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P. Collet
A. Galves
B. Schmitt

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Unpredictability of the occurrence time of a long laminar period in a model of temporal intermittency

by

P. COLLET

Centre de Physique Théorique,
École Polytechnique,
91128 Palaiseau Cedex, France
Laboratoire U.P.R. 14 du C.N.R.S.

A. GALVES

Instituto de Matemática e Estatística,
Universidade de São Paulo,
B.P. 20570 (Ag. Iguatemi),
01498 São Paulo SP, Brasil

and

B. SCHMITT

URA 755, Laboratoire de Topologie,
Université de Bourgogne,
B.P. 138, 21004 Dijon, France

ABSTRACT. — We study a piecewise affine non uniformly hyperbolic map of the interval exhibiting type I intermittency. Using a probabilistic approach we prove that the occurrence time of long laminar periods converges in law when suitably renormalized to a mean one exponential random time.

RÉSUMÉ. — Nous étudions une transformation de l'intervalle affine par morceaux mais non uniformément hyperbolique qui présente une intermittence de type I. En utilisant des méthodes probabilistes nous montrons que le temps d'entrée dans une longue phase laminaire convenablement renormalisé converge en distribution vers une loi exponentielle de moyenne 1.

I. INTRODUCTION

In this paper we obtain the asymptotic law of the occurrence time of a long laminar period in a model of temporal intermittency. We consider a kind of Pomeau-Manneville's type 1 model at the transition point. This is a dynamical system defined by a smooth map of the interval [0, 1], which is hyperbolic except for an indifferent fixed point (*see* [P.M.] for a precise definition).

The Bowen-Ruelle invariant probability measure of this map is the Dirac delta measure concentrated at the non hyperbolic fixed point and Cesaro averages along a typical orbit converge to this measure.

On the other hand, the presence of the non hyperbolic fixed point produces the following phenomena which was conjectured by Manneville [M.] and rigorously proven by Bowen [B.] and Collet & Ferrero [C.F.]. If we consider Cesaro averages rescaled by the logarithm of the time, then for functions whose support does not contain the non hyperbolic fixed point, we will get L^1 (but not almost sure cf. Aaronson [A.]) convergence to the integral of the function with respect to a new invariant measure which is not normalizable. This σ-finite measure describes the statistical properties of intermittent events.

We consider the special case of a piecewise affine Markov map. For this model we give the asymptotic law of the time needed to observe a laminar period longer than a fixed length a (when a diverges). More precisely we prove that when suitably renormalized (by a factor $a \log a$) this time converges in law to a mean one exponential random time. For the dynamical system this is the time it takes for an orbit to get very close to the non hyperbolic fixed point.

Before entering into technical details let us discuss briefly the physical implications of this result. Since the statistics of an exponential random variable is so peculiar, this precise information should provide a sensitive test of the adequacy of the model. Also we want to stress the fact that having exponential law is a rigorous way of expressing unpredictability of the occurrence time of the phenomena.

This result will be proven using a probabilistic pathwise approach. We construct an associated Markov chain which comes out from a symbolic dynamics with infinitely many states labeled by the set of positive integers. The model can be illustrated by a flee jumping in one unit of time from the bottom of a slope to the height k ($k \in \mathbb{N}$) with probability $1/k(k+1)$ and then slipping down at constant speed. This is a recurrent motion but since every jump typically takes the flee very high, the time needed to slip back to the bottom has infinite mean.

In this description the bottom of the slope corresponds to the most turbulent region of the dynamical system and the laminar phase is situated at an infinite altitude. In other words, the dynamical system in $[0, 1]$ is a compactification of the infinite states Markov chain. The same markovian description appeared also in [W.].

As far as we know, [B.H.] and [H.] are the first papers to put in evidence exponential random law as limit distributions of the occurence time of rare events. A similar point of view was developed in the so called pathwise approach to metastability introduced in [C.G.O.V.] (*see* [S.] for a recent review of the subject and [G.S.] for a proof of a similar result in the case of hyperbolic systems).

In the next section we will give a precise definition of the model and state our main result. In section 3 we prove an a-priori lower bound for the occurence time of a laminar interval of length longer than a. Finally, in section 4 we conclude the proof of the main theorem.

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II. MODEL AND RESULTS

Let f be the map of the unit interval $[0, 1]$ defined in the following way:

(i) $f(0)=0$,

(ii) f is affine and increasing on $A_0=[1/2, 1]$ and satisfies $f(A_0)=[0, 1]$,

(iii) For any integer $k \geqq 1$, f is affine and increasing on the interval $A_k=[1/(k+2), 1/(k+1)[$ and satisfies $f(A_k)=A_{k-1}$.

The iterations of the map f define a dynamical system on the unit interval which is a simplified (piecewise affine) version of the mappings with an indifferent fixed point which appear in the type 1 intermittency model of Pomeau and Manneville [P.M].

In the Pomeau-Manneville picture the laminar phase corresponds to the neighborhood $[0, 1/2[$ of the origin, whereas the turbulent phase is described by the rest of the phase space. Our main result gives the asymptotic law of the time it takes for the process to enter a vanishingly small neighborhood of the indifferent fixed point. More precisely, for any integer $a \geqq 1$, and for any $x \in]0, 1]$, let $T_a(x)$ be the integer defined by

$$T_a(x) = \inf\left(n \in \mathbb{N} : f^n(x) \leqq \frac{1}{a}\right).$$

We consider $T_a(.)$ as a random variable defined on the standard Lebesgue probability space (i.e. the interval $[0, 1]$ equiped with the standard Borel σ-field and the Lebesgue probability measure λ). We can now formulate our main result

THEOREM 1. — *The random variable* $T_a/a \log a$ *converges in law to a mean one exponential law as* $a \to \infty$.

We remark that T_a is also the time it takes for the process to start a laminar interval longer than a.

The proof will involve an associated Markov chain defined over $\mathbb{N}^{\mathbb{N}}$. As a first step in the definition of this Markov chain, we construct a coding (defined except on a countable set) of the unit interval to the set $\mathbb{N}^{\mathbb{N}}$. This coding is explicitly defined (except for the origin and all it's preimages) by a map $\varphi(x) = (\omega_i)_{i \in \mathbb{N}}$ where ω_i satisfies

$$f^i(x) \in A_{\omega_i} \qquad \text{for} \quad i = 0, 1, \ldots$$

This coding φ is an isomorphism (in the sense of measure theory) between the standard Lebesgue space and the probability space $\mathbb{N}^{\mathbb{N}}$ with the product σ-field and the probability measure $\mathbb{P}$ defined on the cylinders by

$$\mathbb{P}\{\omega \in \mathbb{N}^{\mathbb{N}} : \omega_0 = i_0, \ldots, \omega_n = i_n\} = \frac{1}{(i_0+1)(i_0+2)} \prod_{j=0}^{n-1} Q(i_j, i_{j+1}),$$

where $Q : \mathbb{N} \times \mathbb{N} \to [0, 1]$ is the probability transition given by:

$$\begin{gathered} Q(j, j-1) = 1 \qquad \text{if} \quad j \geqq 1; \\ Q(0, j) = 1/(j+1)(j+2) \qquad \text{for every} \quad j \in \mathbb{N}; \\ Q(i, j) = 0 \qquad \text{otherwise.} \end{gathered}$$

Note that although φ is an isomorphism in the measure theoretic sense, it's range is not the whole set $\mathbb{N}^{\mathbb{N}}$. More precisely, we can define an incidence matrix $\mathscr{A}: \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ by

$$\mathscr{A}_{j, j-1} = 1 \quad \text{if} \quad j \geqq 1;$$
$$\mathscr{A}_{0, j} = 1 \quad \text{for every} \quad j \in \mathbb{N};$$
$$\mathscr{A}_{i, j} = 0 \quad \text{otherwise.}$$

It is easy to verify that the range of φ is the set Ω of sequences $(\omega_i)_{i \in \mathbb{N}}$ which satisfy for any integer i

$$\mathscr{A}_{\omega_i, \omega_{i+1}} = 1.$$

The map φ defines a bijection between the above set of sequences and the points of the unit interval which are not preimages of zero. Note also that the probability measure $\mathbb{P}$ is supported by Ω. From now on our sample space will be Ω equiped with the restriction of the product σ-field and with the probability measure $\mathbb{P}$.

For every integer n we will denote by X_n the projection on the n-th coordinate of the infinite product $\mathbb{N}^{\mathbb{N}}$ (*i.e.* if $\omega = (\omega_0, \omega_1, \ldots)$, then $X_n(\omega) = \omega_n$). With the above definition of $\mathbb{P}$, the sequence of random variables $(X_n)_{n \in \mathbb{N}}$ is a Markov chain on the integers with transition probability Q and initial measure μ given by

$$\mu(i) = \frac{1}{(i+1)(i+2)}.$$

In the isomorphism φ the action of the map f becomes the shift $\mathscr{S}$. This implies that through the coding the time evolution is the same for the Markov chain and the dynamical system starting with a random uniformly chosen initial condition. We will also denote by T_a the composition with φ of the previously defined function T_a.

For the convenience of the reader we will collect a few elementary but usefull formulae and facts concerning this Markov chain.

Let us define recursively an increasing sequence τ_i of integer valued stopping times by

$$\tau_1 = \inf\{n \in \mathbb{N} : X_n = 0\} \tag{2.1}$$

and for $i \geqq 2$

$$\tau_i = \inf\{n \in \mathbb{N} : n > \tau_{i-1};\ X_n = 0\}. \tag{2.2}$$

This sequence of stopping times is the sequence of successive entrance times in the state 0. It is infinite since the Markov chain is irreducible and null recurrent (as follows from the explicit expression of the transiton probability Q).

We also define a sequence of integer valued random variables by

$$U_i = \tau_{i+1} - \tau_i. \tag{2.3}$$

This notation enables us to describe the chain in the following simple way:

$$X_t = X_0 - t \qquad \text{if} \quad t \leqq \tau_1 \tag{2.4}$$

and

$$X_t = U_n - (t - \tau_n) \qquad \text{if} \quad \tau_n < t \leqq \tau_{n+1}, \tag{2.5}$$

for $n \geqq 1$.

It follows in particular that if $X_0 < a$, then

$$T_a(\omega) = T_a(\mathscr{S}^{\tau_1}\omega) + \tau_1, \tag{2.5}$$

and

$$T_a = \tau_1 + 1 + \sum_{j=1}^{J-1} U_j = \tau_J + 1 \tag{2.6}$$

where

$$J = \inf\{j \geqq 1 : U_j - 1 \geqq a\},$$

and the convention that if $J = 1$, then $\sum_{j=1}^{J-1} U_j = 0$.

The Markov property implies that the random variables U_1, U_2, ... are independent, identically distributed and independent of X_0. In particular for any $x < a$ we have

$$\mathbb{P}\{T_a > t \mid X_0 = x\} = \mathbb{P}\left\{x + 1 + \sum_{j=1}^{J-1} U_j > t\right\}. \tag{2.7}$$

It is easy to verify that the common law of these random variables is given by

$$\mathbb{P}(U_i = k) = \frac{1}{k(k+1)} = Q(0, k-1) \qquad \text{for} \quad k \geqq 1.$$

Note that $\mathbb{E}(U_i) = \infty$, and therefore the Markov chain is null recurrent. This is a main difference with the case of uniformly hyperbolic Markovian dynamical systems where one gets positive recurrent chains.

III. PRELIMINARIES

We first introduce some additional notations. Let us call N_n the number of returns to the state 0 up to time n. This number is given by

$$N_n = \sum_{i=1}^{\infty} 1_{\{\tau_i \leqq n\}}.$$

Note also that for $n \geqq a$

$$\{X_0 < a\} \cap \cup_{j=1}^{\infty} \{U_1 \leqq a, \ldots, U_j \leqq a\} \cap \{N_n = j\} \subset \{T_a > n\}. \quad (3.1)$$

We now define a time scale β_a associated with the state a by

$$\beta_a = \min\{n \in \mathbb{N} : \mathbb{P}(T_a \geqq n) \leqq e^{-1}\}.$$

This number is finite since the chain is recurrent. Theorem 1 will follow from the following result

THEOREM 2:

$$\lim_{a \to \infty} \mathbb{P}(T_a > \beta_a t) = e^{-t}, \quad (3.2)$$

$$\lim_{a \to \infty} \frac{\mathbb{E}(T_a)}{\beta_a} = 1, \quad (3.3)$$

$$\lim_{a \to \infty} \frac{\mathbb{E}(T_a)}{a \log a} = 1. \quad (3.4)$$

The main idea of the proof is the following. The time T_a is much larger than the time needed to loose memory from the initial condition. Therefore every unsuccessful trial to overrun level a after the process starts afresh a new run from the origin. In order to fullfill this program we first need an a-priory lower bound for T_a which is derived in Proposition 3.

PROPOSITION 3. — *There exists an increasing positive function* γ *defined on the integers such that*

$$\lim_{a \to \infty} \frac{\gamma(a)}{a} = +\infty, \quad (3.5)$$

and

$$\lim_{a \to \infty} \inf_{x < a} \mathbb{P}(T_a > \gamma(a) \mid X_0 = x) = 1. \quad (3.5)$$

In order to prove this we first need two auxilliary lemmata.

LEMMA 4. — *There exists an increasing integer valued function* l *defined on the integers such that*

$$\lim_{r \to \infty} \frac{l(r)}{r} = \infty,$$

and

$$\lim_{r \to \infty} \mathbb{P}(U_1 + U_2 + \ldots + U_r \leqq l(r)) = 0.$$

Proof. — We consider the Laplace transform of the random variable

$$W_r = \sum_{i=1}^{r} U_i.$$

From the independence of the U_i's and Chebychev's inequality we have for $t \in [0, 1[$

$$\mathbb{P}(W_r \leqq l) \leqq t^{-l}[\mathbb{E}(t^{U_1})]^r \leqq t^{-l}\left(1+\frac{1-t}{t}\log(1-t)\right)^r.$$

If we set $t=1-u$ with $u>0$ small we get

$$\mathbb{P}(W_r \leqq l) \leqq e^{lu+ru\log u}\, e^{\mathcal{O}(1)(lu^2+ru^2(\log u)^2)}.$$

A nearly optimal choice for u is

$$u=e^{-1-l/r},$$

and one can check that the choice

$$l(r)=\left[\frac{r\log r}{\log\log r}\right]$$

satisfies the conditions of the lemma.

LEMMA 5:

$$\inf_{x<a} \mathbb{P}\{T_a>t \mid X_0=x\} = \mathbb{P}\{T_a>t \mid X_0=0\}.$$

Proof. – Using formula (2.7) we have for $x<a$

$$\begin{aligned}\mathbb{P}\{T_a>t \mid X_0=x\} &= \mathbb{P}\left\{\tau_1+1+\sum_{j=1}^{J-1} U_j>t \mid X_0=x\right\}\\ &\geqq \mathbb{P}\left\{1+\sum_{j=1}^{J-1} U_j>t \mid X_0=x\right\}\\ &= \sum_{j=1}^{t} \mathbb{P}\{1+U_1+\ldots U_{j-1}>t;\\ &\qquad U_1 \leqq a, \ldots, U_{j-1} \leqq a, U_j>a \mid X_0=x\}.\end{aligned}$$

It follows from the Markov property and the homogeneity of the chain that this last expression is equal to

$$\begin{aligned}&\sum_{j=1}^{t} \mathbb{P}\{1+U_1+\ldots U_{j-1}>t;\ U_1 \leqq a, \ldots, U_{j-1} \leqq a, U_j>a \mid X_0=0\}\\ &\qquad = \mathbb{P}\{T_a>t \mid X_0=0\},\end{aligned}$$

and this concludes the proof of the lemma.

Proof of Proposition 3. – From formula (3.1) we have for any integers r and $\gamma>r$

$$\begin{aligned}\mathbb{P}(T_a>\gamma \mid X_0=0) &\geqq \sum_{j=1}^{r} \mathbb{P}[\{U_1 \leqq a, \ldots, U_j \leqq a\} \cap \{N_\gamma=j\} \mid X_0=0]\\ &\geqq \mathbb{P}[\{U_1 \leqq a, \ldots, U_r \leqq a\} \cap \{N_\gamma \leqq r\} \mid X_0=0].\end{aligned}$$

We will now choose $r=r(a)$ such that both sets which appear in the above formula have a probability which converges to one when a diverges.

For the first set we use the independence of the random variables U_i and X_0 to obtain

$$\mathbb{P}\{U_1 \leqq a, \ldots, U_r \leqq a \mid X_0=0\} = \left(1 - \frac{1}{a+1}\right)^r.$$

Therefore this probability will converge to 1 if we choose $r(a)$ in such a way that

$$\lim_{a \to \infty} r(a)/a = 0.$$

For the second set, we first remark that

$$\mathbb{P}(N_\gamma \leqq r(a) \mid X_0=0) = \mathbb{P}(U_1 + \ldots + U_{r(a)} > \gamma).$$

By Lemma 4, this last probability goes to one if we choose

$$\gamma = \gamma(a) = l(r(a)).$$

We remark that one can simultaneously impose the conditions

$$\lim_{a \to \infty} \frac{r(a)}{a} = 0 \quad \text{and} \quad \lim_{a \to \infty} \frac{l(r(a))}{a} = \lim_{a \to \infty} \frac{l(r(a))}{r(a)} \frac{r(a)}{a} = +\infty.$$

Using Lemma 5 we conclude the proof of proposition 3.

COROLLARY 6:

$$\lim_{a \to \infty} \frac{\beta_a}{a} = +\infty.$$

Proof. — We first remark that

$$\mathbb{P}\{T_a > \gamma(a)\} = \sum_{x=0}^{a-1} \mathbb{P}\{X_0 = x\} \mathbb{P}\{T_a > \gamma(a) \mid X_0 = x\}$$
$$\geqq \mathbb{P}\{X_0 < a\} \inf_{x<a} \mathbb{P}\{T_a > \gamma(a) \mid X_0 = x\}.$$

By proposition 3, this last expression converges to 1 as a diverges. This implies that $\beta_a \geqq \gamma(a)$, and the corollary follows from (3.5).

COROLLARY 7. — *For any fixed time s and fixed state $x<a$, we have*

$$\lim_{a \to \infty} \sup_{x<a} \mathbb{P}\{\beta_a s - a - 1 < T_a \leqq \beta_a s + 1 \mid X_0 = x\} = 0,$$

Proof. — Using Markov property we deduce for a large

$$\begin{aligned}\mathbb{P}\{\beta_a s-a-1<\mathrm{T}_a\leqq\beta_a s+1\,|\,\mathrm{X}_0=x\}\\ &\leqq\sum_{y=0}^{a-1}\mathbb{P}\{\mathrm{T}_a>\beta_a s-a-1,\mathrm{X}_{[\beta_a s-a-1]}=y\,|\,\mathrm{X}_0=x\}\\ &\quad\times\mathbb{P}\{\mathrm{T}_a\leqq a+3\,|\,\mathrm{X}_0=y\}\leqq\sup_{0\leqq y<a}\mathbb{P}\{\mathrm{T}_a\leqq a+3\,|\,\mathrm{X}_0=y\}.\end{aligned}$$

By proposition 3, this last quantity converges to 0 as a diverges.

COROLLARY 8. — *There is a positive real number $e^{-1}<\rho<1$ such that for a large enough and any integer n we have*

$$\mathbb{P}\{\mathrm{T}_a\geqq\beta_a n\}\leqq\rho^n.$$

Proof. — The proof is by induction. For $n=1$ the result is obvious from the definition of β_a. Assume now that the inequality holds for the integer n. We will prove it for $n+1$. Using Markov property we get

$$\begin{aligned}\mathbb{P}\{\mathrm{T}_a>\beta_a(n+1)\}\\ &=\sum_{x=0}^{a-1}\mathbb{P}\{\mathrm{T}_a>\beta_a n,\mathrm{X}_{\beta_a n}=x\}\,\mathbb{P}\{\mathrm{T}_a>\beta_a\,|\,\mathrm{X}_0=x\}\\ &\leqq\mathbb{P}\{\mathrm{T}_a>\beta_a n\}\sup_{x<a}\mathbb{P}\{\mathrm{T}_a>\beta_a\,|\,\mathrm{X}_0=x\},\end{aligned}$$

which from the induction hypothesis is smaller than

$$\rho^n\sup_{x<a}\mathbb{P}\{\mathrm{T}_a>\beta_a\,|\,\mathrm{X}_0=x\}.$$

On the other hand it follows from formula (2.7) that

$$\sup_{x<a}\mathbb{P}\{\mathrm{T}_a>\beta_a\,|\,\mathrm{X}_0=x\}=\mathbb{P}\{a+\sum_{j=1}^{\mathrm{J}-1}\mathrm{U}_j>\beta_a\}=\mathbb{P}\{\mathrm{T}_a>\beta_a-a+1\,|\,\mathrm{X}_0=0\}.$$

Using again Markov property and formula 2.6 we obtain

$$\begin{aligned}|\,\mathbb{P}\{\mathrm{T}_a>\beta_a-a+1\,|\,\mathrm{X}_0=0\}-\mathbb{P}\{\mathrm{T}_a>\beta_a\}\,|\\ &\leqq\mathbb{P}\{\mathrm{X}_0>a\}+\mathbb{P}\{\beta_a-a<\mathrm{T}_a\leqq\beta_a\,|\,\mathrm{X}_0=0\}.\end{aligned}$$

This last quantity converges to zero by Corollary 7. Therefore it becomes smaller than $\rho-e^{-1}$ for a large enough and this concludes the proof.

We conclude this section by a lemma concerning the time scale β_a.

LEMMA 9:

$$\lim_{a\to\infty}\mathbb{P}\{\mathrm{T}_a\geqq\beta_a\}=e^{-1}.$$

Proof. — By the definition of β_a we have

$$\mathbb{P}\{\mathrm{T}_a\geqq\beta_a\}\leqq e^{-1}<\mathbb{P}\{\mathrm{T}_a\geqq\beta_a-1\}.$$

Since

$$0 \leqq \mathbb{P}\{T_a \geqq \beta_a - 1\} - \mathbb{P}\{T_a \geqq \beta_a\} \leqq \mathbb{P}\{\beta_a - 1 \leqq T_a < \beta_a\},$$

we conclude the proof by using the Markov property, namely we have the inequalities

$$\mathbb{P}\{\beta_a - 1 \leqq T_a < \beta_a\} = \mathbb{P}\{X_{\beta_a - 1} = 0\}\, \mathbb{P}\{T_a = 1 \mid X_0 = 0\} \leqq \mathbb{P}\{U_1 \geqq a\},$$

and this last quantity converges to 0 when a diverges.

IV. PROOF OF THE MAIN RESULT

We come now to the proof of the first part of Theorem 2. The distinctive feature of the exponential law is its factorization property. In the next Lemma we will prove that the same property holds asymptotically for the random variable T_a/β_a.

LEMMA 10. — *Let s and t be two fixed positive real numbers, then the following holds*

$$\lim_{a \to \infty} |\mathbb{P}\{T_a > \beta_a (t+s)\} - \mathbb{P}\{T_a > \beta_a t\}\, \mathbb{P}\{T_a > \beta_a s\}| = 0. \tag{4.1}$$

Proof. — Using Markov property the above expression is equal to

$$\Bigg| \sum_{x=0}^{x=a-1} \mathbb{P}\{T_a > \beta_a t, X_{[\beta_a t]} = x\} \\ \times [\mathbb{P}\{T_a > \beta_a s + \beta_a t - [\beta_a t] \mid X_0 = x\} - \mathbb{P}\{T_a > \beta_a s\}]\Bigg|.$$

This quantity is smaller than

$$\mathbb{P}\{X_0 \geqq a\} + \sum_{x=0}^{x=a-1} \mathbb{P}\{T_a > \beta_a t, X_{[\beta_a t]} = x\} \\ \times \sum_{y=0}^{y=a-1} \mathbb{P}\{X_0 = y\} \,|\, \mathbb{P}\{T_a > \beta_a s + \beta_a t - [\beta_a t] \mid X_0 = x\} \\ - \mathbb{P}\{T_a > \beta_a s \mid X_0 = y\}\,|,$$

which is bounded by

$$\sup_{0 \leqq x,\, y < a} |\mathbb{P}\{T_a > \beta_a s + \beta_a t - [\beta_a t] \mid X_0 = x\} \\ - \mathbb{P}\{T_a > \beta_a s \mid X_0 = y\}| + \mathbb{P}\{X_0 \geqq a\}.$$

The second term in the above expression decreases obviously to 0 as a diverges. On the other hand, using formulas (2.6) and (2.7) we can rewrite the first term as:

$$\sup_{0\leqq x,\, y<a}\left|\mathbb{P}\left\{x+1+\sum_{j=1}^{J-1}U_j>\beta_a s+\beta_a t-[\beta_a t]\right\}\right.$$
$$\left.-\mathbb{P}\left\{y+1+\sum_{j=1}^{J-1}U_j>\beta_a s\right\}\right|$$
$$\leqq\mathbb{P}\left\{\beta_a s-a-1<1+\sum_{j=1}^{J-1}U_j\leqq\beta_a s+1\right\}.$$

Using again formula (2.6) the last expression is equal to

$$\mathbb{P}\{\beta_a s-a-1<T_a\leqq\beta_a s+1\,|\,X_0=0\},$$

which converges to 0 by Corollary 7.

Lemma 10 insures that if the law of T_a/β_a converges, as $a\to\infty$ then the limit must be an exponential law (perhaps degenerate). On the other hand, Lemmata 9 and 10 imply that if t is a positive rational number, then the limit

$$\lim_{a\to\infty}\mathbb{P}\{T_a\geqq\beta_a t\}$$

does exist and is equal to e^{-t}. Since the exponential law is continuous this is enough to prove the convergence for all positive real t and this concludes the proof of (3.2).

The proof of (3.3) is based on Lebesgue's Dominated Convergence Theorem. We have

$$\frac{\mathbb{E}(T_a)}{\beta_a}=\frac{1}{\beta_a}\int_0^\infty\mathbb{P}\{T_a>t\}\,dt=\int_0^\infty\mathbb{P}\left\{\frac{T_a}{\beta_a}>t\right\}dt.$$

Corollary 8 enables us to use Lebesgue's theorem and we get

$$\lim_{a\to\infty}\int_0^\infty\mathbb{P}\left\{\frac{T_a}{\beta_a}>t\right\}dt=1.$$

This concludes the proof of the assertion (3.3).

In order to prove (3.4) we use again formula (2.6) to get the inequalities

$$\mathbb{E}\left\{\sum_{j=1}^{J-1}U_j\right\}\mathbb{P}\{X_0<a\}\leqq\mathbb{E}\{T_a\}\leqq\left(a+1+\mathbb{E}\left\{\sum_{j=1}^{J-1}U_j\right\}\right)\mathbb{P}\{X_0<a\}.$$

Since the random variables U_i are independent and identically distributed we can rewrite the expectation appearing in the lower and upper bound in the following way:

$$\mathbb{E}\left\{\sum_{j=1}^{J-1}U_j\right\}=\sum_{n=2}^{\infty}(n-1)\,\mathbb{E}\{U_1;U_1\leqq a\}\,\mathbb{P}(U_1\leqq a)^{n-2}\,\mathbb{P}(U_1>a).$$

Assertion (3.4) follows from a straight forward computation using the law of U_1.

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