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On the renormalization group iteration of a two-dimensional hierarchical non-linear $O(N)$ σ-model (*)

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ABSTRACT. — We investigate the behaviour of a simplified two-dimensional non-linear $O(N)$ σ-model under Wilson renormalization group transformations in the small coupling region. It can be controlled rigorously by suitable bounds on polymer activities uniformly for all field configurations, *i.e.* there is no need for a separate discussion of large and small field domains. This investigation provides us some preliminary insights to the renormalization group flow of and to phase space expansion methods applied to the complete σ-model.

RÉSUMÉ. — Nous décrirons le comportement par le groupe de renormalization de Wilson d'un model σ $O(N)$ simplifié à deux dimensions dans la région de couplage faible. Le modèle peut être contrôlé rigoureusement par des bornes sur les activités de polymères uniformément pour toutes

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les configurations de champ, *i.e.* il n'y a pas besoin d'une analyse séparée pour les domaines de grands champs et de petits champs. Ce travail apporte des indications préliminaires sur la coulée du groupe de renormalisation et sur les méthodes de développement en espace de phase appliquées au modèle σ complet.

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1. INTRODUCTION

Two dimensional non-linear $O(\mathrm{N})$ σ-models for $\mathrm{N} \geqq 3$ serve as toy models for non-Abelian gauge theories in four dimensions. They share renormalizability and ultraviolet asymptotic freedom, at least within perturbation theory ([1]-[3], [22], [25]). One should expect that the most powerful non-perturbative investigation is by the Wilson renormalization group (RG) [4]. To be specific, let us put the model on a lattice. For a single RG step the lattice field $\varphi(x)$ is decomposed in to a block spin field $\Phi(y)$ and a fluctuation field $\eta(x)$ of block mean zero according to [5], [6]

$$\varphi(x)=\sum_{y} \mathscr{A}(x, y)\,\Phi(y)+\eta(x), \tag{1.1}$$

where the sum is over the blocks, and x denotes sites on the basic lattice Λ_n of lattice spacing $a=\mathrm{L}^{-n}$, say. Because of the non-locality of the kernel $\mathscr{A}$ (although exponentially decaying) ([7], and references therein), long range correlations must be controlled very carefully, and this makes the investigation unduly complicated.

For a first principal discussion of the behaviour of the full model under the RG it is meaningful to consider mathematically simplified models, so called hierarchical or ultralocal ones ([8], [9]) which are more easily to be dealt with and which are expected to share the properties of interest with the full model, such as ultraviolet asymptotic freedom (for hierarchical approximations of other than σ-models *c.f.* e.g. [10]-[17], [24]). Such a hierarchical non-linear $O(\mathrm{N})$ σ-model is presented in Section 2. It has already been investigated by K. Gawedzki and A. Kupiainen [8] (and corresponds to the "ultralocal" σ-model studied by P. K. Mitter and T. R. Ramadas [9]). They gave a rigorous construction of the continuum limit and proved ultraviolet asymptotic freedom. Their approach is based on a partition of field space into "small and large field" domains. In the small field region, where perturbation theory applies, bounds on the effective action are given, whereas in the large field domain appropriate stability bounds for the Boltzmannian allow one a systematic discussion of the flow of the effective couplings under successive RG transformations.

Here we follow a different strategy ([5], [6], [18]). Instead of partitioning configuration space we apply a polymer expansion to the effective partition functions and give suitable bounds on the activities in this expansion. For the hierarchical model under consideration, only monomer activities are different from zero. Each of them is written as a sum of a "relevant" and an "irrelevant" contribution. For field configurations near the minimum of the effective action, the first one gives the corresponding perturbation expansion to a given order in the running coupling constant, whereas the latter is small of higher order, in the coupling as well as in the deviation of the fields from the minimizing configurations. On the other hand, both terms have bounds uniform for all possible field configurations. In order to make successive RG transformations managable, the bounds must be such that they are stable under the RG. In this paper we state such uniform bounds and prove that they are actually stable, at least if the effective coupling constant is sufficiently small, *i.e.* in a neighbourhood of the expected Gaussian fixed point.

The method used in this paper is in the spirit of phase space expansions, in particular polymer expansions on the multigrid ([5], [6], [19]). An investigation of a hierarchical SU(2) gauge model in four dimensions by these methods can be found in [16], [17]. We were motivated by the hope that the method can be refined (as e.g. for φ^4 theory [20]) to such a degree that it applies also to the full non-linear $O(\mathrm{N})$ σ-model, beyond the perturbative level [21]. This paper will be continued by a further one which contains the polymer representation of the hierarchical nonlinear $O(\mathrm{N})$ σ-model on the multigrid without ultraviolet cutoff. The existence of and bounds on Green functions will be investigated. Polymer activities will be estimated by using the same methods as presented here.

2. THE HIERARCHICAL MODEL

First of all we introduce some notations. For integer j, n, $0 \leqq j \leqq n$, let Λ_j be a two-dimensional lattice of lattice spacing $a_j = \mathrm{L}^{-j}$ and volume V^2, say, for some $\mathrm{V} \in \mathbf{N}$. L is some large integer to be specified later on. We identify Λ_j as the $(n-j)$th L^2-block lattice of Λ_n, *i.e.* every $x \in \Lambda_j$ is identified with a block of size $(\mathrm{L}^{n-j} a)^2 = \mathrm{L}^{-2j}$. Furthermore, for any $x \in \Lambda_n$ let $\bar{x}^i$ be the unique element of Λ_{n-i} which contains x in the sense just explained.

The partition function of the full non-linear $O(\mathrm{N})$ σ-model is given by

$$Z=\int \prod_{x \in \Lambda_n} d^{\mathrm{N}} \varphi(x) . \mathrm{F}(\varphi)$$

$$\times \exp \left\{\frac{1}{2} \frac{\mathrm{Z}(a)}{g(a)} \sum_{x, i, \mu} \varphi_i(x) \times\left[\varphi_i(x+a \hat{\mu})+\varphi_i(x-a \hat{\mu})-2 \varphi_i(x)\right]\right\}, \quad (2.1\,a)$$

where

$$\mathrm{F}(\varphi)=\exp \left(-\tilde{\lambda} \sum_x\left(\mathrm{Z}(a) \varphi^2(x)-1\right)^2\right). \quad (2.1\,b)$$

The vector field φ is real valued. $\hat{\mu}$ denotes a unit vector in μ-direction, $\mu=1,2$. $g(a)$ and $Z(a)$ are bare coupling constant and wave function renormalization, respectively, to be chosen in such a way that the $a \rightarrow 0$, *i.e.* $n \rightarrow \infty$ limit exists in some specified sense. The normal δ-constraint form of F which would fix $\varphi(x)$ to the $(\mathrm{N}-1)$-sphere $\mathrm{Z}(a) \varphi^2(x)=1$ has been replaced by a "δ-iterating" function ([8], [9]) because anyhow it would be destroyed after the first RG step. It is reproduced for large $\tilde{\lambda}$. Action and measure have a global $O(\mathrm{N})$ symmetry.

Now we introduce effective actions on the coarser lattices Λ_j (scale j), $0 \leqq j \leqq n$, and the corresponding block spin decomposition.

$$\left.\begin{aligned} \varphi(x)=\sum_{i=0}^n \eta_i(x), \qquad \eta_i(x)=\sum_{z \in \Lambda_i} \mathscr{A}_i(x, z) \varphi_i(z) \\ v(x, y)=\sum_{i=0}^n w_i(x, y), \\ w_i(x, y)=\sum_{z, w \in \Lambda_i} \mathscr{A}_i(x, z) v_i(z, w) \mathscr{A}_i(y, w), \end{aligned}\right\} \quad (2.2)$$

where v is the covariance (propagator) derived from equation (2.1). The effective Boltzmannian or partition function on "scale j" is then given by

$$\mathrm{Z}_j(\rho_j)=\int \prod_{i=j+1}^n d \mu_{w_i}(\eta_i) \mathrm{F}\left(\rho_j+\sum_{i=j+1}^n \eta_i\right), \quad (2.3)$$

where $d\mu_{w_i}$ denotes the Gaussian measure of covariance w_i and mean zero. These effective partition functions are recursively related by the RG transformation

$$\left.\begin{aligned} \mathrm{Z}_{j-1}(\rho_{j-1})=\int d \mu_{w_j}(\eta_j) \mathrm{Z}_j(\rho_{j-1}+\eta_j), \\ \mathrm{Z}_n=\mathrm{F}. \end{aligned}\right\} \quad (2.4)$$

As alluded to in the introduction we now replace the exact covariance by a simpler one which is not too far away from the exact propagator. For

$x, y \in \Lambda_n$ set

$$v(x, y) = \gamma\, \mathrm{M}(x, y), \tag{2.5a}$$

where $\gamma > 0$ and $\mathrm{M}(x, y) \leqq n$ denotes the number of blocks on all coarser lattices which contain both x and y. Then

$$v(x, y) = \gamma \sum_{i=0}^{n} \delta_{\bar{x}^i, \bar{y}^i}. \tag{2.5b}$$

For "almost all" $x, y \in \Lambda_n$, $|x-y| \sim a\,\mathrm{L}^p = \mathrm{L}^{p-n}$, we have

$$v(x, y) \sim \gamma (n-p) = -\gamma \frac{\log|x-y|}{\log \mathrm{L}}.$$

thus not too a bad approximation of the true propagator.

The resulting hierarchical model has no independent wave function renormalization. We set

$$\left.\begin{aligned} &\mathrm{Z}(a) = g(a) \\ &f_n^2 = g(a)^{-1} \\ &\lambda = 8\, g(a)\, \tilde{\lambda}, \end{aligned}\right\} \tag{2.6}$$

to get most notational coincidence to the approach of Gawedzki and Kupiainen [8].

We are led to the following iteration

$$\left.\begin{aligned} \mathrm{Z}_{j-1}(\psi) = c_{0,j} \left\{ \int \frac{d^{\mathrm{N}}\eta}{(2\pi\gamma)^{\mathrm{N}/2}} \exp\left(-\frac{(\eta-\psi)^2}{2\gamma} \right) \mathrm{Z}_j(\eta) \right\}^{\mathrm{L}^2}, \\ j = n, \ldots, 1 \end{aligned}\right\} \tag{2.7}$$

with initial condition

$$\mathrm{Z}_n(\psi) = \exp\left(-\frac{\lambda}{8 f^2} (\psi^2 - f^2)^2 \right). \tag{2.8}$$

Because of $O(\mathrm{N})$ symmetry we write for real fields $\psi = r\hat{\psi}$, $|\hat{\psi}| = 1$, $r \geqq 0$. If we parametrize in (2.7) $\eta = \sigma\hat{\psi} + \pi$, $\pi \cdot \hat{\psi} = 0$, the RG iteration becomes

$$\begin{aligned} \mathrm{Z}_{j-1}(r_{j-1}, 0) = c_{0,j} \Bigg\{ \int & \frac{d\sigma\, d^{\mathrm{N}-1}\pi}{(2\pi\gamma)^{\mathrm{N}/2}} \\ & \times \exp\left(-\frac{(\sigma - r_{j-1})^2 + \pi^2}{2\gamma} \right) \mathrm{Z}_j(\sigma, \pi) \Bigg\}^{\mathrm{L}^2}, \end{aligned} \tag{2.9}$$

$$\mathrm{Z}_n(r_n, 0) = \exp\left\{ -\frac{\lambda}{8 f^2} (r_n^2 - f^2)^2 \right\}. \tag{2.10}$$

The normalization factor $c_{0,j}$ is chosen in such a way that at the minimum f_{j-1} of the effective action $-\log \mathrm{Z}_{j-1}(r_{j-1}, 0)$, we have $\mathrm{Z}_{j-1}(f_{j-1}, 0) = 1$.

Complex analyticity in field space for different field theory models has been pioneered by Gawedzki and Kupiainen (e. g. [7], [8], [10], [11]). In the following we will use complex analyticity in the "radial components", exploiting $O(N)$ symmetry (reasonable bounds for arbitrary oriented complex fields do not seem to iterate in our approach without additional effort). This is sufficient for our investigation. Anyhow in the end only real fields are of interest (in this model). Considered as a function of r_n, $Z_n(r_n, 0)$ is complex analytic. It will be shown here that if for some j, $0 \leqq j \leqq n$, $Z_j(r_j, 0)$ is analytic in r_j, and if $Z_j(\sigma, \pi)$ is smooth and satisfies suitable bounds, so does Z_{j-1}, at least for sufficiently large f_j (running f on scale j). In what follows we consider $Z_j(r_j, 0)$ on the domain $\mathrm{Re}\, r_j \geqq -1$ (in order to allow application of Cauchy formula for proving iteration of stability bounds).

To prove UV-convergence on each physical (= finite length) scale we have to show that for an appropriate choice of the inverse bare coupling $g^{-1}(a) = f_n^2$

$$\mathscr{Z}_j(\psi) = \lim_{\substack{n \to \infty \\ (n \geqq j)}} Z_j(\psi)$$

exists (at least pointwise), where Z_j derives from Z_n by $n-j$ RG iterations (dependence on f_n is implicit). It is expected [8] that ($a = L^{-n}$)

$$g^{-1}(a) = f_n^2 \sim n, \tag{2.11}$$

which is denoted as asymptotic freedom. In this case the bare Boltzmanian $F(\varphi)$ reproduces the δ-constraint because of (2.6), already for finite λ. In this paper we are mainly concerned to investigate the RG transformation without performing a separation of large and small field domains.

Our treatment is as follows. We split the partition function Z_j into a perturbatively determined, so called "relevant" part and a remainder in such a way that both contributions have a reasonable bound for all possible field configurations. Near the configurations where the effective action takes its minimum, the relevant part should reproduce the standard perturbative expansion [to a given order $O(f^{-4})$]. It is then shown that this pattern is preserved under RG transformations. In particular the bounds are stable. We may write the RG step (2.7) as

$$Z_{j-1} = I_{RG} Z_j. \tag{2.12}$$

Because of bounds on Z_j, the RG transformation I_{RG} is continuous (with respect to pointwise convergence) on the set of such functions. That we succeed in avoiding a partition into large and small field domains in this toy model confirms our hope to successfully iterate the complete σ-model on the multigrid and to investigate it by methods similar to those in [20].

Before we are going to study the RG iteration in detail, let us note that the fields ψ_j are dimensionless in two dimensions. That means interactions with arbitrary powers of the fields are marginal under (2.7). Actually the minimum of the effective action $-\log Z_j(r_j, 0)$ for real r_j is located at some non-zero f_j, at least for the starting action (where it is f). To make perturbation theory available for studying the RG flow we should consider an expansion about this minimum.

Thus let f_i be the location of the minimum of the effective action $-\log Z_j(r_j, 0)$,

$$c_{2,j} = -\frac{1}{2}\frac{\partial^2}{\partial r_j^2}\log Z_j(r_j, 0)\Big|_{r_j=f_j}, \tag{2.13}$$

and $\mathscr{L}_j = (1+2\gamma c_{2,j})^{-1}$. Write in (2.9) $r_{j-1}=f_j+\rho$ and substitute $\sigma \to f_j + \sigma$. We get

$$Z_{j-1}(f_j+\rho, 0) = c_{0,j}\exp(-L^2\mathscr{L}_j c_{2,j}\rho^2).$$
$$\left\{\int \frac{d\sigma\, d^{N-1}\pi}{(2\pi\gamma)^{N/2}}\exp\left(-\frac{(\sigma-\mathscr{L}_j\rho)^2+\pi^2}{2\gamma} - c_{2,j}(\sigma-\mathscr{L}_j\rho)^2\right)\right.$$
$$\left.\times\exp(c_{2,j}\sigma^2)Z_j(f_j+\sigma, \pi)\right\}^{L^2}. \tag{2.14}$$

We take care of not to write Z_j $(f_j+\sigma+\mathscr{L}\rho, \pi)$ for complex ρ because we only prove bounds if imaginary and real parts of the argument are collinear. We don't need prove analyticity of Z_j for arbitrary oriented complex fields. On the finest lattice $j=n$, we have $f_n=f$, $c_{2,n}=\lambda/2$ and

$$\exp\left(\frac{\lambda}{2}\sigma^2\right)Z_n(f+\sigma, \pi) = \exp(O((\sigma, \pi)^3, f^{-1})).$$

That means the saddle point expansion is an expansion in powers of f^{-1}. For $f\to\infty$ there is a fixed point

$$L^2\mathscr{L}^* c_2^* = c_2^* \qquad \text{or} \quad c_2^* = \frac{L^2-1}{2\gamma}.$$

For finite f corrections must be taken into account, which will come out to be of order $O(f^{-2})$. At least justified by perturbation theory, we see that close to the critical point,

$$\mathscr{L} \sim L^{-2}.$$

For large L this is a small number. That means the RG transformation (2.14) has only one marginal direction, which described the Gaussian fluctuations about the minimum on each scale.

3. PROPERTIES OF THE PARTITION FUNCTIONS Z_j

We list properties of effective partition functions Z_j which are maintained under RG transformations (2.7), (2.14) (with the obvious replacements of the running couplings), as described by the theorem below. Let $0 \leqq j \leqq n$.

(A0) $Z_j(\psi)$ is an $O(N)$ invariant smooth function of

$$\psi = (\psi_1, \ldots, \psi_N), \qquad N \geqq 3,$$
$$Z_j(\psi) = Z_j^{\mathrm{rel}}(\psi) + R_j(\psi). \tag{3.1}$$

For $\psi = r\hat{\psi}$, $|\hat{\psi}| = 1$, $r \geqq 0$, we write $Z_j(\psi) = Z_j(r, 0)$ and for Z_j^{rel} and R_j in analogy.

(A 1)

$$Z_j^{\mathrm{rel}}(\psi) = \exp\left\{-\frac{c_{2,j}}{4f_j^2}(\psi^2 - f_j^2)^2\right\}\left\{1 - \sum_{\substack{\nu=3 \\ (\nu \neq 8)}}^{9} c_{\nu,j}\left[\frac{\psi^2 - f_j^2}{2f_j}\right]^\nu\right\}. \tag{3.2}$$

f_j and $c_{2,j}$ are defined by

$$0 = \left.\frac{\partial Z_j(r, 0)}{\partial r}\right|_{r = f_j} \tag{3.3}$$

and

$$c_{2,j} = -\frac{1}{2}\left.\frac{\partial^2 Z_j(r, 0)}{\partial r^2}\right|_{r = f_j}, \tag{3.4}$$

respectively, where we have assumed normalization $Z_j(f_j, 0) = 1$. For $\nu = 3, \ldots, 9$, $(\nu \neq 8)$ let

$$\tilde{c}_{\nu,j} = -\frac{1}{\nu!}\frac{\partial^\nu}{\partial r^\nu}[Z_j(r, 0)\exp(c_{2,j}(r - f_j)^2)]_{r = f_j}. \tag{3.5}$$

Then $c_{\nu,j}$ are related to $\tilde{c}_{\nu,j}$ according to Lemma 3.1 below. The coefficients $\tilde{c}_{\nu,j}$ satisfy the bounds

$$\left.\begin{aligned} \frac{1}{4}\frac{L^2 - 1}{2\gamma} \leqq c_{2,j} \leqq \frac{L^2}{\gamma}, \\ |\tilde{c}_{3,j}| \leqq \frac{L^2}{f_j}, \\ |\tilde{c}_{4,j}|, |\tilde{c}_{6,j}| \leqq \frac{L^4}{f_j^2}, \\ |\tilde{c}_{5,j}|, |\tilde{c}_{7,j}|, |\tilde{c}_{9,j}| \leqq \frac{L^6}{f_j^3}. \end{aligned}\right\} \tag{3.6}$$

(A 2) The remainder $R_j(\psi)=R_j(f_j+\rho, 0)$ is a complex analytic function in $\rho=r-f_j$ for $\operatorname{Re}\rho>-f_i-1$, $|\operatorname{Im}\rho|<f_j/6$, is of order $O(\rho^8)$ as $\rho\to 0$, and satisfies for those ρ the upper bound

$$\begin{aligned}|R_j(f_j+\rho, 0)|&\\ \leqq\frac{K_1}{f_j^4}\exp&\left(-\frac{c_{2,j}}{50}(\operatorname{Re}\rho)^2+10\,c_{2,j}(\operatorname{Im}\rho)^2\right)\\ &+\frac{K_2}{f_j^4}\exp\left(-\left(c_{2,j}-\frac{1}{2\gamma}\right)(\operatorname{Re}\rho)^2\right.\\ &\left.+\left(c_{2,j}+\frac{1}{2\gamma}\right)(\operatorname{Im}\rho)^2\right),\end{aligned}\tag{3.7}$$

with appropriate constants K_1, $K_2(N, \gamma, L^2)$.

The relations between $c_{\nu,j}$ and $\tilde{c}_{\nu,j}$ are arranged in such a way that

$$Z_j^{\mathrm{rel}}(f_j+\rho, 0)=\exp(-c_{2,j}\rho^2)\left\{1-\sum_{\substack{\nu=3\\(\nu\neq 8)}}^{9}\tilde{c}_{\nu,j}\rho^\nu\right\}+R_j^{\mathrm{rel}}(f_j+\rho),\tag{3.8}$$

and that $R_j^{\mathrm{rel}}(f_j+\rho)=O(\rho^8)$ and satisfies $O(f_j^{-4})$ bounds uniformly in ρ. The order of the coefficients $\tilde{c}_{\nu,j}$ are given by (3.6). The reason for absence of a ρ^8-term is that it is of the order $O(f_j^{-4})$, and we take coefficients only to the order f_j^{-3}. The specific form (3.2) for the relevant part has been chosen mainly for two reasons. First, it yields exponential bounds for large real fields ψ, and secondly it is manifestly analytic in the fields. If for instance we had chosen as the relevant part the first term on the right hand side of equation (3.8), *i.e.* an expansion in $\rho=r-f_j$, the analyticity properties would not iterate via (2.14).

We are allowed to omit the restriction $|\operatorname{Im}\rho|\leqq f_j/6$ for validity of the bound (3.7) by simultaneously including into the first term on the right hand side of (3.7) a factor

$$\exp\left\{c_2\frac{17}{4f_j^2}(\operatorname{Im}\rho)^4\right\},$$

but this will be of no use to us in the following. We remark that the second term of the bound appears because we use analyticity by estimating on Cauchy circles, and because of the fact that R_j increases very rapidly with increasing imaginary fields. This is in contrast to Φ^4 theory [20].

For the starting partition function (2.8), *i.e.* for $j=n$, these conditions are satisfied with

$$\left.\begin{aligned}R_n=0,\qquad f_n=f,\\ c_{2,n}=\frac{\lambda}{2},\qquad c_{\nu,n}=0\quad\text{for }\nu\geqq 3,\end{aligned}\right\}\tag{3.9}$$

if we suppose that

$$\frac{1}{2}\,\frac{L^2-1}{2\gamma}\leqq\lambda\leqq 2\frac{L^2}{\gamma}. \tag{3.10}$$

The upper bound on λ is chosen just for simplicity in order to avoid estimates uniform in (arbitrary large) λ in the first RG step. Actually, as the bare inverse coupling $g^{-1}(a)$ increases according to (2.11), $\tilde{\lambda}=\lambda/8\,g(a)\sim n$ as well, and that restricts the support of the bare partition function to an arbitrary small neighbourhood of the sphere $\psi^2=g^{-1}(a)$.

We prove that the properties (A 0)-(A 2) iterate under the RG transformation (2.7), (2.14) if the f_j are not too small. This statement is given in the following

THEOREM 3.1. – *There exists* $L_0(N, \gamma)>0$ *and for all* $L\geqq L_0$ *there is* $F_0(N, \gamma, L^2)>0$ *so that the following statement holds.*

Suppose that for some $j\leqq n$ *the effective partition function* $Z_j(\psi)$ *satisfies the conditions* (A 0)-(A 2) *with some sufficiently large* $K_1(N, \gamma, L^2)$ *and* $K_2(N, \gamma, L^2)$ *and with* $f_j\geqq F_0$. *Then these conditions are true also for* $Z_{j-1}(\psi)$ *as defined by equation* (2.7), *with new coupling constants* f_{j-1}, $c_{2,j-1}$, $\tilde{c}_{\nu,j-1}$ *given as follows:*

$$f_j-\frac{\gamma(N-1)}{f_j}<f_{j-1}=f_j-\frac{G_{1,j}}{f_j}-\frac{G_{2,j}}{f_j^3}+R_{1,j}<f_j, \tag{3.11}$$

$$\left.\begin{aligned} c_{2,j-1}&=\Delta_{2,j}+R_{2,j}\\ \tilde{c}_{\nu,j-1}&=\Delta_{\nu,j}+R_{\nu,j}, \qquad \nu\geqq 3\,(\nu\neq 8).\end{aligned}\right\} \tag{3.12}$$

$G_{1,j}>0$, $G_{2,j}$ *and* $\Delta_{\nu,j}$ *are polynomials of* $\tilde{c}_{\nu,j}$ *and rational functions of* $c_{2,j}$. *There are* $\kappa(N, \gamma, L^2)$ *and* $\tau(N, \gamma, L^2)$ *so that*

$$|G_{1,j}|,\ |G_{2,j}|\leqq\kappa, \qquad |\Delta_{\nu,j}|\leqq\frac{\kappa}{f_j^{\mathcal{O}(\nu)}}, \tag{3.13}$$

where $\mathcal{O}(\nu)$ *denotes the order of* c_ν, *resp.* $\tilde{c}_\nu$ *in* f_j^{-1}, i.e. $\mathcal{O}(2)=0$, $\mathcal{O}(3)=1$, $\mathcal{O}(4)=\mathcal{O}(6)=2$, $\mathcal{O}(5)=\mathcal{O}(7)=\mathcal{O}(9)=3$. *The remainders* $R_{\nu,j}$ *are uniformly bounded by*

$$|R_{\nu,j}|\leqq\frac{\tau}{f_j^4}, \qquad \nu\geqq 1\,(\nu\neq 8). \tag{3.14}$$

$G_{1,j}$ is given by

$$G_{1,j}=\gamma\left(\frac{N-1}{2}+\frac{3}{2}\,\frac{\mathcal{L}_j}{c_{2,j}}\,f_j\,\tilde{c}_{3,j}\right)>0, \tag{3.15}$$

in agreement with [8]. By uniform bounds we mean that τ, κ, C, etc. do not depend on the specific values of f_j, $c_{2,j}$, $\tilde{c}_{\nu,j}$, *i.e.* they are independent of the RG step and depend only on external parameters N, γ and on L^2.

K_1 and K_2 sufficiently large means that they are fixed above a lower bound. For instance, on the first lattice $R_n=0$, so we could have set $K_1=K_2=0$, but this cannot hold on the next coarser lattice. Similarly, on the next to the finest lattice we could have chosen $K_1\neq 0$, but $K_2=0$, but this again would not iterate. The theorem just says that K_1, $K_2(N, \gamma, L^2)$ are not too small.

Before we are going to prove this theorem we state the already announced lemma which relates $c_{\nu,j}$ and $\tilde{c}_{\nu,j}$ and gives bounds on R_j^{rel}, *cf.* (3.8).

LEMMA 3.1. — *Consider*

$$Z_j^{\mathrm{rel}}(f_j+\rho, 0)=\exp\left\{-c_{2,j}\left[\frac{(f_j+\rho)^2-f_j^2}{2f_j}\right]^2\right\}\times\left\{1-\sum_{\substack{\nu=3\\(\nu\neq 8)}}^{9} c_{\nu,j}\left[\frac{(f_j+\rho)^2-f_j^2}{2f_j}\right]^{\nu}\right\}. \quad (3.16)$$

Then

$$Z_j^{\mathrm{rel}}(f_j+\rho, 0)=\exp(-c_{2,j}\rho^2)\left\{1-\sum_{\substack{\nu=3\\(\nu\neq 8)}}^{9}\tilde{c}_{\nu,j}\rho^{\nu}\right\}+R_j^{\mathrm{rel}}(\rho), \quad (3.17)$$

where $R_j^{\mathrm{rel}}(\rho)=O(\rho^8)$ *is an analytic function of* ρ, *and the coefficients* $\tilde{c}_{\nu,j}$ *are given by*

$$\left.\begin{aligned}
\tilde{c}_{3,j}&=c_{3,j}+\frac{c_{2,j}}{f_j}\\
\tilde{c}_{4,j}&=c_{4,j}+\frac{3}{2}\frac{c_{3,j}}{f_j}+\frac{c_{2,j}}{4f_j^2}\\
\tilde{c}_{5,j}&=c_{5,j}+2\frac{c_{4,j}}{f_j}+\frac{3}{4}\frac{c_{3,j}}{f_j^2}\\
\tilde{c}_{6,j}&=c_{6,j}-\frac{c_{2,j}c_{3,j}}{f_j}-\frac{c_{2,j}^2}{2f_j^2}+\left(\frac{5}{2}\frac{c_{5,j}}{f_j}+\frac{3}{2}\frac{c_{4,j}}{f_j^2}+\frac{c_{3,j}}{8f_j^3}\right)\\
\tilde{c}_{7,j}&=c_{7,j}+3\frac{c_{6,j}}{f_j}-\frac{c_{2,j}c_{4,j}}{f_j}-\frac{7}{4}\frac{c_{2,j}c_{3,j}}{f_j^2}\\
&\quad-\frac{c_{2,j}^2}{4f_j^3}+\left(\frac{c_{4,j}}{2f_j^3}+\frac{5}{2}\frac{c_{5,j}}{f_j^2}+3\frac{c_{6,j}}{f_j}\right)\\
\tilde{c}_{9,j}&=c_{9,j}-\frac{c_{2,j}c_{6,j}}{f_j}+\frac{c_{2,j}^2c_{3,j}}{2f_j^2}+\frac{c_{2,j}^3}{6f_j^3}.
\end{aligned}\right\} \quad (3.18)$$

Suppose in addition that the coefficients satisfy the bounds (3.6) *and that* $f_j \geqq 2$. *Then there is a* $C(N, \gamma, L^2) > 0$ *so that*

$$|R_j^{rel}(\rho)| \leqq \frac{C}{f_j^4} \exp\left(-\frac{1}{40} c_{2,j} (\mathrm{Re}\,\rho)^2 + \frac{5}{4} c_{2,j} (\mathrm{Im}\,\rho)^2\right), \tag{3.19}$$

for all ρ *satisfying* $\mathrm{Re}\,\rho \geqq -f_j - 1, |\mathrm{Im}\,\rho| \leqq f_j/6$.

$\tilde{c}_{6,j}$ and $\tilde{c}_{7,j}$ contain terms which are of order f_j^{-4}. These may be absorbed in the remainder, yielding $R_j^{rel}(\rho) = O(\rho^6)$, but the bound (3.19) still holds. That would be sufficient for our further investigations.

Proof. – We omit the scale index j.

$$Z^{rel}(f+\rho, 0) = \exp(-c_2 \rho^2) \exp\left(-c_2\left(\frac{\rho^3}{f} + \frac{\rho^4}{4f^2}\right)\right) \times \left\{1 - \sum_{\substack{\nu=3 \\ (\nu \neq 8)}}^{9} c_\nu \left(\rho + \frac{\rho^2}{2f}\right)^\nu\right\}.$$

Expanding the second exponential and collecting equal powers of ρ (up to 9th order) yields (3.17) with coefficients given by equation (3.18). The remainder R^{rel} is analytic in ρ, of order ρ^8 and gets its contributions from two terms. The first one collects higher orders of the expansion,

$$|R_1^{rel}| \leqq \frac{1}{f^4} |\exp(-c_2 \rho^2) P(\rho) \rho^8|,$$

where P is a polynomial with bounded coefficients [because of (3.6)]. For an appropriate $C_1(N, \gamma, L^2)$

$$|R_1^{rel}(\rho)| \leqq \frac{C_1}{f^4} \exp\left(-\frac{c_2}{2} (\mathrm{Re}\,\rho)^2 + \frac{5}{4} c_2 (\mathrm{Im}\,\rho)^2\right),$$

where we have used (I.3), *cf.* Appendix A. The other one is the Taylor remainder of the exponential and is bounded by

$$|R_2^{rel}(\rho)| \leqq \frac{1}{f^4} \left|\exp(-c_2 \rho^2) \exp\left(-\Theta c_2 \left(\frac{\rho^3}{f} + \frac{\rho^4}{4f^2}\right)\right) P'(\rho) \rho^{12}\right|,$$

where $0 \leqq \Theta(\rho, f) \leqq 1$ and P′ is some polynomial. Thus

$$
\begin{aligned}
|R_2^{\mathrm{rel}}(\rho)| &= \frac{1}{f^4}\left| \exp\left\{ -c_2(1-\Theta)\rho^2 \right.\right.\\
&\qquad \left.\left. -\Theta c_2 \left[\frac{(f+\rho)^2 - f^2}{2f}\right]^2 \right\} P'(\rho)\rho^{12} \right| \\
&\leqq \frac{1}{f^4}\exp\left(-c_2(1-\Theta)((\mathrm{Re}\,\rho)^2 - (\mathrm{Im}\,\rho)^2) - \Theta \frac{c_2}{32}(\mathrm{Re}\,\rho)^2 \right) \\
&\qquad \times \exp\left(\Theta c_2 \frac{9}{8}(\mathrm{Im}\,\rho)^2 \right) |P'(\rho)\rho^{10}| \\
&\leqq \frac{C_2(N, \gamma, L^2)}{f^4}\exp\left(-\frac{c_2}{40}(\mathrm{Re}\,\rho)^2 + \frac{5}{4}c_2(\mathrm{Im}\,\rho)^2 \right),
\end{aligned}
$$

for some C_2, and we have used (I1), (I2) of Appendix A. Choosing $C = C_1 + C_2$ proves the lemma. ∎

4. PROOF OF THE THEOREM. FIRST PRINCIPLES

Let us write $Z, f, c_2, \tilde{c}_v$, etc. for $Z_j, f_j, c_{2,j}, \tilde{c}_{v,j}$ and $Z', f', c_2', \tilde{c}_v'$ etc. for $Z_{j-1}, f_{j-1}, c_{2,j-1}, \tilde{c}_{v,j-1}$. In all what follows we assume $L \geqq 2$ and $f \geqq 2$, and we use the inequalities stated in Appendix A without explicitly referring to them. For any nonnegative f we define a subset $\Omega(f) \subseteqq \mathbf{C}$ by

$$\Omega(f) = \{ \rho \in \mathbf{C} \mid \mathrm{Re}\,\rho \geqq -f-1, \ |\mathrm{Im}\,\rho| \leqq f/6 \}. \tag{4.1}$$

First of all we notice that the conditions (A0)-(A2) together with Lemma 3.1 and Lemma B1 state integrability of the partition function Z. Hence smoothness of $Z(\psi)$ implies smoothness of $Z'(\psi)$. Also, Z' has $O(N)$ symmetry. Let us introduce the Gaussian measure

$$d\mu_s(\sigma, \pi \mid \lambda) = \prod_{x=1}^{L^2} \frac{d\sigma\, d^{N-1}\pi}{(2\pi\gamma)^{N/2}\mathscr{L}^{1/2}s^{N/2}} \exp\left(-\frac{(\sigma_x - \lambda)^2}{2\gamma\mathscr{L}s} - \frac{\pi_x^2}{2\gamma s} \right) \tag{4.2a}$$

for $\lambda \in \mathbf{C}$, and

$$d\mu_s(\sigma, \pi) \equiv d\mu_s(\sigma, \pi \mid 0), \tag{4.2b}$$

where $0 \leqq s \leqq 1$ and $\mathscr{L} = (1 + 2\gamma c_2)^{-1}$. Then the RG transformation (2.14) can be written as

$$
\begin{aligned}
Z'(f+\rho, 0) &= c_0 \mathscr{L}^{L^2/2}\exp(-L^2\mathscr{L}c_2\rho^2) \\
&\qquad \times \int d\mu_1(\sigma, \pi \mid \mathscr{L}\rho) \prod_{x=1}^{L^2} \exp(c_2\sigma_x^2) Z(f+\sigma_x, \pi_x).
\end{aligned} \tag{4.3}
$$

We write $Z=Z^{rel}+R$ and expand the term which has only factors of Z^{rel} about the Gaussian measure, which yields

$$Z'(f+\rho,0)=c_0\,\mathscr{L}^{L^2/2}\sum_{i=1}^{3}Z'_i(\rho), \tag{4.4}$$

where

$$Z'_1(\rho)=\exp(-L^2\mathscr{L}c_2\rho^2)\sum_{\nu=0}^{4}\frac{1}{\nu!}\frac{\partial^\nu}{\partial s^\nu}\int d\mu_s(\sigma,\pi)\times\prod_{x=1}^{L^2}\exp(c_2(\sigma_x+\mathscr{L}\rho)^2)\,Z^{rel}(f+\sigma_x+\mathscr{L}\rho,\pi_x)\big|_{s=0}, \tag{4.5}$$

$$Z'_2(\rho)=\exp(-L^2\mathscr{L}c_2\rho^2)\frac{1}{4!}\int_0^1 ds\,(1-s)^4\frac{\partial^5}{\partial s^5}\int d\mu_s(\sigma,\pi)\times\prod_{x=1}^{L^2}\exp(c_2(\sigma_x+\mathscr{L}\rho)^2)\,Z^{rel}(f+\sigma_x+\mathscr{L}\rho,\pi_x), \tag{4.6}$$

and

$$Z'_3(\rho)=\exp(-L^2\mathscr{L}c_2\rho^2)\sum_{\varnothing\neq P\subseteq\{1,\ldots,L^2\}}\int d\mu_1(\sigma,\pi\,|\,\mathscr{L}\rho)\times\prod_{x\notin P}\{\exp(c_2\sigma_x^2)\,Z^{rel}(f+\sigma_x,\pi_x)\}\times\prod_{x\in P}\{\exp(c_2\sigma_x^2)\,R(f+\sigma_x,\pi_x)\}. \tag{4.7}$$

Derivatives of the Gaussian measure with respect to s are dealt with by the formula [23]

$$\begin{aligned}\frac{\partial}{\partial s}d\mu_s(\sigma,\pi)\,I(\sigma,\pi)&=d\mu_s(\sigma,\pi)\frac{\gamma}{2}\sum_{x=1}^{L^2}\left\{\mathscr{L}\frac{\partial^2}{\partial\sigma_x^2}+\sum_{i=1}^{N-1}\frac{\partial^2}{\partial\pi_{i,x}^2}\right\}I(\sigma,\pi)\\&=d\mu_s(\sigma,\pi)\frac{\gamma}{2}\sum_{x=1}^{L^2}\left\{\mathscr{L}\frac{\partial^2}{\partial\sigma_x^2}+2(N-1)\times\frac{\partial}{\partial\pi_x^2}+4\pi_x^2\frac{\partial^2}{\partial(\pi_x^2)^2}\right\}I(\sigma,\pi),\end{aligned} \tag{4.8}$$

for any differentiable function I which depends on π only by π^2. This is satisfied for Z^{rel}. The differentiations can be done without problems because Z^{rel} is manifestly differentiable by construction.

Using the bounds (3.6) it is easy to see that a least every third derivative with respect to the fields produce a factor of f^{-1}. Thus to get f', c'_2 etc.

out of f, c_2 to order f^{-3}, only Z'_1 must be calculated explicitly, Z'_2 and Z'_3 being of order $O(f^{-4})$. All contributions to Z' satisfy suitable bounds for small as well as large fields ρ, as will be shown below. In Section 5 we do the perturbation theory for Z'_1 and derive universal bounds on Z'_1 and Z'_2. In Section 6 estimates of Z'_3 and its derivatives are given. These considerations provide the basis for proving that Z' satisfies the conditions (A0)-(A2) with the obvious replacements. In Section 7 the values of the coupling constants on the next scale, f', c'_2, $\tilde{c}'_v$, . . . are derived within the order of interest, and the conditions (A1) are reproduced. Finally, in Section 8 we prove that the stability bound (A2) iterates, by using the same methods as in Section 6. That completes the proof of the theorem.

5. PERTURBATION THEORY IN THE RUNNING COUPLING CONSTANT f^{-1}

We collect the results of this and the next section in

PROPOSITION 1. — *There is* $L_1(N, \gamma)>0$ *and for* $L \geqq L_1$ *there are* $F_1(N, \gamma, L^2)>0$ *and* $K_1(N, \gamma, L^2)$, $K_2(N, \gamma, L^2)>0$ *so that the conditions* (A 0)-(A 2) *imply that for all* $f \geqq F_1$

$$\begin{aligned} Z'(f+\rho, 0) = {} & c_0 \mathscr{L}^{L^2/2} \exp(-L^2 \mathscr{L} c_2 \rho^2) \\ & \times \{ d_0 + d_{11} \mathscr{L} \rho + d_{22} \mathscr{L}^2 \rho^2 + d_{31} \mathscr{L}^3 \rho^3 + d_{42} \mathscr{L}^4 \rho^4 + d_{53} \mathscr{L}^5 \rho^5 \\ & + d_{62} \mathscr{L}^6 \rho^6 + d_{73} \mathscr{L}^7 \rho^7 + d_{93} \mathscr{L}^9 \rho^9 \} + c_0 \mathscr{L}^{L^2/2} \tilde{R}(\rho). \end{aligned} \quad (5.1)$$

The d_0, d_{ij} *are polynomials of* $\tilde{c}_v$, *rational functions of* c_2 *and* $d_0 = 1 + O(f^{-2})$, $d_{ij} = O(f^{-j})$. *They are uniformly bounded by*

$$\left. \begin{aligned} |d_0 - 1| &\leqq \frac{C}{f^2} \\ |d_{11}|, |d_{31}| &\leqq \frac{D}{f} L^4, \\ |d_{22}|, |d_{42}|, |d_{62}| &\leqq \frac{D}{f^2} L^8, \\ |d_{53}|, |d_{73}|, |d_{93}| &\leqq \frac{D}{f^3} L^{12}, \end{aligned} \right\} \quad (5.2)$$

for some $D(N, \gamma) > 0$ *and* $C(N, \gamma, L^2) > 0$. $\tilde{R}$ *is analytic in* ρ *and satisfies the bound*

$$|\tilde{R}(\rho)| \leq \frac{1}{f^4}\left\{\frac{3}{4} K_1 \exp\left(-\frac{1}{40}[L^2-1]\mathscr{L} c_2 (Re\,\rho)^2 + 10 L^2 \mathscr{L} c_2 (Im\,\rho)^2\right) + \frac{1}{2} K_2 \exp(-[L^2-1]\mathscr{L} c_2 (Re\,\rho)^2 + L^2 \mathscr{L} c_2 (Im\,\rho)^2)\right\} \quad (5.3)$$

for $\rho \in \Omega(f)$. *Furthermore, there is* $K_3(N, \gamma, L^2) > 0$ *so that for all* $\nu = 0, 1, \ldots, 10$

$$\left|\frac{\partial^\nu}{\partial \rho^\nu} \tilde{R}(\rho)\right| \leq \frac{1}{f^4} K_3 \exp\left(-\frac{1}{50} L^2 \mathscr{L} c_2 (Re\,\rho)^2 + 8 L^2 \mathscr{L} c_2 (Im\,\rho)^2\right). \quad (5.4)$$

Remember that $\Omega(f)$ is defined in equation (4.1). Large coefficients K_1 and K_2 in the bound (3.7) means they stay fixed (for fixed L^2) above a lower bound. For K_1 it is determined by perturbation theory, whereas $K_2 = K_1 . H(L^2)$ for an appropriate $H \geq 1$, *cf.* below. Note that there is no 8th order term in ρ because it is of order $O(f^{-4})$. It follows from Proposition 1 that the minimum of $-\log Z'(f+\rho, 0)$ is at some $\rho_0 = O(f^{-1})$. That means $c_0 \mathscr{L}^{L^2/2} = 1 + O(f^{-2})$ [remember that c_0 is chosen in such a way that $Z'(f+\rho_0, 0) = 1$].

5.1. Perturbation expansion for Z'_1

Using the identity

$$\int d\mu_s(\sigma, \pi) F(\sigma, \pi)|_{s=0} = F(0, 0), \quad (5.5)$$

Z'_1 can be calculated explicitly. The computation to order f^{-3} is rather long but straight-forward. The result is the following

LEMMA 5.1

$$Z'_1(\rho) = Z'_{1,\,\text{pert}}(\rho) + R'_1(\rho), \quad (5.6)$$

where

$$Z'_{1,\,\text{pert}}(\rho) = \exp(-L^2 \mathscr{L} c_2 \rho^2) \times \{d_0 + d_{11} \mathscr{L} \rho + d_{22} \mathscr{L}^2 \rho^2 + d_{31} \mathscr{L}^3 \rho^3 + d_{42} \mathscr{L}^4 \rho^4 + d_{53} \mathscr{L}^5 \rho^5 + d_{62} \mathscr{L}^6 \rho^6 + d_{73} \mathscr{L}^7 \rho^7 + d_{93} \mathscr{L}^9 \rho^9\}. \quad (5.7)$$

The d_0, d_{ij} *are polynomials in* $\tilde{c}_\nu$ *and rational in* c_2. *The second index refers to the leading order in* f^{-1} *and* $d_0 = 1 + O(f^{-2})$. *There are* $F(N, \gamma, L^2)$ *and* $D(N, \gamma)$, $C(N, \gamma, L^2)$ *so that for* $f \geq F$, *the* d_0, d_{ij} *satisfy the inequalities*

(5.2), *and in the domain* $\Omega(f)$, $R'_1(\rho)$ *is analytic and bounded according to*

$$\left|\frac{\partial^\nu}{\partial\rho^\nu} R'_1(\rho)\right| \leqq \frac{\hat{C}(N, \gamma, L^2)}{f^4} \times \exp\left(-\frac{1}{40} L^2 \mathscr{L} c_2 (Re\,\rho)^2 + \frac{5}{4} L^2 \mathscr{L} c_2 (Im\,\rho)^2\right), \quad (5.8)$$

for appropriate $\hat{C}>0$ *and for all* $\nu=0, \ldots, 10$.

$Z'_{1,\,\mathrm{pert}}$ results from an explicit calculation. All contributions in Z'_1 which are of order $O(f^{-4})$ are collected into R'_1. The proof of the bound (5.8) for $\nu=0$ is just a repitition of the proof of Lemma 3.1. The bound could be made even better because of additional factors of $\mathscr{L}$ and $\mathscr{L} \leqq 4/L^2 \leqq 1$, but we don't need do so. Also evident from the form of R'_1 we can choose $\hat{C}$ so large that all derivatives to 10th order, say, satisfy the same bound. Notice also that explicit powers of L^2 appear as combinatorical factors, so that $\hat{C}$ depends explicity of them. Below it is shown that Z'_2 and Z'_3 are uniformly bounded in ρ and are of order $O(f^{-4})$.

5.2. Estimate for Z'_2

This part is the remainder of the perturbative expansion. It is given by

$$Z'_2(\rho) = \frac{1}{4!}\int_0^1 ds\,(1-s)^4 \exp(-L^2 \mathscr{L} c_2 \rho^2) \int d\mu_s(\sigma, \pi) \times \left\{\frac{\gamma}{2}\sum_{y=1}^{L^2}\left[\mathscr{L}\frac{\partial^2}{\partial\sigma_y^2} + 2(N-1)\frac{\partial}{\partial(\pi_y^2)} + 4\pi_y^2\frac{\partial^2}{\partial(\pi_y^2)^2}\right]\right\}^5 \times \prod_{x=1}^{L^2}\left\{\exp(c_2(\sigma_x+\mathscr{L}\rho)^2) Z^{\mathrm{rel}}(f+\sigma_x+\mathscr{L}\rho, \pi_x)\right\}$$

$$= \frac{1}{4!}\int_0^1 ds\,(1-s)^4 \frac{1}{f^4}\prod_{i=1}^{L^2} I_i(\rho; s), \quad (5.9)$$

where for each $i=1, \ldots, L^2$

$$I_i(\rho; s) = \exp(-\mathscr{L} c_2 \rho^2) \int \frac{d\sigma\, d^{N-1}\pi}{(2\pi\gamma)^{N/2}\mathscr{L}^{1/2}} \frac{1}{s^{N/2}} \exp\left(-\frac{\sigma^2}{2\gamma\mathscr{L}_s} - \frac{\pi^2}{2\gamma s}\right) \times \exp\left(c_2(\sigma+\mathscr{L}\rho)^2 - \frac{c_2}{4f^2}[(f+\sigma+\mathscr{L}\rho)^2+\pi^2-f^2]^2\right) \times P_i(\sigma+\mathscr{L}\rho, \pi). \quad (5.10)$$

The P_i are polynomials with coefficients bounded by some $G(N, \gamma, L^2)>0$. Derivatives of Z'_2 with respect to ρ up to an order 10, say, are given by a sum of at most $(L^2)^{10}$ terms of the form (5.9). We can choose G so large that their coefficients are still bounded by G (actually most differentiations introduce powers of f^{-1} or $\mathscr{L}$).

Bounds on integrals of the form (5.10) has been given in Appendix B. According to Lemma B.2 there are $\tilde{L}(N, \gamma)$ and $\tilde{C}(N, \gamma, L^2)$ so that for $L \geqq \tilde{L}$ and $f \geqq 10$ for any $0 \leqq s \leqq 1$

$$|I_i(\rho; s)| \leqq \tilde{C} \exp\left(-\frac{1}{40} \mathscr{L} c_2 (Re\,\rho)^2 + \frac{2}{\gamma} (\mathrm{Im}\,\rho)^2\right), \tag{5.11}$$

for all $\rho \in \Omega(f)$. Thus

LEMMA 5.2. – *There is* $\tilde{L}(N, \gamma)$ *and for* $L \geqq \tilde{L}$ *there is* $C(N, \gamma, L^2)$ *so that for* $f \geqq 10$ *and all* $\nu = 0, \ldots, 10$

$$\left|\frac{\partial^\nu}{\partial \rho^\nu} Z'_2(\rho)\right| \leqq \frac{C}{f^4} \exp\left(-\frac{1}{40} L^2 \mathscr{L} c_2 (Re\,\rho)^2 + 8 L^2 \mathscr{L} c_2 (Im\,\rho)^2\right), \tag{5.12}$$

for all $\rho \in \Omega(f)$.

This completes the perturbative part of Proposition 1.

6. UNIFORM BOUND ON Z'_3

In this section we state bounds on possible nonperturbative corrections to Z', namely on Z'_3. Special care is needed in order to control combinatorical prefactors, which tend the expressions to grow by the RG iteration. We have to get additional factors which compensate for such effects. These so-called downstairs factors result from a careful analysis of the "irrelevance" condition of the remainder, *i.e.* $R(f+\rho, 0) = O(\rho^8)$ as $\rho \to 0$ (A.2), together with analyticity in a strip around the real ρ-axis. In contrast to the analogous discussion in Φ^4-theory [24] derivatives of R cannot be reasonably bounded by applying just Cauchy's formula with a big radius. This is because bound on R are worse away from the real axis. Instead, powers of ρ combined with a fraction of fast decreasing exponential prefactors produce the downstars factors needed, e. g. for real ρ according to

$$|\rho^8 \exp(-c_2 \rho^2)| \leqq \frac{\text{const}}{c_2^4}$$

and by $c_2 \sim L^2$.

We consider

$$Z'_3(\rho) = \sum_{\nu=1}^{L^2} \binom{L^2}{\nu} (I_1(\rho))^{L^2-\nu} (I_2(\rho))^\nu, \tag{6.1}$$

where

$$I_1(\rho)=\exp(-\mathcal{L}c_2\rho^2)\times\int\frac{d\sigma\,d^{N-1}\pi}{(2\pi\gamma)^{N/2}\mathcal{L}^{1/2}}\exp\left\{-\frac{\sigma^2}{2\gamma\mathcal{L}}-\frac{\pi^2}{2\gamma}\right\}\times\exp(c_2(\sigma+\mathcal{L}\rho)^2)Z^{rel}(f+\sigma+\mathcal{L}\rho,\pi)\quad(6.2a)$$

and

$$I_2(\rho)=\exp(-\mathcal{L}c_2\rho^2)\times\int\frac{d\sigma\,d^{N-1}\pi}{(2\pi\gamma)^{N/2}\mathcal{L}^{1/2}}\exp\left\{-\frac{(\sigma-\mathcal{L}\rho)^2}{2\gamma\mathcal{L}}-\frac{\pi^2}{2\gamma}\right\}\times\exp(c_2\sigma^2)R(f+\sigma,\pi).\quad(6.2b)$$

Both I_1 and I_2 are analytic in ρ for Re $\rho>-f-1$, Im $\rho<f/6$ because of the assumptions (A 0)-(A 2), Lemmas 3.1 and B.1. Z^{rel} is known explicitly. On the other hand, $R(f+\rho, 0)$ satisfies the conditions (A 2). In particular, it is $O(f^{-4})$ and $O(\rho^8)$. First of all we consider those terms in (6.1) with $\nu\geqq 2$, *i.e.* where there are at least two integrals of the type (6.2*b*). The term with $\nu=1$ is more tricky.

6.1. Bounds on I_1 and I_2. The case $\nu\geqq 2$

We first of all consider I_1. In order not to get too a bad bound (I_1 is needed for the term with $\nu=1$ also) we do not straightforwardly apply Lemma B.2. Instead we expand I_1 about the Gaussian measure. This way we get the leading term explicitly plus corrections down by f^{-1}.

LEMMA 6.1. – *There are* $\hat{L}(N,\gamma)$, $F_{R,1}(N,\gamma,L^2)$ *and* $G_1(N,\gamma,L^2)$ *so that for* $\mu=1,\ldots,L^2-1$ *for all* $L\geqq\hat{L}$, $f\geqq F_{R,1}$

$$[I_1(\rho)]^{\mu}=\exp(-\mu\mathcal{L}c_2\rho^2)+\delta_{1,\mu}(\rho),\quad(6.3)$$

where $\delta_{1,\mu}$ *is analytic and satisfies*

$$|\delta_{1,\mu}(\rho)|\leqq\frac{G_1}{f}\exp\left(-\frac{1}{40}\mu\mathcal{L}c_2(Re\,\rho)^2+\frac{5}{2\gamma}\mu(\mathrm{Im}\,\rho)^2\right),\quad(6.4)$$

for all $\rho\in\Omega(f)$.

Proof. – $\delta_{1,\mu}(\rho)$ is obviously analytic in ρ. Suppose that $\rho\in\Omega(f)$. We first consider the case $\mu=1$. Let us write

$$I_1(\rho)=I_a(\rho)+I_b(\rho),$$

where

$$I_a(\rho)=\exp(-\mathcal{L}c_2(1-\mathcal{L})\rho^2)Z^{rel}(f+\mathcal{L}\rho,0),$$

$$
\begin{aligned}
I_b(\rho) &= \exp(-\mathscr{L} c_2 \rho^2) \int_0^1 ds \frac{\partial}{\partial s} \int \frac{d\sigma\, d^{N-1}\pi}{(2\pi\gamma)^{N/2}\mathscr{L}^{1/2}} \frac{1}{s^{N/2}} \\
&\quad \times \exp\left(-\frac{\sigma^2}{2\gamma\mathscr{L}s} - \frac{\pi^2}{2\gamma s}\right) \\
&\quad \times \exp(c_2(\sigma+\mathscr{L}\rho)^2) Z^{\text{rel}}(f+\sigma+\mathscr{L}\rho, \pi) \\
&= \exp(-\mathscr{L} c_2 \rho^2) \int_0^1 ds \int \frac{d\sigma\, d^{N-1}\pi}{(2\pi\gamma)^{N/2}\mathscr{L}^{1/2}} \frac{1}{s^{N/2}} \\
&\quad \times \exp\left(-\frac{\sigma^2}{2\gamma\mathscr{L}s} - \frac{\pi^2}{2\gamma s}\right) \\
&\quad \times \frac{\gamma}{2}\left(\mathscr{L}\frac{\partial^2}{\partial\sigma^2} + 2(N-1)\frac{\partial}{\partial(\pi)^2} + 4\pi^2\frac{\partial^2}{\partial(\pi^2)^2}\right) \\
&\quad \times \exp(c_2(\sigma+\mathscr{L}\rho)^2) Z^{\text{rel}}(f+\sigma+\mathscr{L}\rho, \pi).
\end{aligned}
$$

For sufficiently large $L \geqq \hat{L}(N, \gamma)$ there is $D_1(N, \gamma, L^2)$ so that

$$
|I_b(\rho)| \leqq \frac{D_1}{f} \exp\left(-\frac{1}{40}\mathscr{L} c_2 (Re\,\rho)^2 + \frac{2}{\gamma}(\operatorname{Im}\rho)^2\right), \tag{6.5}
$$

for all $f \geqq 10$. The proof of this inequality is just a repitition of the proof of Lemma 5.2. Furthermore, there is $D_2(N, \gamma)$ so that for those f

$$
\left.
\begin{aligned}
I_a(\rho) &= \exp(-\mathscr{L} c_2 \rho^2) + R_a(\rho), \\
|R_a(\rho)| &\leqq \frac{D_2}{f} \exp\left(-\frac{1}{40}\mathscr{L} c_2 (\operatorname{Re}\rho)^2 + \frac{1}{\gamma}(\operatorname{Im}\rho)^2\right).
\end{aligned}
\right\} \tag{6.6}
$$

This is a straightward consequence of the assumption (A1) and Lemma 3.1. Finally, this result for $\mu = 1$ extends to any $\mu \in \{1, \ldots, L^2 - 1\}$.

$$
[I_1(\rho)]^{\mu} = \exp(-\mu \mathscr{L} c_2 \rho^2) + S(\rho), \tag{6.7 a}
$$

where for $L \geqq \hat{L}$ and sufficiently large f

$$
|S(f)| \leqq \frac{G_1}{f} \exp\left(-\frac{1}{40}\mu \mathscr{L} c_2 (\operatorname{Re}\rho)^2 + \frac{5}{2\gamma}\mu (\operatorname{Im}\rho)^2\right). \tag{6.7 b}
$$

with some $G_1(N, \gamma, L^2) > 0$. ■

Now we come to the remainder integral I_2. For real σ, π, the induction hypothesis (A2) tells us that $R(f+\sigma, \pi)$ is bounded by

$$
|R(f+\sigma, \pi)| \leqq \frac{\bar{K}}{f^4} \exp\left(-\frac{c_2}{50}\lambda^2\right), \tag{6.8}
$$

with $\lambda=((f+\sigma)^2+\pi^2)^{1/2}-f$ and $\bar{K}=K_1+K_2$, *cf.* (3.7). Substituting $\sigma\to\sigma+\operatorname{Re}\rho$, I_2 is easily bounded by

$$|I_2(\rho)|\leqq\frac{\bar{K}}{f^4}\exp\left(\frac{1}{2\gamma}(\operatorname{Im}\rho)^2\right)\times\int\frac{d\sigma\, d^{N-1}\pi}{(2\pi\gamma)^{N/2}\mathscr{L}^{1/2}}\exp\left(-\frac{\sigma^2+\pi^2}{2\gamma}-\frac{c_2}{50}\times\left(\left[(f+\sigma+\operatorname{Re}\rho)^2+\pi^2\right]^{1/2}-f\right)^2\right).$$

To the integral we apply Lemma B.1 by writting $\psi=(f+\operatorname{Re}\rho)\hat{\psi}$, $|\hat{\psi}|=1$. With some $K(N,\gamma)$ we get for sufficiently large $L\geqq\tilde{L}(N,\gamma)$

$$|I_2(\rho)|\leqq\frac{\bar{K}K}{f^4}\left(\frac{1+2\gamma c_2}{1+\gamma c_2/50}\right)^{1/2}\times\exp\left(-\frac{1}{250}\,\frac{c_2}{1+2\gamma c_2/50}(|\psi|-f)^2+\frac{1}{2\gamma}(\operatorname{Im}\rho)^2\right)$$
$$\leqq 8\frac{\bar{K}K}{f^4}\exp\left(-\frac{2}{750}\,\frac{c_2}{1+2\gamma c_2/50}(\operatorname{Re}\rho)^2+\frac{1}{2\gamma}(\operatorname{Im}\rho)^2\right),$$

for $\operatorname{Re}\rho\geqq -f-1$ and $f\geqq 10$, where we have used that

$$(|\psi|-f)^2\geqq\frac{2}{3}(\operatorname{Re}\rho)^2\qquad\text{for}\quad\operatorname{Re}\rho\geqq -f-1,\quad f\geqq 10,$$

cf. the proof of Lemma B.2. For $L^2\geqq 49$

$$\frac{2}{750}\,\frac{c_2}{1+2\gamma(c_2/50)}\geqq\frac{1}{40}\,\frac{c_2}{1+2\gamma c_2}.$$

Thus

LEMMA 6.2. – *There are* $\tilde{L}(N,\gamma)>0$ *and* $G_2(N,\gamma)$ *so that for* $L\geqq\tilde{L}$ *and* $f\geqq 10$

$$|I_2(\rho)|\leqq\frac{G_2}{f^4}(K_1+K_2)\exp\left(-\frac{1}{40}\mathscr{L}c_2(\operatorname{Re}\rho)^2+\frac{1}{2\gamma}(\operatorname{Im}\rho)^2\right)\quad(6.9)$$

for all ρ *satisfying* $\operatorname{Re}\rho\geqq -f-1$.

As a corollary of the last two lemmas we get a bound on the sum (6.1) with the term $\nu=1$ omitted. Assume the bounds (3.7) to hold for given constants K_1 and K_2.

COROLLARY 6.1. – *There are* $\tilde{L}(N, \gamma)$ *and* $F_{R,2}(N, \gamma, L^2, K_1, K_2)$ *so that for* $L \geqq \tilde{L}, f \geqq F_{R,2}$

$$\left| \sum_{\nu=2}^{L^2} \binom{L^2}{\nu} [I_1(\rho)]^{L^2-\nu} [I_2(\rho)]^{\nu} \right| \leqq \frac{K_1}{6 f^6} \exp\left(-\frac{1}{40} L^2 \mathscr{L} c_2 (\operatorname{Re} \rho)^2 + \frac{5}{2\gamma} L^2 (\operatorname{Im} \rho)^2 \right), \quad (6.10)$$

for all $\rho \in \Omega(f)$.

A fraction f^{-2} of the original f^{-8} is used to absorb "bad" coefficients. $K_1(N, \gamma, L^2)$ and $K_2(N, \gamma, L^2)$ are given by assumption (A2). Clearly, the larger we choose K_1 and K_2, the larger must be $F_{R,2}$, *i.e.* f. We retain f^{-6} instead of f^{-4} (which we finally want to have) just to simplify the management of derivatives of the sum, *cf.* Section 8.

6.2. Improved bounds on I_2. The case $\nu = 1$

We now consider the more delicate term of (6.1) with $\nu = 1$,

$$L^2 [I_1(\rho)]^{L^2-1} . I_2(\rho), \quad (6.11)$$

where I_1 and I_2 are given by (6.2). If compared to the other terms dealt with in Subsection 6.1 above there are no more additional factors of f^{-4}, which could be used to bend down big coefficients, in particular those which grow as some function of L. Powers of I_1 are controlled by Lemma 6.1. The prefactor L^2 must be compensated by so-called downstairs factors. They result from a more careful analysis of I_2 than we have done before in order to derive the bound of Lemma 6.2. We have to exploit the inductive assumption of "irrelevance" of R, *cf.* (A2). As will come out below, the first term of the bound on R with not too a good exponent does not give us such downstairs factors, but the second does. On the other hand, the first one mainly gets its contributions from the perturbation expansion of the effective Boltzmannian Z and its remainder, *cf.* Lemma 5.1 and Lemma 5.2. It cannot be made better because part of the exponential decay must be used to control increase of the expressions by powers of the fields. Also, away from the real axis, the bound is rather bad, so that we don't get downstairs factors by applying Cauchy's formula, integrating on a circle of big radius.

Actually, this part of the stability bound behaves under the RG in such a way that on the next scale it is bounded by an expression of the form of the second part in (3.7). This term is well behaved under subsequent iterations, as will be shown below.

The improved bound on I_2 is as follows.

LEMMA 6.3. – *Let* $L^2 \geqq 150$, $f \geqq 10$. *There are* $G_3(N, \gamma) > 0$ *and* $H(L^2, \gamma) \geqq 1$ *so that*

$$|I_2(\rho)| \leqq \frac{1}{L^8} \frac{G_3}{f^4} (K_1 \cdot H + K_2) \times \exp\left(-\frac{1}{50} \mathscr{L} c_2 (\operatorname{Re} \rho)^2 + \mathscr{L} c_2 (\operatorname{Im} \rho)^2\right), \quad (6.12)$$

for all $\rho \in \Omega(f)$.

Notice the downstairs factor L^{-8}. For large L the coefficient of K_2 can be made arbitrary small (but normally the coefficient of K_1 cannot) without using a fraction of f^4.

Proof. – Set

$$\lambda = ([f + \sigma]^2 + \pi^2)^{1/2} - f$$

and for real $\lambda \geqq -f$,

$$\tilde{R}(\lambda) = R(f + \sigma, \pi).$$

By induction hypothesis (A2), considered as a function of complex λ, $\tilde{R}(\lambda)$ is analytic in $\Omega(f)$, satisfies the "irrelevance" condition $\tilde{R}(\lambda) = O(\lambda^8)$ (as $\lambda \to 0$) and the corresponding bound (3.7). Assume the following statement holds: There are $D(\gamma) > 0$ and $H(L^2, \gamma) \geqq 1$ so that for all $L^2 \geqq 150, f \geqq 10$

$$|\tilde{R}(\lambda)| \leqq \frac{1}{L^8} \frac{D}{f^4} (K_1 \cdot H + K_2) \exp\left(-\frac{c_2}{100} \lambda^2\right). \quad (6.13)$$

Then Lemma 6.3 follows in exactly the same as we proved Lemma 6.2. Thus we have to prove (6.13). Because of

$$\tilde{R}(\lambda) \exp(c_2 \lambda^2) = O(\lambda^8),$$

we have by Taylor's formula

$$\tilde{R}(\lambda) = \exp(-c_2 \lambda^2) \int_0^1 ds \frac{(1-s)^7}{7!} \lambda^8 \frac{\partial^8}{\partial \eta^8} [\exp(c_2 \eta^2) \tilde{R}(\eta)]_{\eta = s\lambda}.$$

We apply Cauchy's formula, estimating on a circle of unit radius with midpoint $s\lambda \geqq -f$. Because of (A2) we get

$$\left| \frac{\partial^8}{\partial \eta^8} [\exp(c_2 \eta^2) \tilde{R}(\eta)]_{\eta = s\lambda} \right| \leqq \frac{8!}{f^4} \exp\left(c_2 \left(1 - \frac{1}{50}\right)\left(1 + \frac{1}{L^2}\right) \lambda^2 + \frac{3}{2\gamma}\right) \times \left(K_1 \exp\left(\frac{2}{\gamma} L^4\right) + K_2\right).$$

Hence for $L^2 \geqq 150$

$$|\tilde{R}(\lambda)| \leqq \frac{8!}{f^4}|\lambda^8| \exp\left(-\frac{c_2}{75}\lambda^2 + \frac{3}{2\gamma}\right) \times \left(K_1 . \exp\left(\frac{2}{\gamma}L^4\right) + K_2\right) \leqq \frac{1}{L^8}\frac{D(\gamma)}{f^4}\exp\left(-\frac{1}{100}c_2\lambda^2\right)(K_1 . H(L^2, \gamma) + K_2),$$

for appropriate D and H. That proves (6.13) and hence the lemma. ■

With the improved bound on I_2 by Lemma 6.3 and those on powers of I_1 by Lemma 6.1 we get the desired bound on the contribution to Z'_3 given by the $\nu = 1$ term.

COROLLARY 6.2. — *There are* $\bar{L}(N, \gamma) > 0$, $H(L^2, \gamma) \geqq 1$, $K_1(N, \gamma, L^2)$ *and* $F_{R,3}(N, \gamma, L^2) > 0$, *so that with* $K_2 = H . K_1$ *for all* $L \geqq \bar{L}$, $f \geqq F_{R,3}$ *and all* $\rho \in \Omega(f)$

$$|L^2 I_1(\rho)^{L^2-1} I_2(\rho)| \leqq \frac{1}{f^4}\frac{K_1}{3} \times \exp\left(-\frac{1}{40}(L^2-1)\mathscr{L}c_2(\mathrm{Re}\,\rho)^2 + 8L^2\mathscr{L}c_2(\mathrm{Im}\,\rho)^2\right) + \frac{1}{f^4}\frac{K_2}{2}\exp(-(L^2-1)\mathscr{L}c_2(\mathrm{Re}\,\rho)^2 + L^2\mathscr{L}c_2(\mathrm{Im}\,\rho)^2). \quad (6.14)$$

Remark. — "Bad" contributions coming from the iteration of the first part of the bound (3.7) are absorbed in a term of the second form for sufficiently large H. This latter part is stable under the RG.

Proof. — We apply to $(I_1)^{L^2-1}$ Lemma 6.1 and consider both of the resulting terms separately. By Lemma 6.3 there are $G(N, \gamma)$, $H(L^2, \gamma)$ and $F_{R,1}(N, \gamma, L^2) \geqq 10$ so that

$$|L^2 \exp(-(L^2-1)\mathscr{L}c_2\rho^2) I_2(\rho)| \leqq \frac{G}{f^4 L^6}(K_1 . H + K_2) \times \exp(-(L^2-1)\mathscr{L}c_2(\mathrm{Re}\,\rho)^2 + L^2\mathscr{L}c_2(\mathrm{Im}\,\rho)^2) \leqq \frac{K_2}{2f^4}\exp(-(L^2-1)\mathscr{L}c_2(\mathrm{Re}\,\rho)^2 + L^2\mathscr{L}c_2(\mathrm{Im}\,\rho)^2),$$

where we have fixed $K_2 = K_1 . H$ and $L^2 \geqq (2G)^{1/3}$. Here the downstairs factors are of great importance. Similarly

$$|L^2 \delta_{1, L^2-1}(\rho) I_2(\rho)| \leqq \frac{K_1}{3f^4}\exp\left(-\frac{1}{40}(L^2-1)\mathscr{L}c_2(\mathrm{Re}\,\rho)^2 + 8L^2\mathscr{L}c_2(\mathrm{Im}\,\rho)^2\right),$$

for sufficiently large f. ■

The results of Section 6 can be summarized in the following

COROLLARY 6.3. – *There exists* $L_R(N, \gamma)>0$ *so that for all* $L \geqq L_R$ *there are* $H(L^2, \gamma) \geqq 1$, $K_1(N, \gamma, L^2)>0$ *and* $F_R(N, \gamma, L^2)>0$ *such that with* $K_2 = K_1 \cdot H$ *for all* $f \geqq F_R$

$$\begin{aligned} |Z_3'(\rho)| \leqq \frac{K_1}{2f^4} & \\ & \times \exp\left(-\frac{1}{40}(L^2-1)\mathscr{L}c_2(\operatorname{Re}\rho)^2 + 8L^2\mathscr{L}c_2(\operatorname{Im}\rho)^2\right) \\ & + \frac{K_2}{2f^4}\exp(-(L^2-1)\mathscr{L}c_2(Re\,\rho)^2 + L^2\mathscr{L}c_2(\operatorname{Im}\rho)^2), \quad (6.15) \end{aligned}$$

for all $\rho \in \Omega(f)$. *Furthermore, there is* $C_R(N, \gamma, L^2)$ *so that for* $\nu = 0, 1, \ldots, 10$ *for all those* ρ

$$\left|\frac{\partial^\nu}{\partial\rho^\nu} Z_3'(\rho)\right| \leqq \frac{C_R}{f^4}\exp\left(-\frac{1}{50}L^2\mathscr{L}c_2(\operatorname{Re}\rho)^2 + 8L^2\mathscr{L}c_2(\operatorname{Im}\rho)^2\right). \quad (6.16)$$

Proof. – We collect the Corollaries 6.1 and 6.2 and notice that

$$\frac{1}{3\gamma} \leqq \mathscr{L}c_2 \leqq \frac{1}{2\gamma}.$$

Derivatives of Z_3' with respect to ρ are dealt with in a similar way, using the representations (6.1), (6.2), compare Section 5. ■

Proof of Proposition 1. – $Z'(f+\rho)$ decomposes according to (4.4). Lemma 5.1 states the asserted properties of the coefficients d_0, d_{ij} of the expression (5.1). They result from Z_1' alone. By the Lemmas 5.1, 5.2 and Corollary 6.3 there are $L_1(N, \gamma)>0$, $F_1(N, \gamma, L^2)>0$, so that for $L \geqq L_1$, $f \geqq F_1$ the remainder $\tilde{R}(\rho)$ [*cf.* (5.1)] is analytic in $\Omega(f)$ and bounded by

$$\begin{aligned} |\tilde{R}(\rho)| \leqq \frac{1}{f^4}\left(\hat{C} + C + \frac{K_1}{2}\right) & \\ & \times \exp\left(-\frac{1}{40}(L^2-1)\mathscr{L}c_2(\operatorname{Re}\rho)^2 + 8L^2\mathscr{L}c_2(\operatorname{Im}\rho)^2\right) \\ & + \frac{K_2}{2f^4}\exp(-(L^2-1)\mathscr{L}c_2(\operatorname{Re}\rho)^2 + L^2\mathscr{L}c_2(\operatorname{Im}\rho)^2), \end{aligned}$$

where $K_2 = K_1 . H$ for some $H(L^2, \gamma) \geqq 1$. If we set K_1 so large that it satisfies

$$K_1(N, \gamma, L^2) \leqq 4(\hat{C} + C),$$

the bound (5.3) of Proposition 1 follows. Finally, derivatives of Z' with respect to ρ are bounded according to (5.4), where $K_3 \geqq \hat{C}+C+C_R$. This completes the proof of Proposition 1.

7. RUNNING COUPLINGS ON THE NEXT SCALE, ITERATION OF (A1)

We are going to determine the values of the running coupling constants f, c_2, $\tilde{c}_v$ on the new scale with the help of Proposition 1, up to order $O(f^{-4})$. According to this proposition, $Z'(f+\rho, 0)$ is analytical in ρ for $\operatorname{Re}\rho \geqq -f-1$, $(\operatorname{Im}\rho)^2 \leqq f/6$, and its derivatives with respect to ρ are uniformly bounded. The coupling constants f', c_2', $\tilde{c}_v'$ are defined according to (A1) with the obvious replacements. Let us denote the $O(f^{-4})$ corrections to $c_0 \mathscr{L}^{L^2/2}$, f', c_2', $\tilde{c}_3'$, ..., $\tilde{c}_9'$ by $\mathscr{R}_0$, $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$, ..., $\mathscr{R}_9$, respectively. We suppose that f is sufficiently large, so that the equation 7.1 below has a unique solution. The location of the minimum of the effective action $-\log Z'(r, 0)$ on the nonnegative reals, $r=f'$, is determined by the equation

$$\left(\frac{\partial}{\partial\rho} Z'(f+\rho, 0)\right)_{\rho=f'-f} = 0. \tag{7.1}$$

This implicit equation is solved iteratively in f^{-1}, with the result

$$f' = f - \frac{G_1}{f} - \frac{G_2}{f^3} + \mathscr{R}_1. \tag{7.2}$$

G_1 and G_2 are polynomials in $\tilde{c}_v$, rational in c_2, and because of Equation (3.6) they are bounded by some $\kappa_1(N, \gamma, L^2) > 0$. In particular,

$$G_1 = \gamma\left(\frac{N-1}{2} + \frac{3\mathscr{L}}{2c_2} f\tilde{c}_3\right). \tag{7.3}$$

The normalization factor $c_0 \mathscr{L}^{L^2/2}$ is chosen in such a way that

$$1 = Z'(f', 0).$$

We get

$$c_0 \mathscr{L}^{L^2/2} = 1 - \left[(d_0 - 1) + \frac{\mathscr{L} d_{11}^2}{2L^2 c_2 d_0}\right] + \mathscr{R}_0. \tag{7.4}$$

According to Proposition 1 (bounds on derivatives of Z with respect to ρ) and the implicit function theorem, the remainder $\mathscr{R}_0$, $\mathscr{R}_1$, ... are uniformly bounded by

$$|\mathscr{R}_i| \leqq \frac{T}{f^4} \quad \text{for all } i=0, 1, \ldots, 7, 9, \tag{7.5}$$

with some constant $T(N, \gamma, L^2)$. Because of this, we infer from (7.2), (7.3) for sufficiently large $L > \tilde{L}(N, \gamma)$ that $G_1 > 0$. Hence for sufficiently large f

$$f - \frac{\gamma(N-1)}{f} < f' < f. \tag{7.6}$$

Furthermore $c'_2 = -\frac{1}{2}\frac{\partial^2}{\partial\rho^2} Z'(f+\rho, 0)_{\rho=f'-f}$ is given by

$$c'_2 = \Delta_2 + \mathscr{R}_2, \tag{7.7}$$

with $\Delta_2 = L^2 \mathscr{L} c_2$. Hence

$$\frac{1}{4}\frac{L^2-1}{2\gamma} \leqq c'_2 \leqq \frac{L^2}{\gamma}.$$

In particular that means $c'_2 > 0$, *i.e.* the effective action $-\log Z'(f+\rho, 0)$ actually has a minimum at $\rho = f' - f$.

The coupling constants $\tilde{c}'_\nu$, $\nu = 3, \ldots, 7, 9$ are determined by

$$\tilde{c}'_\nu = -\frac{1}{\nu!}\frac{\partial^\nu}{\partial\rho^\nu}[Z'(f+\rho, 0)\exp(c'_2(f+\rho-f')^2)]_{\rho=f'-f}. \tag{7.8}$$

They are computed according to Appendix C in terms of derivatives of Z'. As a result

$$\tilde{c}'_\nu = \Delta_\nu + \mathscr{R}_\nu \qquad \text{for} \quad \nu = 3, \ldots, 7, 9, \tag{7.9}$$

where $\Delta_\nu = \mathscr{L}^\nu \eta_\nu = O(f^{-\mathcal{O}(\nu)})$, and η_ν are polynomials of at most third degree in d_0, d_{ij} as given in Proposition 1, which themselves are polynomials in $\tilde{c}_\nu$ and rational in c_2, satisfying the bound (5.2) for c_2, $\tilde{c}_\nu$ in the domain (3.6). According to this, for sufficiently large f, the Δ_ν are uniformly bounded by

$$|\Delta_\nu| \leqq \frac{\kappa_2}{f^{\mathcal{O}(\nu)}}, \qquad \nu = 2, 3, \ldots, 7, 9, \tag{7.10}$$

with some $\kappa_2(N, \gamma, L^2) > 0$. $\mathcal{O}(\nu)$ denotes the order of c_ν respectively $\tilde{c}_\nu$ in f^{-1}, *i.e.* $\mathcal{O}(2) = 0$, $\mathcal{O}(3) = 1$, $\mathcal{O}(4) = \mathcal{O}(6) = 2$ and $\mathcal{O}(5) = \mathcal{O}(7) = (9) = 3$. Furthermore, the bounds (3.6) on c'_2, $\tilde{c}'_\nu$, hold, with f replaced by $f' < f$.

We collect the results of our perturbative discussion in

PROPOSITION 2. — *There exists* $L_2(N, \gamma) > 0$ *and for* $L \geqq L_2$ *there is* $F_2(N, \gamma, L^2) > 0$ *so that for all* $f \geqq F_2$ *the following relations and bounds hold.*

$$\left.\begin{aligned} f - \frac{\gamma(N-1)}{f} \leqq f' = f - \frac{G_1}{f} - \frac{G_2}{f^3} + \mathscr{R}_1 < f,& \\ c'_2 = \Delta_2 + \mathscr{R}_2,& \\ \tilde{c}'_\nu = \Delta_\nu + \mathscr{R}_\nu, \qquad \nu = 3, \ldots, 7, 9.& \end{aligned}\right\} \tag{7.11}$$

G_1, G_2, Δ_ν are polynomials in $\tilde{c}_\nu$, rational in c_2. There are constants $\kappa(N, \gamma, L^2)$ and $C(N, \gamma)$ so that

$$\left.\begin{aligned} |G_1|,\ |G_2| &\leqq \kappa, \\ |\Delta_\nu| &\leqq \frac{\kappa}{f^{\mathcal{O}(\nu)}}, \qquad \nu=2, 3, \ldots, 7, 9, \end{aligned}\right\} \tag{7.12}$$

where $\mathcal{O}(\nu)$ denotes the order of c_ν, resp. $\tilde{c}_\nu$ in f^{-1}. The new coupling constants satisfy

$$\left.\begin{aligned} \frac{1}{4}\,\frac{L^2-1}{2\gamma} \leqq c_2' \leqq \frac{L^2}{\gamma}, \\ |\tilde{c}_3'| \leqq \frac{L^2}{f'}, \\ |\tilde{c}_4'|,\ |\tilde{c}_6'| \leqq \frac{L^4}{(f')^2}, \\ |\tilde{c}_5'|,\ |\tilde{c}_7'|,\ |\tilde{c}_9'| \leqq \frac{L^6}{(f')^3}, \\ \frac{7}{8} \leqq c_0\, \mathcal{L}^{L^2/2} \leqq \frac{9}{8}. \end{aligned}\right\} \tag{7.13}$$

This proves that the assertions (A1) hold for $Z'(\psi)$, where the new "relevant" part is given by

$$Z'^{\,\mathrm{rel}}(\psi) = \exp\left\{-\frac{c_2'}{4(f')^2}(\psi^2-(f')^2)^2\right\} \times \left\{1 - \sum_{\substack{\nu=3\\(\nu\neq 8)}}^{9} c_\nu' \left[\frac{\psi^2-(f')^2}{2f'}\right]^\nu\right\}, \tag{7.14}$$

for all $\psi \in \mathscr{R}^N$.

8. ITERATION OF THE STABILITY BOUND (A2)

It remains to prove that

$$R'(f'+\eta, 0) = Z'(f'+\eta, 0) - (Z')^{\mathrm{rel}}(f'+\eta, 0) \tag{8.1}$$

satisfies the conditions (A2) with the coupling constants f, c_2, ... replaced by the new ones f', c_2', ... Here we have defined the new ("block spin") field η by

$$f'+\eta = f+\rho, \tag{8.2}$$

so that

$$(Z')^{\mathrm{rel}}(f'+\eta,0)=\exp\left[-\frac{c'_2}{4(f')^2}[(f'+\eta)^2-(f')^2]^2\right]\times\left\{1-\sum_{\substack{\nu=3\\(\nu\neq 8)}}^{9}c'_\nu\left[\frac{(f'+\eta)^2-(f')^2}{2f'}\right]^\nu\right\}. \quad (8.3)$$

The coupling constants c'_ν, $\nu\geqq 3$ are related to the $\tilde{c}'_\nu$ as defined according to (A1) by Lemma 3.1 with the obvious replacements.

The results of this section are summarized in

PROPOSITION 3. — $R'(f'+\eta, 0)$ *is analytic for* $\eta\in\Omega(f')$ *and satisfies* $R'(f'+\eta,0)=O(\eta^8)$ *as* $\eta\to 0$. *Furthermore, there are* $L_3(N,\gamma)>0$, K_1, $K_2(N,\gamma,L^2)>0$ *and* $F_3(N,\gamma,L^2)>0$ *so that for* $L\geqq L_3$, $f\geqq F_3$ *and all* $\eta\in\Omega(f')$

$$|R'(f'+\eta,0)|\leqq\frac{K_1}{(f')^4}\exp\left(-\frac{1}{50}c'_2(\mathrm{Re}\,\eta)^2+10\,c'_2(\mathrm{Im}\,\eta)^2\right)+\frac{K_2}{(f')^4}\exp\left(-\left(c'_2-\frac{1}{2\gamma}\right)(\mathrm{Re}\,\eta)^2+\left(c'_2+\frac{1}{2\gamma}\right)(\mathrm{Im}\,\eta)^2\right). \quad (8.4)$$

Combining the Propositions 2 and 3 we see that the properties (A0)-(A2) of the effective partition function iterate, and that the bounds on the coefficients stated in the theorem hold. This proves the theorem with $L_0=\max(L_2, L_3)$ and $F_0=\max(F_2, F_3)$. In the remainder of this section we prove Proposition 3.

8.1. Outline of the proof of Proposition 3

By Proposition 1, $R'(f'+\eta,0)$ as given by (8.1) is analytic in $\eta\in\Omega(f')$. Furthermore, again by Lemma 3.1 and the definitions of f', c'_2, $\tilde{c}'_\nu$

$$(Z')^{\mathrm{rel}}(f'+\eta,0)=\exp(-c'_2\eta^2)\times\sum_{\substack{\nu=0\\(\nu\neq 8)}}^{9}\frac{\eta^\nu}{\nu!}\left[\frac{\partial^\nu}{\partial\eta^\nu}\exp(c'_2\eta^2)Z'(f'+\eta,0)\right]_{\eta=0}-(R')^{\mathrm{rel}}(\eta), \quad (8.5)$$

where $(R')^{\mathrm{rel}}(\eta)=O(\eta^8)$ as $\eta\to 0$, is obviously analytic in η and satisfies for some positive $C(N,\gamma,L^2)$

$$|(R')^{\mathrm{rel}}(\eta)|\leqq\frac{C}{(f')^4}\exp\left(-\frac{1}{40}c'_2(\mathrm{Re}\,\eta)^2+\frac{5}{4}c'_2(\mathrm{Im}\,\eta)^2\right), \quad (8.6)$$

for all $\eta \in \Omega(f')$. From (8.5) we get

$$R'(f'+\eta,0)=\exp(-c_2'\eta^2)\left[1-\sum_{\substack{\nu=0\\(\nu\neq 8)}}^{9}\frac{\eta^\nu}{\nu!}\left[\frac{\partial^\nu}{\partial\eta^\nu}\right]_{\eta=0}\right]$$
$$\times\exp(c_2'\eta^2)\,Z'(f'+\eta,0)+(R')^{\mathrm{rel}}(\eta). \quad (8.7)$$

Now it is obvious that $R'(f'+\eta,\,0)=O(\eta^8)$. It remains to prove the desired bound on R'. This however is straightforward, by the same methods as developed in Section 6 above, using analyticity of Z' and the bounds of the Sections 5, 6 and Proposition 2 (on $c_0\,\mathscr{L}^{L^2/2}$). We are thus very scetchy in the following.

Let us introduce the notations

$$\left.\begin{aligned}\Delta&=f-f',\\ \delta&=c_2'-L^2\,\mathscr{L}\,c_2.\end{aligned}\right\} \quad (8.8)$$

For some $K_0(N,\,\gamma,\,L^2)>0$,

$$|\delta|\leqq\frac{K_0}{f^2}. \quad (8.9)$$

Furthermore, there are $L'(N,\,\gamma)>0$, $F'(N,\,\gamma,\,L^2)>0$, so that for $L\geqq L'$ and $f\geqq F'$

$$\left.\begin{aligned}&0<\Delta<\frac{\gamma(N-1)}{f},\\ &\frac{7}{8}\leqq c_0\,\mathscr{L}^{L^2/2}\leqq\frac{9}{8}.\end{aligned}\right\} \quad (8.10)$$

Remember that $f'\leqq f$, $\mathrm{Re}\,\rho=\mathrm{Re}\,\eta-\Delta$, $\mathrm{Im}\,\rho=\mathrm{Im}\,\eta$, and that $\eta\in\Omega(f')$ implies $\rho\in\Omega(f)$. According to the decomposition (4.4) of Z' we write

$$R'(f'+\eta,0)=c_0\,\mathscr{L}^{L^2/2}\sum_{i=1}^{3}\tilde{R}_i(f'+\eta)+(R')^{\mathrm{rel}}(\eta), \quad (8.11)$$

with

$$\tilde{R}_i(f'+\eta,0)=\exp(-c_2'\eta^2)\left[1-\sum_{\substack{\nu=0\\(\nu\neq 8)}}^{9}\frac{\eta^\nu}{\nu!}\left[\frac{\partial^\nu}{\partial\rho^\nu}\right]_{\rho=-\Delta}\right]$$
$$\times\exp(c_2'(\Delta+\rho)^2)\,Z_i'(\rho). \quad (8.12)$$

All four terms in (8.11) are well behaved separately, as will be seen below. $(R')^{\mathrm{rel}}$ is already done, *cf.* (8.6).

8.2. The perturbative part again

LEMMA 8.1. — *There exists* $L_\alpha(N, \gamma)$ *so that for* $L \geqq L_\alpha$ *there are* $F_\alpha(N, \gamma, L^2)$ *and* $C_1(N, \gamma, L^2)$ *such that for* $f > F_\alpha$ *and for all* $\eta \in \Omega(f')$

$$|\tilde{R}_1(f'+\eta)+\tilde{R}_2(f'+\eta)| \leqq \frac{C_1}{(f')^4}\exp\left(-\frac{c_2'}{50}(\operatorname{Re}\eta)^2+8c_2'(\operatorname{Im}\eta)^2\right). \quad (8.13)$$

Proof. — According to Lemma 5.1 we decompose

$$Z_1'(\rho)+Z_2'(\rho)=Z'_{1,\,\text{pert}}(\rho)+R_1'(\rho)+Z_2'(\rho) \quad (8.14)$$

and make use of

$$\left[1-\sum_{\substack{\nu=0\\(\nu\neq 8)}}^{9}\frac{\eta^\nu}{\nu!}\left(\frac{\partial^\nu}{\partial\rho^\nu}\right)_{\rho=-\Delta}\right]Z'_{1,\,\text{pert}}(\rho) = \int_0^1 ds\frac{(1-s)^9}{9!}\eta^{10}\left[\frac{\partial^{10}}{\partial\lambda^{10}}F(\lambda)\right]_{\lambda=s\eta-\Delta} + \frac{\eta^8}{8!}\left[\frac{\partial^8}{\partial\rho^8}F(\rho)\right]_{\rho=-\Delta}. \quad (8.15)$$

Both of these terms as well as derivatives of R_1' and Z_2' are bounded according to the Lemmas 5.1 and 5.2. ■

8.3. The nonperturbative part

LEMMA 8.2. — *There are* $L_\beta>0$, $H(L^2)\geqq 1$, $K_1(N, \gamma, L^2)$ *and* $F_\beta(N, \gamma, L^2)>0$ *so that with* $K_2=K_1\cdot H$ *for* $L\geqq L_\beta$ *and* $f\geqq F_\beta$

$$|\tilde{R}_3(f'+\eta)| \leqq \frac{2}{3}\frac{K_1}{(f')^4}\exp\left(-\frac{c_2'}{50}(\operatorname{Re}\eta)^2+10c_2'(\operatorname{Im}\eta)^2\right) + \frac{8}{9}\frac{K_2}{(f')^4}\exp\left(-\left(c_2'-\frac{1}{2\gamma}\right)(\operatorname{Re}\eta)^2 + \left(c_2'+\frac{1}{2\gamma}\right)(\operatorname{Im}\eta)^2\right) \quad (8.16)$$

for all $\eta\in\Omega(f')$.

The Lemma is a consequence of the two following inequalities.

$$|Z_3'(\eta-\Delta)| \leqq \frac{2}{3}\frac{K_1}{(f')^4}\exp\left(-\frac{c_2'}{50}(\operatorname{Re}\eta)^2+10\,c_2'(\operatorname{Im}\eta)^2\right) + \frac{2}{3}\frac{K_2}{(f')^4}\exp\left(-\left(c_2'-\frac{1}{2\gamma}\right)(\operatorname{Re}\eta)^2 + \left(c_2'+\frac{1}{2\gamma}\right)(\operatorname{Im}\eta)^2\right) \quad (8.17)$$

and

$$\exp(-c_2\eta^2)\sum_{\substack{\nu=0\\(\nu\neq 8)}}^{9}\frac{\eta^9}{\nu!}\left[\frac{\partial^\nu}{\partial\rho^\nu}\exp(c_2'(\Delta+\rho)^2)\,Z_3'(\rho)\right]_{\rho=-\Delta} \leqq \frac{2}{9}\frac{K_2}{(f')^4}\exp\left(-\left(c_2'-\frac{1}{2\gamma}\right)(\operatorname{Re}\eta)^2 + \left(c_2'+\frac{1}{2\gamma}\right)(\operatorname{Im}\eta)^2\right), \quad (8.18)$$

for all η satisfying $\eta\in\Omega(f')$. The first one follows from Corollary 6.3, noticing that for sufficiently large f

$$L^2\mathscr{L}\,c_2\leqq c_2'+\frac{1}{2\gamma}, \quad (8.19)$$

and that because of (8.9) and (8.10),

$$(L^2-1)\,\mathscr{L}\,c_2(\operatorname{Re}\rho)^2\geqq\left(c_2'-\frac{1}{2\gamma}\right)(\operatorname{Re}\eta)^2-\log\left(\frac{4}{3}\right). \quad (8.20)$$

So we have to prove the bound (8.18). We cannot just refer to Proposition 1 and Corollary 6.3 because we have to be careful in order to get the right downstairs factors. Nevertheless, (8.18) is a straightforward consequence of

LEMMA 8.3. — *Let* $\Lambda>0$ *be a constant. There are* $L_\beta'(N,\gamma,\Lambda)>0$ *and for* $L\geqq L_\beta'$ *there are* $H(L^2,\gamma)\geqq 1$, $K_1(N,\gamma,L^2)$ *and* $F_\beta'(N,\gamma,L^2,\Lambda)>0$ *so that with* $K_2=K_1\cdot H$ *for* $f\geqq F_\beta'$ *and for all* $0\leqq\nu\leqq 9$

$$\left|\frac{\partial^\nu}{\partial\rho^\nu}\exp(c_2'(\Delta+\rho)^2)\,Z_3'(\rho)\right|_{\rho=-\Delta}\leqq\frac{1}{\Lambda}\frac{K_2}{f^4}\nu!. \quad (8.21)$$

Proof. — We split Z_3' into three parts, *cf.* (6.1):

$$Z_3'(\rho)=\sum_{i=1}^{3}E_i(\rho), \quad (8.22)$$

where

$$\left.\begin{aligned} E_1(\rho) &= \sum_{v=2}^{L^2} \binom{L^2}{v} [I_1(\rho)]^{L^2-v} I_2(\rho)^v, \\ E_2(\rho) &= L^2 \delta_{1, L^2-1}(\rho) I_2(\rho), \\ E_3(\rho) &= L^2 \exp(-(L^2-1) \mathscr{L} c_2 \rho^2) I_2(\rho). \end{aligned}\right\} \tag{8.23}$$

I_1, I_2 are defined in (6.2), and we have used the expansion of $[I_1(\rho)]^{(L^2-1)}$ according to Lemma 6.1. All the E_i are manifestly analytic. Only E_3 needs some care (we have to exploit irrelevance again). E_1 and E_2 get additional powers of f^{-1} and are straightforwardly bounded.

Applying Lemma 6.3, we get for sufficiently large L and for $f \geqq 10$

$$\begin{aligned} &\left| \frac{\partial^v}{\partial \rho^v} [E_3(\rho) \exp(c_2' (\Delta + \rho)^2)]_{\rho = -\Delta} \right| \\ &\qquad \leqq L^2 v! \max_{|\rho + \Delta| = 1} |I_2(\rho) \exp(c_2' \Delta^2 + 2 c_2' \Delta \rho + (\delta + \mathscr{L} c_2) \rho^2)| \\ &\qquad\qquad \leqq L^2 v! \frac{G_3(N, \gamma)}{L^8 f^4} (K_1 \cdot H + K_2) \\ &\qquad \times \exp(\mathscr{L} c_2 + c_2' \Delta^2 + 2 c_2' \Delta + (\delta + \mathscr{L} c_2)(\Delta + 1)^2) \leqq \frac{1}{3\Lambda} \frac{K_2}{f^4} v!, \end{aligned}$$

where we have used $K_2 = K_1 \cdot H$. ■

Completion of the proof of Proposition 3. – Combining the Lemmas 8.1, 8.2 and the inequality (8.6) and noticing that $|c_0 \mathscr{L}^{L^2/2}| \leqq 9/8$, the bound of Proposition 3 follows for $K_1 \geqq 4(C + 9 C_1/8)$ and $K_2 = K_1 H$.

9. CONCLUSIONS

In the present paper we investigated a hierarchical model which serves as a guide to two dimensional nonlinear $O(N)$ sigma models for $N \geqq 3$. It is defined in such a way that it does not show independent wave function renormalization. This implies a considerable simplification for a study of the model by Wilson renormalization group transformations, without being too far away from the full sigma model.

A renormalization group study implies determination of the flow of effective partition functions on different lenght scales, which result from systematically integrating out high frequency modes. Of particular interest is the determination of the corresponding effective actions and running coupling constants. This can be done systematically at least in the neighborhood of an ultraviolet stable fixed point, where the running coupling constants are small down to a normalizing length scale.

We were particularly interested to avoid a separate treatment of field configurations which are in some sense out of range of perturbations theory, the so called large field domains. The separate treatment of these regions is at least partly responsible for the complex combinatorics in renormalization group investigations of models with full wave function renormalization. Instead, our study is in the spirit of polymer expansions in field space. Typically in such an approach, effective partition functions are expressed as a sum over products of activities corresponding to disjoint subsets of the current "block lattice". These activities have to satisfy appropriate bound, uniform in the fields, in order to guarantee convergence of the expansion. Furthermore, it must be made sure that these bounds do not destabilize under successive renormalization group iterations.

At least in the hierarchical model under investigation, this procedure is managable, although still nontrivial. This is because only monomer activities are different from zero, *i. e.* those which depend on the fields only at one point. Bounds on them can be given explicitly, which are uniform in the fields and which are actually stable under successive renormalization group iterations.

Our results coincide with those obtained by Gawedzki and Kupiainen, who investigated this model by renormalization group methods with a separation of large and small field domains. In particular, the flow of the effective coupling which determines the location of the minimum of the effective action is exactly the same, as should be. Gawedzki and Kupiainen also proved ultraviolet asymptotic freedom of the model. With some more effort we can prove ultraviolet stability by our methods as well. For the existence of the ultraviolet limit on each finite length scale, the bare coupling constant behaves as

$$\frac{1}{g(a)} \sim b - \alpha \log a + \frac{\delta}{\alpha} \log\left(1 - \frac{\alpha}{b} \log a\right),$$

with b, $\alpha > 0$. We have not investigated this here explicitly because our main interest was to get a feeling for a different method to investigate the renormalization group behavior of sigma models in two dimensions.

For the hiearchical model the choice of methods seems to be just a matter of taste, but we expect that in full models where we have to face with the complete spectrum of mathematical difficulties of local field theories, polymer expansions allow us a systematic investigation with considerable reduced complexity if compared to other methods. Our aim here was to get first principal insights into the difficulties which encounter by applying polymer expansion methods to the twodimensional nonlinear $O(N)$ sigma model, considered for instance as a statistical system on the multigrid, without facing at a whole the full mathematical complexity of the model.

Iteration of the hierarchical renormalization group equation leads to an expansion on the multigrid $\Lambda_{\leqq n}=\Lambda_0+\ldots+\Lambda_n$ (phase space expansion). This expansion is represented in terms of polymer activities. The problem of performing the continuum limit $n\to\infty$ is related to the convergence of the polymer representation on the infinitegrid $\lim_{n\to\infty}\Lambda_{\leqq n}=\Lambda_0+\ldots$. Green functions are expressed in terms of corresponding activities. These topics will be investigated in a further paper as the next step towards a treatment of the full σ-model.

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APPENDIX A. SOME USEFUL INEQUALITIES

The following simple inequalities are used throughout. Let ρ be any complex number, $f>0$.

(I1) $\operatorname{Re}[(f+\rho)^2-f^2]^2 \geqq \frac{1}{2}[(f+\operatorname{Re}\rho)^2-f^2]^2-[4f^2(\operatorname{Im}\rho)^2+17(\operatorname{Im}\rho)^4]$.

For $|\operatorname{Im}\rho|\leqq f/6$,

$$\operatorname{Re}[(f+\rho)^2-f^2]^2 \geqq \frac{1}{2}[(f+\operatorname{Re}\rho)^2-f^2]^2-\frac{9}{2}f^2(\operatorname{Im}\rho)^2.$$

(I2) For $\operatorname{Re}\rho \geqq -f-1$, $f\geqq 2$,

$$[(f+\operatorname{Re}\rho)^2-f^2]^2 \geqq \frac{f^2}{4}(\operatorname{Re}\rho)^2.$$

Hence if in addition $|\operatorname{Im}\rho|\leqq f/6$,

$$\operatorname{Re}[(f+\rho)^2-f^2]^2 \geqq \frac{f^2}{8}(\operatorname{Re}\rho)^2-\frac{9}{2}f^2(\operatorname{Im}\rho)^2.$$

(I3) For any $\alpha>0$ and any integer n there is $C>0$ so that

$$|\rho^n| \leqq C(1+|\operatorname{Re}\rho|^n)\exp(\alpha(\operatorname{Im}\rho)^2).$$

(I4) For any real numbers a, b, positive α, ε and integer n

(i) $|a^n \exp(-\alpha a^2)| \leqq \left(\frac{n}{2\alpha e}\right)^{n/2}$;

(ii) $2ab \leqq a^2 + b^2$

$(a+b)^2 \leqq 2(a^2+b^2)$;

(iii) $(a+b)^2 \geqq a^2\left(1-\frac{1}{\varepsilon}\right)+b^3(1-\varepsilon)$

$(a+b)^2 \geqq \frac{1}{2}a^2 - b^2.$

APPENDIX B. INTEGRATION OF HIGH FREQUENCY MODES

We give strong bounds on the integration over the fluctuation fields of a given scale. Integrals of this type occur frequently. The first one, Lemma B.1, cares about big angles between the block mean field and the fluctuations. They effect a smearing of the partition about the minimizing orbit. We take special care here not to introduce powers of (large) f in the estimates (as e. g. in [8]). This we have to do because the bounds should be effective for large and small fields simultaneously. Lemma B.2 states bounds on fluctuation integrals of the relevant part of the partition function on that scale. These estimates, too, hold for all real fields.

Consider the O(N) invariant integral

$$\mathrm{I}_s(\psi) = \int \frac{d^{\mathrm{N}}\varphi}{(2\pi\gamma)^{\mathrm{N}/2}} \frac{1}{\mathscr{L}^{1/2} s^{\mathrm{N}/2}} \exp\left(-\frac{\varphi^2}{2\gamma s} - d(|\varphi+\psi|-f)^2\right), \quad (\mathrm{B}.1)$$

where $\psi \in \mathbf{R}^{\mathrm{N}}$, $\mathrm{N} \geqq 2$, $0 \leqq s \leqq 1$, $|\psi| = \left\{\sum_{i=1}^{\mathrm{N}} \psi_i^2\right\}^{1/2}$, $\mathscr{L} = (1+2\gamma d)^{-1}$ and γ, f, d are positive numbers.

LEMMA B.1. — *Let* $d \geqq 1/6\gamma$. *There is* $\mathrm{K}(\mathrm{N}, \gamma)$ *so that*

$$|\mathrm{I}_s(\psi)| \leqq \mathrm{K}(\mathrm{N}, \gamma)\left(\frac{1+2\gamma d}{1+2\gamma ds}\right)^{1/2} \exp\left(-\frac{1}{5}\frac{d}{1+2\gamma ds}(|\psi|-f)^2\right), \quad (\mathrm{B}.2)$$

for all $0 \leqq s \leqq 1$.

$\mathrm{I}_s(\psi)$ is uniformly bounded in $0 \leqq s \leqq 1$ by

$$|\mathrm{I}_s(\psi)| \leqq \mathrm{K}(\mathrm{N}, \gamma)(1+2\gamma d)^{1/2} \exp\left(-\frac{1}{5}\mathscr{L} d(|\psi|-f)^2\right). \quad (\mathrm{B}.3)$$

In applications, $d \sim \mathrm{L}^2$, where L is a big positive constant.

Before proving the lemma we first of all give two preliminary statements. The first one takes into account the angular distribution more carefully in order not to generate a constant which grows with some power of f.

SUBLEMMA 1. — *Let* $x \in \mathbf{R}^N$, $r=|x|$ *and* $r^{N-1}\,dr\,d\Omega_{N-1}$ *volume form on* $\mathbf{R}^N$. *There is a constant* $C(N)$ *so that for any positive* f *and any* $\psi \in \mathbf{R}^N$

$$\left|\int d\Omega_{N-1} \exp(\psi . x)\right| \leqq \frac{C(N)}{(rf)^{(N-1)/2}} \exp(r[\,|\,|\psi|-f\,|+f]).$$

Using the mean value theorem of calculus,

$$\int d\Omega_{N-1} \exp(\psi . x)$$
$$= R(N) \int_0^{\pi} d\theta \sin^{N-2}\theta \, exp(r\,|\psi|\, cos\,\theta)$$
$$= R(N) \exp(r[\,|\psi|-f]\,\mu) \int_0^{\pi} d\theta \sin^{N-2}\theta \, exp(rf \cos\theta),$$

where $-1 \leqq \mu(r|\psi|, f) \leqq 1$. The integral can be expressed by a Bessel function of imaginary argument. Using corresponding bounds,

$$\left|\int d\Omega_{N-1} \exp(\psi . x)\right|$$
$$\leqq C(N) \exp(r[\,|\psi|-f]\,\mu) \frac{1}{(rf)^{(N-1)/2}} \exp(rf)$$
$$\leqq C(N) \frac{1}{(rf)^{(N-1)/2}} \exp(r[\,|\,|\psi|-f\,|+f]).$$

That proves the sublemma.

Big angles are suppressed in the sense that their integration yields the f^{-1} factors. They compensate corresponding factors of the radial integration, *cf.* below.

SUBLEMMA 2. — *Suppose* $n \geqq 8$ *and* $d \geqq 1/6\gamma$. *For any* $r \geqq 0$ *and* $0 \leqq s \leqq 1$,

$$\frac{r}{\gamma s}\left(\left|\,|\psi|-\frac{f}{n}\right|+\frac{f}{n}\right)+2\,\mathrm{drf}$$
$$\leqq \frac{r}{4\gamma s}(|\psi|+2\gamma f\,ds)+\frac{3}{4}\left(\frac{1+2\gamma ds}{2\gamma s}r^2+df^2+\frac{\psi^2}{2\gamma s}\right). \quad (B.4)$$

To prove this we have to check that

$$\frac{3}{4}\,\frac{r}{\gamma}\left(\frac{\tau}{s}|\psi|+2\gamma df+\lambda f\right)-\frac{3}{4}\left(\frac{1+2\gamma ds}{2\gamma s}r^2+df^2+\frac{\psi^2}{2\gamma s}\right) \leqq 0 \quad (B.5)$$

holds for $n \geqq 8$, where

$$\tau=1, \quad \lambda=0 \qquad \text{for} \quad |\psi| \geqq \frac{f}{n},$$

$$\tau=-\frac{5}{3}, \quad \lambda=\frac{8}{3n} \qquad \text{for} \quad |\psi| < \frac{f}{n}.$$

The left hand side of (B.5) is equal to

$$-\frac{3}{4}\,\frac{1+2\gamma ds}{2\gamma s}\left(r-\frac{\tau|\psi|+2\gamma dfs+\lambda fs}{1+2\gamma ds}\right)^2$$
$$-\frac{3}{4}\,\frac{1+2\gamma ds-\tau^2}{2\gamma s(1+2\gamma ds)}\left(|\psi|-\frac{s\tau(2\gamma df+\lambda f)}{1+2\gamma ds-\tau^2}\right)^2$$
$$-\frac{3}{4}df^2\left(1-\frac{2\gamma ds}{1+2\gamma ds}\left(1+\frac{\lambda}{2\gamma d}\right)^2\left(1+\frac{\tau^2}{1+2\gamma ds-\tau^2}\right)\right).$$

For $\tau=1$, $\lambda=0$ the last line vanishes, *i. e.* (B.5) holds. On the other hand, for $\tau<0$ and $\lambda=8/3n$, we first observe that those two terms of order τ^2 of the second and third line cancel each other, and that the remaining terms of the first two lines are not greater than zero. Finally note that

$$0 \leqq 1-\frac{2\gamma ds}{1+2\gamma ds}\left(1+\frac{\lambda}{2\gamma d}\right)^2, \tag{B.6}$$

for $6\gamma d \geqq 1$ and $\lambda=8/3n \leqq 1/3$.

Proof of Lemma B.1. – We write $I_s(\psi)$ in the form

$$I_s(\psi)=\int \frac{d^N\varphi}{(2\pi\gamma s)^{N/2}}\,\frac{1}{\mathscr{L}^{1/2}}\exp\left(-\frac{(\varphi-\psi)^2}{2\gamma s}-d(|\varphi|-f)^2\right)$$
$$=\int \frac{d^N\varphi}{(2\pi\gamma s)^{N/2}}\,\frac{1}{\mathscr{L}^{1/2}}$$
$$\times\exp\left(-\frac{1+2\gamma ds}{2\gamma s}\varphi^2+\frac{1}{\gamma s}\varphi.\psi-\frac{\psi^2}{2\gamma s}-df^2+2d|\varphi|f\right).$$

Using Sublemma 1 with f replaced by f/n and with $r=(\gamma s)^{-1}$ we get

$$|I_s(\psi)| \leqq C(N,\gamma)\int_0^\infty dr\left(\frac{r}{f}\right)^{(N-1)/2}$$
$$\times\frac{1}{(s\mathscr{L})^{1/2}}\exp\left(-r^2\frac{1+2\gamma ds}{2\gamma s}\right)$$
$$\times\exp\left(\frac{r}{\gamma s}\left(\left||\psi|-\frac{f}{n}\right|+\frac{f}{n}\right)-\frac{\psi^2}{2\gamma s}-df^2+2drf\right).$$

To the exponent we apply Sublemma 2 and get

$$|I_s(\psi)| \leqq C(N,\gamma) \int_0^\infty dr \left(\frac{r}{f}\right)^{(N-1)/2} \frac{1}{(s\mathscr{L})^{1/2}} \exp\left(-\frac{1+2\gamma ds}{8\gamma s}\left(r - \frac{|\psi|+2\gamma f\, ds}{1+2\gamma ds}\right)^2 - \frac{d}{4(1+2\gamma ds)}(|\psi|-f)^2\right). \quad (B.7)$$

The remaining Gaussian integral is easily estimated. As a result,

$$|I_s(\psi)| \leqq K(N,\gamma)\left(\frac{1+2\gamma d}{1+2\gamma ds}\right)^{1/2} \times P\left(\frac{1}{f}\frac{|\psi|-f}{1+2\gamma ds}+1\right) \times \exp\left(-\frac{d}{4(1+2\gamma ds)}(|\psi|-f)^2\right),$$

where P is a polynomial of order equal to the least integer greater or equal to $(N-1)/2$. Using a fraction of the exponential to bound the polynomial we finally get (B.2). ■

When integrating relevant parts of partition functions, integrals of the following type occur frequently.

$$J_s(\rho) = \exp(-\mathscr{L} c_2 \rho^2) \int \frac{d\sigma\, d^{N-1}\pi}{(2\pi\gamma s)^{N/2}\mathscr{L}^{1/2}} \times \exp\left(-\frac{\sigma^2+\pi^2}{2\gamma s} - \frac{c_2}{s}\sigma^2\right) \times P(\sigma+\mathscr{L}\rho,\pi)\exp\left(c_2(\sigma+\mathscr{L}\rho)^2 - \frac{c_2}{4f^2} \times [(f+\sigma+\mathscr{L}\rho)^2+\pi^2-f^2]^2\right) \quad (B.8)$$

where ρ is a complex number, $0 \leqq s \leqq 1$, $N \geqq 2$, γ, f, c_2 are positive numbers and $\mathscr{L} = (1+2\gamma c_2)^{-1}$. P is a polynomial with coefficients depending only on N, γ. Because of $\mathscr{L} \sim L^{-2}$, it is evident for RG recursions represented by integrals of the form (B.8) that there is only one marginal direction. The other ones are irrelevant and contracted by the RG steps.

LEMMA B.2. – *Let* $c_2 \geqq (L^2-1)/8\gamma$, $f \geqq 10$. *There is* C(N, γ) *so that for* $L^2 \geqq 12$

$$|J_s(\rho)| \leqq C(N,\gamma)(1+2\gamma c_2)^{1/2} \times \exp\left(-\frac{1}{40}\mathscr{L} c_2 (\mathrm{Re}\,\rho)^2 + \frac{2}{\gamma}(Im\,\rho)^2\right) \quad (B.9)$$

for all $0 \leqq s \leqq 1$ *and all* ρ *satisfying* $\operatorname{Re}\rho \geqq -f-1$, $|\operatorname{Im}\rho| \leqq f/6$.

Proof. — Substituting in (B.8)

$$\sigma \to \sigma + \frac{2\gamma c_2 s}{1+2\gamma c_2(1-s)}\mathcal{L}\rho,$$

we get

$$\begin{aligned} J_s(\rho) = \exp\left(-c_2\mathcal{L}\frac{2\gamma c_2(1-s)}{1+2\gamma c_2(1-s)}\rho^2\right)\int\frac{d\sigma\, d^{N-1}\pi}{(2\pi\gamma s)^{N/2}\mathcal{L}^{1/2}} \\ \times\exp\left(-\frac{1+2\gamma c_2(1-s)}{2\gamma s}\sigma^2 - \frac{\pi^2}{2\gamma s} - \frac{c_2}{4f^2}\right. \\ \left.\times\left[\left(\sigma+f+\frac{\rho}{1+2\gamma c_2(1-s)}\right)^2+\pi^2-f^2\right]^2\right) \\ \times P\left(\sigma+\frac{\rho}{1+2\gamma c_2(1-s)},\pi\right). \end{aligned}$$

This is bounded as follows. For $0 \leqq s \leqq 1/2$ and $|\operatorname{Im}\rho| \leqq f/6$ we use the estimate

$$\begin{aligned} \operatorname{Re}\left[-\frac{c_2}{4f^2}\left(\left(\sigma+f+\frac{\rho}{1+2\gamma c_2(1-s)}\right)^2+\pi^2-f^2\right)^2\right] \\ \leqq -\frac{c_2}{8}\left(\left|\left(\sigma+f+\frac{\operatorname{Re}\rho}{1+2\gamma c_2(1-s)},\pi\right)\right|-f\right)^2+\frac{2}{\gamma}(\operatorname{Im}\rho)^2, \end{aligned}$$

whereas for $1/2 \leqq s \leqq 1$ we substitute in the integral

$$\sigma \to \sigma - i\frac{\operatorname{Im}\rho}{1+2\gamma c_2(1-s)}.$$

We get a factor of $\exp((\operatorname{Im}\rho)^2/\gamma)$, and in the polynomial P, ρ is replaced by $\operatorname{Re}\rho$. Furthermore, we bound P according to

$$\begin{aligned} \left|P\left(\sigma+\frac{\rho}{1+2\gamma c_2(1-s)},\pi\right)\right| \\ \leqq K(N,\gamma)(1+|\operatorname{Re}\rho|^q)(1+(\sigma^2+\pi^2)^q)\exp\left(\frac{(\operatorname{Im}\rho)^2}{4\gamma}\right), \end{aligned}$$

where q denotes the degree of P. In summary

$$\begin{aligned} |J_s(\rho)| \leqq K(N,\gamma)(1+|\operatorname{Re}\rho|^q) \\ \times\exp\left(-c_2\mathcal{L}\frac{2\gamma c_2(1-s)}{1+2\gamma c_2(1-s)}(\operatorname{Re}\rho)^2\right) \\ \times\exp\left(\frac{2}{\gamma}(\operatorname{Im}\rho)^2\right)\int\frac{d^N\varphi}{(2\pi\gamma s)^{N/2}\mathcal{L}^{1/2}} \\ \times\exp\left(-\frac{(\varphi-\psi)^2}{4\gamma s}-\frac{c_2}{8}(|\varphi|-f)^2\right), \end{aligned}$$

where we have used that $\mathscr{L} c_2 \leqq 1/2\gamma$, and we have set

$$\psi = \left(f + \frac{\operatorname{Re}\rho}{1+2\gamma c_2(1-s)}\right)\hat{\psi},$$

where $\hat{\psi}$ denotes an arbitrary oriented unit vector in $\mathbf{R}^N$. Now for $c_2 \geqq (L^2-1)/8\gamma$ and $L^2 \geqq 12$ we have $c_2/8 \geqq 1/6\gamma$. We can thus apply Lemma B.1. Using that for $\operatorname{Re}\rho \geqq -f-1, f \geqq 10$

$$\frac{1}{10}(|\psi|-f)^2 \geqq \frac{1}{15}\frac{(\operatorname{Re}\rho)^2}{(1+2\gamma c_2(1-s))^2},$$

we get

$$|J_s(\rho)| \leqq \tilde{K}(N,\gamma)(1+2\gamma c_2)^{1/2}(1+|\operatorname{Re}\rho|^q)\exp\left(\frac{2}{\gamma}(\operatorname{Im}\rho)^2\right)$$
$$\times \exp\left(-\frac{\mathscr{L} c_2(\operatorname{Re}\rho)^2}{15}\left(\frac{30\gamma c_2(1-s)}{1+2\gamma c_2(1-s)} + \frac{1+2\gamma c_2}{4+\gamma c_2 s}\frac{1}{(1+2\gamma c_2(1-s))^2}\right)\right).$$

For $\gamma c_2 \geqq 11/8$, the last bracket is always greater or equal 1/2. Finally, using a fraction of the exponential to bound powers of $\operatorname{Re}\rho$, the lemma flows. ■

APPENDIX C.
DEFINITION OF RUNNING COUPLING CONSTANTS

Suppose we expand a (sufficiently smooth) function $Z(\eta)$, $\eta \in \mathbf{C}$ (which may be a partition function on a given scale) in the following form.

$$Z(f+\rho) = Z(f)\exp(-d_2\rho^2)\left\{1 - \sum_{v=3}^{9} d_v \rho^v + O(\rho^{10})\right\},$$

where f denotes the location of the minimum of "the effective action" $-\log Z(\eta)$, and we assume that $Z(f)>0$. For the cases under consideration in this paper, f is real and unique above a lower bound. Let us assume Z to be normalized by $Z(f)=1$. f is determined by

$$0 = \left.\frac{\partial Z(\eta)}{\partial\eta}\right|_{\eta=f}.$$

Furthermore,

$$d_2 = -\frac{1}{2}\left.\frac{\partial^2}{\partial\eta^2}\log Z(\eta)\right|_{\eta=f}.$$

The coupling constants d_ν are given by the following expressions.

$$d_3 = -\frac{1}{3!}\frac{\partial^3}{\partial\eta^3}Z(\eta)\bigg|_{\eta=f},$$

$$d_4 = -\frac{1}{4!}\left(\frac{\partial^4}{\partial\eta^4}Z(\eta)\bigg|_{\eta=f} - 12\,d_2^2\right),$$

$$d_5 = -\frac{1}{5!}\left(\frac{\partial^5}{\partial\eta^5}Z(\eta)\bigg|_{\eta=f} - 120\,d_2\,d_3\right),$$

$$d_6 = -\frac{1}{6!}\left(\frac{\partial^6}{\partial\eta^6}Z(\eta)\bigg|_{\eta=f} - 4!\,30\,d_2\,d_4 + 120\,d_2^3\right),$$

$$d_7 = -\frac{1}{7!}\left(\frac{\partial^7}{\partial\eta^7}Z(\eta)\bigg|_{\eta=f} - 5!\,42\,d_2\,d_5 + 3!\,420\,d_2^2\,d_3\right),$$

$$d_8 = -\frac{1}{8!}\left(\frac{\partial^8}{\partial\eta^8}Z(\eta)\bigg|_{\eta=f} - 6!\,56\,d_2\,d_6 + 4!\,840\,d_2^2\,d_4 - 1680\,d_2^4\right),$$

$$d_9 = -\frac{1}{9!}\left(\frac{\partial^9}{\partial\eta^9}Z(\eta)\bigg|_{\eta=f} - 7!\,72\,d_2\,d_7 + 5!\,1512\,d_2^2\,d_5 - 3!\,10080\,d_2^3\,d_3\right).$$

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