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A Nekhoroshev-type theorem for the Pauli-Fierz model of classical electrodynamics

by

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ABSTRACT. – We consider the dynamical system composed of a rigid extended charged particle and the electromagnetic field (Pauli-Fierz model); the particle is also constrained on a plane and subjected to a central potential. For this system, we prove a Nekhoroshev-type theorem according to which the purely mechanical circular motions of the particle are stable up to times growing exponentially with the ratio of the radius of the particle to its "classical radius", provided the frequency of revolution of the particle is large enough. The order of magnitude of such a frequency is the same as that of the radiationless motions already discussed by Bohm and Weinstein in the context of the reduced Abraham-Lorentz-Dirac equation for the electron. From the purely mathematical point of view, the present result constitutes a nontrivial extension of Nekhoroshev-type results to a certain class of infinite dimensional systems.

RÉSUMÉ. – Nous considérons le système dynamique formé d'une particule chargée rigide étendue et du champ électromagnétique (modèle de Pauli-Fierz), la particule se déplaçant dans un plan et étant soumise aussi à un potentiel central. Pour ce système nous démontrons un théorème de type Nekhoroshev qui assure que les mouvements circulaires purement mécaniques sont stables sur des temps qui croissent exponentiellement avec le rapport du rayon de la particule à son « rayon classique », pourvu que la fréquence de révolution soit assez grande. L'ordre de grandeur de cette fréquence est le même que celui dans le mouvement sans radiation discuté par Bohm et Weinstein dans le contexte de l'équation réduite de Abraham-Lorentz-Dirac pour l'électron. D'un point de vue purement mathématique, ce résultat constitue une extension non triviale du théorème de Nekhoroshev à une classe de systèmes en dimension infinie.

1. INTRODUCTION

In this paper, following ([1], [2]), we continue the study of the interaction between the classical electromagnetic field and matter, as described by the nonrelativistic Pauli-Fierz [3] [or Abraham ([4], [5])] model: namely, the coupled system composed of an extended rigid charged particle (whose rotational degrees of freedom are neglected) obeying Newton equation with the Lorentz force, and of the electromagnetic field, satisfying Maxwell equations with a current due to the particle's motion; the point of view is that of the theory of dynamical systems (*see* [6]), in which the Cauchy problem for the complete system is studied. Here we concentrate on the problem of the radiation emitted by the charged particle in an external central field of forces, and in particular on the possible existence of radiationless motions; some results in the spirit of Nekhoroshev's theorem for infinite dimensional systems are obtained.

Problems of this type were studied long ago using what might be called standard radiation theory ([7], [8]), namely the procedure originally developed in order to calculate the radiation emitted by antennas; such a theory is essentially characterized by the fact that the current is prescribed *a priori*, so that the difficulty due to the radiation reaction on the current is eliminated, and the problem is essentially reduced to the linear one of describing the field "created by a given current"; in particular, use is made of the retarded potentials, which corresponds to a situation with initially (or asymptotically) vanishing fields; moreover, the so called dipole approximation is often made.

With such a procedure one finds that a point-like particle radiates energy with a rate proportional to the square of the instantaneous acceleration (Larmor formula), and thus should fall onto the center of attraction (*see* e. g. [7], p. 274, problem 1). However, it is quite well known that classical radiation theory predicts the possibility of a different behaviour in the case of an extended particle, and in fact the possibility of periodic radiationless motions was pointed out by many authors ([9], [10]) (*see* also [11]). In particular Schott [9] studied the motion of a rigid uniformly charged sphere, and showed that periodic radiationless motions can exist, but only if the radius a of the sphere and the frequency ω of the motion are related by $\omega = n\, 2\pi\, c/a$, where n is any positive integer number (c being the speed of light).

An intermediate treatment of the radiation problem is that of the classical work of Bohm and Weinstein [10]. Indeed these authors intend to take into account the role of the radiation reaction; but, due to the difficulty of a rigorous treatment of the complete nonlinear system, they limit themselves to the study of a reduced equation for an extended rigid charged particle, which is obtained in some approximations. Within such a framework, they concentrate on the problem of the motion of a "free" particle, subjected only to "its own" field, and obtain the result that there exist special conditions under which periodic motions of the particle are possible. Such conditions connect the shape of the particle, its mass m, its charge e_0, and the frequency ω of the periodic motion. In particular, for a uniformly charged sphere of radius a, the frequency of the periodic oscillation can assume only one value ($\omega^2 = 2\, e_0^2/ma^3$); moreover, such motions can exist only if the equation $\mathrm{tg}\,(2\,\bar{a}/a) = (2\,\bar{a}/a)$ is satisfied, where $\bar{a}$ is the so called "classical radius" $\bar{a} := e_0^2/(mc^2)$ of the particle.

In the present paper, we take into full account the role of the radiation reaction, by studying the Cauchy problem for the complete nonlinear coupled system describing particle and field. In particular, we study the motion of a particle in an external central field of forces, and prove the stability in Nekhoroshev sense of some purely mechanical circular motions. The main idea is to make use of two known facts, namely: on the one hand, the exchange of energy between the particle and the field is restricted essentially to a small range of frequencies around the instantaneous angular frequency ω of the motion; on the other hand, as shown in [1] (theorem 2.4), the rate of energy exchange of the field modes decreases exponentially with their frequency. Thus, one can conjecture that the particle will not be able to radiate, and therefore to fall towards the center, within accessible times (as in Nekhoroshev theorem), provided ω is sufficiently large; in fact, we show that the circular orbits are stable in Nekhoroshev sense, if $\omega > c/a$. Precisely, we assume the charge distribution of the particle to be given by a gaussian; moreover, we consider the particle to be constrained on a plane, and subjected to a central potential of the form $\alpha\, r^b/b$, with positive α. Then we prove that, if the ratio $a/\bar{a}$ is large, where a is here the dispersion of the gaussian describing the charge distribution of the particle, and $\bar{a}$ the classical radius, there exists an open set of initial data such that the particle's motion remains close to a uniform circular motion up to times which increase exponentially with a power of the above ratio. The result is obtained for parameters b in the interval $-2 < b < 2$, satisfying the technical condition that $\sqrt{b+2}$ should be diophantine. Moreover, we find that the allowed set of initial data is characterized by the fact that the frequency ω of the corresponding circular orbits is larger than c/a.

Such a result is qualitatively in agreement with that of Schott recalled above, but is here deduced as a rigorous theorem for the nonlinear coupled system composed of particle and field. Bohm and Weinstein's oscillations are instead a different phenomenon; in fact, their condition on the shape of the particle excludes the case of an analytical exponentially localized form factor, which is essential for our proof. However, we point out that the order of magnitude of the frequencies of the periodic motions is the same in all these works.

For what concerns the mathematical aspects of the problem, the proof of the theorem given here is based on the rigorous methods of classical perturbation theory for infinite dimensional systems, and in our opinion might constitute in itself an interesting contribution in such a domain. In fact, while perturbation theory of finite dimensional systems is now quite well developed, the corresponding theory for infinite dimensional systems is still very incomplete; indeed almost all known results refer to the extension of KAM theorem to infinite systems with discrete spectrum ([12]-[18]), while very little is known for systems with a continuous spectrum ([19], [20]), and in general on the possibility of extending Nekhoroshev theory to infinite dimensional systems [21] (*see* also [22]). The present result is a Nekhoroshev type theorem for a systems of partial differential equations with continuous spectrum; we think that this is the first example of a new class of systems that can be studied studied using perturbation theory [23].

Our theorem is an improvement of a result recently obtained by Benettin Galgani and Giorgilli (BGG) in [24], for the case of finite dimensional systems. They proved that if a Hamiltonian system is composed of two subsystems, one of which has motions characterized by frequencies much higher than the other one, then there is essentially no exchange of energy between the two subsystems, up to times growing exponentially with the ratio of the two characteristic frequencies. Their estimates depend strongly on the number of degrees of freedom of the high frequency system, but the dependence on the dimension of the low frequency system turns out to be very good, since use is made of a diophantine condition with an exponent completely independent of the dimension of the low frequency system.

The idea of the present paper is to consider the purely mechanical system as the high frequency one, and the free field as the low frequency system. A first improvement of BGG's theorem which is needed to obtain our result consists in eliminating completely the dependence of their estimates on the dimension of the low frequency system. This can be obtained using techniques borrowed from complex analysis in Banach spaces, which allow to deal with the variables defining the state of the low frequency system as a single object. In such a way one obtains a result of BGG kind, but valid also in the case of an infinite dimensional low frequency system [for example a system whose unperturbed dynamics is given by finite difference equations ([19], [20]).

However, the above preliminary result is not enough to deal with the Pauli-Fierz model, because the high frequency degrees of freedom of the field are to be taken into account. On the other hand, such degrees of freedom are essentially isolated, due to the analyticity of the form factor, and this suggests the idea of dealing with the whole electromagnetic field as if it were a low frequency system. This is actually obtained using techniques similar to those currently used for systems with short range interaction [18], however, also these techniques require some extensions that we will discuss in section 3.

The paper is organized as follows. In section 2 we recall the model and state our result in a somehow simplified way. In section 3 we outline the main ideas of the proof. The formal scheme of proof is given in section 4. The analytic scheme needed to exploit the particular structure of the interaction and to obtain estimates independent of the dimension of the low frequency system is given in section 5, together with the main steps of the proof and with the most general formulation of our result (thm. 5.3). The technical part of the proof is finally given in section 6.

2. STATEMENT OF THE RESULTS

As usual in dealing with the Pauli-Fierz model, we work in the Coulomb gauge, so that the only dynamically relevant unknown for the field is the vector potential A, with the constraint div $A=0$. Fix a cartesian coordinate system (x_1, x_2, x_3) with the origin coinciding with the center of attraction of the particle. Let (r, θ) denote the polar coordinates of the particle in the plane $x_3=0$ where we assume it is constrained, and let $q=q(r, \theta)$ denote its cartesian coordinates. The charge density at x is given by $e_0\,\rho(x-q)$, where the "form factor" ρ is an assigned normalized L^1 function while e_0 is the total charge of the particle; the Hamiltonian of the system is given by [1], [3], [7],

$$H=\hat{h}(E, A)+\frac{1}{2m}\left[(p-A_r)^2+\left(\frac{M}{r}-A_\theta\right)^2\right]+V(r), \qquad (2.1)$$

where

$$\left.\begin{aligned}\hat{h}(\mathrm{E}, \mathrm{A}) &:= \int \left(2\pi c^2 \mathrm{E}^2(x) - \frac{1}{8\pi}\langle \mathrm{A}(x), \Delta \mathrm{A}(x)\rangle\right) d^3 x \\ \mathrm{A}_r &:= \int \frac{e_0}{c} \rho(x-q)[\cos\theta\, \mathrm{A}_1(x) + \sin\theta\, \mathrm{A}_2(x)]\, d^3 x, \\ \mathrm{A}_\theta &:= \int \frac{e_0}{c} \rho(x-q)[-\sin\theta\, \mathrm{A}_1(x) + \cos\theta\, \mathrm{A}_2(x)]\, d^3 x;\end{aligned}\right\} \quad (2.2)$$

here $\mathrm{E}=(\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3)$ is the momentum conjugate to $\mathrm{A}=(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3)$, p is the momentum conjugate to r, M is the momentum conjugate to θ, and $\mathrm{V}(r)$ is an external potential. The momentum E coincides, up to a factor, with the electric field; the momentum p coincides with radial component of the total momentum of the particle.

In order to simplify the statement of the theorem we will assume that V has the form

$$\mathrm{V}(r) := \frac{\alpha}{b} r^b;$$

the most general assumptions on the potential under which our theorem holds will be given at the beginning of section 3.

We will consider the function space $\mathrm{H}_*^{\{s\}} = \mathrm{H}_*^{\{s\}}(\mathbb{R}^3, \mathbb{R}^3)$, $s \geqq 0$ which is defined as the completion of C_c^∞ (the lower index c stands for "compactly supported") vector fields with vanishing divergence in the norm $\|\mathrm{A}\|_{\{s\}} := \| |\Delta|^{s/2} \mathrm{A}\|_{\mathrm{L}^2}$ (we recall that, with respect to such a norm, $\mathrm{H}_*^{\{s\}}$ is a Hilbert space and that, by the Sobolev inequality, one has that $\mathrm{H}_*^{\{s\}}$ is continuously imbedded in L^6 [25], [26]).

As proved in [1] the appropriate phase space for the system is

$$\mathrm{F} := \mathrm{H}_*^0 \times \mathrm{H}_*^{\{s\}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbf{T} \ni (\mathrm{E}, \mathrm{A}, p, r, \mathrm{M}, \theta), \tag{2.3}$$

where the Cauchy problem for the equations of motion of (2.1) is well posed provided $\rho \in \mathrm{L}^1 \cap \mathrm{H}^1$. In fact, we shall need much more smoothness of ρ, in order to prove our result. The precise requirement needed for the charge distribution will be given in section 5 [*cf.* (5.26)]; here, just to fix ideas, we take ρ to be a gaussian

$$\rho(x) = \frac{1}{\pi^{3/2} a^3} \exp\left(-\frac{x^2}{a^2}\right). \tag{2.4}$$

Consider now the "mechanical" case (with $e_0 = 0$), in which there is no interaction between the particle and the electromagnetic field. We fix our attention on the uniform circular motions; they exist with any angular frequency ω, if $b \neq 0,2$; correspondingly, the radius r and the angular momentum M are given as functions of ω by $r=r_c(\omega)$, $M=M_c(\omega)$, where

$$r_c(\omega)=\left(\frac{1}{\omega}\sqrt{\frac{\alpha}{m}}\right)^{2/(2-b)}, \qquad M_c(\omega)=\sqrt{\alpha m}\left(\frac{1}{\omega}\sqrt{\frac{\alpha}{m}}\right)^{(b+2)/(2-b)}, \quad (2.5)$$

(while obviously the radial component p of the momentum is zero). For the complete system ($e_0\neq 0$) we can prove the following.

THEOREM 2.1. – *Consider the dynamical system described by Hamiltonian* (2.1), (2.2), *in the phase space* F [*cf. eq.* (2.3)]. *Let the parameter b of the external potential be in the range* $-2<b<2$, $b\neq 0$ *and satisfy the diophantine condition*

$$|\sqrt{b+2}\,k_1+k_2|\geq\frac{\nu'}{(|k_1|+|k_2|)^2}, \qquad \forall k\in\mathbf{Z}^2\setminus\{0\},$$

for some positive ν'. *For a given* ω, *define the pure numbers* ε_1, ε_2, *according to*

$$\varepsilon_1:=\left(\frac{\bar{a}}{a}\right)^{1/4}, \qquad \varepsilon_2:=\frac{c/a}{\omega}, \qquad (2.6)$$

where a is the dispersion of ρ [*cf.* (2.4)], *and* $\bar{a}:=e_0^2/(mc^2)$ *is the "classical radius" of the particle. Then there exist strictly positive numerical constants* ε_*, $k_1, \ldots, k_6$ *which depend only on b, with the following property: if* $[r_c(\omega)/a]\leq 2^7$, *and*

$$0<\varepsilon\leq\frac{1}{3^7}\varepsilon_*,$$

where

$$\varepsilon:=\varepsilon_1+k_1\,\varepsilon_2=\left(\frac{\bar{a}}{a}\right)^{1/4}+k_1\frac{c/a}{\omega},$$

then, for all initial data $(E_0, A_0, p_0, r_0, M_0, \theta_0)\in F$ *satisfying*

$$\frac{\hat{h}(E_0, A_0)}{\omega M_c}<\frac{1}{2}k_2|\log(\varepsilon)|\varepsilon_1^2, \qquad \frac{|p_0|}{m\omega r_c}<\frac{4}{5}k_3|\log(\varepsilon)|^{1/2}\varepsilon_1,$$

$$\frac{|r_0-r_c|}{r_c}<\frac{4}{5}k_4|\log(\varepsilon)|^{1/2}\varepsilon_1, \qquad \frac{|M_0-M_c|}{M_c}<\frac{29}{30}k_5|\log(\varepsilon)|\varepsilon_1^2,$$

one has, along the corresponding solution of the problem, the bounds

$$\left.\begin{aligned}&\frac{\hat{h}(E(t), A(t))}{\omega M_c}<k_2|\log(\varepsilon)|\varepsilon_1^2, \qquad \frac{|p(t)|}{m\omega r_c}<k_3|\log(\varepsilon)|^{1/2}\varepsilon_1,\\ &\frac{|r(t)-r_c|}{r_c}<k_4|\log(\varepsilon)|^{1/2}\varepsilon_1,\\ &\frac{|M(t)-M_c|}{M_c}<k_5|\log(\varepsilon)|\varepsilon_1^2,\end{aligned}\right\} \quad (2.7)$$

for all times t with

$$|t| \leq k_6 \frac{1}{\omega \varepsilon^{3/4}} \exp\left[2\left(\frac{\varepsilon_*}{\varepsilon}\right)^{1/5}\right].$$

This result can also be stated informally in the following way. If (i) the ratio of the so called classical radius of the particle, *i.e.*, $e_0^2/(mc^2)$, to its "true radius" a is small, and (ii) the revolution time of the particle is small compared to the time taken by light to cross the particle, then, given initial data near those allowing the purely mechanical circular motion, up to times exponentially long with $1/\varepsilon$ one has that (1) the motion of the particle remains close to a uniform circular motion, and (2) the energy radiated by the particle is small compared to its mechanical energy; by the way, this implies that the mean power radiated by the particle is exponentially small with $1/\varepsilon$.

This theorem makes precise the idea that in some situations the interaction with the electromagnetic field just perturb a little the purely mechanical motions. From this point of view, the present result is closely related to classical radiation theory according to which radiation reaction is neglected; in particular, it makes precise its range of validity (at least for the study of radiationless periodic motions). By the way, as far as we know the radiation emitted by an extended particle was calculated (using classical radiation theory) only in the work by Schott, who, as recalled in the introduction, found that radiationless motions are very special, since they must have a frequency which is exactly an integer multiple of the inverse of the time taken by light to cross the particle. The present precise treatment which takes into account also the radiation reaction, shows that, provided the coupling is small ($\bar{a}/a \ll 1$), Schott's result is essentially correct; however it shows that the only important property of essentially radiationless motions is the order of magnitude of their frequency and not its precise value. Concerning the phenomenon discovered by Bohm and Weinstein, it is clear that it is quite different from the one analyzed here; in fact these authors study the case in which the motions of the charged particle are significantly different from the purely mechanical ones. So it is not astonishing that their phenomenon appears only in the case of strong coupling ($\bar{a}/a$ has to be significantly different from zero), and is thus somehow complementary to the one found here.

Another comment concerns the order of magnitude of the frequency of the radiationless motions found in classical electrodynamics (those of Schott, those of Bohm and Weinstein, and the present ones). Indeed, such a frequency has to be larger than c/a, so that the corresponding motions appear as a little awkward physically: in fact, orbits of this kind, satisfying also the condition that no points of the particle move with velocity larger than c, have the property that their radius should be smaller than the radius of the particle.

A last comment concerns the nonrelativistic character of the Pauli-Fierz model, and the possibility of a relativistic extension of the present result. A study of the fully covariant model of a rigid electron (*see* [27], [28]) from the dynamical point of view would be very interesting, but we think that it would be rather difficult. However, we point out that the present result can be extended quite easily, as a corollary of the general theorem 5.3, to the semirelativistic model described by the Hamiltonian

$$H = c\sqrt{(p-A_r)^2 + \left(\frac{M}{r} - A_\theta\right)^2 + m^2 c^2} + \frac{\alpha}{b} r^b,$$

with A_r and A_θ given by equation (2.2), and b in a suitable range. In such a model the particle's position $q = q(r, \theta)$ satisfies the equation

$$\frac{d}{dt}\frac{m\dot{q}}{\sqrt{1-(\dot{q}^2/c^2)}} = -\frac{\partial V}{\partial q} + F_L(q, A),$$

where the Lorentz force F_L is calculated as in the nonrelativistic case, *i. e.*, using a non covariant definition of rigid charge distribution: we recall that in this case the k-th component of F_L is given by

$$-\int \rho(x-q)\,\dot{A}_k(x)\,d^3x + \sum_{l=1}^{3} \dot{q}_l \int \rho(x-q)\left(\frac{\partial}{\partial x_k}A_l(x) - \frac{\partial}{\partial x_l}A_k(x)\right)d^3x.$$

The interest of such a non covariant model rests on the fact that it presents some features characteristic of relativistic theories, as typically the fact that c is a limit velocity.

3. PRELIMINARIES AND SCHEME OF THE PROOF

First we specify the general properties of the potential $V(r)$ which are required in order to prove our theorem. The point is that we apply classical perturbation theory considering the purely mechanical system as a high frequency one. So, since we are interested in the neighbourhood of a circular orbit, we have to require that such an orbit exists and that it has good properties from the point of view of perturbation theory.

Formally we assume that there exists a positive M_c such that the corresponding effective potential

$$V^*(r) := \frac{M_c^2}{2\,mr^2} + V(r),$$

has at least one strict minimum r_c. Then, we denote by ω_r the frequency of the small oscillations in the radial direction

$$\omega_r := \frac{1}{m} \frac{\partial^2 V^*}{\partial r^2}(r_c),$$

and by $\omega := M_c/(mr_c^2)$ the frequency of revolution of the particle; we assume also that ω_r/ω is a diophantine number.

By the way, in case $V(r) = \alpha r^b/b$ it turns out that $\omega_r = \omega\sqrt{b+2}$. A further assumption, that will be formulated in a quantitative way in the statements of the forthcoming theorems and proposition, is that both ω and ω_r have to be large.

We come now to perturbation theory; since we are interested in a neighbourhood of a circular orbit it is useful to make the following canonical coordinate transformation

$$\left.\begin{aligned} p &= \sqrt{\frac{m\omega_r}{2}}(\pi + i\xi), \\ r &= \frac{i}{\sqrt{2m\omega_r}}(\pi - i\xi) + r_c, \qquad \tilde{M} := M - M_c, \\ \tilde{A}_k(x) &:= \frac{1}{\sqrt{4\pi c}} |\Delta|^{1/4} A_k(x), \qquad \tilde{E}_k(x) := \sqrt{4\pi c}\,|\Delta|^{-1/4} E_k(x) \end{aligned}\right\} \tag{3.1}$$

(the transformation on the field variables is introduced for future convenience). In terms of the new variables, the phase space turns out to be

$$\tilde{F} := H_*^{\{1/2\}} \times H_*^{\{1/2\}} \times \mathbf{C} \times \mathbf{C} \times \mathbf{C} \times \mathbf{T}^{\mathbf{C}} \ni (E, A, \pi, \xi, M, \theta) = y$$

($\mathbf{T}^{\mathbf{C}}$ denotes the complexification of the torus); as usual, we consider also complex values of M and θ. For simplicity, we shall drop the tilde from F. A relevant domain of the new variables, which is invariant under the Hamiltonian flow, is the one which corresponds to real values of the original variables; by abuse of language, we will simply qualify such a domain as "real".

Then, we expand the Hamiltonian in powers of π, ξ, $\tilde{M}$ and in Fourier series in θ; omitting constant terms and tildes, it takes the form

$$H = h_\omega(M, \pi, \xi) + \hat{h}(E, A) + H_{\text{int}}(E, A, \pi, \xi, M, \theta), \tag{3.2}$$

where

$$\left.\begin{aligned} h_\omega(M, \pi, \xi) &:= \omega M + i\omega_r \pi\xi, \\ \hat{h}(E, A) &:= \frac{c}{2}\int (\langle E, |\Delta|^{1/2} E\rangle + \langle A, |\Delta|^{1/2} A\rangle d^3x) \end{aligned}\right\} \tag{3.3}$$

and H_{int} is a function which (in the case where $V(r) = \alpha r^b/b$) will be given explicitly in section 6. Actually, *all the following developments do not depend* *on the explicit form of the interaction* H_{int}, *but only on some of its properties that will be stated in the hypotheses of the forthcoming theorems and propositions.*

Then, we aim to proceed as in reference [24], considering h_ω as describing a high frequency system interacting with the "low" frequency system described by $\hat{h}$, and obtaining a normal form theorem which can be used in order to bound the time derivatives of the actions M and $\pi\xi$. However, as pointed out in the introduction, $\hat{h}$ describes a system of oscillators (think of the Fourier transform of the fields) whose frequencies belong to an unbounded set. So we need to take into account the fact that, due to the special form of the interaction, there exists a cutoff frequency, such that the oscillators with higher frequency do not interact with the rest of the system (*see* [1]). This information will be inserted in the perturbation scheme by identifying the special class of the functions which enter in a relevant way in the perturbation procedure, and introducing a suitable weighted norm for these functions. This is quite usual in dealing with system with short range interaction (*see* e. g. [15], [18]), however the extension of such a method to the present case is non-trivial, since it turns out that $\hat{h}$, do not belong to the above class and, in particular, has an infinite norm. This problem can be overcome since (*see* sect. 4) $\hat{h}$ does not enter directly in the perturbation procedure: the only relevant quantity is the Poisson bracket of $\hat{h}$ with any function of the above class, and the norm of such a Poisson bracket can be estimated.

Using the above procedure, one succeeds in obtaining a normal form theorem. In order to deduce, as in Nekhoroshev theorem, bounds on the variations of the actions up to exponentially long times, one has to ensure that all the variables do not leave the domain where the above theorem holds. This is a crucial point, since the normal form theorem gives no information on the time evolution of the field variables E and A. So, their motion can be controlled only throught energy conservation. This means that we are forced to choose the domains for these variables to be spheres in the space of states with finite energy.

In section 4 we will recall the formal procedure used to normalize the Hamiltonian, referring to [24] for further details. Then, in section 5 we shall define the class of functions entering the perturbation procedure, the corresponding domains, and the norms.

4. FORMAL THEORY

Our perturbative scheme is based on the idea that M, ξ and π are infinitesimal together with the interaction H_{int}, while ω is large. Since all these quantities are not dimensionally homogeneous, it seems difficult to identify *a priori* one dimensionless perturbative parameter; therefore we shall work without specifying what we mean by perturbative order, but acting as if it were a well defined concept; the developments of the next section will show that the present formal theory is coherent and can be made rigorous. Moreover, it will turn out that there is a natural perturbative parameter, which is essentially the ratio of the size of H_{int} to the size of h_ω plus the ratio of the cutoff frequency to the angular frequency of the mechanical motion.

First, we recall that, according to the algebraic approach of [29], a near to identity canonical transformation can be defined as follows. Consider a sequence of functions $\{\chi_s\}_{s\geqq 1}$ on the phase space, and define a corresponding linear operator T_χ acting on functions by

$$T_\chi f := \sum_{r\geqq 0} f_r, \tag{4.1}$$

where

$$f_0 := f, \qquad f_r := \sum_{l=1}^{r} \frac{l}{r}\{\chi_l, f_{r-l}\}, \quad r\geqq 1. \tag{4.2}$$

Letting this operator act on the coordinates (with respect to an arbitrary canonical basis), we obtain a transformation of the phase space synthetically written as

$$y = T_\chi y', \qquad y, y' \in F.$$

This transformation turns out to be canonical, and the following identity holds [29]:

$$(T_\chi f)(y') = f(T_\chi y').$$

So, we look for a finite generating sequence $\chi = \{\chi_s\}_{s=1}^{r}$ such that the transformed Hamiltonian turns out to be in normal form up to a small remainder $\mathscr{R}^{(r)}$, *i.e.* of the form

$$(T_\chi H)(y') = H(y') = h_\omega + \hat{h} + Z + \mathscr{R}^{(r)}, \tag{4.3}$$

where Z is such that $\{\pi'\xi', Z\} \equiv \{M', Z\} \equiv 0$, with the standard notation for Poisson brackets. In order to write down the equations for χ, we first decompose the interaction in the form

$$H_{int} = \sum_{l\geqq 1} H_l, \tag{4.4}$$

with H_l of order l (some other requirements on this decomposition will be specified in the next section); furthermore, we denote $T_\chi h_\omega = \sum_{s\geqq 0} h_s$, $T_\chi \hat{h} = \sum_{s\geqq 0} \hat{h}_s$, $T_\chi H_l = \sum_{s\geqq 0} H_{l,s}$, $Z = \sum_{s=1}^{r} Z_s$ with Z_s of order s, and assume that h_s, $\hat{h}_s$, and $H_{s,l}$ are of order s, $s+1$ and $s+l$ respectively. So, equating terms of the same order in (4.3), we obtain for χ_s and Z_s the following equations:

$$\{h_\omega, \chi_s\} + Z_s = \Psi_s, \qquad 1 \leqq s \leqq r, \tag{4.5}$$

with

$$\left.\begin{aligned} &\Psi_1 = H_1 \\ &\Psi_s = \hat{h}_{s-1} + \sum_{l=1}^{s-l} \frac{l}{s}\{\chi_l, h_{s-l}\} + \sum_{l-1}^{s} H_{l,s-l}, \qquad 2 \leqq s \leqq r. \end{aligned}\right\} \tag{4.6}$$

These equations can be solved recursively provided ω_r/ω irrational, thus obtaining the generating sequence χ.

5. ANALYTIC THEORY

In order to make rigorous the procedure of the previous section we need to exploit the particular structure of the interaction between matter and electromagnetic field. The point is that the interaction is described by a function H_{int} in which the field variables enter only through expressions of the form

$$\int \langle \phi(x), A(x) \rangle_{\mathbb{R}^3} d^3 x, \tag{5.1}$$

where ϕ is a smooth function; so, the high frequency Fourier components of the field are always multiplied by a small coefficient. Before entering into details, let us explain the idea of the scheme we are going to build up. First, we need to identify the class of functions whose size have to be controlled in order to ensure the convergence of all the series and to get quantitative estimates. It is clear that such a class has to be invariant under Poisson bracket, and under the operation of calculating the Poisson bracket with $\hat{h}$ which are the operations involved in the recursion described in the previous section; thus, denoting by $\mathscr{E}$ this class of functions, it has to satisfy

$$\{\mathscr{E}, \mathscr{E}\} \subset \mathscr{E}, \qquad \{\hat{h}, \mathscr{E}\} \subset \mathscr{E}. \tag{5.2}$$

If ϕ is analytical (in a suitable sense), then this class is obtained simply as the algebra generated by H_{int} under the operations (5.2), and obviously coincides with the set of the polynomials in quantities of the form (5.1), with coefficients which are functions of all the dynamical variables apart from E and A.

We come now to the formal definitions. We begin by specifying the smoothness properties needed for ϕ (which will be proved to be implied by simple assumptions on the charge distribution, *see* lemma 6.11). To this end, we introduce the function space $H^{\{s\}}$, $s<0$, which is defined as the dual of $H^{\{-s\}}$ (closure of C_c^∞ in the norm of $H_*^{\{s\}}$); we assume that the application

$$\mathbf{C}^3 \to H_*^{\{1/2\}}(\mathbb{R}^3, \mathbf{C}^3) \cap H^{\{-1/2\}}(\mathbb{R}^3, \mathbf{C}^3)$$
$$z \mapsto \phi_z,$$

where

$$\phi_z(x) := \phi(x-z), \tag{5.3}$$

is well defined and analytic in a neighbourhood of the origin. A function having this property will be called $H^{\{1/2\}}$-$H^{\{-1/2\}}$-analytic. An example of $H^{\{1/2\}}$-$H^{\{-1/2\}}$-analytic function is given by any analytical (in the usual sense) function from $\mathbb{R}^3$ to $\mathbb{R}^3$, which decays exponentially at infinity. By the way, a more natural requirement on ϕ would involve the use of the space dual to $H_*^{\{1/2\}}$, but we made our different choice in order to simplify the identification of $H^{\{1/2\}}$-$H^{\{-1/2\}}$-analytic functions. We also point out that, if we were interested in the study of the interaction of the electromagnetic field with matter in a bounded domain of $\mathbb{R}^3$, then the above definition could have been substituted by the simple requirement of ordinary analyticity of ϕ.

Consider now a linear functional B on $H^{\{1/2\}}$ of the form

$$B(W) := \int \langle \phi(x), W(x) \rangle_{\mathbb{R}^3} d^3 x, \tag{5.4}$$

with a ϕ which is $H^{\{1/2\}}$-$H^{\{-1/2\}}$-analytic, $W \in H^{\{1/2\}}$; and its multilinear generalization, $g(E, A)$

$$g(E, A) = \sum_{k_1 \dots k_l} g_{k_l \dots k_l} B_{k_1}(W_{k_1}) \dots B_{k_l}(W_{k_l}), \tag{5.5}$$

where B_k is of the form (5.4) and W_k is either A or E.

Then, we give the following

DEFINITION. — *(Class $\mathscr{E}$) an analytical functional f : F $\to$ **C** is said to be of classe $\mathscr{E}$ if it is of the form*

$$f(y) = \sum_{mlnk} f_{mlnk}(A, E) \pi^m \xi^l M^n e^{ik\theta}, \tag{5.6}$$

with the coefficients $f_{mlnk}(E, A)$ of the form (5.5).

We point out that the above definition makes sense also for functions taking values in Fréchet spaces [take the coefficients $g_{k_1 \dots k_l}$ of equation (5.5) to be elements of such a space]; we shall use it also in this context.

In order to specify the norms we shall use, it is necessary to premise the definition of the domains $\Delta_{R, d}$ where perturbation theory will be developed. This is given through a fixed vector of positive quantities

$R := (R_1, R_2, R_3, R_4)$:

$$\Delta_{R,d} := \{(E, A, \pi, \xi, M, \theta) \in F : \sqrt{a}\|E\|_{\{1/2\}} \leqq R_1(1-d),$$
$$\sqrt{a}\|A\|_{\{1/2\}} \leqq R_1(1-d),\ |\pi| \leqq R_2(1-d),\ |\xi| \leqq R_2(1-d),$$
$$|M| \leqq R_3(1-d),\ \theta \in \mathbf{T} + iR_4(1-d)\},$$

where $d<1$ is a real parameter, a a parameter having the dimension of a length (*e.g.* the radius of the particle), and

$$\mathbf{T} + iR_4(1-d) := \{\theta \in \mathbf{T}^C : |\operatorname{Im}\theta| \leqq R_4(1-d)\}$$

($\mathbf{T}^C$ being the complexification of the torus). We shall also denote by

$$\mathscr{A} := \operatorname{Min}\{R_1^2, R_2^2, R_3 R_4\}$$

a typical action which will enter in the perturbative developments.

The size of a function $f : \Delta_{R,d} \to \mathbf{C}$ will be measured using a norm $N_{d,\varkappa}(.)$ which depends on R, on d and on a further positive parameter κ. This norm is defined according to the following procedure.

If ϕ is a $H^{\{1/2\}}$-$H^{\{-1/2\}}$-analytic function, we put [*cf.* eq. (5.3)]

$$\|\phi\|_\varkappa := \sup_{\|z\|<\varkappa} \left\{\sqrt{a}\|\phi_z\|_{\{1/2\}}, \frac{1}{\sqrt{a}}\|\phi_z\|_{\{-1/2\}}\right\}. \tag{5.7}$$

If $g = g(E, A)$ is a function of the form (5.5), we put

$$N_{d,\varkappa}(g) := \sum_{k_1 \dots k_l} |g_{k_1 \dots k_l}|\,\|\phi_{k_1}\|_\varkappa \cdots \|\phi_{k_l}\|_\varkappa (R_1(1-d))^l, \tag{5.8}$$

where ϕ_j are defined by

$$B_j(W_j) = \int \langle \phi_j(x), W_j(x) \rangle_{\mathbb{R}^3}\, d^3x.$$

Finally, if f is a function of class $\mathscr{E}$, we write it in the form (5.6), and put

$$N_{d,\varkappa}(f) := \sum_{mlnk} N_{d,k}(f_{mlnk})(R_2(1-d))^{m+l}(R_3(1-d))^n e^{|k|R_4(1-d)}. \tag{5.9}$$

We point out that

$$\sup_{y \in \Delta_{R,d}} |f(y)| \leqq N_{d,\varkappa}(f), \qquad \forall d, \kappa.$$

It is useful to introduce also another norm which measures the size of the Hamiltonian vector field associated to a function of class $\mathscr{E}$. In order to do that, first notice that the definition of the norm $N_{d,\varkappa}(.)$ makes sense also for functions which take values in the space of $H^{\{1/2\}}$-$H^{\{-1/2\}}$-analytic functions. In fact, in this case one can write the decomposition (5.5), (5.6) with the coefficients $g_{k_1 \dots k_l}$ which are $H^{\{1/2\}}$-$H^{\{-1/2\}}$-analytic functions, and therefore, substitute in definition (5.8) $\|g_{k_1 \dots k_l}\|_\varkappa$ to $g_{k_1 \dots k_l}|$, obtaining the wanted quantity. Then, we define the gradient $\nabla_E f \in H^{\{-1/2\}}$ with respect to E of a function f of class $\mathscr{E}$, by

$$\langle \nabla_E f(y), W \rangle_{L^2} = d_E f(y) W, \qquad \forall W \in H^{\{1/2\}},$$

where $d_E f(y)$ is the differential of f at y with respect to the variable E (all other variables being considered as parameters). It is easy to see that, if $f \in \mathscr{E}$, then $\nabla_E f(y)$ is a $H^{\{1/2\}}$-$H^{\{-1/2\}}$-analytic function for any $y \in F$, and that it is of class $\mathscr{E}$. So, we can define $N_{d,\varkappa}(\nabla_E f)$, $N_{d,\varkappa}(\nabla_A f)$, and introduce the norm

$$N^{\nabla}_{d,\varkappa}(f) := R_1 (N_{d,\varkappa}(\nabla_E f) + N_{d,\varkappa}(\nabla_A f)) + R_2\left(N_{d,\varkappa}\left(\frac{\partial f}{\partial \pi}\right) + N_{d,\varkappa}\left(\frac{\partial f}{\partial \xi}\right)\right) + R_3 N_{d,\varkappa}\left(\frac{\partial f}{\partial M}\right) + R_4 N_{d,\varkappa}\left(\frac{\partial f}{\partial \theta}\right). \tag{5.10}$$

We point out that the following estimate holds:

$$N^{\nabla}_{0,\varkappa}(f) \leqq \frac{4\, N_{-d,\varkappa}(f)}{d}, \qquad 0 < d \tag{5.11}$$

(for the proof *see* sect. 6).

Using these definition and notations we can state a proposition concerning the behaviour of the series defining the transformation T_χ.

PROPOSITION 5.1. – *Let* $\{\chi_l\}_{l \geqq 1}$, $\chi_l \in \mathscr{E}$, *be a generating sequence with*

$$N^{\nabla}_{d,\varkappa}(\chi_l) \leqq \frac{\beta^{l-1}}{l} \Phi, \tag{5.12}$$

for some positive β, Φ *and* $d < 1/2$; *assume*

$$e \frac{\Phi}{\mathscr{A}\, d} + \beta \leqq \frac{e}{e+1}. \tag{5.13}$$

Then, the transformation $y = T_\chi(y')$ [*cf.* (4.1), (4.2)] *analytically maps* $\Delta_{R,d}$ *in* $\Delta_{R,0}$, *and one has* $T_\chi \Delta_{R,d} \supset \Delta_{R,2d}$; *moreover, one has*

$$\left.\begin{array}{ll} \sqrt{a}\,\|E - E'\|_{\{1/2\}} \leqq R_1 d, & \sqrt{a}\,\|A - A'\|_{\{1/2\}} \leqq R_1 d, \\ |\pi - \pi'| \leqq R_2 d, & |\xi - \xi'| \leqq R_2 d, \\ |M - M'| \leqq R_3 d, & |\theta - \theta'| \leqq R_4 d. \end{array}\right\} \tag{5.14}$$

The proof is deferred to the technical section 6.

We are interested in the use of the particular generating sequence defined by equations (4.5), (4.6); therefore we must show that the norms of such a χ_s satisfy an inequality of the kind of (5.12). This is ensured by

PROPOSITION 5.2. – *Consider an Hamiltonian of the form* (3.2), *with* $H_{int} \in \mathscr{E}$, *and decompose* H_{int} *according to*

$$H_{int} = \sum_{s \geqq 1} H_s,$$

where H_s *has the property that there exist positive constants* $K \geqq 1$, $\mathscr{F}$, γ *such that*

$$H_s(A, \pi, \xi, M, \theta) = \sum_{((|k|/K)+m+l)<2(s+2)} H_{s,mlk}(A, M)\, \pi^l \xi^m e^{ik\theta}, \tag{5.15}$$

and

$$N^{\nabla}_{0,\varkappa}(H_s) \leqq \mathscr{F} \gamma^{s-1}. \tag{5.16}$$

Assume also that there exists a non increasing sequence of positive numbers α_s, $(s \geqq 1)$, *such that*

$$\left| k_1 + k_2 \frac{\omega_r}{\omega} \right| \geqq \alpha_s, \qquad \forall k \in \mathbf{Z}^2 \setminus \{0\} \quad \textit{with } |k| \leqq 2(s+2)K. \tag{5.17}$$

Then, for each natural integer r, *there exists a generating sequence* $\{\chi_s\}_{s=1}^r$ *which solves system* (4.5); *moreover, for any* $0<d<1$ *the estimate*

$$N^{\nabla}_{d,\varkappa(1-d)}(\chi_s) \leqq \frac{\Phi \beta^{s-1}}{s},$$

holds, with

$$\Phi = \frac{\mathscr{F}}{\alpha_r \omega}, \qquad \beta = \frac{6(r-1)^2}{\omega \alpha_r d} \left(\frac{\mathscr{F}}{\mathscr{A}} + \frac{\sqrt{3}\, c}{\kappa d} \right) + 2\gamma. \tag{5.18}$$

The proof will be given in section 6. We point out that the proof of this proposition is quite delicate, since it is here that the problem of the infinite norm of $\hat{h}$ causes significant troubles.

This proposition together with proposition 5.1 allows the use of the above generating sequence in order to transform the Hamiltonian. Then we have to estimate the norm of the transformation (*cf.* eq. (5.14)) and the norm of the remainder $\mathscr{R}_{(r)}$. Finally, we choose the order of normalization r with the aim of minimizing the remainder and bounding the diffusion of the actions up to exponentially long times. So we get

THEOREM 5.3. – *In the same hypotheses of proposition* 5.2, *and with the additional assumption that there exists positive constants* ν, and $\mathscr{F}'$, *such that*

$$\left.\begin{aligned} &\left| k_1 + k_2 \frac{\omega_r}{\omega} \right| \geqq \frac{\nu}{((s+2)K)^2}, \ \forall k \in \mathbf{Z}^2 \setminus \{0\} \text{ with } |k| \leqq 2(s+2)K, \\ &\qquad \sup_{\Delta_R, 1/2} |H_{\text{int}}| \leqq \mathscr{F}', \end{aligned}\right\} \tag{5.19}$$

define

$$\mu := \frac{4K^2}{\nu} \left[(e+6) \frac{\mathscr{F}}{\mathscr{A}\omega} + \frac{6\sqrt{3}}{d} \frac{c/\kappa}{\omega} \right] + 2\gamma, \tag{5.20}$$

and assume also

$$\mu \leqq \frac{1}{(3e)^4}, \tag{5.21}$$

$$\frac{c R_1^2}{16 a} > 2 \mathscr{F}' + \frac{3 K^2 (e+1)}{R_4 \nu e^2} \left(\sqrt{b+2} + \frac{1}{R_4} \right) \frac{\mathscr{F}}{\mu^{1/2}}. \tag{5.22}$$

Consider a solution of the Cauchy problem for system (3.2) *with real initial data* $(E_0, A_0, \pi_0, \xi_0, M_0, \theta_0)$ *such that*

$$\left.\begin{aligned} I(\pi_0, \xi_0) &< \frac{R_2^2}{16} - \frac{\mathscr{F}}{\omega \mu^{1/2}} \frac{3 K^2 (e+1)}{\nu e^2}, \\ |M_0| &< \frac{R_3}{4} - \frac{\mathscr{F}}{\omega \mu^{1/2}} \frac{3 K^2 (e+1)}{R_4 \nu e^2}, \\ \hat{h}(E_0, A_0) &< \frac{c R_1^2}{16 a} - \left[2 \mathscr{F}' + \frac{3 K^2 (e+1)}{R_4 \nu e^2} \left(\sqrt{b+2} + \frac{1}{R_4} \right) \frac{\mathscr{F}}{\mu^{1/2}} \right], \end{aligned}\right\} \tag{5.23}$$

where $I(\pi, \xi) := i \pi \xi$ *is the radial action; then, for all times t with*

$$|t| \leqq T := \frac{1}{\omega \mu^{3/4}} \frac{K^2}{\nu e^{11}} \exp\left[\frac{4}{e} \left(\frac{1}{\mu} \right)^{1/4} \right], \tag{5.24}$$

one has that the solution $y(t) = (E(t), A(t), \pi(t), \xi(t), M(t), \theta(t))$ *exists and belongs to* $\Delta_{R, 3/4}$; *moreover one has the bounds*

$$\left.\begin{aligned} |I(t) - I(0)| &\leqq \frac{\mathscr{F}}{\omega \mu^{1/2}} \frac{3 K^2 (e+1)}{\nu e^2}, \\ |M(t) - M_0| &\leqq \frac{\mathscr{F}}{\omega \mu^{1/2}} \frac{3 K^2 (e+1)}{R_4 \nu e^2}. \end{aligned}\right\} \tag{5.25}$$

Here we used the obvious notation $I(t) := I(\pi(t), \xi(t))$. We point out that (as will be proved in section 6) the right hand side of the first two equations of (5.23) is automatically positive.

In order to obtain theorem 2.1 from theorem 5.3 we shall proceed as follows (for more details *see* sect. 6). First we consider Hamiltonian (2.1) in the case $V(r) = \alpha r^b / b$, and perform the development of H_{int} in powers and Fourier series in the variables π, ξ, M, θ. Then, we prove that, if the charge distribution is such that the application ρ_z [*cf.* eq. (5.3)] is analytic, and, if

$$|\rho|_\sigma := \operatorname{Max} \left\{ \sup_{|\operatorname{Im} z_k| < 2\sigma} \| \rho_z \|_{L^2}, \frac{1}{a} \sup_{|\operatorname{Im} z_k| < 2\sigma} \| \rho_z \|_{\{-1\}} \right\} < \infty, \tag{5.26}$$

then it turns out that H_{int} is of class $\mathscr{E}$. Subsequently, we identify the dominant part of the interaction, and estimate its norm $N_{-1/4, \sigma}(.)$ in terms of $|\rho|_\sigma$, R_1, R_2, R_3, R_4. Then we specialize to the case of small initial data for the field and choose R_1 in order to satisfy equation (5.22); it turns out that a good choice is

$$R_1^2 = \frac{\omega}{c/a}\left(\sqrt{b+2} + \frac{1}{R_4}\right) R_2^2. \tag{5.27}$$

Subsequently we insert this expression in the above norm, and divide it by $\omega \mathscr{A}$ obtaining a good estimate for $\mathscr{F}/\mathscr{A}\omega$. Then we put

$$\mathscr{A} = R_2^2 = R_3 R_4, \tag{5.28}$$

and choose R_2 in order to minimize our estimate of $\mathscr{F}/(\mathscr{A}\omega)$. The next step is the evaluation of $|\rho|_\sigma$ for the case

$$\rho(x) := \frac{1}{a^3 \pi^{3/2}} \exp\left(-\frac{x^2}{a^2}\right),$$

which is done by Sobolev embedding theorem. Finally, we put $\sigma = a$, and insert all these estimates in the expression of μ and in the bounds (5.25), obtaining theorem 2.1.

6. TECHNICAL LEMMAS AND PROOFS

In this section we shall denote by $\mathscr{S}$ a space which is either **C** or the space of $H^{\{1/2\}}$-$H^{\{-1/2\}}$-analytic functions; correspondingly the symbol $\|\cdot\|_{\mathscr{S}}$ will mean $|\cdot|$ or $\|\cdot\|_\varkappa$.

Moreover, in order to simplify the proofs, it is useful to define the Poisson brackets of a **C**-valued function f' with an $\mathscr{S}$-valued function f. To this end we put

$$\{f', f\}(y) := d_{(A,\xi,\theta)} f(y) \nabla_{(E,\pi,M)} f'(y) - d_{(E,\pi,M)} f(y) \nabla_{(A,\xi,\theta)} f'(y); \tag{6.1}$$

it is easy to verify that, if v_n is any basis of $\mathscr{S}$, and $f_n(y)$ are the corresponding components of f, then

$$\{f', f\}(y) = \sum_n \{f', f_n\}(y)\, v_n. \tag{6.2}$$

LEMMA 6.1. – *Let $f, f' \in \mathscr{E}$, with $f : \Delta_{R,d} \to \mathscr{S}$ and $f' : \Delta_{R,d+d'} \to$ **C**, be two functions depending only on* E *and* A; *then*

$$N_{d+d',\varkappa}(\{f', f\}) \leq \frac{1}{R_1 d'} N_{d,\varkappa}(f)\,(N_{d+d',\varkappa}(\nabla_E f') + N_{d+d',\varkappa}(\nabla_A f')). \tag{6.3}$$

Proof. – We consider the case where f depends only on A, and f' on E, from which the general case follows easily. Write f, f' in the form (5.5):

$$f(\mathrm{A}) = \sum_{k_1 \ldots k_l} f_{k_1 \ldots k_l} \mathrm{B}_{k_1}(\mathrm{A}) \ldots \mathrm{B}_{k_l}(\mathrm{A}),$$

$$f'(\mathrm{E}) = \sum_{j_1 \ldots j_n} f'_{j_1 \ldots j_n} \mathrm{B}'_{j_1}(\mathrm{E}) \ldots \mathrm{B}'_{j_n}(\mathrm{E});$$

then

$$\nabla_{\mathrm{E}} f'(\mathrm{E}) = \sum_{j_1 \ldots j_n} n f'_{k_1 \ldots k_l} \mathrm{B}'_{j_1}(\mathrm{E}) \ldots \mathrm{B}'_{j_{n-1}}(\mathrm{E}) \phi'_{j_n},$$

and there follows

$$\{f', f\}(y) = \sum_{k_1 \ldots k_l j_1 \ldots j_n} nl f_{k_1 \ldots k_l} f'_{j_1 \ldots j_n} \mathrm{B}_{k_1}(\mathrm{A}) \ldots \mathrm{B}_{k_{l-1}}(\mathrm{A}) \mathrm{B}'_{j_1}(\mathrm{E}) \ldots \mathrm{B}'_{j_{n-1}}(\mathrm{E}) \times \int \langle \phi_{k_l}(x), \phi'_{j_n}(x) \rangle_{\mathbb{R}^3} d^3 x. \quad (6.4)$$

The modulus of the integral is less than $\| \phi_{k_l} \|_{\varkappa} \| \phi'_{j_n} \|_{\varkappa}$, and therefore the norm of (6.4) is less than

$$\left(\sum_{k_1 \ldots k_l} \frac{1}{\mathrm{R}_1 d'} \| f_{k_1 \ldots k_l} \|_{\mathscr{S}} [\mathrm{R}_1 (1-d)]^l \| \phi_{k_1} \|_{\varkappa} \ldots \| \phi_{k_l} \|_{\varkappa} \right) \times \left(\sum_{j_1 \ldots j_n} n | f'_{j_1 \ldots j_n} | [\mathrm{R}_1 (1-d-d')]^{n-1} \| \phi'_{j_1} \|_{\varkappa} \ldots \| \phi'_{j_n} \|_{\varkappa} \right)$$

which coincides with the r.h.s. of (6.3). Here we used the inequality

$$l (1-d-d')^{l-1} \leqq \frac{(1-d)^l}{d'}. \quad (6.5)$$

□

LEMMA 6.2. – *Let $f, f' \in \mathscr{E}$, with $f : \Delta_{\mathrm{R}, d} \to \mathscr{S}$ and $f' : \Delta_{\mathrm{R}, d+d'} \to \mathbf{C}$, be two functions depending only on* M *and* θ*; then*

$$\mathrm{N}_{d+d', \varkappa}(\{f', f\}) \leqq \mathrm{N}_{d, \varkappa}(f) \left[\frac{1}{\mathrm{R}_3 d'} \mathrm{N}_{d+d', \varkappa}\left(\frac{\partial f'}{\partial \theta} \right) + \frac{1}{\mathrm{R}_4 e d'} \mathrm{N}_{d+d', \varkappa}\left(\frac{\partial f'}{\partial \mathrm{M}} \right) \right]. \quad (6.6)$$

Proof. – Write

$$f = \sum_{kn} f_{kn} e^{ik\theta} \mathrm{M}^n, \qquad f' = \sum_{k' n'} f'_{k' n'} e^{ik'\theta} \mathrm{M}^{n'};$$

then, the l.h.s. of (6.6) is less than

$$\sum_{knk'n'} |f_{kn}| \| f'_{k'n'} \|_{\mathcal{S}} (n|k'|+n'|k|) \, e^{R_4(1-d-d')(|k|+|k'|)} [R_3(1-d-d')]^{n+n'-1}$$
$$\leqq \sum_{kn} n e^{R_4(1-d)|k|} [R_3(1-d-d')]^{n-1} |f_{kn}|$$
$$+ \sum_{k'n'} \| k' f'_{k'n'} \|_{\mathcal{S}} e^{R_4(1-d-d')|k'|} [R_3(1-d-d')]^{n'}$$
$$+ \sum_{kn} |k| e^{R_4(1-d-d')|k|} [R_3(1-d)]^{n} |f_{kn}|$$
$$\times \sum_{k'n'} \| f'_{k'n'} \|_{\mathcal{S}} n' e^{R_4(1-d-d')|k|} [R_3(1-d-d')]^{n'-1},$$

which, using again inequality (6.5) and the trivial inequality

$$x e^{-xR_4 d'} \leqq \frac{1}{e R_4 d'}, \qquad \forall x>0, \tag{6.7}$$

is easily seen to be smaller than the r.h.s. of (6.6).

□

LEMMA 6.3. – *Let $f, f' \in \mathcal{E}$, with $f : \Delta_{R,d} \to \mathcal{S}$ and $f' : \Delta_{R,d+d'} \to \mathbf{C}$; then*

$$N_{d+d',\varkappa}(\{f', f\}) \leqq \frac{1}{\mathcal{A} d'} N_{d,\varkappa}(f) N^{\nabla}_{d+d',\varkappa}(f'). \tag{6.8}$$

Moreover, if $\mathcal{S} = \mathbf{C}$, then

$$N^{\nabla}_{d+d',\varkappa}(\{f', f\}) \leqq \frac{2}{\mathcal{A} d'} N^{\nabla}_{d,\varkappa}(f(N^{\nabla}_{d,\varkappa}(f'). \tag{6.9}$$

Proof. – Equation (6.8) is obtained by a straightforward calculation from the results of lemmas 6.1, 6.2. In order to obtain (6.9) it is enough to remark that

$$N_{d+d',\varkappa}(\nabla_E \{f, f'\}) \leqq N_{d+d',\varkappa}(\{\nabla_E f, f'\}) + N_{d+d',\varkappa}(\{f, \nabla_E f')$$
$$\leqq \frac{1}{\mathcal{A} d'} N_{d,\varkappa}(\nabla_E f) N^{\nabla}_{d+d',\varkappa}(f') + \frac{1}{\mathcal{A} d'} N^{\nabla}_{d+d',\varkappa}(f) N_{d,\varkappa}(\nabla_E f'),$$

and so on. Then, applying the very definition of $N^{\nabla}_{.,.}(.)$, one immediately obtains (6.9). □

We point out that using a technique which is almost identical to that used for the proofs of the above lemmas, it is easy to prove equation (5.11)

LEMMA 6.4. – *Let ϕ be an $H^{\{1/2\}}$-$H^{\{-1/2\}}$-analytic function; then*

$$\| |\Delta|^{1/2} \phi \|_{\varkappa(1-\delta)} \leqq \frac{\sqrt{3}}{\delta\kappa} \| \phi \|_{\varkappa}.$$

Proof. – First notice that the differential $d\phi_z$ of the application ϕ_z (*see* (5.3)) at z is given by

$$d\phi_z: \quad \mathbf{C}^3 \to \mathrm{H}^{\{1/2\}} \cap \mathrm{H}^{\{-1/2\}}$$
$$w \mapsto \sum_{k=1}^{3} w_k \partial_k \phi(x-z),$$

and therefore, using Cauchy inequality [30],

$$\|(\partial_k \phi)\|_{\varkappa(1-\delta)} \leqq \sup_{|z|<\varkappa(1-\delta)} \|d\phi_z\| \leqq \frac{1}{\kappa\delta} \|\phi\|_{\varkappa}. \tag{6.10}$$

On the other hand, we have

$$\||\Delta|^{1/2}\phi\|^2_{\{1/2\}} = \langle |\Delta|^{1/2}\phi_z, |\Delta|^{1/2}\phi_z \rangle_{\{1/2\}}$$
$$= \langle \phi_z, -\Delta\phi_z \rangle_{\{1/2\}} = \sum_k \|\partial_k \phi_z\|^2_{\{1/2\}}. \tag{6.11}$$

In order to get the thesis we have to prove a similar relation for the norm $\mathrm{H}^{\{-1/2\}}$. To this end, consider the following set

$$\{ f \in \mathrm{H}^{\{1/2\}}_* : \exists g \in \mathrm{H}^{\{1/2\}}_* \text{ with } |\Delta|^{1/2} g = f \},$$

which is dense in $\mathrm{H}^{\{1/2\}}_*$. For f in such a set, we have

$$\langle |\Delta|^{1/2}\phi, f \rangle_{\mathrm{L}^2} = \sum_k \langle i\partial_k \phi, i\partial_k g \rangle_{\mathrm{L}^2}$$
$$\leqq \sqrt{\sum_k \|\partial_k \phi\|^2_{\{-1/2\}}} \sqrt{\sum_k \|\partial_k g\|^2_{\{1/2\}}} = \sqrt{\sum_k \|\partial_k \phi\|^2_{\{-1/2\}}} \, \||\Delta|^{1/2} g\|_{\{1/2\}},$$

where the last equality is due to (6.11). It follows that

$$\||\Delta|^{1/2}\phi\|_{\{-1/2\}} \leqq \sqrt{\sum_k \|\partial_k \phi\|^2_{\{-1/2\}}},$$

which, together with (6.11), proves the thesis, □

LEMMA 6.5. – *Let* $f \in \mathscr{E}, f : \Delta_{\mathrm{R},d} \to \mathbf{C}$ *then*

$$\mathrm{N}^{\nabla}_{d+d', \varkappa(1-\delta)}\left(\left\{ \hat{h}, f \right\} \right) \leqq \frac{2\sqrt{3}\,c}{\kappa\delta\, d'} \mathrm{N}^{\nabla}_{d,\varkappa}(f). \tag{6.12}$$

Proof. – We shall consider only the case where f depends only on A. Write

$$f(\mathrm{A}) = \sum_{k_1 \dots k_l} f_{k_1 \dots k_l} \mathrm{B}_{k_1}(\mathrm{A}) \dots \mathrm{B}_{k_l}(\mathrm{A});$$

then, we have

$$\{\hat{h}, f\} = \sum_{k_1 \dots k_l} l f_{k_1 \dots k_l} \mathrm{B}_{k_1}(\mathrm{A}) \dots \mathrm{B}_{k_{l-1}}(\mathrm{A}) \{\hat{h}, \mathrm{B}_{k_l}(\mathrm{A})\}.$$

But

$$\{\hat{h}, \mathrm{B}_{k_l}(\mathrm{A})\} = c \langle |\Delta|^{1/2} \mathrm{E}, \phi_{k_l} \rangle_{\mathrm{L}^2} = \langle \mathrm{E}, c|\Delta|^{1/2} \phi_{k_l} \rangle_{\mathrm{L}^2},$$

so there follows

$$\nabla_{A}\{\hat{h},f\}=\sum_{k_1\dots k_l} l(l-1) f_{k_1\dots k_l}\langle A,\phi_{k_1}\rangle\dots\langle A,\phi_{k_{l-2}}\rangle \times\phi_{k_{l-1}}\langle cE,|\Delta|^{1/2}\phi_{k_l}\rangle,$$

$$\nabla_{E}\{\hat{h},f\}=\sum_{k_1\dots k_l} l(l-1) f_{k_1\dots k_l}\langle A,\phi_{k_1}\rangle\dots\langle A,\phi_{k_{l-1}}\rangle\, c|\Delta|^{1/2}\phi_{k_l},$$

where all scalar products are taken in L^2. Then, the l.h.s. of (6.12) is less than

$$2c\sum_{k_1\dots k_l} 1(l-1)|f_{k_1\dots k_l}|[R_1(1-d-d')]^{l-1} \times\|\phi_{k_1}\|_{\varkappa(1-\delta)}\cdots\|\phi_{k_{l-1}}\|_{\varkappa(1-\delta)}\||\Delta|^{1/2}\phi_{k_l}\|_{\varkappa(1-\delta)},$$

which, using inequality (6.5) and lemma 6.4, is seen to be less than the r.h.s. of (6.12). □

LEMMA 6.6. – *Let* $\{\chi_l\}_{l\geq 1}$ *be a generating sequence with*

$$N^{\nabla}_{d,\varkappa}(\chi_l)\leqq\frac{\beta^{l-1}}{l}\Phi,$$

and let g be E, A, π, ξ, M, θ *or a generic function of class* $\mathscr{E}$*; then for the r-th term* ($r\geqq 1$) *of the sequence* $T_\chi g$ [*cf.* (4.1)] *one has*

$$N_{d,\varkappa}(g_r)\leqq\Phi\left(\frac{e\Phi}{\mathscr{A} d}+\beta\right)^{r-1}A_1, \tag{6.13}$$

where A_1 *is a constant determined by the condition*

$$\frac{N_{d,\varkappa}(\{\chi,g\})}{N^{\nabla}_{d,\varkappa}(\chi)}\leqq A_1,\qquad \forall\chi\in\mathscr{E}\quad\text{with}\quad N^{\nabla}_{d,\varkappa}(\chi)<\infty. \tag{6.14}$$

Moreover, concerning the "gradient norm" $N^{\nabla}_{\cdot,\cdot}(.)$ *of the r-th terms of* $T_\chi\hat{h}$ *and of* $T_\chi f$ *(for a function* $\mathscr{E}\ni f:\Delta_{R,d}\to C$), *we have*

$$\left.\begin{aligned} N^{\nabla}_{d+d',\varkappa(1-\delta)}(\hat{h}_r)&\leqq\Phi\left(\frac{2e\Phi}{\mathscr{A}d'}+\beta\right)^{r-1}\frac{2\sqrt{3}c}{\kappa\delta d'},\\ N^{\nabla}_{d+d',\varkappa}(f_r)&\leqq\Phi\left(\frac{2e\Phi}{\mathscr{A}d'}+\beta\right)^{r-1}\frac{2N^{\nabla}_{d,\varkappa}(f)}{\mathscr{A}d'}.\end{aligned}\right\} \tag{6.15}$$

Proof. – Exploiting the results of lemmas 6.3, 6.5, the proof becomes a little variant of the proof of lemma 10.3 of reference [24], and therefore is omitted. □

Proof of Proposition 5.1. – First notice that

$$N_{d,\varkappa}(\{E,\chi\})\leqq\frac{N^{\nabla}_{d,\varkappa}(\chi)}{R_1}.$$

Using the result of lemma 6.6, it is immediate to obtain

$$\sqrt{a}\,\| E - E' \|_{\{1/2\}} \leqq \frac{\Phi}{R_1}\left[1-\left(\frac{e\Phi}{\mathscr{A}\,d}+\beta\right)\right]^{-1}.$$

The l.h.s. of this relation is surely less than $R_1\,d$ if

$$\frac{\Phi}{R_1^2\,d}\left[1-\left(\frac{e\Phi}{\mathscr{A}\,d}+\beta\right)\right]^{-1} \leqq 1,$$

which is evidenced by equation (5.13). The other inequalities (5.14) can be proved in the same way. □

LEMMA 6.7. – *Let* Ψ_s *be defined by* (4.6), *assume that the diophantine condition* (5.17) *holds, and that* (5.15) *is satisfied; then, the homological equation* (4.5) *has solutions* χ_s, Z_s *satisfying*

$$\left.\begin{aligned} N^{\nabla}_{d,\varkappa}(\chi_s) &\leqq \frac{1}{\omega\alpha_s} N^{\nabla}_{d,\varkappa}(\Psi_s), \\ N^{\nabla}_{d,\varkappa}(Z_s) &\leqq N^{\nabla}_{d,\varkappa}(\Psi_s). \end{aligned}\right\} \tag{6.16}$$

Proof. – The proof is a straightforward generalization of the usual one. □

Proof of proposition 5.2. – First we transform (4.6), putting it in the more suitable form

$$\left.\begin{aligned} \Phi_1 &= H_1 \\ \Psi_s &= \sum_{l=1}^{s-1} \frac{l}{s}\{\chi_l, Z_{s-l}\} + \frac{s-1}{s}\{\chi_{s-1}, \hat{h}\} + \sum_{l=1}^{s} \frac{l}{s} H_{l,s-l} + \frac{1}{s}\hat{h}_{s-1}. \end{aligned}\right\} \tag{6.17}$$

For the proof of the equivalence of (6.17) and (4.6) *see* reference [24] (p. 593). Then, following reference [24], we introduce the sequence

$$d_s := \left(\frac{s-1}{r-1}\right) d, \qquad s = 1, \ldots, r;$$

and look for sequences $\{\eta_s\}_{1 \leqq s \leqq r}$, $\{\zeta_s\}_{0 \leqq s \leqq r-1}$, $\{\bar{\zeta}_{l,s}\}_{1 \leqq l \leqq r}^{0 \leqq s \leqq r-1}$, such that one has

$$\begin{aligned} N^{\nabla}_{d_s,\varkappa(1-d_s)}(\Psi_s) &\leqq \eta_s \mathscr{F}, && 1 \leqq s \leqq r, \\ N^{\nabla}_{d_s+1,\varkappa(1-d_{s+1})}(\hat{h}_s) &\leqq \zeta_s E_0, && 1 \leqq s \leqq r-1, \\ N^{\nabla}_{d_{l+s},\varkappa(1-d_{l+s})}(H_{l,s}) &\leqq \bar{\zeta}_{l,s} \mathscr{F}, && 1 \leqq l \leqq r, \quad 0 \leqq s \leqq r-l, \end{aligned}$$

with

$$E_0 := \frac{\sqrt{3}\,c\,\mathscr{A}}{\kappa\,d}.$$

Using the results of lemmas 6.3, 6.5, and 6.7 it is easy to see that the above sequences can be defined by:

$$\left.\begin{aligned} \zeta_{l,0} &= \gamma^{l-1}, \qquad 1 \leqq l \leqq r, \\ \zeta_0 &= 1, \\ \eta_1 &= 1, \end{aligned}\right\} \tag{6.18}$$

and

$$\left.\begin{aligned} \tilde{\zeta}_{l,s} &= \frac{\tilde{C}\mathscr{F}}{s} \sum_{j=1}^{s} j \eta_j \tilde{\zeta}_{l,s-j}, \qquad 1 \leqq l \leqq r-1, \qquad 1 \leqq s \leqq r-l, \\ \zeta_s &= \frac{\tilde{C}\mathscr{F}}{s} \frac{\tilde{C}\mathscr{F}}{s} \sum_{j=1}^{s} j \eta_j \zeta_{s-j}, \qquad 1 \leqq s \leqq r-1, \\ \eta_s &= \frac{\tilde{C}\mathscr{F}}{s} \sum_{l=1}^{s} l \eta_l \eta_{s-l} + \frac{(s-1)\tilde{C}E_0}{s} \eta_{s-1} \\ &\quad + \frac{1}{s} \sum_{l=1}^{s} l \tilde{\zeta}_{l,s-l} + \frac{E_0}{s\mathscr{F}} \zeta_{s-1}, \qquad 2 \leqq s \leqq r, \end{aligned}\right\} \tag{6.19}$$

with

$$\tilde{C} = \frac{2(r-1)^2}{\omega \alpha_r \mathscr{A} d}.$$

This sequence coincides with that defined by equations (11.9), (11.10) of reference [24], and therefore can be estimated in the same way (p. 595-596 of that paper), obtaining the bound

$$\eta_s \leqq \frac{1}{s} \left(\frac{6(r-1)^2}{\omega \alpha_r d} \left(\frac{\mathscr{F}}{\mathscr{A}} + \frac{\sqrt{3}\, c}{\kappa d} \right) + 2\gamma \right)^{s-1}. \quad \square$$

LEMMA 6.8. – *For any positive $d < 1/2$, and $s \geqq 1$, we have*

$$N^{\nabla}_{2d, \varkappa(1-d)}(h_s) \leqq \frac{1}{s} \left(\frac{2e\Phi}{\mathscr{A} d} + \beta \right)^{s-1} \mathscr{F}. \tag{6.20}$$

Proof. – The proof is identical to that of lemma 11.3 of reference [24], and therefore is omitted. $\square$

LEMMA 6.9. – *Under the hypotheses of proposition* 5.1, *assume also that the sequence* α_s (cf. (5.17)) *has the form* $\alpha_s = \nu/((s+2)\,K)^2$*; define* μ *according to* (5.20) *and assume* (5.21)*; then there exists a real analytical canonical transformation* T_χ *from* $\Delta_{R,2d}$ *to* $\Delta_{R,d}$ *which puts the Hamiltonian* *into normal form up to some optimal order* r_{opt}. *Moreover, for this transformation, one has the bounds*

$$\left.\begin{aligned} N^{\nabla}_{2d,\,\varkappa(1-d)}(\mathcal{R}^{r_{\text{opt}}}) &\leqq 2\,\mathcal{F}\,\mu^{1/4}(e+1)\,e^9\exp\left[-\frac{4}{e}\left(\frac{1}{\mu}\right)^{1/4}\right] \\ |M-M'| &\leqq \frac{\mathcal{F}}{\omega\mu^{1/2}}\,\frac{K^2(e+1)}{\nu e^2 R_4} \\ |I(\pi,\xi)-I(\pi',\xi')| &\leqq \frac{\mathcal{F}}{\omega\mu^{1/2}}\,\frac{K^2(e+1)}{\nu e^2}, \end{aligned}\right\} \tag{6.21}$$

where $I(\pi,\xi):=i\pi\xi$ *is the radial action.*

Proof. — The proof of this lemma can be obtained easily by exploiting the results of lemmas 6.6, 6.8, and of proposition 5.2, and following closely the proof of theorems 7.1, 7.3 of reference [24]: it is a long but straightforward calculation which is omitted; we just recall that equation (5.21) ensures that $r_{\text{opt}}\geqq 1$ and that the transformation T_χ converges. □

Proof of Theorem 5.3. — Fix $d=1/4$; then we prove that the r.h.s. of the first two equations of (5.23) are positive. In fact, this is surely true if

$$\frac{1}{16}>\frac{4\,\mathcal{F}K^2}{\omega\mathcal{A}\nu}\,\frac{3(e+1)}{e^2}\,\frac{1}{4\mu^{1/2}}.$$

But the r.h.s. is smaller than $\mu^{1/2}/4$ (*see* the definition of μ), and moreover, due to (5.21) this is always less than the l.h.s.

We come to the bounds (5.25). Just calculating the Poisson brackets of I′ and M′ with the remainder, using the definition of T [*cf.* (5.24)], and observing that

$$|I(t)-I(0)|\leqq|I'(t)-I(t)|+|I'(t)-I'(0)|+|I'(0)-I(0)|,$$

(and similarly for M), and using (6.21), one obtains that bounds (5.25) hold for a time which is the minimum between T and the escape time of the variables from the domain $\Delta_{R,\,3/4}$. We shall prove that this escape time is larger than T.

Denote by T_E, T_A, T_π, T_ξ, T_M the escape times of E, A, π, ξ, M respectively, and assume for simplicity that they are different (the case of some equalities is essentially identical to this one). Then, one has to distinguish two cases:

1) the smallest escape time is T_π, T_ξ or T_M,

2) the smallest escape time is T_E or T_A.

Consider first case 1); we shall give the details of the proof only in the subcase where $T_\pi<T_\xi$, T_M, the proof being similar in the other subcases.

Assume that $T_\pi<T$, and notice that, for real initial data, we have

$$|\pi(t)|^2=I(t),$$

and therefore

$$|\pi(t)|^2 \leqq |I(t)-I(0)| + |\pi(0)|^2. \tag{6.22}$$

Let $\vartheta > 0$ be such that

$$\vartheta < \frac{R_2^2}{16} - \frac{\mathscr{F}}{\omega\mu^{1/2}} \frac{3\,K^2(e+1)}{e^2 \nu} - |\pi(0)|^2$$

[this ϑ exists in virtue of (5.23)]. Then, by virtue of the continuity in time of $\pi(t)$ (*cf.* theorem 2.1 of reference [1]), there exists a t_1 less than T_π such that

$$|\pi(t_1)|^2 = \frac{R_2^2}{16} - \vartheta;$$

but, at time t_1 inequality (5.24) still holds, and this, together with (6.22) and (5.25), implies

$$|\pi(t_1)|^2 < \frac{R_2^2}{16} - \vartheta,$$

which is a contradiction.

Consider now case 2), in the subcase $T_E < T_A$. We proceed in way similar to case 1). Let $\vartheta > 0$ be such that

$$\vartheta < \frac{c}{a} \frac{R_1^2}{16} - \left(2\,\mathscr{F}' + \frac{3\,K^2(e+1)}{\nu e^2}\left(\sqrt{b+2} + \frac{1}{R_4}\right)\frac{\mathscr{F}}{\mu^{1/2}}\right) - \hat{h}(0);$$

then there exists a time $t_1 < T_E$ such that

$$\hat{h}(t_1) \geqq c\,\|E(t_1)\|_{\{1/2\}}^2 = \frac{c}{a} \frac{R_1^2}{16} - \vartheta;$$

but at time t_1 we are still in the considered domain, and inequalities (5.25) still hold. Then, by energy conservation there follows

$$\hat{h}(t_1) \leqq \hat{h}(0) + \omega_r |I(t)-I(0)| + \omega |M(t) - M(0)| + |H_{int}(0)| + |H_{int}(t)| < \frac{c}{a} \frac{R_2^2}{16} - \vartheta. \quad \square$$

In the following lemmas we shall make use of real variables $\hat{\pi}$, $\hat{\xi}$ defined by

$$\hat{\pi} := \frac{1}{\sqrt{m\,\omega_r}} p, \qquad \hat{\xi} := \sqrt{m\,\omega_r}\,(r - r_c). \tag{6.23}$$

When we pass from the variables $\hat{\pi}$, $\hat{\xi}$ to the variable π, ξ, the norm of a function changes according to

LEMMA 6.10. — *Let* $\hat{f}(\mathrm{E}, \mathrm{A}, \hat{\pi}, \hat{\xi}, \mathrm{M}, \theta) \in \mathscr{E}$ *be a function of the form*

$$\hat{f} = \sum_{m+l<s} \hat{f}_{ml}(\mathrm{E}, \mathrm{A}, \mathrm{M}, \theta)\, \hat{\pi}^l \hat{\xi}^m,$$

and let $f(\mathrm{E}, \mathrm{A}, \pi, \xi, \mathrm{M}, \theta)$ *be the transformed function under the canonical transformation*

$$\hat{\pi} = \frac{1}{\sqrt{2}}(\pi + i\xi), \qquad \hat{\xi} = \frac{i}{\sqrt{2}}(\pi - i\xi);$$

then, we have

$$\mathrm{N}_{d,\varkappa}(f) \leqq 2^{s/2}\, \mathrm{N}_{d,\varkappa}(\hat{f}). \tag{6.24}$$

Proof. — *See* reference [24], proof of lemma 9.1. □

The following lemma essentially shows that, if $|\rho|_\sigma < \infty$, then $\mathrm{H}_{\mathrm{int}}$ is of class $\mathscr{E}$.

LEMMA 6.11. — *Let* ρ *be a function such that the application* ρ_z cf. *equation* (5.3) *can be extended to a complex analytic mapping on the strip* $|\mathrm{Im}\, z_k| < 2\sigma$ *for some positive* σ, *and assume* $|\rho|_\sigma$ [cf. (5.26)] *is finite. Fix* r_c *and define*

$$\left.\begin{aligned} \mathrm{R}_4 &:= \frac{1}{1+d}\min\left\{1, \frac{1}{3}\log\left[\frac{\sigma}{2r_c} + \sqrt{\frac{\sigma^2}{4r_c^2}+1}\right]\right\}, \\ \zeta &:= \sqrt{m\,\omega_r}\,\frac{\sigma}{4e^3}; \end{aligned}\right\} \tag{6.25}$$

assume also that there exists $\mathrm{C}_1 > 1$ *such that*

$$\frac{\mathrm{R}_2(1+d)}{\zeta} \leqq \mathrm{C}_1\, e^{-(\mathrm{K}+1)\mathrm{R}_4(1+d)}, \qquad \mathrm{C}_1\, e^{-\mathrm{KR}_4(1+d)} < 1. \tag{6.26}$$

Consider the functional

$$f(\mathrm{A}, \hat{\xi}, \theta) = \int \langle \sqrt{4\pi c}\,|\Delta|^{-1/4}\rho(x - q(\hat{\xi}, \theta)), \mathrm{A}(x)\rangle_{\mathbb{R}^3}\, d^3x,$$

with

$$\begin{aligned} q(\hat{\xi}, \theta) &= (q_1(\hat{\xi}, \theta), q_2(\hat{\xi}, \theta), 0) \\ &:= \left(\left(r_c + \frac{\hat{\xi}}{\sqrt{m\,\omega_r}}\right)\cos\theta, \left(r_c + \frac{\hat{\xi}}{\sqrt{m\,\omega_r}}\right)\sin\theta, 0\right), \end{aligned}$$

and decompose it in the form

$$f(\mathrm{A}, \hat{\xi}, \theta) = \sum_{s \geqq 0} f_s(\mathrm{A}, \hat{\xi}, \theta), \tag{6.27}$$

with

$$f_s(\mathrm{A}, \xi, \theta) = \sum_{2s \leqq (|k|/\mathrm{K}+l) < 2(s+1)} e^{ik\theta} \xi^l \int \langle \rho_{kl}(x), \mathrm{A}(x) \rangle_{\mathbb{R}^3} d^3 x. \quad (6.28)$$

Then, we have

$$\mathrm{N}_{-d,\sigma}(f_s) \leqq 2\,\mathrm{K}\sqrt{4\pi c a}\,\frac{e^{-2\mathrm{R}_4(1+d)}}{(1-e^{-2\mathrm{R}_4(1+d)})^2} \times |\rho|_\sigma \mathrm{R}_1(1+d)(\mathrm{C}_1 e^{-\mathrm{K}(1+d)\mathrm{R}_4})^{2s}. \quad (6.29)$$

Proof. — First notice that, according to the definition of q, one has

$$\sup_{|\operatorname{Im}\theta| < 3\mathrm{R}_4(1+d)|} \sup_{|\xi| \leqq \zeta} |\operatorname{Im} q_k(\xi, \theta)| \leqq \sigma,$$

and therefore, by the exponential estimate of the Fourier coefficients of an analytic function and by Cauchy inequality, writing

$$\sqrt{4\pi c}\,|\Delta|^{-1/4} \rho(x - q(\xi, \theta)) = \sum_{kl} \rho_{kl}(x) e^{ik\theta} \xi^l,$$

one has

$$\max\left\{\sup_{|z|<\sigma} \sqrt{a}\,\|(\rho_{kl})_z\|_{\{1/2\}}, \sup_{|z|<\sigma} \frac{1}{\sqrt{a}}\|(\rho_{kl})_z\|_{\{-1/2\}}\right\} \leqq \sqrt{4\pi c a}\,|\rho|_\sigma \frac{e^{-3\mathrm{R}_4(1+d)|k|}}{\zeta^l}.$$

From this one has

$$\begin{aligned}
\frac{\mathrm{N}_{-d,\sigma}(f_s)}{\sqrt{4\pi c a}} &\leqq \mathrm{R}_1(1+d)|\rho|_\sigma \sum_{2s \leqq (|k|/\mathrm{K}+l) < 2(s+1)} \left(\frac{\mathrm{R}_2(1+d)}{\zeta}\right)^l e^{-2\mathrm{R}_4(1+d)|k|} \\
&< \mathrm{R}_1(1+d)|\rho|_\sigma \sum_{2s \leqq n+l < 2(s+1)} \left(\frac{\mathrm{R}_2(1+d)}{\zeta}\right)^l \sum_{n\mathrm{K} \leqq |k| < (n+1)\mathrm{K}} e^{-2\mathrm{R}_4(1+d)|k|} \\
&< \mathrm{R}_1(1+d)|\rho|_\sigma \sum_{2s \leqq n+l < 2(s+1)} \left(\frac{\mathrm{R}_2(1+d)}{\zeta}\right)^l 2\,\mathrm{K} e^{-2\mathrm{K}\mathrm{R}_4(1+d)n} \\
&\leqq \mathrm{R}_1(1+d)|\rho|_\sigma \sum_{2s \leqq m} 2\,\mathrm{K}\mathrm{C}_1^m e^{-\mathrm{K}\mathrm{R}_4(1+d)m} m e^{-\mathrm{R}_4(1+d)m},
\end{aligned}$$

which is less than the r.h.s. of (6.29). □

LEMMA 6.12. — *Let*

$$\rho(x) := \frac{1}{a^3 \pi^{3/2}} \exp\left(-\frac{x^2}{a^2}\right);$$

then we have

$$|\rho|_\sigma \leq \frac{1}{(2\pi)^{3/4} a^{3/2}} \exp\left(12\frac{\sigma^2}{a^2}\right).$$

Proof. – First calculate, for $|\operatorname{Im} z_k| < 2\sigma$,

$$\|\rho_z\|_{L^p} = \frac{1}{(p\,\pi^{p-1})^{3/(p-1)}} \frac{1}{a^{3(p-1)/p}} \exp\left(12\frac{\sigma^2}{a^2}\right),$$

and then notice that, by Sobolev embedding theorem,

$$\|\rho_z\|_{\{-1\}} \leq K_1 \|\rho_z\|_{L^{6/5}}$$

for some numerical K_1. □

LEMMA 6.13. – *In terms of the variables* E, A, $\hat{\pi}$, $\hat{\xi}$, M, θ, *the quantity* H_{int} *has the form*

$$\begin{aligned} H_{int} = \frac{M_c^2}{2mr_c^2} \sum_{k\geq 3} (-1)^k (k+1) \left(\frac{\hat{\xi}}{r_c\sqrt{m\,\omega_r}}\right)^k \\ + a\frac{r_c^b}{b} \sum_{k\geq 3} \frac{1}{k!} \left(\prod_{l=0}^{k-1} (b-l)\right) \left(\frac{\hat{\xi}}{r_c\sqrt{m\,\omega_r}}\right)^k \\ + \frac{M_c}{mr_c^2} M \sum_{k\geq 1} (-1)^k (k+1) \left(\frac{\hat{\xi}}{r_c\sqrt{m\,\omega_r}}\right)^k \\ + \frac{M^2}{2mr_c^2} \sum_{k\geq 0} (-1)^k (k+1) \left(\frac{\hat{\xi}}{r_c\sqrt{m\,\omega_r}}\right)^k \\ + \frac{M_c A_\theta}{mr_c} \sum_{k\geq 0} (-1)^k \left(\frac{\hat{\xi}}{r_c\sqrt{m\,\omega_r}}\right)^k + \frac{MA_\theta}{mr_c} \sum_{k\geq 0} (-1)^k \left(\frac{\hat{\xi}}{r_c\sqrt{m\,\omega_r}}\right)^k \\ + \frac{\hat{\pi} A_r \sqrt{\omega_r}}{\sqrt{m}} + \frac{1}{2m} A_r^2 + \frac{1}{2m} A_\theta^2. \end{aligned} \quad (6.30)$$

Proof. – A trivial calculation. □

Proof of theorem 2.1. – Consider the function

$$\begin{aligned} H_{red} = -\frac{2M_c^2}{mr_c^2}\left(\frac{\hat{\xi}}{r_c\sqrt{m\,\omega_r}}\right)^3 + \frac{ar_c^b}{6}(b-1)(b-2)\left(\frac{\hat{\xi}}{r_c\sqrt{m\,\omega_r}}\right)^3 \\ + 2\omega M \left(\frac{\hat{\xi}}{r_c\sqrt{m\,\omega_r}}\right) + \frac{M_c (A_\theta)_0}{mr_c}, \end{aligned} \quad (6.31)$$

where $(A_\theta)_0$ is the first term of the development of A_θ according to (6.27). Function (6.31) will turn out to be the dominant part of H_{int}. We fix $\mathscr{A}$, R_1, R_3 according to (5.27), (5.28), and notice that, since $(A_\theta)_0$ is linear in A, we can put

$$N_{-d,\sigma}((A_\theta)_0) := A_0 R_1,$$

with A_0 independent of R_1. Then we calculate

$$\frac{N_{-d,\sigma}(H_{red})}{\mathscr{A}\psi},$$

as a function of R_2, and minimize it over R_2, obtaining

$$\left.\begin{aligned} R_2^2 &= k_7 \frac{r_c^2\,\omega\,A_0\sqrt{m}}{\sqrt{c/a}} R_4^2, \\ \frac{N_{-d,\sigma}(H_{red})}{\mathscr{A}\,\omega} &= \sqrt{\frac{k_8\,A_0}{\sqrt{mc/a}}}, \end{aligned}\right\} \tag{6.32}$$

where k_7, k_8 are dimensionless constants depending only on b. In what follows $k_9, \ldots$ will always denote numerical constants with the above properties.

Using lemmas 6.11, 6.12 one can estimate A_0 obtaining (keep in mind that we have performed the transformation (3.1))

$$A_0 \leqq k_9 \exp\left(12\frac{\sigma^2}{a^2}\right) K \frac{1}{R_4^2} \frac{e_0}{a\sqrt{c}}. \tag{6.33}$$

Substituting in (6.32), and choosing $\sigma := a$, we get

$$\left.\begin{aligned} R_2^2 &= k_{10}\,K\,m\,\omega\,r_c^2\,\varepsilon_1^2, \\ \frac{N_{-1/4,\sigma}(H_{red})}{\mathscr{A}\,\omega} &\leqq k_{11}\frac{1}{R_4^2}\varepsilon_1, \end{aligned}\right\} \tag{6.34}$$

where

$$\varepsilon_1^4 = \frac{e_0^2}{mc^2}\,\frac{1}{a}$$

is the ratio of the classical radius of the particle to its true radius. It is now easy (but long) to evaluate $\mathscr{F}$ [using (6.30), (6.24), (5.11)]; then, with the choice

$$K := \frac{1}{R_4(1+d)}\,|\log(\varepsilon)|,$$

we obtain the statement of theorem 2.1. □

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