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The simplified Wheeler-DeWitt equation: The Cauchy problem and some spectral properties

by

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ABSTRACT. — In this paper we consider the simplified Wheeler-DeWitt equation which describes the minisuperspace model for the wave function ψ of a closed universe (*cf.* [4], [1]). We study this equation as an evolution equation in the scalar field $y \in \mathbf{R}$ with a scale factor $x \in]0, \mathbf{R}[$. We solve the Cauchy problem for the initial data $\psi(x, 0)$ and $\frac{\partial \psi}{\partial y}(x, 0)$ and we study the spectrum of a differential operator related to the equation (*cf.* [4]).

RÉSUMÉ. — Dans ce papier nous considérons l'équation de Wheeler-DeWitt simplifiée qui décrit le modèle de mini-superspace pour la fonction d'onde ψ d'un univers fermé (*cf.* [4], [1]). Nous étudions cette équation comme une équation d'évolution dans le champ scalaire $y \in \mathbf{R}$ avec le facteur d'échelle $x \in]0, \mathbf{R}[$. Nous résolvons le problème de Cauchy pour des données initiales $\psi(x, 0)$ et $\frac{\partial \psi}{\partial y}(x, 0)$ et nous étudions le spectre d'un opérateur différentiel associé à l'équation (*cf.* [4]).

1. INTRODUCTION

The simplified Wheeler-DeWitt equation can be written as follows (cf. [4], [1]):

$$\frac{1}{x^2} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} - \frac{p}{x} \frac{\partial \psi}{\partial x} + x^2 \psi - k^2 x^4 \psi = 0 \quad (1.1)$$

where $y \in \mathbf{R}$ is the scalar field, $x \in]0, \mathbf{R}[$ ($\mathbf{R} > 0$) is a scale factor, $p \in \mathbf{R}$ and $k^2 > 0$ are given constants (p reflects the factor-ordering ambiguity and k^2 is a mass factor) and $\psi :]0, \mathbf{R}[\times \mathbf{R} \rightarrow \mathbf{C}$ is the wave function of the universe for the minisuperspace model. The equation (1.1) is equivalent, in the sense of distributions, to the following equation:

$$\frac{\partial^2 \psi}{\partial y^2} - x^2 \frac{\partial^2 \psi}{\partial x^2} - px \frac{\partial \psi}{\partial x} + x^4 \psi - k^2 x^6 \psi = 0 \quad (1.2)$$

which can be considered as an evolution equation in $y \in \mathbf{R}$.

Given $\psi(x, 0)$ and $\frac{\partial \psi}{\partial y}(x, 0)$, we will solve the corresponding Cauchy problem, assuming that the initial data belong to some suitable weighted Sobolev spaces.

In second part of the paper we study the following eigenvalue problem (cf. [4], V):

$$-\frac{\partial^2 \psi}{\partial x^2} - \frac{p}{x} \frac{\partial \psi}{\partial x} + x^2 \psi - k^2 x^4 \psi = \lambda \psi \quad (1.3)$$

for ψ in a suitable domain.

In order to solve these problems we must introduce some weighted Sobolev spaces related to those introduced by P. Grisvard (cf. [2]) in the case where $]0, \mathbf{R}[$ is replaced by \mathbf{R}_+ .

2. FUNCTION SPACES

Let $\varphi \in \mathcal{D}(]0, \mathbf{R}[)$ (extended by zero to \mathbf{R}_+) and put

$$F(x) = \int_0^x |\varphi'(t)| dt, \quad G(x) = \int_x^{+\infty} |\varphi'(t)| dt, \quad \varphi' = \frac{d\varphi}{dt}.$$

We have, by theorem 330 in [3],

$$\int_0^{+\infty} x^{-r} F^2(x) dx \leq \frac{4}{|r-1|^2} \int_0^{+\infty} x^{-r} (x |\varphi'(x)|)^2 dx, \quad \text{if } r > 1,$$

$$\int_0^{+\infty} x^{-r} G^2(x) dx \leq \frac{4}{|r-1|^2} \int_0^{+\infty} x^{-r} (x |\varphi'(x)|)^2 dx, \quad \text{if } r < 1.$$

Hence, we obtain

$$\int_0^{\mathbf{R}} x^{p-2} |\varphi(x)|^2 dx \leq \frac{4}{|p-1|^2} \int_0^{\mathbf{R}} x^p |\varphi'(x)|^2 dx, \quad \text{if } p \neq 1 \quad (2.1)$$

In the special case $p = 1$, applying (2.1) to $p = 1 + \varepsilon$, $\varepsilon > 0$, we get

$$\int_0^{\mathbf{R}} x^{-1+\varepsilon} |\varphi(x)|^2 dx \leq \frac{4}{\varepsilon^2} \int_0^{\mathbf{R}} x^{1+\varepsilon} |\varphi'(x)|^2 dx \leq \frac{4\mathbf{R}^\varepsilon}{\varepsilon^2} \int_0^{\mathbf{R}} x |\varphi'(x)|^2 dx \quad (2.2)$$

We define, for $p \in \mathbf{R}$,

$$\mathbf{H}_p^1 = \mathbf{H}_p^1(]0, \mathbf{R}[) = \left\{ u \in \mathbf{L}_p^2 \left| \frac{du}{dx} \in \mathbf{L}_p^2 \right. \right\}$$

with its natural norm, where $\mathbf{L}_p^2(]0, \mathbf{R}[)$ is the \mathbf{L}^2 space for the measure $d\mu = x^p dx$. We denote by $\mathbf{H}_{p,0}^1$, the closure of $\mathcal{D}(]0, \mathbf{R}[)$ in \mathbf{H}_p^1 . By (2.1), (2.2), we get that $\left| \frac{du}{dx} \right|_{\mathbf{L}_p^2}$ is, in $\mathbf{H}_{p,0}^1$, an equivalent norm to $|u|_{\mathbf{H}_p^1}$ denoted by $|u|_{\mathbf{H}_{p,0}^1}$.

By (2.1) we can also conclude that, if $p \neq 1$, we have $\mathbf{H}_{p,0}^1 \hookrightarrow \mathbf{L}_{p-2}^2$, where \hookrightarrow means continuous injection.

We can finally remark that, following the ideas of [2], the functions belonging to $\mathbf{H}_{p,0}^1$, for $p < 1$, are the functions of \mathbf{H}_p^1 that are equal to zero at the boundary of $]0, \mathbf{R}[$.

We need the following lemmas:

LEMMA 2.1. — We have $\mathcal{D}(]0, \mathbf{R}[)$ dense in $\mathbf{H}_{1,0}^1 \cap \mathbf{L}_{-1}^2$ for its natural norm.

Proof. — Let $u \in \mathbf{H}_{1,0}^1 \cap \mathbf{L}_{-1}^2$, $\varphi_n \in \mathcal{D}(]0, \mathbf{R}[)$ such that $\varphi_n \rightarrow u$ in $\mathbf{H}_{1,0}^1$. We can choose φ_n such that

$$|\varphi_n - u|_{\mathbf{H}_{1,0}^1} \leq \frac{1}{n^5}, \quad n = 1, 2, \dots \quad (2.3)$$

Let $\psi_n = x^{1/n} \varphi_n$, $n = 1, 2, \dots$. We have $\psi_n \in \mathcal{D}(]0, \mathbf{R}[)$.

With $v_n = \varphi_n - u$, we get, by (2.1) and (2.3),

$$\begin{aligned} \int_0^{\mathbf{R}} x^{-1} |x^{1/n} v_n|^2 dx &\leq \mathbf{R}^{1/n^2} \int_0^{\mathbf{R}} x^{-1-(1/n^2)} |x^{1/n} v_n|^2 dx \\ &\leq \mathbf{R}^{1/n^2} 4n^4 \left[\int_0^{\mathbf{R}} x^{1-(1/n^2)} |x^{1/n} v_n'|^2 dx + \frac{1}{n^2} \int_0^{\mathbf{R}} x^{1-(1/n^2)} |x^{(1/n)-1} v_n|^2 dx \right] \\ &\leq \mathbf{R}^{1/n^2} 4n^4 \left[\int_0^{\mathbf{R}} x |v_n'|^2 dx + \frac{1}{n^2} \int_0^{\mathbf{R}} x^{-1+(2/n)-(1/n^2)} |v_n|^2 dx \right] \\ &\leq \mathbf{R}^{1/n^2} 4n^4 \left(\frac{1}{n^5} + \frac{c}{n^2} \int_0^{\mathbf{R}} x |v_n'|^2 dx \right) \rightarrow 0. \end{aligned}$$

Hence, since $u \in L^2_{-1}$, we obtain

$$|\Psi_n - u|_{L^2_{-1}}^2 \leq 2 [|x^{1/n}(\varphi_n - u)|_{L^2_{-1}}^2 + |(x^{1/n} - 1)u|_{L^2_{-1}}^2] \rightarrow 0.$$

Furthermore, we have

$$\begin{aligned} |\Psi_n - u|_{H^1_{1,0}}^2 &\leq 2 \left[\int_0^{\mathbf{R}} x |u' - x^{1/n} \varphi'_n|^2 dx + \int_0^{\mathbf{R}} x \left| \frac{1}{n} x^{(1/n)-1} \varphi_n \right|^2 dx \right] \\ &\leq 4 \left[\int_0^{\mathbf{R}} x (1 - x^{1/n})^2 |u'|^2 dx \right. \\ &\quad \left. + \int_0^{\mathbf{R}} x |x^{1/n} (u' - \varphi'_n)|^2 dx + \frac{1}{n^2} \int_0^{\mathbf{R}} x^{-1} |x^{1/n} \varphi_n|^2 dx \right]. \end{aligned}$$

But

$$\int_0^{\mathbf{R}} x |x^{1/n} (u' - \varphi'_n)|^2 dx \leq \mathbf{R}^{1/n^2} |u - \varphi_n|_{H^1_{1,0}}^2 \rightarrow 0$$

and

$$\int_0^{\mathbf{R}} x^{-1} |x^{1/n} \varphi_n|^2 dx = |\Psi_n|_{L^2_{-1}}^2 \text{ is bounded.}$$

Hence, $|\Psi_n - u|_{H^1_{1,0}}^2 \rightarrow 0$ and we conclude that

$$\Psi_n \rightarrow u \text{ in } H^1_{1,0} \cap L^2_{-1} \quad \square$$

LEMMA 2.2. — We have $\{u | x^{p/2} u \in H^1_0\} \Subset H^1_{p,0} \cap L^2_{p-2}$, where $H^1_0 = H^1_{0,0}$.

Proof. — Let u be such that $x^{p/2} u \in H^1_0$ and take $\varphi_n \in \mathcal{D}]0, \mathbf{R}[$, $\Psi_n \rightarrow x^{p/2} u$ in H^1_0 .

Hence, $x^{-1} \varphi_n \rightarrow x^{(p/2)-1} u$ in L^2 and, if $u_n = x^{-p/2} \varphi_n$, we have

$$\begin{aligned} u_n &\in \mathcal{D}]0, \mathbf{R}[, \quad u_n \rightarrow u \text{ in } L^2_p \\ x^{-p/2} \varphi'_n &\rightarrow \frac{p}{2} x^{-1} u + u' \text{ in } L^2_p \\ x^{-1} u_n &= x^{-1-(p/2)} \varphi_n \rightarrow x^{-1} u \text{ in } L^2_p. \end{aligned}$$

Hence,

$$u'_n = -\frac{p}{2} x^{-(p/2)-1} \varphi_n + x^{-(p/2)} \varphi'_n \rightarrow -\frac{p}{2} x^{-1} u + \frac{p}{2} x^{-1} u + u' = u' \text{ in } L^2_p.$$

We get $u \in H^1_{p,0}$ and $|u|_{H^1_{p,0}} \leq c |x^{p/2} u|_{H^1_0}$.

By (2.1) (with $p=0$) it follows that $\{u | x^{1/2} u \in H^1_0\} \Subset L^2_{-1}$. \square

LEMMA 2.3. — We have

$$H^1_{p,0} \cap L^2_{p-2} \Subset \{u | x^{p/2} u \in H^1_0\}$$

and hence by lemma 2.2

$$H^1_{p,0} \cap L^2_{p-2} = \{u | x^{p/2} u \in H^1_0\}.$$

For $p=1$, we have $H_{1,0}^1 \subset \{u \mid x^{q/2} u \in H_0^1\}$, $\forall q > 1$.

Proof. — Assume $p \neq 1$ and let $u \in H_{p,0}^1 \cap L_{p-2}^2 = H_{p,0}^1$, $\varphi_n \in \mathcal{D}(]0, R[)$, $\varphi_n \rightarrow u$ in $H_{p,0}^1$. By (2.1) we get $\varphi_n \rightarrow u$ in L_{p-2}^2 and so $x^{(p/2)-1} \varphi_n \rightarrow x^{(p/2)-1} u$ in L^2 .

Since

$$x^{p/2} \psi'_n \rightarrow x^{p/2} u' \quad \text{in } L^2$$

we get

$$(x^{p/2} \varphi_n)' = \frac{p}{2} x^{(p/2)-1} \varphi_n + x^{p/2} \varphi'_n \rightarrow \frac{p}{2} x^{(p/2)-1} u + x^{p/2} u' = (x^{p/2} u)' \quad \text{in } L^2.$$

Hence, $x^{p/2} u \in H_0^1$ and $|x^{p/2} u|_{H_0^1} \leq c_1 |u|_{H_{p,0}^1}$.

For $p=1$, by lemma 2.1, we can choose $\varphi_n \in \mathcal{D}(]0, R[)$ such that $\varphi_n \rightarrow u$ in $H_{1,0}^1 \cap L_{-1}^2$ and the proof is the same. Furthermore,

$$H_{1,0}^1 \subset H_{q,0}^1 \subset \{u \mid x^{q/2} u \in H_0^1\}, \quad \forall q > 1. \quad \square$$

We can now prove the following:

PROPOSITION 2.1. — *The injection $H_{p,0}^1 \subset L_p^2$ is compact, for every $p \in \mathbf{R}$.*

Proof. — Assume first $p \neq 1$. Since the injection $H_0^1 \subset L^2$ is compact, the result is a consequence of lemmas 2.2 and 2.3 and of the mappings

$$\begin{aligned} u &\rightarrow x^{p/2} u \rightarrow x^{p/2} u \rightarrow u \\ (H_{p,0}^1 &\rightarrow H_0^1 \rightarrow L^2 \rightarrow L_p^2). \end{aligned}$$

Suppose now $p=1$ and let $u \in H_{1,0}^1$: we have

$$\int_0^{\mathbf{R}} x |u|^2 dx = \int_0^{\mathbf{R}} x^{-1/2} x^{3/2} |u|^2 dx \leq \left(\int_0^{\mathbf{R}} x^{-3/4} dx \right)^{2/3} \left(\int_0^{\mathbf{R}} x^{9/2} |u|^6 dx \right)^{1/3}$$

Hence $|u|_{L_1^2} \leq c |x^{3/4} u|_{L^6}$. Furthermore, by applying lemma 2.3 with $q = \frac{3}{2}$, we get

$$|x^{3/4} u|_{H_0^1} \leq c_1 |u|_{H_{1,0}^1}.$$

The result is now a consequence of the compact injection $H_0^1 \subset L^6$ and of the mappings

$$\begin{aligned} u &\rightarrow x^{3/4} u \rightarrow x^{3/4} u \rightarrow u \\ (H_{1,0}^1 &\rightarrow H_0^1 \rightarrow L^6 \rightarrow L_1^2). \quad \square \end{aligned}$$

3. THE CAUCHY PROBLEM

In order to study the Cauchy problem for the equation (1.2) let us put $\psi_1 = \frac{\partial \psi}{\partial y}$,

$$B\psi = x^2 \frac{\partial^2 \psi}{\partial x^2} + px \frac{\partial \psi}{\partial x} \quad (3.1)$$

$$V\psi = -x^4 \psi + k^2 x^6 \psi \quad (3.2)$$

The equation (1.2) can be written as follows:

$$\frac{\partial}{\partial y} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} \quad (3.3)$$

We consider the Hilbert space $H = H_{p,0}^1 \times L_{p-2}^2$.

If $p \neq 1$ we have $H_{p,0}^1 \subset L_{p-2}^2$.

We put

$$D(A) = D(B) \times (H_{p,0}^1 \cap L_{p-2}^2),$$

where

$$D(B) = \{ u \in H_{p,0}^1 \mid Bu \in L_{p-2}^2 \}.$$

THEOREM 3.1. — *The operator $A : D(A) \rightarrow H$ defined by*

$$A \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ B\psi \end{pmatrix}$$

is skew-self-adjoint in H .

Proof. — Let $(w, h) \in D(A^*)$: there exists $(q, l) \in H$ such that, for every $(u, v) \in D(A)$, we have

$$(A(u, v), (w, h)) = ((u, v), (q, l))$$

that is

$$(v, w)_{H_{p,0}^1} + (Bu, h)_{L_{p-2}^2} = (u, q)_{H_{p,0}^1} + (v, l)_{L_{p-2}^2} \quad (3.4)$$

In particular, with $u=0$, we get

$$\int_0^{\mathbf{R}} x^p v' \bar{w}' dx = \int_0^{\mathbf{R}} x^{p-2} v \bar{l} dx, \quad \forall v \in \mathcal{D}([0, \mathbf{R}[)$$

which implies

$$-Bw = l \quad \text{in } \mathcal{D}'([0, \mathbf{R}[)$$

Since $w \in H_{p,0}^1$, we get $w \in D(B)$.

Now, put $v=0$ in (3.4). We obtain

$$(Bu, h)_{L_{p-2}^2} = \int_0^{\mathbf{R}} x^p u' \bar{q}' dx, \quad \forall u \in D(B)$$

We have

$$\int_0^{\mathbf{R}} x^p |f'|^2 dx = \|f\|_{\mathbf{H}_{p,0}^1}^2, \quad f \in \mathbf{H}_{p,0}^1.$$

Hence, if $\varphi \in \mathcal{D}([0, \mathbf{R}])$, there exists $u \in \mathbf{H}_{p,0}^1$ such that

$$\int_0^{\mathbf{R}} x^p u' \bar{f}' dx = \int_0^{\mathbf{R}} x^{p-2} \varphi \bar{f} dx, \quad \forall f \in \mathbf{H}_{p,0}^1,$$

that is $\mathbf{B}u = \varphi$ in $\mathcal{D}'([0, \mathbf{R}])$. Then $u \in \mathbf{D}(\mathbf{B})$ and

$$\begin{aligned} \langle \bar{h}, x^{p-2} \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} &= (\varphi, h)_{\mathbf{L}_{p-2}^2} \\ &= \int_0^{\mathbf{R}} x^p u' \bar{q}' dx = \int_0^{\mathbf{R}} x^{p-2} \varphi \bar{q} dx = \langle \bar{q}, x^{p-2} \varphi \rangle_{\mathcal{D}' \times \mathcal{D}}. \end{aligned}$$

Since φ is arbitrary we conclude that $h = q \in \mathbf{H}_{p,0}^1$. Hence, $(w, h) \in \mathbf{D}(\mathbf{A})$ and so $\mathbf{D}(\mathbf{A}^*) \subset \mathbf{D}(\mathbf{A})$. Furthermore, if $w \in \mathbf{D}(\mathbf{B})$ and $v \in \mathbf{H}_{p,0}^1 \cap \mathbf{L}_{p-2}^2$, it is easy to see that $(v, w)_{\mathbf{H}_{p,0}^1} = -(v, \mathbf{B}w)_{\mathbf{L}_{p-2}^2}$, by lemma 2.1.

This implies $\mathbf{D}(\mathbf{A}) \subset \mathbf{D}(\mathbf{A}^*)$ and that, in $\mathbf{D}(\mathbf{A})$, $\mathbf{A}^* = -\mathbf{A}$. This achieves the proof of theorem 3.1. \square

PROPOSITION 3.1. — *The operator $\mathbf{D} : \mathbf{H} \rightarrow \mathbf{H}$ defined by*

$$\mathbf{D} \begin{pmatrix} \Psi \\ \Psi_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \mathbf{V} & 0 \end{pmatrix} \begin{pmatrix} \Psi \\ \Psi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{V}\Psi \end{pmatrix}$$

is continuous.

Proof. — We have

$$\|\mathbf{V}\Psi\|_{\mathbf{L}_{p-2}^2} \leq \|x^4 \Psi\|_{\mathbf{L}_{p-2}^2} + k^2 \|x^6 \Psi\|_{\mathbf{L}_{p-2}^2} \leq c \|\Psi\|_{\mathbf{H}_{p,0}^1}$$

by (2.1) and (2.2). \square

By applying a well known result (cf. [5]) we get

COROLLARY 3.1. — *The operator $\mathbf{T} = \mathbf{A} + \mathbf{D} : \mathbf{D}(\mathbf{A}) \rightarrow \mathbf{H}$ is the infinitesimal generator of a strongly continuous group of operators in \mathbf{H} , denoted by $e^{y\mathbf{T}}$, $y \in \mathbf{R}$.*

Hence, for

$$\begin{pmatrix} \Psi_0 \\ \Psi_{1,0} \end{pmatrix} \in \mathbf{D}(\mathbf{A}) = \mathbf{D}(\mathbf{B}) \times (\mathbf{H}_{p,0}^1 \cap \mathbf{L}_{p-2}^2),$$

there exists a unique

$$\begin{pmatrix} \Psi \\ \Psi_1 \end{pmatrix} \in \mathcal{C}(\mathbf{R}; \mathbf{D}(\mathbf{A})) \cap \mathcal{C}^1(\mathbf{R}; \mathbf{H}),$$

such that $\begin{pmatrix} \Psi(0) \\ \Psi_1(0) \end{pmatrix} = \begin{pmatrix} \Psi_0 \\ \Psi_{1,0} \end{pmatrix}$ and verifying (3.3) for $y \in \mathbf{R}$.

We have $\begin{pmatrix} \Psi(y) \\ \Psi_1(y) \end{pmatrix} = e^{yT} \begin{pmatrix} \Psi_0 \\ \Psi_{1,0} \end{pmatrix}$, $y \in \mathbf{R}$.

Returning to the equation (1.2) we obtain

THEOREM 3.2. — *Assume $\Psi_0 \in \mathbf{D}(\mathbf{B})$ and $\Psi_{1,0} \in \mathbf{H}_{p,0}^1 \cap \mathbf{L}_{p-2}^2$. Then, there exists a unique*

$$\Psi \in \mathcal{C}(\mathbf{R}; \mathbf{D}(\mathbf{B})) \quad \text{with} \quad \frac{\partial \Psi}{\partial y} \in \mathcal{C}(\mathbf{R}; \mathbf{H}_{p,0}^1 \cap \mathbf{L}_{p-2}^2) \cap \mathcal{C}^1(\mathbf{R}; \mathbf{L}_{p-2}^2)$$

such that

$$\Psi(x, 0) = \Psi_0(x), \quad \frac{\partial \Psi}{\partial y}(x, 0) = \Psi_{1,0}(x), \quad x \in]0, \mathbf{R}[$$

and verifying (1.2).

We have also the following

PROPOSITION 3.2. — *Assume $p \neq 1$. Then we have*

$$\mathbf{D}(\mathbf{B}) = \{u \in \mathbf{H}_{p,0}^1 \mid x^{(p/2)+1} u \in \mathbf{H}^2\}.$$

Proof. — Suppose $u \in \mathbf{D}(\mathbf{B})$: we have $u \in \mathbf{H}_{p,0}^1$ (and hence $x^{p/2} u \in \mathbf{H}_0^1$ by lemma 2.3),

$$x^2 u'' + pxu' \in \mathbf{L}_{p-2}^2$$

Hence

$$\begin{aligned} \frac{d^2}{dx^2} (x^{(p/2)+1} u) &= x^{(p/2)+1} u'' + (p+2) x^{p/2} u' + \left(\frac{p}{2} + 1\right) \frac{p}{2} x^{(p/2)-1} u \\ &= (x^{(p/2)+1} u'' + x^{p/2} pu') + 2x^{p/2} u' + \left(\frac{p}{2} + 1\right) \frac{p}{2} x^{(p/2)-1} u \in \mathbf{L}^2, \end{aligned}$$

and

$$x^{(p/2)+1} u \in \mathbf{H}^2.$$

Conversely, if $u \in \mathbf{H}_{p,0}^1$ and $x^{(p/2)+1} u \in \mathbf{H}^2$, we get

$$x^{(p/2)+1} u'' + px^{p/2} u' = \frac{d^2}{dx^2} (x^{(p/2)+1} u) - \left(\frac{p}{2} + 1\right) \frac{p}{2} x^{(p/2)-1} u - 2x^{(p/2)} u' \in \mathbf{L}^2$$

Hence,

$$x^{(p-2)/2} (x^2 u'' + pxu') \in \mathbf{L}^2,$$

that is $x^2 u'' + pxu' \in \mathbf{L}_{p-2}^2$ and so $u \in \mathbf{D}(\mathbf{B})$. \square

4. SPECTRAL PROPERTIES OF A RELATED OPERATOR

We want to find $\lambda \in \mathbf{C}$ such that there exists u verifying

$$u \in \mathbf{D}(\mathbf{B}), \quad u \neq 0, \quad -\mathbf{B}u - \mathbf{V}u = \lambda(x^2 u) \quad \text{in } \mathcal{D}'(]0, \mathbf{R}[) \quad (4.1)$$

where B and V are defined by (3.1), (3.2).

Since $u \in H_{p,0}^1$ implies $Vu, x^2 u \in L_{p-2}^2$, we can replace condition (4.1) by the following

$$u \in H_{p,0}^1, \quad u \neq 0, \quad -\frac{1}{x^2}(B+V)u = \lambda u \quad \text{in } \mathcal{D}'(]0, R[) \quad (4.2)$$

The equation in (4.2), which is the same in (1.3), has an important physical meaning, namely if we can find sufficient conditions such that $\lambda = 0$ is not a possible eigenvalue (cf. [4], V).

In order to solve this problem, we consider the following hermitian continuous sesquilinear form over $H_{p,0}^1$:

$$a_\gamma(u, v) = \int_0^R x^p u' \bar{v}' dx + \int_0^R x^{p+2} u \bar{v} dx - k^2 \int_0^R x^{p+4} u \bar{v} dx + \gamma \int_0^R x^p u \bar{v} dx, \quad \text{where } \gamma \geq 0 \quad (4.3)$$

By (2.1), (2.2), we have

$$\beta = \sup_{v \in H_{p,0}^1, v \neq 0} \frac{\int_0^R x^p |v|^2 dx}{\int_0^R x^p |v'|^2 dx} < +\infty \quad (4.4)$$

and $\beta \leq \frac{4R^2}{|p-1|^2}$ for $p \neq 1$, $\beta \leq R^2$ for $p = 1$.

It is easy to check that if

$$\theta = \gamma - k^2 R^4 > -\beta^{-1} \quad (4.5)$$

then, with $\alpha = \min[t, (1 + \beta\theta - t)\beta^{-1}]$, $0 < t < \min(1, 1 + \beta\theta)$, we have

$$a_\gamma(u, u) \geq \alpha |u|_{H_{p,0}^1}^2, \quad \forall u \in H_{p,0}^1 \quad (4.6)$$

Furthermore, by proposition 2.1, the injection $H_{p,0}^1 \hookrightarrow L_p^2$ is compact. Hence, by a well known result we obtain:

PROPOSITION 4.1. — *Let $\gamma \geq 0$ be such that (4.5) is verified. Then there exists an increasing sequence $\{\theta_n\}$, $\theta_n > 0$, $\theta_n \rightarrow +\infty$ such that, for each n , the equation*

$$a_\gamma(u, v) = \theta_n \int_0^R x^p u \bar{v} dx, \quad \forall v \in H_{p,0}^1 \quad (4.7)$$

has a nontrivial solution $u \in H_{p,0}^1$.

It is easy to prove that (4.7) is equivalent to

$$u \in D(B), \quad -Bu - Vu = (\theta_n - \gamma)(x^2 u).$$

Hence, we can find a nontrivial solution $u \in D(\mathbf{B})$ for (4.1) if

$$\lambda = \lambda_n = \theta_n - \gamma, \quad n = 1, 2, \dots$$

Then, since $\theta_n > 0$, we have $\lambda_n > -\gamma$, $n = 1, 2, \dots$

If $k^2 \mathbf{R}^4 < \beta^{-1}$ we can choose $\gamma = 0$ (a sufficient condition is $k^2 < \frac{|p-1|^2}{4\mathbf{R}^6}$, if $p \neq 1$, or $k^2 < \frac{1}{\mathbf{R}^6}$, if $p = 1$).

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