\quad + (\Psi_{KCDV}\, \varepsilon_{K'C'}\, \varepsilon_{D'V'} + \overline{\Psi}_{K'C'D'V'}\, \varepsilon_{KC}\, \varepsilon_{DV}) \\ &= -\Psi_{ABCU}\, \varepsilon_{DV}\, \overline{\Psi}_{'A'B'C'V}\, \varepsilon_{D'U'} - \text{c.c.} \end{aligned} \tag{B.26}$$

Now applying the spinorial identity (B.19) on indices U and V we get

$$C^k{}_{abu}\,C_{kcdv} \leftrightarrow -\Psi_{ABCV}\,\varepsilon_{DU}\overline{\Psi}_{A'B'C'V'}\,\varepsilon_{D'U'}$$
$$\Psi_{ABCD}\,\varepsilon_{UV}\,\overline{\Psi}_{V'A'B'C'}\,\varepsilon_{D'U'} + \text{c.c.} \tag{B.27}$$

Next, consider the quantity $C^k{}_{ab}{}^l\,C_{cklu}\,g_{vd}$, which expands in spinorial form to be

$$C^k{}_{ab}{}^l\,C_{cklu}\,g_{vd} \leftrightarrow \big(\Psi^K{}_{AB}{}^L\,\varepsilon^{K'}{}_{A'}\,\varepsilon_{B'}{}^{L'} + \overline{\Psi}^{K'}{}_{A'B'}{}^{L'}\,\varepsilon^K{}_A\,\varepsilon_B{}^L\big)$$
$$\Psi_{CKLU}\,\varepsilon_{C'K'}\,\varepsilon_{L'U'} + \overline{\Psi}_{C'K'L'U'}\,\varepsilon_{CK}\,\varepsilon_{LU}\big)\,\varepsilon_{VD}\,\varepsilon_{V'D'}$$
$$= \Psi_{CABU}\overline{\Psi}_{C'A'B'U'}\,\varepsilon_{VD}\,\varepsilon_{V'D'} + \text{c.c.} \tag{B.28}$$

Adding (B.27) and (B.28) one gets

$$C^k{}_{abu}\,C_{kcdv} + C^k{}_{ab}{}^l\,C_{cklu}\,g_{vd}$$
$$\leftrightarrow \Psi_{ABCD}\,\varepsilon_{UV}\,\overline{\Psi}_{V'A'B'C'}\,\varepsilon_{D'U'} + \text{c.c.} \tag{B.29}$$

Symmetrizing (B.29) on the indices u, v, we obtain

$$C^k{}_{ab(u}C_{|kcd|v)} + C^k{}_{ab}{}^l\,C_{ckl(u}g_{v)d}$$
$$\leftrightarrow \frac{1}{2}\,\Psi_{ABCD}\,\overline{\Psi}_{A'B'C'V'}\,\varepsilon_{D'U'}\,\varepsilon_{UV}$$
$$+ \frac{1}{2}\,\Psi_{ABCD}\,\overline{\Psi}_{A'B'C'U'}\,\varepsilon_{D'V'}\,\varepsilon_{VU} + \text{c.c.}$$
$$= \Psi_{ABCD}\,\overline{\Psi}_{A'B'C'[U'}\varepsilon_{|D'|V']}\,\varepsilon_{UV} + \text{c.c.} \tag{B.30}$$

But using identity (B.19) once more, this is nothing but

$$C^k{}_{ab(u}C_{|kcd|v)} + C^k{}_{ab}{}^l\,C_{ckl(u}g_{v)d}$$
$$\leftrightarrow \frac{1}{2}\,\Psi_{ABCD}\,\overline{\Psi}_{A'B'C'D'}\,\varepsilon_{UV}\,\varepsilon_{U'V'} + \text{c.c.} \tag{B.31}$$

Thus, since one may easily verify that

$$\begin{aligned}
& C^k{}_{ab}{}^l\, C_{kcdl}\, g_{uv} \\
& \leftrightarrow (\Psi^K{}_{AB}{}^L\, \varepsilon^{K'}{}_{A'}\, \varepsilon_{B'}{}^{L'} + \Psi^{K'}{}_{A'B'}{}^{L'}\, \varepsilon^K{}_A\, \varepsilon_B{}^L) \\
& (\Psi_{KCDL}\, \varepsilon_{K'C'}\, \varepsilon_{D'L'} + \overline{\Psi}_{K'C'D'L'}\, \varepsilon_{KC}\, \varepsilon_{DL})\, \varepsilon_{UV}\, \varepsilon_{U'V'} \\
& = \Psi_{ABCD}\overline{\Psi}_{A'B'C'D'}\, \varepsilon_{UV}\, \varepsilon_{U'V'} + \text{c.c.},
\end{aligned} \tag{B.32}$$

we have, combining (B.31) and (B.32)

$$C^k{}_{ab(u}C_{|kcd|v)} + C^k{}_{ab}{}^l\, C_{ckl(u}g_{v)d} = \frac{1}{2}\, C^k{}_{ab}{}^l\, C_{kcdl}\, g_{uv}. \tag{B.33}$$

Contracting (B.33) with L_{ab} and recalling (1.10) we get

$$C^k{}_{ab}{}^l\, (C_{kcd}{}^m\, L_{lm} + C_{ckl}{}^m\, L_{md}) = -\frac{1}{6}\, RC^k{}_{ab}{}^l\, C_{kcdl}, \tag{B.34}$$

which is the eighth identity.

Our last identity comes from [19]. Consider first the quantity $TS\,(C^k{}_{abl}\, C^l{}_{cmn}\, C^n{}_d{}^m{}_k)$. In spinorial form it can be written as

$$\begin{aligned}
& TS\,(C^k{}_{abl}\, C^l{}_{cmn}\, C^n{}_d{}^m{}_k) \\
& \leftrightarrow S\,[(\Psi^K{}_{ABC}\, \overline{\Psi}_{B'C'M'N'}\, \varepsilon^{K'}{}_{A'}\, \varepsilon_{MN} \\
& + \overline{\Psi}^{K'}{}_{A'B'C'}\, \Psi_{BCMN}\, \varepsilon^K{}_A\, \varepsilon_{M'N'}) \\
& (\Psi^N{}_D{}^M{}_K\, \varepsilon^{N'}{}_{D'}\, \varepsilon^{M'}{}_{K'} + \overline{\Psi}^{N'}{}_{D'}{}^{M'}{}_{K'}\, \varepsilon^N{}_D\, \varepsilon^M{}_K)] \\
& = S\,(\Psi_{AB}{}^{KL}\, \Psi_{KLCD}\, \overline{\Psi}_{A'B'C'D'} + \text{c.c.}).
\end{aligned} \tag{B.35}$$

Similar calculations (which the reader should now be well acquainted with) for the quantities $TS\,(C^k{}_{abl}\, C^l{}_{men}\, C^n{}_d{}^m{}_k)$ and $TS\,(C^k{}_{ab}{}^l\, C^m{}_{cd}{}^n\, C_{kmnl})$ yield respectively

$$\begin{aligned}
& TS\,(C^k{}_{abl}\, C^l{}_{men}\, C^n{}_d{}^m{}_k) \\
& \leftrightarrow 2\, S\,(\Psi_{AB}{}^{KL}\, \Psi_{KLCD}\, \overline{\Psi}_{A'B'C'D'} + \text{c.c.}),
\end{aligned} \tag{B.36}$$

$$TS\,(C^{k}{}_{ab}{}^{l}\,C^{m}{}_{cd}{}^{n}\,C_{kmnl})$$
$$\leftrightarrow 3\,S\,(\Psi_{AB}{}^{KL}\,\Psi_{KLCD}\,\overline{\Psi}_{A'B'C'D'} + \text{c.c.}). \qquad \text{(B.37)}$$

From (B.35)-(B.37) we infer

$$TS\,(C^{k}{}_{abl}\,C^{l}{}_{cmn}\,C^{n}{}_{d}{}^{m}{}_{k}) = \frac{1}{2}\,TS\,(C^{k}{}_{abl}\,C^{l}{}_{mcn}\,C^{n}{}_{d}{}^{m}{}_{k})$$
$$= \frac{1}{3}\,TS\,\left(C^{k}{}_{ab}{}^{l}\,C^{m}{}_{cd}{}^{n}\,C_{kmnl}\right). \qquad \text{(B.38)}$$

which is the ninth and final identity that we will derive;

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