ANNALES DE L'I. H. P., SECTION A

P. COLLET
P. FERRERO

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Annales de l'I. H. P., section A, tome 52, n° 3 (1990), p. 283-301
<http://www.numdam.org/item?id=AIHPA_1990__52_3_283_0>

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Ann. Inst. Henri Poincaré,

Vol. 52, n° 3, 1990, p. 283-301. *Physique théorique*

Some limit ratio theorem related to a real endomorphism in case of a neutral fixed point

by

P. COLLET

Centre de Physique Théorique, U.P.R.A. 14 du C.N.R.S.,
École Polytechnique, 91128 Palaiseau Cedex, France

and

P. FERRERO

Centre de Physique Théorique de Marseille, U.P.R. 7061 du C.N.R.S.,
CNRS-Centre de Luminy, Case 907, 13288 Marseille Cedex 9, France

ABSTRACT. – The unique ergodic non finite absolutely continuous invariant measure is obtained via the convergence of some average of the iterates of the Perron-Frobenius operator for endomorphisms of the interval which are smooth enough and expanding except at a neutral fixed point.

RÉSUMÉ. – L'unique mesure ergodique non normalisable, absolument continue par rapport à la mesure de Lebesgue est obtenue par une moyenne des itérés de l'opérateur de Perron-Frobenius pour des endomorphismes de l'intervalle réguliers et dilatants sauf en un point fixe neutre.

1. INTRODUCTION

The construction of invariant measures for expanding dynamical systems has been the object of many developments in the past few years. In

particular, for one dimensional systems one has now rather detailed results at least in the case of uniformly expanding systems. There are however important situations where one meets non uniformly expanding one dimensional systems. A typical example is the case of quadratic maps which have a critical point. In this paper we shall consider a different (and milder) case of non uniform expansiveness, namely maps of the interval which are expanding except in one point. It is easy to see that in order to prevent some iterate of the mapping to be expanding, one has to impose that the non expanding point be a fixed point. In other words, we shall consider maps of the interval [0, 1] which are continuously differentiable except at finitely many points where their derivatives have limits on both sides, and such that the slope is in modulus everywhere larger than one except at a fixed point where it is one (we shall make more precise hypothesis below, in particular, we shall only consider Markov maps). One of the simplest example of such maps is given by the following formula

$$f(x)=\begin{cases} x/1-x, & \text{for} \quad x\leqq 1/2 \\ 2x-1, & \text{for} \quad x>1/2. \end{cases}$$

In their original paper about expanding maps of the interval [L-Y], Lasota and Yorke observed that in the above situation the map cannot have an absolutely continuous invariant probability. It is now known [Me] that for such maps the Bowen-Ruelle measure is the delta measure at the marginal fixed point (here $x=0$). This class of maps which are expanding except at a marginal fixed points appear in several contexts [Man]. It was shown by P. Manneville [Ma] [G-W] that they are a model for the dynamics that takes place just at the point of an intermittency transition. They also recently appeared in the renormalisation group analysis of critical diffeomorphisms of the circle with general rotation number [F]. As we have already mentioned, the Bowen-Ruelle measure is trivial for the map that will be discussed in this paper. It was discovered by Manneville however that the transient behavior of these dynamical systems is far from trivial. He gave very strong arguments for the existence of another (non normalisable) invariant measure which disappears from the ultimate time asymptotic results but is responsible for interesting finite time results. In particular, he showed that the occupation time of a set which does not contain the fixed point has an unusual behavior proportional to $n/\log n$ (instead of n). He also derived consequences for the behavior of the spectrum of correlations.

We shall give below a proof of existence and uniqueness of a (non normalisable) absolutely continuous invariant measure for a class of maps similar to the above example. Our proof is based on direct estimates for the dynamics of the map. The key argument for proving the existence is a

compactness result based on an unusual normalisation of the Cesaro average of the iterates of the Perron-Frobenius operator.

In fact, using the standard normalisation, one would get back the trivial Dirac measure at the fixed point. The complete argument is given in section 2 and uses some technical results which are collected in the appendix. Our method allows us to recover Bowen's result [B]. Section 3 is devoted to a detailed analysis of the ergodic properties of the invariant measure. We first give a precise estimation of the singularity of the measure. We then prove that the measure is unique and ergodic. In appendix B we shall give a precise estimate of the abnormal ergodic normalisation using a tauberian theorem. Throughout this paper, if f is a map of the interval, f^n will denote the n-th iterate of f.

2. EXISTENCE OF AN ABSOLUTELY CONTINUOUS INVARIANT MEASURE

We shall be interested in invariant measures which are absolutely continuous with respect to the Lebesgue measure λ (a.c.i.m. for short). Their density h must be a fixed point of the Perron-Frobenius operator P given by

$$\mathrm{P}h(x) = \sum_{y : f(y) = x} \frac{h(y)}{|f'(y)|}$$

If $|f'|$ is bounded below by a number larger than one, it is well known that under the above hypothesis there is a unique a.c.i.m., and if f has an indifferent fixed point, there may not be any normalizable one [L-Y]. We shall denote by $(h_n)_{n \in \mathbf{N}}$ the sequence of functions defined by

$$h_n = \mathrm{P}^n 1$$

One of the most interesting property of these functions is summarized by the following formula

$$\int_{\mathrm{A}} h_n(x)\, dx = \lambda(f^{-n}(\mathrm{A})),$$

for any Borel subset A of [0, 1]. We shall prove the following theorem

MAIN THEOREM. — *Suppose that*

(1) $\qquad f\colon\ [0,1] \to [0,1], \qquad f(0) = 0, \qquad f(1) = 1$

(2) $\qquad f([0,1/2]) = [0,1[, \qquad f(]1/2,1]) =]0,1]$ *or* $[0,1[.$

We denote by f_- *and* f_+ *the inverses of the restriction of* f *to* $[0,1/2]$ *and* $]1/2,1]$.

(3) *f is C^3 except at $x=1/2$ where its first three derivatives have left and right limits, and also it is monotone on $[0, 1/2[$ and on $]1/2, 1]$.*

(4) *There is some number $\alpha > 1$ such that*

$$|f'|_{]1/2,1]} \geqq \alpha, \qquad f'(0)=1, \qquad f'_{]0,\,1/2[} > 1, \qquad f''(0) \neq 0$$

which implies $f''(0) > 0$.

Then

(1) P *has a fixed point e, which is unique under some additional assumptions given below.*

$$\Omega_n \equiv \sum_{k=0}^{n-1} h_k(1/2) \tag{2}$$

is of order $n/\mathrm{Log}\, n$.

(3) *If λ denote the Lebesgue measure, and u is any real function, bounded away from 0 and $+\infty$, 1/2 Hölder, then*

$$\lim_{n \to \infty} \Omega_n^{-1} \sum_{k=0}^{n-1} \mathrm{P}^k u = \lambda(u)\, e$$

for the uniform convergence on compact subset of $]0, 1]$. Moreover f is ergodic for the measure $e\lambda$.

$$|e(x) - f'_+(0)/x| \leqq O(1)\, x^{-1/2} \tag{4}$$

$$\frac{x^2 e(x)}{y^2 e(y)} \leqq \exp A\, |x-y|^{1/2}. \tag{5}$$

Note however that it follows from a result of Aaronson [A] that for a measurable function u integrable with respect to the invariant measure, one cannot have convergence almost everywhere of the ergodic average

$$\Omega_n^{-1} \sum_{k=0}^{n-1} u(f^k(x))$$

toward $e\lambda(u)$.

On the other hand it is shown by [Me] that δ_0 is the Bowen-Ruelle measure, that is

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} u(f^k(x)) = u(0)$$

for all continuous fonction u, and almost every x.

We shall investigate the large n behavior of the sequence $(h_n)_{n \in \mathbf{N}}$ by looking at the ratios

$$\frac{h_n(x)}{h_n(y)}$$

for x and y in $]0, 1]$.

It is easy to see that if F denotes the partition $\{[0, 1/2[,]1/2, 1]\}$ (modulo λ), we shall have to estimate quantities of the form

$$\prod_{i=0}^{n} \frac{f'(f^i(x))}{f'(f^i(y))}$$

where x and y belong to the same atom of $\bigvee_{i=0}^{n} f^{-i}\mathrm{F}$.

LEMMA 2.1. — *There is a number* $\varepsilon>0$, *and a number* $c>0$ *such that if* K *is an atom of* $\bigvee_{0\leqq i\leqq n} f^{-i}\mathrm{F}$ *with* $f^i(\mathrm{K}) \subset [0, \varepsilon]$ *for* $0\leqq i\leqq n$, *then, if* x *and* y *are in* K

$$\prod_{i=0}^{n} \frac{f'(f^i(x))}{f'(f^i(y))} \leqq \left(\frac{1+naf^n(x)}{1+naf^n(y)}\right)^2 e^{c\,|f^n x - f^n y|^{1/2}}.$$

Proof. — We have

$$\log\left[\frac{f'(f^i(x))}{f'(f^i(y))}\right] = \int_{f^i(y)}^{f^i(x)} \frac{f''(t)}{f'(t)}\,dt.$$

Since f is in $C^3([0, 1/2])$, there is a constant A such that

$$\left|\frac{f''(t)}{f'(t)} - 2a\right| \leqq \mathrm{A}\,t,$$

where $2a=f''(0)$. Therefore

$$\left|\log\left[\frac{f'(f^i(x))}{f'(f^i(y))}\right] - 2a[f^i(x)-f^i(y)]\right| \leqq \mathrm{A}\,|f^i(x)-f^i(y)|^2.$$

This implies

$$\sum_0^n \log\left[\frac{f'(f^i(x))}{f'(f^i(y))}\right] = 2a\sum_0^n [f^i(x)-f^i(y)] + O(1)\sum_0^n |f^i(x)-f^i(y)|^2.$$

We now observe that since $|f^{n-i'}| \geqq 1$, we have $|f^i(x)-f^i(y)| \leqq |f^n(x)-f^n(y)|$. Therefore, using Lemma A2 we obtain

$$\begin{aligned}
&\sum_0^n |f^i(x)-f^i(y)|^2 \\
&\leqq |f^n(x)-f^n(y)|^{1/2} \sum_0^n (|f^i(x)|^{3/2} + |f^i(y)|^{3/2}) \\
&\leqq 4\,|f^n(x)-f^n(y)|^{1/2} \sum_0^n (ap+1)^{-3/2} \\
&\leqq O(1)\,|f^n(x)-f^n(y)|^{1/2}.
\end{aligned}$$

Using now Lemma A.3 we obtain

$$\sum_0^n [f^i(x)-f^i(y)]$$

$$=\sum_0^n [(p+(af^n(x))^{-1})^{-1}-(p+(af^n(y))^{-1})^{-1}].$$

$$+O(1)\,|f^n(x)-f^n(y)|^{1/2}.$$

Let $\alpha=af^n(x)$, $\beta=af^n(y)$, we have

$$\left|\int_p^{p+1}(|s+\alpha^{-1}]^{-1}-[s+\beta^{-1}]^{-1})\,ds-[p+\alpha^{-1}]^{-1}+[p+\beta^{-1}]^{-1}\right|$$

$$\leqq O(1)\,[|\alpha-\beta|]^{1/2}\,[1+p]^{-3/2}.$$

The result follows now by summation over p.

LEMMA 2.2. – *There is a positive number* ε_1 *and a real positive valued function* D *defined on* $]0,\varepsilon_1]$ *which satisfies* $D(\varepsilon)=O(\varepsilon^2)$ *such that if* K *is an atom of* $\bigvee_0^n f^{-i}F$ *and if* x *and* y *in* K *satisfy*

$$0<x<\varepsilon<y,\quad f^i(x)\leqq\varepsilon,\quad \text{and}\quad \varepsilon<f^i(y)\leqq 1/2,\quad \text{for } 0\leqq i\leqq n,$$

then

$$\left|\log\prod_0^n\frac{f'(f^i(x))}{f'(f^i(y))}\right|\leqq D(\varepsilon)\,|f^n(x)-f^n(y)|.$$

Proof. – We shall first consider the case $|f^n(x)-f^n(y)|>|f^n(y)|/2$. From the hypothesis we have

$$\frac{f'(f^i(x))}{f'(f^i(y))}\leqq\exp(c_2\,|f^i(y)|)$$

where c_2 depends only on f (we have used $f'(f^i(x))\geqq 1$).

We shall now estimate the quantity $\sum_0^n f^i(y)$. Let

$$\eta(\alpha)=\inf_{x\in\,]\alpha,\,1/2[}[f(x)-x].$$

If α is small enough, we have $\eta(\alpha)=f(\alpha)-\alpha$.

Note also that $\eta(\alpha)=O(\alpha^2)$ if $\alpha\to 0$. We now observe that if $f^i(y)\geqq\varepsilon$, then $f^{i-1}(y)\leqq f^i(y)-\eta(\varepsilon)$. Therefore if $\varepsilon<f^i(y)\leqq 1/2$ for $0\leqq i\leqq n$ we must have $n\leqq f^n(y)\,\eta(\varepsilon)^{-1}$, and $f^i(y)\leqq f^n(y)-(n-i)\,\eta(\varepsilon)$ which implies

$$\sum_0^n f^i(y)\leqq[f^n(y)]^2\,\eta(\varepsilon)^{-1}\leqq 2\,\eta(\varepsilon)^{-1}\,|f^n(y)-f^n(x)|.$$

If $|f^n(x)-f^n(y)| \leqq f^n(y)/2$ we must have $f^n(y) \leqq 2f^n(x) \leqq 2\varepsilon$, and there is a number $c_3>0$ such that $|\log f'(f^i(y))/f'(f^i(x))| \leqq c_3 |f^i(x)-f^i(y)|$. The result now follows from lemma A.3.

LEMMA 2.3. — *There is a number $c_4>0$ such that if* K *is an atom of* $\bigvee_0^n f^{-i}\mathrm{F}$ *with*

$$f^i(\mathrm{K}) \subset [1/2, 1] \qquad \text{for} \quad 0 \leqq i \leqq n,$$

and if $x, y \in \mathrm{K}$ then

$$\prod_0^n \frac{f'(f^i(x))}{f'(f^i(y))} \leqq \exp c_4 |f^n(x)-f^n(y)|^{1/2}.$$

Proof. — Since f is expanding on $[1/2, 1]$, we have

$$|f^i(x)-f^i(y)| \leqq \alpha^{-(n-i)} |x-y|$$

The result follows at once from

$$\frac{f'(f^i(x))}{f'(f^i(y))} \leqq \exp O(1) |f^i(x)-f^i(y)|.$$

We shall now derive a uniform estimate for the ratio of the derivatives of the n-th iterate of our map.

LEMMA 2.4. — *There is a number $c>0$ such that if* K *is an atom of*

$$\bigvee_0^n f^{-i}\mathrm{F}$$

and if x and y belong to K, *then*

$$\prod_0^n \frac{f'(f^i(x))}{f'(f^i(y))} \leqq \mathrm{C}_n(f^n(x); f^n(y)) \exp |f^n(x)-f^n(y)|^{1/2}$$

where C_n is defined below. Moreover, for any $b \in [0, 1]$, there is a constant $\mathrm{C}(b)$ such that for all $x, y \in [b, 1]$, we have if $\{u_s(\ ),\ 1<s<n\}$ denotes the set of preimages of order p_0 a fixed number to be defined below

$$|\log \mathrm{C}_n(x; y)|$$

$$\leqq 2a \max_{1<s<n_0} \left| \mathrm{Log} \frac{1+a(n-p_0)u_s(x)}{1+a(n-p_0)u_s(y)} \right| + O(1) |u_s(x)-u_s(y)|^{1/2}$$

$$\leqq \mathrm{C}(b) |x-y|.$$

Proof. — Let $\varepsilon>0$ small enough to be chosen later on. ε will be smaller than the constant ε_1 in Lemma 2.2. Let $j_0 \leqq k_0 \leqq l_0 \leqq j_1 \leqq \ldots \leqq j_s \leqq k_s \leqq l_s$ be a sequence of integers defined for ε small enough by

$$f^i(\mathrm{K}) \subset [0, \varepsilon] \qquad \text{for} \quad j_q \leqq i < k_q$$

$$f^i(\mathrm{K}) \not\subset [0,\varepsilon] \quad \text{and} \quad f^i(\mathrm{K}) \subset [0,1/2] \quad \text{for } k_q \leqq i < l_q$$
$$f^i(\mathrm{K}) \subset [1/2,1] \quad \text{for } l_q \leqq i < j_{q+1}.$$

We may have j_0 and k_0 equal to zero, h_s and l_s equal to s. Notice that if L belongs to $\bigvee_0^n f^{-i}\mathrm{F}$, $s \geqq 1$, $\mathrm{L} \cap [0,\varepsilon] \neq \varnothing$, and $f(\mathrm{L}) \cap [0,\varepsilon[= \varnothing$ then $f(\mathrm{L}) \subset [0,1/2]$. From Lemma 2.1, 2.2 and 2.3, we have

$$\prod_0^n \frac{f'(f^i(x))}{f'(f^i(y))} \leqq \mathrm{D}_1 \mathrm{D}_2 \mathrm{D}_3,$$

where

$$\log \mathrm{D}_1 = \sum_{q=0}^{s} \sum_{p=0}^{k_q - j_q - 1} \log [f'(f^{jq+p}(x))/f'(f^{jq+p}(y))]$$

$$\log \mathrm{D}_2 \leqq \mathrm{D}(\varepsilon) \sum_{q=0}^{s} |f^{l_q-1}(x) - f^{l_q-1}(y)| \leqq \mathrm{A}[1-\alpha^{-1/2}]^{-1} |f^n(x) - f^n(y)|^{1/2}.$$

We now have to complete the estimation of $\log \mathrm{D}_2$. Using Lemma 2.1 and the above estimate, we have

$$\mathrm{A} \sum_{q=0}^{s} \sum_{p=0}^{k_q - j_q - 1} |f^{j_q+p}(x) - f^{j_q+p}(y)| = \sum_{q=0}^{s} \sum_{p=0}^{k_q - j_q - 1} \lambda(f^{j_q+p}[x,y])$$

(we have assumed $x < y$ for definiteness).

We shall first treat a special case.

Assume first that $f^n(x), f^n(y) \in [0,\varepsilon]$. Then, we conclude by Lemma 2.1

$$\sum_{q=0}^{s} \sum_{p=0}^{k_q - j_q - 1} \lambda(f^{j_q+p}[x,y]) \leqq \sum_{j=0}^{n} \frac{1}{aj + [f^n x]^{-1}} - \frac{1}{aj + [f^n y]^{-1}} + \mathrm{A}|f^n x - f^n y|^{1/2}$$

and the result follows in this case.

The same argument can be used in the general situation if f has some special properties.

In the general case for f, one has first to be sure that the preimages are in $[0,\varepsilon]$ to be able to apply the above estimates.

Let x and y be two points in $[0,1]$ and let J denote the interval with endpoints $\{x,y\}$. We shall determine the largest connected component of $f^{-1}(\mathrm{J})$.

Let p_0 be an integer large enough so that $f^{p_0}(1) < \varepsilon$, and let $n_0 = 2^{p_0}$. For $l > \log_2 n_0$ we shall construct a family of intervals $\mathrm{J}_1, \ldots, \mathrm{J}_{n_0}$, such that $f^l(\mathrm{J}_s) = \mathrm{J}$ (for $1 \leqq s \leqq n_0$) and the largest connected component of $f^{-l}(\mathrm{J})$ is one of these intervals. We proceed as follows.

Let $K_1, \ldots, K_{n_0}$ be the ordered family of connected components of $f^{-p_0}(J)$. We define $J_s (1 \leqq s \leqq n_0)$ by

$$J_s = f^{l-p_0}(K_s).$$

Let U be the largest connected components of $f^{-p_0}(J)$. If $U \subset [0, \varepsilon]$, then $f_-(U)$ is bigger than $f_+(U)$, and it is easy to see recursively (using $f'_{[0,\varepsilon[} < f'_{[\varepsilon,1]}$) that $f^{l-p_0}(U)$ is the largest connected component of $f^{-l}(J)$.

Assume now $f^i(U) \not\subset [0, \varepsilon]$ for $0 \leqq i \leqq p_0$. Then using lemma A4, there is a number $\theta > 1$ such that $\lambda(U) \leqq \theta^{-p_0} \lambda(f^{p_0}(U))$. However by Lemma A3,

$$\lambda(f^{p_0}(f^{p_0}(U)) \geqq A \lambda(f^{p_0}(U)) p_0^{-2}.$$

This is a contradiction, and we conclude that the largest connected component of $f^{-1}(J)$ is one of the set $\{J_1, \ldots, J_{n_0}\}$.

We now define $K_n(x, y)$ by replacing x and y by extreme points of some K_s, namely

$$\log K_n(x, y) = 2a \max_{1 \leqq s \leqq n_0} \sum_{j=0}^{n-p_0} \frac{1}{aj + \alpha_s^{-1}} - \frac{1}{aj + \beta_s^{-1}},$$

where $K_s = [\beta_s, \alpha_s]$ and $f^{p_0}(K_s) = f^n([x, y])$.

LEMMA 2.5. — *The sequence* $h_n(1)$ *satisfies*

$$h_n(1) \geqq A/n.$$

Proof. — If we set $y = 1$ in the bound of Lemma 2.4, we get

$$h_n(x) \leqq h_n(1) \left(\frac{1 + na}{1 + nax} \right)^2 \exp A |x - 1|^{1/2}.$$

We integrate over x, and use the fact that $\int_0^1 h_n(x)\, dx = 1$ to get the estimate.

We now introduce a normalisation for the density of the invariant measure. We define a sequence e_n of functions by

$$e_n = \Omega_n^{-1} \sum_0^{n-1} h_l.$$

THEOREM 2.6. — *The sequence* e_n *is precompact in* $C^0(]0, 1])$. *Every accumulation point e is a fixed point of* P, i.e. *the density of an invariant measure. Moreover it satisfies the estimate*

$$\frac{x^2 e(x)}{y^2 e(y)} \leqq \exp A |x - y|^{1/2},$$

for $x, y \in [0, 1[$.

Proof. – From Lemma 2.4 we have

$$\left(\frac{1+nax}{1+nay}\right)^2 \frac{e_n(x)}{e_n(y)} \leqq \exp O(1)\,|x-y|^{1/2}.$$

Therefore for any $1>b>0$, using $e_n(1/2)=1$, we conclude that the sequence e_n is equicontinuous on $[b,1]$, and compactness follows from the Stone-Weierstrass theorem. The uniform bound follows by taking the limit. We also have (recall that $h_0=1$)

$$\mathrm{P}\,e_n = \frac{\Omega_{n+1}}{\Omega_n} e_{n+1} - \frac{1}{\Omega_n}$$

and using lemma B we see that $\mathrm{P}\,e=e$.

3. PROPERTIES OF THE INVARIANT MEASURE

In this section we shall prove some results about the density of the invariant measure and derive its uniqueness and ergodic properties. We shall first give a precise estimate on the singularity of the density of the a.c.i.m. at the fixed point.

LEMMA 3.1. – *Any accumulation point e of the sequence e_n satisfies*

$$|e(x)-f'_+(0)/x| \leqq O(1)\,x^{-1/2}.$$

Proof. – From lemma 2.4 we have for $x\in[1/2,1]$

$$|h_n(x)-h_n(1/2)| \leqq O(1)\,h_n(1/2)\,|x-1/2|^{1/2}.$$

It is easy to show recursively that

$$h_n = f_-^{n\prime} + \sum_{l=0}^{n-1} [f_+ \circ f^l]' [h_{n-l-1} \circ f_+ \circ f_-^l].$$

We now use this decomposition in order to estimate $\sum_{q=0}^{n} h_q(x)$. We shall first replace the contributions of the sum in the above expression by the more manageable quantity (this corresponds to preimages whose orbit is not entirely contained in A_-)

$$\sum_{q=0}^{n} \sum_{l=0}^{q-1} f_-^{l\prime}(x) f'_+(0)\, h_{q-l-1}(1/2).$$

We first estimate the correction coming from this replacement. It is given by

$$\sum_{q=0}^{n}\sum_{l=0}^{q-1} f_-^{l'}(x)\,[f'_+ \circ f_-^{l}]\,\big|h_{q-l-1}\circ f_+\circ f_-^{l}-f'_+(1/2)\,h_{q-l-1}(1/2)\big|$$

$$\leqq O(1)\sum_{q=0}^{n}\sum_{l=0}^{q-1} f_-^{l'}(x)\,\big|h_{q-l-1}\circ f_+\circ f_-^{l}(x)-h_{q-l-1}(1/2)\big|$$

$$+\big|f'_+(0)-f'_+\circ f_-^{l}(x)\big|\,h_{q-l-1}(1/2)$$

$$\leqq O(1)\sum_{q=0}^{n}\sum_{l=0}^{q-1} f_-^{l'}(x)\,\big|f_+\circ f^{l}(x)-1/2\big|^{1/2}$$

$$\leqq O(1)\sum_{q=0}^{n}\sum_{l=0}^{q-1} f_-^{l'}(x)\,\big|f_-^{l}(x)\big|^{1/2}\,h_{q-l-1}(1/2)$$

$$\leqq O(1)\sum_{k=0}^{n-1} h_k(1/2)\sum_{l=0}^{+\infty}\big|f_-^{l}(x)\big|^{1/2} f_-^{l'}(x).$$

From Lemma A.2 and Lemma A.3, we have $f^{l}(x)\leqq O(1)\,x/(alx+1)$ and $f_-^{l'}(x)\leqq O(1)\,x/(alx+1)^{1/2}$. We now have an estimate on the sequence of functions e_n

$$e_n(x)=\frac{\sum_{0}^{n}(f_-^{l'}(x)+\sum_{0}^{l-1}(f_-^{j'}(x)f'_+(0)\,h_{i-j-1}(1/2))}{\sum_{l=0}^{n}h_l(1/2)}+O(1)\,x^{-1/2}.$$

Using the result of appendix B, we have

$$\sum_{q=0}^{+\infty} t^q\sum_{i=0}^{q-1} f_-^{l'}(x)\,h_{q-l-1}(1/2)=\left(\sum_{l=0}^{+\infty} f_-^{l'}(x)\,t^{l}\right)\left(\sum_{l=0}^{+\infty} t^{l}\,h_l(1/2)\right)$$

$$\sim\frac{\sum_{l=0}^{+\infty} f_-^{l'}(x)}{-(1-t)\log(1-t)}$$

for all fixed nonzero x if $t\to 1_-$. Using Karamata's theorem (*see* [Ti]) and theorem B we obtain

$$\sum_{q=0}^{n}\sum_{l=0}^{q-1} f_-^{l'}(x)\,h_{q-l-1}(1/2)\sim\left(\sum_{l=0}^{+\infty} f_-^{n'}(x)\right)\left(\sum_{q=0}^{n} h_q(1/2)\right)$$

The result follows easily from the fact that $\sum_{l=0}^{+\infty} f_-^{l'}(x)$ has a pole of order one at zero.

We now show that the absolutely continuous invariant measure is unique.

LEMMA 3.2. – *Let* g_i, $i=1, 2$ *be such that*

(1) $\mathrm{P}\, g_i = g_i$

(2) $g_i(1/2)=1$

(3) *the numbers* α_i *are some positive constants such that*

$$|g_i - \alpha_i/x| \leqq O(1)\, x^{-1/2}.$$

Then $g_1 = g_2$.

Proof. – We have $|g_i - \alpha_i/x| \leqq \beta_i x^{-1/2}$ by hypothesis. Multiplying g_1 by a constant, we can assume that $\alpha_1 = \alpha_2$. Let $g = g_1 - g_2$, then obviously $\mathrm{P}\, g = g$ and $|g(x)| \leqq O(1)\, x^{-1/2}$. This implies

$$\int_0^1 x\,|g(x)|\,dx = \int_0^1 x\,|\mathrm{P}^j g(x)|\,dx \leqq \int_0^1 x\,\mathrm{P}^j\,|g(x)|\,dx = \int_0^1 f^j(x)\,|g(x)|\,dx.$$

Therefore

$$\int_0^1 x\,|g(x)|\,dx \leqq \int_0^1 n^{-1} \sum_0^{n-1} f^j(x)\,|g(x)|\,dx \leqq O(1) \int_0^1 x^{-1/2}\, n^{-1} \sum_0^{n-1} f^j(x)\,dx.$$

The sequence of functions

$$n^{-1} \sum_0^{n-1} f^j(.)$$

is uniformly bounded and converges to zero Lebesgue almost everywhere since the Bowen-Ruelle measure is the Dirac measure at 0.

It follows from the Lebesgue convergence theorem that $g=0$, *i.e.* $g_1 = g_2$.

THEOREM 3.3:

$$\lim_{n \to \infty} \Omega_n^{-1} \sum_{l=0}^{n-1} \mathrm{P}^l 1 = e$$

for the topology of uniform convergence on compact subset of $]0, 1]$, *and*

$$|e(x) - O(1)/x| \leqq O(1)\, x^{-1/2}.$$

Proof. – The sequence $\Omega_n^{-1} \sum_{l=0}^{n-1} \mathrm{P}^l 1$ has only one accumulation point by lemma 3.2.

THEOREM 3.4. – *Any invariant Borel set* A *such that* $\int_{\mathrm{A}} dx/x < +\infty$ *has Lebesgue measure zero.*

Proof. – According to the definition of the Perron-Frobenius operator we have for all integer k, $\lambda(A)=\int_A h_k\,dt$, which implies

$$\int_A e_j\,dt=\int_A \frac{\sum_{k=0}^{j-1} h_k}{\Omega_j}\,dt=\frac{j\lambda(A)}{\Omega_j}.$$

Therefore

$$\lambda(A)=j^{-1}\Omega_j\int_A e_j\,dt,$$

and

$$\lambda(A)=n^{-1}\sum_{j=0}^{n-1} j^{-1}\Omega_j\int_A e_j\,dt.$$

Now using Lemma B

$$j^{-1}\Omega_j\sim\frac{1}{\log j+2},$$

but on the other hand $|e_j(x)-1/x|\leqq O(1)\,x^{-1/2}$ implies

$$\int_A e_j\,dt\leqq O(1)\int_A dt/t,$$

and therefore

$$\lambda(A)\leqq O(1)\left(\int_A\frac{dt}{t}\right)\frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{\log j+2},$$

which implies $\lambda(A)=0$.

LEMMA 3.5. – *Let u be a real function on* $[0, 1]$ *such that*

(1) *There is some strictly positive constant* C, *such that* $C<u<C^{-1}$.

(2) *u is* C^1.

Then there is a finite constant $\sigma(u)$ such that

$$\lim_{n\to\infty}\Omega_n^{-1}\sum_0^{n-1} P^j u=\sigma(u)\,e,\quad\text{in }]0,1].$$

Proof. – It is easy to modify the proofs of lemma 2.1-2.4 and 3.1, to show that $P^n u$ satisfies a bound similar to $P^n 1$. This follows from the fact that if L is an n-preimage of the segment $[x, y]$ then

$$\lambda(L)\leqq O(1)\,|x-y|.$$

We now apply the Chacon-Ornstein ergodic theorem [F] to the operator P in $L_1([0,1])$ and we deduce that

$$\frac{\sum_0^{n-1} P^j u}{\sum_0^{n-1} P^j 1}$$

converges almost surely. Therefore $\Omega_n^{-1} \sum_0^{n-1} P^j u$ is also almost surely convergent. From the compactness in the topology of compact convergence on $]0,1]$, it follows that we have convergence everywhere. Let e^* be the limit; using lemma 3.2, we conclude that e^* is proportional to e. Notice that it is easy to extend σ to a probability measure, since it is a linear functional on C^1 bounded in the C^0 topology.

LEMMA 3.6. – *Let* $u \in L_1$, *then*

$$\lim_{n \to \infty} \frac{\sum_0^{n-1} P^j u}{\sum_0^{n-1} P^j 1} = \sigma(u)$$

and f is ergodic.

Proof. – By the Chacon-Ornstein identification theorem [K], [N]

$$\lim_{n \to \infty} \frac{\sum_0^{n-1} P^j u}{\sum_0^{n-1} P^j 1} = \frac{E(u \mid \sum_i)}{E(1 \mid \sum_i)}$$

where $E(. \mid \sum_i)$ denotes the conditional expectation on the σ algebra of invariant sets. Using the density of C^0 in L_1 we obtain $E(u \mid \sum_i) = \sigma(u) E(1 \mid \sum_i)$ for all u in L_1. This implies that $E(1 \mid \sum_i)$ is almost surely constant, since all the characteristic functions of level sets of $E(1 \mid \sum_i)$ are proportional to $E(1 \mid \sum_i)$.

APPENDICES

Appendix A

The following result is an obvious consequence of our hypothesis. We state it as an independent lemma for further references. We recall that a is defined by $2a=f''(0)$. The following lemma is an immediate consequence of the regularity of f.

LEMMA A.1. – *There is a* C^1 *function* r *defined on* $[0,1]$, *such that*

$$f(x)=x-ax^2+x^3 r(x).$$

The following result is almost a rephrasing of a result in [D]. We shall give it for the convenience of the reader. We first introduce some notations. For x in $[0,1]$ we set $\theta(x)=x^{-1}$ and $t_n(x)=1/(na+\theta)$.

We also define $u_n=f^n/t_n$ and $w_n=u_n-1$. We shall not indicate the x dependence when there is no ambiguity. From lemma A.1 we have the following recursive relation for w_n

$$w_{n+1}=w_n-at_n w_n-at_n w_n^2-at_n^2(1+w_n)^2+(1+w_n)^3 t_n^2(1+at_n)\,r(t_n+t_n w_n),$$

LEMMA A.2. – *There is a positive real number* Z_0 *such that if* $0<x<Z_0$, *then for every integer n we have*

$$|w_n|\leqq t_n^{3/4}$$

and

$$|f^n_-(x)-[an+x^{-1}]^{-1}|\leqq O(1)[an+x^{-1}]^{-7/4}$$

Proof. – From the above recursion on w, it follows easely that if t_n is small enough, then $|w_n|\leqq t_n^{3/4}$ implies $|w_{n+1}|\leqq t_{n+1}^{3/4}$. The estimate is obvious for $n=0$ since $w_0=0$, and the assertion follows recursively if we start with a small enough x since the sequence t is decreasing.

LEMMA A.3. – *If* x *and* x' *are small enough, then*

$$|f^n_-(x)-[na+\theta]^{-1}-f^n_-(x')+[na+\theta']^{-1}|\leqq O(1)\,n^{-5/4}|x-y|$$

where $\theta=x^{-1}$, *and* $\theta'=x'^{-1}$.

Proof. – Let $\delta_n=w_n(x)-w_n(y)$ we have the following recursion relation for δ_n where $\rho_n=t_n(x)-t_n(y)$ and assuming $x<y$, which implies $t_n(x)\leqq t_n(y)$

$$\delta_{n+1}=\delta_n-a\rho_n(w_n(x)+w_n^2(x))-at_n(y)\,\delta_n(1+w_n(x)+w_n(y))+O(t_n^2(y)|\delta_n|+t_n(y)|\rho_n|).$$

Using lemma A.2 and the obvious estimate

$$1-at_n=(1+O(n^{-3/2}))\,t_{n+1}/t_n,$$

we obtain

$$|\delta_{n+1}| \leqq |\delta_n| (1 + O(n^{-3/2})) t_{n+1}(y)/t_n(y) + |\rho_n| t_n^{3/4}(y).$$

This implies since $\delta_0 = 0$

$$|\delta_{n+1}| \leqq O(1) t_n(y)^{-1} \sum_0^n t_{j+1}(y)^{-1} t_j(y)^{3/4} |\rho_j|.$$

Using $t_n/x < 1$ and $\rho_n = (y-x) t_n(x) t_n(y)/xy$ we get

$$|\delta_{n+1}| \leqq O(1) t_n(y)^{-1} |x-y| \sum_0^n t_j(y)^{3/4} \leqq |x-y|^{1/2} t_j(y)^{1/4}.$$

The result follows at once from the relation

$$u_n(x) - u_n(y) = t_n(x) \delta_n + w_n(y) \rho_n$$

using the above estimate and lemma A.2.

LEMMA A.4. – *Let* U *be an interval such that* 1/2 *is not in the interior of* $f^l(U)$ *and* $f^l(U) \not\subset [0, \varepsilon]$ *for* $0 \leqq l \leqq n$. *Then for some* $\rho > 1$ *which depends only on* ε, *and not on* U, *such that*

$$\lambda(f^n(U)) \leqq \rho(1)$$

we have

$$\lambda(U) \leqq \rho^{-n} \lambda(f^n(U)).$$

Proof. – This is obvious if $f^l(U) \cap [0, \varepsilon] = \varnothing$ for $0 \leqq l \leqq n$. If $f^l(U) \cap [0, \varepsilon] \neq \varnothing$, using Lemma A.3 and $\lambda(f^l(U)) \leqq \rho(1)$, it is easy to check that for some $\varepsilon_1 > 0$ such that $f^l(U) \cap [0, \varepsilon_1] = \varnothing$ for $0 \leqq l \leqq n$ and the result follows.

LEMMA A.5. – *Let* ε *be small enough such that*

$$f' | [0, \varepsilon] \leqq f'(\varepsilon).$$

Then there is a constant A_5 *such that if* U *is an interval contained in* $[0, \varepsilon]$, *then for any integer n*

$$\lambda(f_-^n(U)) \geqq A_5 \lambda(U).$$

Proof. – We have

$$f_-^{n'}(\varepsilon) \lambda(U) \leqq \lambda(f^n(U)).$$

Using lemma A.1 it is easy to show that for ε small enough we have

$$f_-^{n'}(\varepsilon) \geqq \prod_0^{n-1} \left(\frac{t_{j+1}(\varepsilon)}{t_j(\varepsilon)} \right)^3$$

and the result follows from the definition of t.

Appendix B

We define a function ζ by

$$\zeta(t)=\sum_{n=0}^{+\infty} t^n h_n(1/2).$$

We shall be interested in the dominant singularity of $\zeta(t)$ when $t \to 1_-$. We then use a Tauberian theorem to obtain the asymptotic behaviour of $\sum_{p=0}^{n} h_p(1/2)$.

THEOREM B: (1) *Since* $h_n(1/2)$ *is bounded (as follows easily from Lemma* 2.4), $\zeta(t)$ *is analytic in the unit disk.*

$$\zeta(t)\sim \frac{-a/f'_+(0)}{(1-t)\log(1-t)} \tag{2}$$

if $t \to 1_-$.

$$\sum_{p=0}^{n} h_p(1/2)\sim \frac{n}{\log n}. \tag{3}$$

Proof. – We shall first find an equation for $\zeta(t)$. We start with the relation

$$h_n(x)=f_-^{n\prime}(x)+\sum_{l=0}^{n-1}[f_+\circ f_-^l]'(x)\,[h_{n-l-1}\circ f_+\circ f_-^l](x)$$

which is easy to prove recursively. We integrate from 0 to 1 and get

$$1=f_-^n(1)+\sum_{l=0}^{n-1}(f_+\circ f_-^l(1)-1/2)\,h_{n-l-1}(1/2)+\sum_{l=0}^{n-1} a_{n,l}$$

where

$$a_{n,l}=\int_{1/2}^{f_+\circ f_-^l(1)}(h_{n-l-1}(x)-h_{n-l-1}(1/2))\,dx.$$

However

$$f_+\circ f_-^n(1)-1/2=f'_+(0)f_-^n(1)$$

up to the first order. Multiplying by t^n and summing over n we get for $|t|<1$

$$\frac{1}{1-t}=\sum_{n=0}^{+\infty} t^n f_-^n(1)+\sum_{n=0}^{+\infty} t^n \sum_{l=0}^{n-1} f'_+(0)f_-^l(1)\,h_{n-l-1}(1/2)+R(t)$$

where

$$\mathrm{R}(t)=\sum_{n=0}^{+\infty} t^n \sum_{l=0}^{n-1} a_{n,l}.$$

This implies

$$\sum_{n=0}^{+\infty} t^n \sum_{l=0}^{n-1} f'_+(0) f^l_-(1)\, h_{n-l-1}(1/2)$$

$$=\sum_{l=0}^{+\infty}\sum_{p=0}^{+\infty} t^{l+p+1} f'_+(0) f^l_-(1)\, h_p(1/2)$$

$$=f'_+(0)\, t\left[\sum_{p=0}^{+\infty} t^p h_p(1/2)\right]\left[\sum_{p=0}^{+\infty} t^p f^p_-(1)\right]=-f'_+(0)\,\zeta(t)\log(1-t).$$

We define a function v by $v(t)=\sum_{l=0}^{+\infty} t^l f^l_-(1)$, and we obtain

$$1=(1-t)\,v(t)+1/2\,(1-t)\,tv(t)\,\zeta(t)+(1-t)\,\mathrm{R}(t).$$

We shall now give a bound on $\mathrm{R}(t)$. We have

$$|h_p(x)-h_p(1/2)|\leqq O(1)\,|x-1/2|^{1/2}\,h_p(1/2)$$

when $t\in[1/2,1[$, and therefore

$$|a_{n,l}|\leqq O(1)\,h_n(1/2)\,|f_+\circ f^l_-(1)-1/2|^{3/2}.$$

Hence if $t\in[1/2,1[$ there exists a positive constant

$$\mathrm{K}=O(1)f'_+(0)^{3/2}\sum_{l=0}^{+\infty} f^l_-(1)^{3/2}$$

such that

$$-\mathrm{K}\,\zeta(t)\leqq\mathrm{R}(t)\leqq\mathrm{K}\,\zeta(t).$$

Therefore if t is such that $1-(1-t)\,v(t)$ is positive

$$\frac{1-(1-t)\,v(t)}{f'_+(0)\,t(1-t)\,v(t)+(1-t)\,\mathrm{K}}\leqq\zeta(t)\leqq\frac{1-(1-t)\,v(t)}{f'_+(0)\,t(1-t)\,v(t)-(1-t)\,\mathrm{K}}.$$

Now we use the fact that (lemma A.2)

$$f^n_-(1)=\frac{1}{1+an}+\frac{O_n}{(1+an)^{7/4}}$$

O_n being a bounded sequence. Henceforth

$$v(t)=-(at)^{-1}\log(1-t)+\sum_{n=0}^{+\infty} t^n\frac{O_n}{(1+an)^{7/4}}+v_1(t)$$

where $v_1(t)$ is uniformly bounded in $[1/2,1]$. We now recall Karamata's theorem.

THEOREM [H], [Ti]. — *Suppose that*

(1) for all n $a_n \geqq 0$,

$$\sum_{n=0}^{+\infty} a_n t^n \sim \frac{1}{1-t} \text{ if } r \to 1_-. \tag{2}$$

Then

$$\sum_{n=0}^{N} a_n \sim N.$$

We conclude using this last theorem for the function $[-\log(1-t)]\zeta(t)$, and a direct estimate of the Taylor coefficients of ζ expressed as convolutions of the Taylor coefficients of the functions $-1/\log(1-t)$ and $-\log(1-t)\zeta(t)$.

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(*Manuscript received September* 13, 1989.)