ANNALES DE L'I. H. P., SECTION A

JEAN BELLISSARD
MICHEL VITTOT

Heisenberg's picture and non commutative geometry of the semi classical limit in quantum mechanics

Annales de l'I. H. P., section A, tome 52, n° 3 (1990), p. 175-235
<http://www.numdam.org/item?id=AIHPA_1990__52_3_175_0>

© Gauthier-Villars, 1990, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section A » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Henri Poincaré,

Vol. 52, n° 3, 1990, p. 175-235. *Physique théorique*

Heisenberg's picture and non commutative geometry of the semi classical limit in quantum mechanics

by

Jean BELLISSARD (*) and Michel VITTOT

Centre de Physique théorique (**), C.N.R.S. Luminy,
Case 907, 13288 Marseille Cedex 09, France

ABSTRACT. – We propose a new framework based upon non commutative geometry to control the semi classical limit in phase space. It leads in particular to uniform estimates of Nekhoroshev's type, as Planck's constant tends to zero, for the perturbation expansion.

RÉSUMÉ. – Nous proposons un cadre nouveau fondé sur la géométrie non commutative, pour contrôler la limite semi classique dans l'espace des phases. Cette approche permet d'obtenir des estimations, uniformes par rapport à la constante de Planck, sur la série de perturbation, semblables à celles de Nekhoroshev en mécanique classique.

0. Introduction.
1. The Harmonic Oscillator: Action-Angle Variables.
2. The Quantal Phase Space.
3. The Semi Classical Observable Algebra.
4. Calculus I: Averaging and Integration.
5. Calculus II: Differential Structure.

(*) Université de Provence, Marseille, (France).
(**) Laboratoire Propre, LP-7061, Centre National de la Recherche Scientifique.

6. Holomorphic Functions on Quantal Phase Space.
7. A Non Commutative Cauchy-Kowaleskaya Theorem.
8. A Convergent Perturbation Theory for One Degree of Freedom.
9. Birkhoff's Expansion and Rayleigh-Schrödinger's Perturbation Series.
10. A Nekhoroshev-type Estimate.
11. Another Birkhoff expansion.

0. INTRODUCTION

During the last ten years many informations on the manifestations of classical deterministic regular or chaotic motions in quantum mechanics have been gathered. One of the most famous observations concerns the universal behaviour of the level spacing distribution of the eigenvalues of a quantum system which follows a Poisson law if the system is classically integrable ([16], [17], [71], [73]), and a Wigner distribution of the type GOE or GUE for classically chaotic ones ([24], [25]). These results have been tested numerically on many examples ([19], [20], [21], [24], [26], [75]), they have also been observed in several experimental results ([23], [32]) and have received some theoretical justifications ([16], [19], [51]).

Another important result was provided by the Selberg trace formula, relating the structure of the spectrum of the Laplacian on a surface of constant negative curvature, to the distribution of the periodic orbits of the corresponding geodesic flow which is known to be ergodic and strongly mixing ([65], [76]). Generalizations to quantum mechanics have been proposed independently by Gutzwiller ([54], [55]) and Balian & Bloch ([3], [4], [5], [6]). Beside some mathematically rigorous results on the asymptotics of the density of states generalizing Weyl's formula ([39], [40], [53]) it provides also a way to associate to each eigenstate packages of classical orbits located in the region where the quantum state lies ([43], [57], [58], [59], [77]). In the same spirit, M. V. Berry proposed recently a formula [19] for the eigenvalues spacing distribution from that kind of semiclassical approach which seems to give a more precise description than the Wigner laws in particular for non generic systems for which the universal statistics do not hold.

Quantum systems with time-dependent potentials periodic in time have been also successfully investigated using comparison with their classical counterpart. The first model was the so-called "kicked rotor" proposed by Taylor [78] and Chirikov [38] as a paradigm for transition to chaos in classical hamiltonian mechanics, and later on again by Casati [30] for comparison between the classical and the quantum model. An important contribution in that respect was the introduction of "Anderson's localization" in that scheme by the Maryland group ([47], [48]). At small coupling

whenever the classical motion is regular, a KAM like theorem has been proved for a smoothly kicked rotor [11] leading to a rigorous version of the so-called EBK quantization condition. At large coupling where the classical motion is chaotic ([52], [71]), it leads to the Chirikov-Shepelyansky relation ([33], [35]) between the classical diffusion constant and the quantum localization length of the eigenstates of the quasi-energy operator. More recently several theoretical and numerical works [50] have been performed on the kicked rotor problem to give a more detailed analysis of the quantum behaviour in term of a semiclassical approach leading to a very precise understanding of the role of classical KAM tori or Cantori in the quantum tunneling effect and the nature of quantal eigenstates of the quasi-energy operator.

These ideas were very successfully applied to explain the multiphotonic ionization of hydrogen atoms by microwaves, an experiment due to J. Bayfield and P. Koch [7] which stayed a puzzling result for years until it was suggested and then verified that the results were a manifestation of the transition to chaos in the corresponding classical system giving rise to quantitative successful predictions [8]. A quantum version of that approach using specifically the localization scheme proved very recently to be quite successfull in predicting the existence of a regime where classical and quantum mechanics disagree due to interference effects ([9], [10], [34]).

Eventhough most of these difficult results have contributed highly in understanding more concretely what has been fashionable to call "quantum chaos", there is still a need for a mathematical background permitting to go beyond the study of special examples, numerical simulations or experiments. In any case it is still to be understood why these semiclassical results are so accurate.

In that respect the situation is in a much better shape for systems which are classically nearly integrable. The Bohr-Sommerfeld quantization rules for the orbits of the hydrogen atom were generalized by Einstein-Brillouin-Kramers ([30], [45], [61], [62]) to provide the semiclassical approximation to eigenvalues whenever classical invariant tori occur in the system.

In modern language, the EBK quantization is successfully used through pseudodifferential calculus ([37], [53], [81]), at least when one is dealing with the Schrödinger operator. For example during the last five years exact results on tunneling have been obtained by several groups ([28], [29], [56]). One of the most remarquable set of results along this line concerns the fractal properties of Hofstadter's spectrum, describing the quantum motion of a Bloch electron in a uniform magnetic field ([14], [56], [73], [82], [83]).

However physicists and quantum chemists ([43], [59], [64], [77]) have developped other approaches in order to get accurate algorithms for the computation of eigenvalues of complicated systems like atoms or molecules. Starting with a classical hamiltonian as a small perturbation of an

integrable one expressed in term of action-angle variables, they compute its Birkhoff expansion up to a finite order by various resummation method, and simply replace each action variable by an integer multiple of Planck's constant $\hbar$. In this way one gets surprinsingly accurate results for the energy spectrum even for the groundstate energy. On the other hand, this approach has the advantage over the pseudodifferential calculus of being canonical, for it uses action-angle variables which are (at least locally in phase space) universal canonical variables. In this respect, such a canonical approach were used in [15] when dealing with Bloch electrons in a magnetic field through the use of a non commutative algebra of operators generalizing the algebra of classical observables on the corresponding classical phase space. It led to a more efficient method when dealing with tunneling in phase space than the pseudodifferential calculus [56], because it never uses localization properties in position space.

In all these problems interference effects play a central role. For classically integrable systems, the WKB approximation consists precisely in computing the phase factor either for the wave function or for the Green function, and the EBK scheme give recipes for getting the result. For classically chaotic systems, trace formulae select classical closed orbits precisely because they are stationnary points of the phase factors occuring in the right-hand side. In much the same way, Anderson localization for the kicked rotor model is precisely due to interference effects trapping the state in a bounded region of the quantum phase space.

However, there are still some difficulties in finding a satisfactory formalism, especially in the chaotic region where no such simple rule as the EBK one is known up to now to give a definite answer to the question of semiclassical quantization. We believe that the Heisenberg point of view using a non commutative algebra of observables as the quantum analog of the set of functions over the classical phase space, can solve that difficulty. The main reason is that it follows closely the canonical formalism in classical mechanics and therefore there is no rule to be found for the quantization. The crucial point then is to find a correct way of expressing interference effects in such a non commutative framework.

This is precisely what has been solved in several recent works using non commutative geometry developped by A. Connes [41].

The first example studied in this way concerned the gap labelling theorem for hamiltonian spectra especially whenever the spectrum is a Cantor set [11]. Gaps occur in quantum mechanics whenever a Bragg condition is violated, again an interference effect. The gap labelling is provided by the value of the so-called "integrated density of states" which is related through Sturm's theorem to the rotation angle per unit length of the phase of the wave function solution of Schrödinger's equation. However, in the non commutative approach, this number can be interpreted as a topological invariant, an element of K-theory, similar to the

Chern class of a fiber bundle. One can then use the machinary of algebraic topology to give exact rules in computing these numbers.

The next example concerns a mathematically complete framework describing the ordinary Quantum Hall Effect [13]. Here interference effects occur in two ways: firstly through the gauge invariance leading to the quantization of the Hall conductance, and then in the Anderson localization due to the disorder, which produce the plateaus observed for the Hall conductance when one change the filling factor. This problem was solved using the machinary of non commutative geometry, permitting to identify the Hall conductance with the Chern class ([2], [12], [63], [79]) of a non commutative fiber bundle, which happens to be an integer.

The last example concerns the property of Hofstadter spectrum, and more generally the behavior of the gap boundaries for the energy spectrum of a 2D Bloch electron in a uniform magnetic field B as a function of B. It turns out that it is a semiclassical problem as first pointed out by Onsager [68] when he explained in a very simple way the de Haas-van Alphen effect ([1], [42]): the magnetic field plays the role of Planck's constant! This remark was used first by M. Wilkinson [82] to give an approximate renormalization scheme describing Hofstadter's spectrum for the Harper equation. It has been mathematically justified recently by Helfer and Sjöstrand [56]. On the other hand Wilkinson also gave a formula [84] for the derivative of the gap boundaries with respect to the magnetic field, which was rederived and physically interpreted by R. Rammal [73 *b*]. As these authors pointed out, this derivative involve a correction term due to the occurence of a Berry phase [18] again an interference effect which can be interpreted as a topological invariant. Unfortunatly the calculations of Rammal and Wilkinson used the specific form of Harper's equation which can be seen as a 1D Schrödinger equation on a lattice. The generalization to the case of a 2D electron in a magnetic field requires different techniques. In [14] one uses precisely non commutative differential geometry to solve that problem, and the Rammal-Wilkinson formula has been given in this general case.

This last result motivated us to systematically develop a non commutative framework to study the semi-classical limit. It is fair to say that the key ideas are based upon the construction of the "tangent groupoid" described privately by A. Connes several years ago in 1983, and a private discussion with S. Graffi in 1983 who explained to one of us (J.B.) how to relate small divisors in classical mechanics and quantum mechanics through a semi-classical limit.

Our strategy is the following: starting from the intuition given by the representation of quantum mechanics in term of annihilation and creation operators (giving rise in the classical limit to action angle variables), we define a C* algebras of observables indexed by the Planck constant $\hbar$, in

such a way that elements of that algebra have a good limit as $\hbar \to 0$. We show that it is then possible to see that algebra as the non commutative analog of the space of continuous functions on the classical phase space (here given as $\mathbf{R}^n_+ \times \mathbf{T}^n$ in action-angle variables) vanishing at infinity. By analogy we then extend to that algebra the usual operations of the canonical formalism namely the "averaging" and the Poisson brackets $\{\ ,\ \}$. In order to get technical results we then define a family of dense subalgebras being the non commutative analog of functions over the classical phase space which can be extended to a complex neighbourhood by holomorphy. The main estimate of our paper concerns the boundedness of Liouville operators (namely the operators $\mathscr{L}_w : f \to \{w, f\}$) on these subalgebras giving rise to the existence and construction of a class of canonical transformations uniformly in $\hbar$.

Then we illustrate the power of this approach by a comparison between perturbation expansions in classical and in quantum mechanics through a Lie formalism using Liouville operators. We get in this way formal power series related to Birkhoff's expansion for $\hbar = 0$ and coinciding with the Rayleigh-Schrödinger one for $\hbar \neq 0$. Moreover estimates on each term of the expansion and on the remainders can be obtained easily uniformly in $\hbar$ leading to an extension of Nekhoroshev's theorem for quantum mechanics. Actually denoting by ε the perturbation parameter, it is shown that the error term after truncating the expansion in an optimal way, is exponentially small in ε. This last result constitutes actually the first rigorous result in our knowledge concerning the use of Birkhoff expansions for computing the eigenvalue spectrum of a complicated system [43]. In addition our estimate permits to understand why eigenstates obtained in this way are so accurate: for indeed if we allow ourself to neglect corrections of order $\hbar^2$, say for the energy spectrum, one gets accurate results for values of ε such that $O(\exp(-1/\varepsilon^{1/N}) = O(\hbar^2)$ (where N is an exponent which is of order of the number of degrees of freedom) namely $\varepsilon = O(1/\ln^N(1/\hbar))$ a value which can be quite big even for small values of $\hbar$.

We believe that our formalism is actually more powerful. First of all we intend to show that such an algebra is "locally universal" namely it represents any algebra of observable constructed in a similar way locally in the "quantum phase space". As a result we expect to be able to study the quantum spectrum near an island of stability of the corresponding classical system. A consequence of such a study will be probably to prove rigorously that in this latter case Poisson's distribution indeed occur for such kind of spectrum [16]. But we also believe that it is especially well fitted for the study of the classically chaotic regime. We intend to go into such directions in the near future.

ACKNOWLEDGEMENTS

We are especially indebted to a discussion with Alain Connes on the construction of the "tangent groupoid" and with Sandro Graffi who introduced us to the semi-classical world.

1. THE HARMONIC OSCILLATOR: ACTION-ANGLE VARIABLES

Let us consider a harmonic oscillator with one degree of freedom. One quantizes it by mean of the Fock space constructed from the vacuum state $|0\rangle$ and the annihilation and creation operators a and a^* satifying the following commutation relation:

$$[a, a^*] = \hbar \mathbf{1} \tag{1.1}$$

where $\hbar = h/2\pi$ and h is the Planck constant. The Fock space is generated by the basis of vectors given by:

$$|n\rangle = \frac{a^{*n}}{\sqrt{n!\,\hbar^n}} |0\rangle. \tag{1.2}$$

Our first result is provided by the following theorem:

THEOREM 1.1. – *Let* $\mathrm{P}(a, a^*)$ *be a polynomial in the operators a and* a^*. *We set:*

$$f_{\mathrm{P}}(\hbar, n\hbar, l) = \langle n | \mathrm{P}(a, a^*) | n-l \rangle, \qquad n \in \mathbf{N}, \quad l \in \mathbf{Z} \tag{1.3}$$

(i) f_{P} *can be decomposed into:*

$$f_{\mathrm{P}}(\hbar, n\hbar, l) = f_{\mathrm{P}}(\hbar, n\hbar, l) \{\theta(\hbar, n\hbar, l)\}^{-1/2} \tag{1.4}$$

where $f_{\mathrm{P}}(\hbar, \mathrm{A}, l)$ *is a polynomial with respect to* $(\hbar, \mathrm{A})$ *vanishing identically for l big enough, and* θ *is a rational function independent of* P *defined by:*

$$\begin{aligned} &(a)\ \theta(\hbar, \mathrm{A}, 0) = 1 \\ &(b)\ \theta(\hbar, \mathrm{A}, 1) = \mathrm{A} \\ &(c)\ \theta(\hbar, \mathrm{A}, l)\, \theta(\hbar, \mathrm{A} - l\hbar, l') = \theta(\hbar, \mathrm{A}, l + l') \end{aligned} \tag{1.5}$$

(ii) f_{P} *depends linearly upon* P *and satisfies:*

$$\begin{aligned} &(a)\ f_{\mathrm{P}_1 \mathrm{P}_2}(\hbar, \mathrm{A}, l) = \sum_{l' \in \mathbf{Z}} f_{\mathrm{P}_1}(\hbar, \mathrm{A}, l')\, f_{\mathrm{P}_2}(\hbar, \mathrm{A} - l'\hbar, l - l') \\ &(b)\ f_{\mathrm{P}^*}(\hbar, \mathrm{A}, l) = \overline{f_{\mathrm{P}}(\hbar, \mathrm{A} - l\hbar, -l)}\, \theta(\hbar, \mathrm{A}, l) \end{aligned} \tag{1.6}$$

(iii) If $\hbar \to 0$, $n \to \infty$ in such a way that $n\hbar \to \mathrm{A}$ one has:

$$\lim_{\hbar \to 0,\ n\hbar \to \mathrm{A}} f_{\mathrm{P}}(\hbar, n\hbar, l) = \int_0^{2\pi} \frac{d\theta}{2\pi} \mathrm{P}(\sqrt{\mathrm{A}}\, e^{i\theta}, \sqrt{\mathrm{A}}\, e^{-i\theta})\, e^{il\theta} \quad \square \tag{1.7}$$

Proof. – From (1.5) we get for $l \geqq 1$:

$$\begin{aligned} \theta(\hbar, A, l) &= A(A-\hbar)\ldots(A-(l-1)\hbar) \\ \theta(\hbar, A, -l) &= \{(A+\hbar)(A+2\hbar)\ldots(A+l\hbar)\}^{-1} \end{aligned} \tag{1.8}$$

Let us now consider the simplest case where $P(a, a^*) = a$. We get:

$$\langle n | a | n-l \rangle = \delta_{l,-1} \{(n+1)\hbar\}^{1/2} = \delta_{l,-1} \{\theta(\hbar, n\hbar, -1)\}^{-1/2}$$

for $\theta(\hbar, A, -1) = \theta(\hbar, A+\hbar, 1)^{-1}$. Thus:

$$f_a(\hbar, A, l) = \delta_{l,-1}. \tag{1.9}$$

If one assumes (1.3) & (1.4) one gets by taking the adjoint of (1.3):

$$\langle n | P(a, a^*)^* | n-l \rangle = \overline{\langle n-l | P(a, a^*) | n \rangle_{\bar{\hbar}}} = \overline{f_P(\bar{\hbar}, (n-l)\bar{\hbar}, -l)}$$

implying (1.6 *b*) immediately. This gives in particular for $P(a, a^*) = a^*$

$$f_{a^*}(\hbar, A, l) = \delta_{l,1} A. \tag{1.10}$$

Given any pair P and P′ of polynomials the matrix multiplication gives:

$$\begin{aligned} &\langle n | P_1 P_2(a, a^*) | n-l \rangle \\ &\quad = \sum_{l'=-\infty}^{n} \langle n | P_1(a, a^*) | n-l' \rangle \langle n-l' | P_2(a, a^*) | n-l \rangle \\ &\quad = \sum_{l' \leqq n} f_{P_1}(\hbar, n\hbar, l') f_{P_2}(\hbar, (n-l')\hbar, l-l') \\ &\qquad\qquad \{\theta(\hbar, n\hbar, l')\theta(\hbar, (n-l')\hbar, l-l')\}^{-1/2} \\ &\quad = \theta(\hbar, n\hbar, l)^{-1/2} \sum_{l' \leqq n} f_{P_1}(\hbar, n\hbar, l') f_{P_2}(\hbar, (n-l')\hbar, l-l') \end{aligned}$$

Actually one has:

$$f_{P_1}(\hbar, n\hbar, l') = \overline{f_{P_1^*}(\hbar, (n-l')\hbar, -l')}\, \theta(\hbar, n\hbar, l').$$

By (1.8), for $l \geqq 1$ we have $\theta(\hbar, n\hbar, l) = n(n-1)\ldots(n-l+1)\hbar^l$ which vanishes if $l \geqq n+1$. Thus

$$f_{P_1}(\hbar, n\hbar, l') = 0 \quad \text{whenever } l' > n$$

leading to formula (1.6 *a*).

Taking the limit $\hbar \to 0$ with A fixed in (1.6) and multiplying both sides by $A^{-1/2}$ which is the limit of $\theta(\hbar, A, l)^{-1/2}$, we see that if the formula (1.7) is true for P_1 and P_2 it is true also for their product and their adjoint. It is therefore sufficient to check it on the polynomial $P(a, a^*) = a$. Thanks to (1.9) one obtains:

$$\lim_{\hbar \to 0} f_a(\hbar, A, l) = \frac{\delta_{l,-1}}{\sqrt{A^l}} = \sqrt{A}\, \delta_{l,-1} = \int_0^{2\pi} \frac{d\theta}{2\pi} \sqrt{A}\, e^{i\theta} e^{il\theta}$$

proving (1.7) at last. □

2. THE QUANTUM PHASE SPACE

What one learns from the previous chapter is that the classical limit is provided by the limit of the kernels $f_P(\hbar, n\hbar, l)$ as $\hbar \to 0$ with $n\hbar \to A$. In that limit this kernel converges to the Fourier transform of the corresponding classical observable obtained by replacing the annihilation operator a by the function $\sqrt{A}\, e^{i\theta}$ and the creation operator a^* by the classical function $\sqrt{A}\, e^{-i\theta}$ of the action A and angle θ. In this section we will make this remark systematic by defining a toplogy on the set of pairs $(\hbar, n\hbar)$ describing properly this semiclassical limit. We will also consider the situation for which one has N degrees of freedom ($N \geqq 1$) described through action-angle variables.

In what follows, if $A=(A_1, \ldots, A_N) \in \mathbf{R}^N$ and $A'=(A'_1, \ldots, A'_N) \in \mathbf{R}^N$ we will set:

$$\left.\begin{aligned} |A|_\infty = \operatorname*{Max}_{1 \leqq \mu \leqq N} |A_\mu|, \qquad |A|_p = \Big\{ \sum_{1 \leqq \mu \leqq N} |A_\mu|^p \Big\}^{1/p} \\ A.A' = A_1 A'_1 + \ldots + A_N A'_N \end{aligned}\right\} \tag{2.1}$$

One denotes by $\Gamma^{(0)}$ the closure in $\mathbf{R}^{N+1}$ of the set of pairs $(\hbar, n\hbar)$ where $0 < \hbar \leqq 1$ and $n \in \mathbf{N}^N$. As one can see from Figure 1 below, $\Gamma^{(0)}$ is the union of the lines (here $n \in \mathbf{N}^N$)

$$\Delta_n = \{(\hbar, n\hbar) \in \mathbf{R}^{N+1};\ 0 \leqq \hbar \leqq 1\}, \qquad \Delta_\infty = \{(0, A) \in \{0\} \times \mathbf{R}^N_+\}.$$

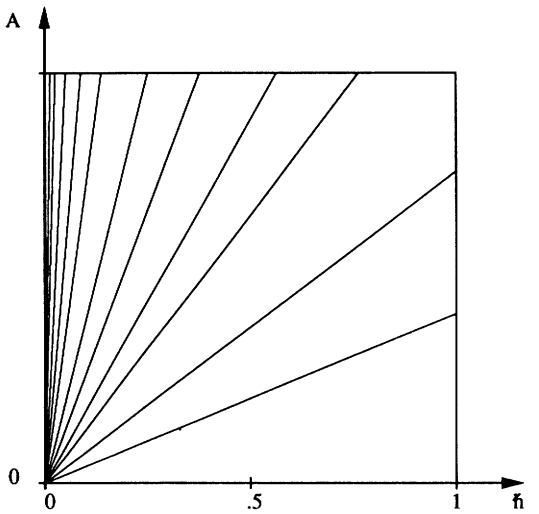

FIG. 1. – The Quantum Phase Space in Action Variables.

If one endows $\Gamma^{(0)}$ with the topology induced by $\mathbf{R}^{N+1}$ one sees that a generating family of neighbourhoods of the classical action (0, A) is

given by the set of points $(\hbar, n\hbar)$ such that for some $\varepsilon > 0$, $|\hbar| < \varepsilon$ and $|A - nfi| < \varepsilon$. This is exactly the topology needed for the semiclassical limit. $\Gamma^{(0)}$ will be called the "state space". Then $\Gamma^{(0)}$ is a locally compact Hausdorff space.

From last section one also sees that angle variables appear through Fourier's transform as dual variables "l" and they give rise for kernels to a generalized convolution. In Heisenberg's language, this convolution can be seen in term of "transitions between states" $|n\rangle$ and $|n-l\rangle$. If we identify the states with points in $\Gamma^{(0)}$ we just need to give a proper definition of transitions between two such points for a given value of $\hbar$. For this purpose we introduce the set Γ which is the closure in $\mathbf{R}^{N+1} \times \mathbf{Z}^N$ of the set of triple $\gamma = (\hbar, A, l)$ such that both $(\hbar, A)$ and $(\hbar, A - l\hbar)$ belong to $\Gamma^{(0)}$. We will interpret γ as an arrow describing a transition between an initial state [its "source" $s(\gamma)$] and a final state [its "range" $r(\gamma)$] exactly as spectroscopists do (see *fig.* 2):

$$s(\gamma) = (\hbar, A - l\hbar), \qquad r(\gamma) = (\hbar, A). \tag{2.2}$$

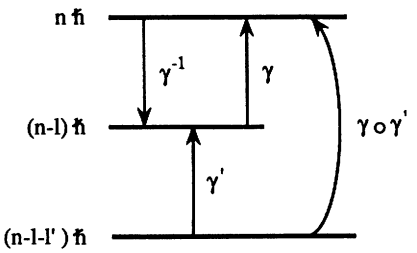

FIG. 2. – As in spectroscopy, the elements of the quantum phase space can be represented by transition arrows. Here the inverse and the composition of arrows are shown, as well as the range and the source of each of them.

The set Γ inherits the topology of $\mathbf{R}^{N+1} \times \mathbf{Z}^N$ and becomes also a locally compact Hausdorff space. Using this transition language one sees that each arrow γ admits a unique inverse γ^{-1} obtained by reversing the source and the range namely:

$$(\hbar, A, l)^{-1} = (\hbar, A - l\hbar, -l). \tag{2.3}$$

At last two arrows γ_1 and γ_2 may be composed provided the source of γ_1 matches the range of γ_2 to give rise to a new one namely:

$$\gamma_1 = (\hbar, A, l_1), \qquad \gamma_2 = (\hbar, A - l_1\hbar, l_2) \Rightarrow \gamma_1 \circ \gamma_2 = (\hbar, A, l_1 + l_2). \tag{2.4}$$

It is easy to check that $\gamma_1 \circ \gamma_2$ belongs indeed to Γ. We will denote by $\Gamma^{(2)}$ the set of pairs (γ_1, γ_2) of composable arrows; it is a closed subset of $\Gamma \times \Gamma$.

Endowed with such a structure Γ is a locally compact "groupoid" [74] namely we get:

PROPOSITION 2.1. – (i) *The mappings r and s from* Γ *into* $\Gamma^{(0)}$ *are continuous.*

(ii) *The mapping* $\gamma \in \Gamma \to \gamma^{-1} \in \Gamma$ *is continuous.*

(iii) *The mapping* $(\gamma_1, \gamma_2) \in \Gamma^{(2)} \to \gamma_1 \circ \gamma_2 \in \Gamma$ *is continuous.*

Endowed with the source and range maps, the inverse and the composition laws, Γ *becomes a locally compact groupoid.* □

Since the proof is essentially elementary we will skip it. We refer the reader to [74] for the definition and properties of groupoids. □

DEFINITION 2.2. – The groupoid Γ will be called the action-angle quantum phase space. □

3. SEMICLASSICAL OBSERVABLE ALGEBRA

The quantum phase space we have defined in the previous section will represent a non commutative generalization of the classical phase space $\mathbf{R}_+^N \times \mathbf{T}^N$ if we use the action-angle variables (here $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ is the torus). A classical observable is usually defined as a continuous function on the classical phase space. In order to avoid inessential difficulties, we will restrict ourself to the space $\mathscr{C}_0(\mathbf{R}_+^N \times \mathbf{T}^N)$ of continuous functions vanishing at infinity endowed with the uniform topology. It turns out that the non commutative analog of such a function space may be defined on a groupoid, the so-called "C*-Algebra" of a locally compact groupoid. In order to make it clear we will describe its construction here. The reader who wants to know more, is referred to [74].

We first consider the space $C_{\mathscr{K}}(\Gamma)$ of continuous functions on Γ with compact support. In what follows if $x = (\hbar, A)$ belongs to $\Gamma^{(0)}$ we will denote by $\Gamma(x)$ the set of arrows with range x. Then a product and a "*" can be defined on $C_{\mathscr{K}}(\Gamma)$ by:

$$\begin{aligned} f_1 f_2(\gamma) &= \sum_{\gamma';\, r(\gamma)=r(\gamma')} f_1(\gamma') f_2(\gamma'^{-1} \circ \gamma), \qquad f_1, f_2 \in C_{\mathscr{K}}(\Gamma) \\ f^*(\gamma) &= \overline{f(\gamma^{-1})}, \qquad f \in C_{\mathscr{K}}(\Gamma). \end{aligned} \tag{3.1}$$

Since these functions have compact support the sum in (3.1) is well defined for the fiber $\Gamma(x)$ of any state is discrete, and only finitely many terms in the sum will contribute. If we use the previous notations we see that (3.1) can be written as:

$$\begin{aligned} f_1 f_2(\hbar, A, l) &= \sum_{l' \in \mathbf{Z}} f_1(\hbar, A, l') f_2(\hbar, A - l'\hbar, l - l'), \quad (\hbar, A, l) \in \Gamma \\ f^*(\hbar, A, l) &= \overline{f(\hbar, A - l\hbar, -l)}, \qquad (\hbar, A, l) \in \Gamma. \end{aligned} \tag{3.2}$$

namely we recover the formulae of theorem 1.1.

We want to introduce now a topology in order to control properly the classical limit. On the other hand algebraic computations will be needed leading to control the convergence of series expansions. At the present stage of the theory we also want to introduce a topology as natural as possible, that is, a construction using no more than the algebraic structure described above.

A standard way consists in defining a C*-norm for it satisfies the fundamental property:

$$\| f \| = \| ff^* \|^{1/2} = \{\text{spectral radius of } ff^*\}^{1/2} \tag{3.3}$$

which gives the norm a purely algebraic origin.

The explicit construction of such norms goes through the choice of one or of a family of representations in Hilbert spaces. In the groupoid case, there is a canonical choice of such representation associated to fibers. For $x \in \Gamma^{(0)}$, $\mathscr{H}_x$ is the Hilbert space $l^2(\Gamma(x))$ of square summable sequences indexed by the fiber. We define the representation π_x acting on $\mathscr{H}_x$ by:

$$f \in C_{\mathscr{K}}(\Gamma), \quad \psi \in \mathscr{H}_x \quad \Rightarrow \quad \pi_x(f)\psi(\gamma) = \sum_{\gamma' \in \Gamma(x)} f(\gamma^{-1} \circ \gamma')\psi(\gamma'). \tag{3.4}$$

A C*-seminorm is then defined as follows:

$$\| f \| = \sup_{x \in \Gamma^{(0)}} \| \pi_x(f) \|. \tag{3.5}$$

One can actually check that this is a norm on $C_{\mathscr{K}}(\Gamma)$ for if $\| f \| = 0$ the matrix elements $f(\gamma^{-1} \circ \gamma')$ vanish for any pair (γ, γ') in $\Gamma(x)$ for every $x \in \Gamma^{(0)}$ and therefore $f = 0$.

We define $C^*(\Gamma)$ as the completion of the algebra $C_{\mathscr{K}}(\Gamma)$ under the previous norm. Clearly any of the representations π_x extends to $C^*(\Gamma)$. Thus the subset $\mathscr{I}_\hbar$ of $C^*(\Gamma)$ of elements such that $\pi_{(\hbar, A)}(f) = 0$ for any possible A's is a closed two-sided ideal. We will denote by $\mathscr{A}_\hbar$ the quotient algebra $\mathscr{A}_\hbar = C^*(\Gamma)/\mathscr{I}_\hbar$ and by $\rho_\hbar$ the canonical projection from $C^*(\Gamma)$ onto $\mathscr{A}_\hbar$. We get the following result:

THEOREM 3.1. – (i) *If $\hbar \neq 0$, the algebra $\mathscr{A}_\hbar$ is isomorphic to the algebra $\mathscr{K}$ of compact operators on the Hilbert space $l^2(\mathbf{N}^N)$.*

(ii) *If $\hbar = 0$, the algebra $\mathscr{A}_0$ is isomorphic to the algebra $\mathscr{C}_0(\mathbf{R}_+^N \times \mathbf{T}^N)$ of continuous functions vanishing at infinity on the classical phase space $\mathbf{R}_+^N \times \mathbf{T}^N$.*

(iii) *If $f = f^* \in C^*(\Gamma)$ and if $\hbar_0 \in [0, 1]$ the spectrum of $\rho_\hbar(f)$ in $\mathscr{A}_\hbar$ is continuous around $\hbar = \hbar_0$ in the following sense: the gap boundaries of the spectrum of $\rho_\hbar(f)$ are continuous functions of $\hbar$ at $\hbar_0$. The eigenvalues of $\rho_\hbar(f)$ at $\hbar \neq 0$ are continuous in $\hbar$ and they accumulate at $\hbar = 0$ on the spectrum of $\rho_0(f)$ which is an interval.*

(iv) *In particular if* $f \in C^*(\Gamma)$ *the norm* $\| \rho_\hbar(f) \|$ *is continuous with respect to* $\hbar$ *and the field of* C*-*algebra* $\{\mathscr{A}_\hbar ; 0 \leqq \hbar \leqq 1\}$ *is continuous* [41]. □

COROLLARY 3.2. – *If* f *belongs to* $C^*(\Gamma)$ *then* f *defines a continuous function vanishing at infinity on* Γ, *and moreover:*

$$\sup_{\gamma \in \Gamma} |f(\gamma)| \leqq \|f\|, \qquad \sup_{(\hbar, A) \in \Gamma^{(0)}} \sum_{l \in \mathbf{Z}^N} |f(\hbar, A, l)|^2 \leqq \|f\|^2. \quad \square$$

Proof of theorem 3.3. – (i) To reinterpret the formula (3.4) in our original notation, let us consider first the case for which $x = (\hbar, n\hbar)$ with $\hbar \neq 0$. Then $\Gamma(x)$ is the set of transition arrows $\gamma = (\hbar, n\hbar, l)$ with $l \in \mathbf{Z}^N$ and $n - l \in \mathbf{N}^N$. The mapping $\gamma \in \Gamma(x) \to n - l \in \mathbf{N}^N$ is a bijection which allows us to identify $\Gamma(x)$ with $\mathbf{N}^N$. Hence:

$$\hbar \neq 0 \quad \Rightarrow \quad \mathscr{H}_x \approx l^2(\mathbf{N}^N) \tag{3.6}$$

giving for the representation:

$$\pi_{\hbar, n\hbar}(f)\psi(m) = \sum_{m' \in \mathbf{N}^N} f(\hbar, m\hbar, m - m')\psi(m'). \tag{3.7}$$

This formula shows that π_x depends only on $\hbar$ and not upon the specific choice of x along the fiber. For short we will denote it by $\pi_\hbar$ giving operators on $l^2(\mathbf{N}^N)$ the matrix elements of which being:

$$\langle m | \pi_\hbar(f) | m' \rangle = f(\hbar, m\hbar, m - m'). \tag{3.8}$$

We therefore recover the formula (1.3) in theorem 1.1. On the other hand, since for $\hbar \neq 0$ all these representations are equivalent, we see immediately that the algebra $\mathscr{A}_\hbar$ is by definition isomorphic to the norm closure of $\pi_\hbar(C_\mathscr{K}(\Gamma))$. The first statement of theorem 3.1 will be a consequence of the following remark:

LEMMA 3.3. – *If* $f \in C_\mathscr{K}(\Gamma)$ *and* $fi \neq 0$ *the operator* $\pi_\hbar(f)$ *acting on* $l^2(\mathbf{N}^N)$ has a finite rank. □

Proof of lemma 3.3. – Since f has compact support, there is $R > 0$ such that if $|m\hbar|_\infty \geqq R$ or if $|m - m'|_1 \geqq R$ the matrix element $f(\hbar, m\hbar, m - m')$ vanishes. Therefore the corresponding matrix has only a finite number of non zero elements. □

Proof of theorem 3.1 (*continued*). – Since the norm closure of the space of finite rank operators is the C*-algebra $\mathscr{K}$ of compact operators on $l^2(\mathbf{N}^N)$ the algebra $\pi_\hbar(C_\mathscr{K}(\Gamma))$ is included in $\mathscr{K}$. On the other hand, let $B(m, m')$ be a finite rank matrix. Let us built a continuous function with compact support f on Γ such that $\pi_\hbar(f)$ coincides with B. In order to do so, let φ be a continuous function on [0, 1] with $\varphi(\hbar) = 1$, $0 \leqq \varphi(s) \leqq 1$ and vanishing if $|s - \hbar| \geqq \hbar/2(M+1)$ where M is the max $\{|m|_1, |m'|_1 ; B(m, m') \neq 0\}$. Then $f(\hbar', m\hbar', 1) = \varphi(\hbar') B(m, m - 1)$ will do it. Therefore any finite rank operator belongs to $\pi_\hbar(C_\mathscr{K}(\Gamma))$ which proves (i).

(ii) Let us now consider the case $\hbar = 0$. Then the fiber of $x = (0, A)$ is given by the set of transition arrows $\gamma = (0, A, l)$ with $l \in \mathbf{Z}^N$. Thus the projection map $\gamma = (0, A, l) \in \Gamma(x) \to l \in \mathbf{Z}^N$ is a bijection which allows us to identify $\Gamma(x)$ with $\mathbf{Z}^N$. Hence:

$$\hbar = 0 \quad \Rightarrow \quad \mathscr{H}_x \approx l^2(\mathbf{Z}^N) \tag{3.9}$$

giving for the representation:

$$\pi_{0,A}(f)\psi(m) = \sum_{m' \in \mathbf{Z}^N} f(0, A, m - m')\psi(m'). \tag{3.10}$$

The Fourier transform permits to identify $l^2(\mathbf{Z}^N)$ with $L^2(\mathbf{T}^N)$ through the formula:

$$\mathscr{F}\psi(\theta) = \sum_{l \in \mathbf{Z}^N} \psi(l) e^{-il\theta}. \tag{3.11}$$

Then $\pi_{(0,A)}(f)$ becomes the operator of multiplication by $\theta \to f_{cl}(A, \theta)$ on $L^2(\mathbf{T}^N)$ where:

$$f_{cl}(A, \theta) = \sum_{l \in \mathbf{Z}^N} f(0, A, l) e^{-il\theta}. \tag{3.12}$$

In particular we recover the situation of theorem 1.1 (1.7). Moreover the norm of $\pi_{(0,A)}(f)$ is

$$\| \pi_{(0,A)}(f) \| = \sup_{\theta \in \mathbf{T}^N} | f_{cl}(A, \theta) |. \tag{3.13}$$

Since f has compact support, the function $f_{cl}(A, \theta)$ is continuous with respect to A and is a trigonometric polynomial in the angle variables.

Now, by definition, the algebra $\mathscr{A}_0$ is the completion of $\rho_0[C_{\mathscr{K}}(\Gamma)]$ under the semi-norm:

$$\| f \|_0 = \| \rho_0(f) \| = \sup_A \| \pi_{(0,A)}(f) \| = \sup_{A,\theta} | f_{cl}(A, \theta) |. \tag{3.14}$$

Therefore $\mathscr{A}_0$ is included into $\mathscr{C}_0(\mathbf{R}_+^N \times \mathbf{T}^N)$. Conversely, any function in $\mathscr{C}_0(\mathbf{R}_+^N \times \mathbf{T}^N)$ can be uniformly approximated by a continuous function with compact support with respect to A which is at the same time a trigonometric polynomial in the angle. Let therefore F be such a function, and let $\underline{F}(A, l)$ be its Fourier sequence according to formula (3.12). Let also χ be the characteristic function of $\mathbf{R}_+^N$. Then the function $f(\hbar, A, l) = \underline{F}(A, l)\chi(A - l\hbar)$ is continuous with compact support on Γ and by construction f_{cl} coincides with F. Therefore $C_{\mathscr{K}}(\Gamma)$ is dense in $\mathscr{C}_0(\mathbf{R}_+^N \times \mathbf{T}^N)$ in the norm $\| \cdot \|_0$ proving (ii).

(iii) & (iv) In order to prove the remaining part of the theorem, let us introduce the following operator on $C_{\mathscr{K}}(\Gamma)$:

$$\eta_\theta(f)(\hbar, A, l) = f(\hbar, A, l) e^{il\theta}. \tag{3.15}$$

One can check immediately that it is a *-homomorphism. Moreover if $\mathcal{N}=(\mathcal{N}_1, \ldots, \mathcal{N}_N)$ is the operator of multiplication by n on $l^2(\mathbf{N}^N)$, η_θ is implemented in the representation $\pi_\hbar$ ($\hbar \neq 0$) by the unitary operator $e^{i\mathcal{N}\theta}$, showing that $\|\pi_\hbar(\eta_\theta(f))\| = \|\pi_\hbar(f)\|$ whenever $\hbar \neq 0$. For $\hbar = 0$, η_θ acts on the Fourier transform as translation of the angles. Thus again the norm is conserved. All together, η_θ is isometric and extends to the full C*-algebra as a *-homomorphism. In much the same way it is easy to see that $\eta_\theta \circ \eta_{\theta'} = \eta_{\theta+\theta'}$ and therefore we get a group of automorphisms. By a 3ε argument one can easily show that η_θ is norm pointwise continuous with respect to θ.

Let now F_k be a sequence of non negative trigonometric polynomials on the torus $\mathbf{T}^N$ with non negative Fourier coefficients such that (the Fejer kernel will do it [60]):

$$\int_{\mathbf{T}^N} \frac{d^N\theta}{(2\pi)^N} F_k(\theta) = 1, \qquad \lim_{k\to\infty} \int_{|\theta| \geqq \varepsilon} \frac{d^N\theta}{(2\pi)^N} F_k(\theta) = 0, \quad \forall \varepsilon > 0.$$

We define:

$$f_k = \int_{\mathbf{T}^N} d^N\theta/(2\pi)^N F_k(\theta)\, \eta_\theta(f) \tag{3.16}$$

and thanks to the definition of F_k, $\lim_{k\to\infty} \|f - f_k\| = 0$. Moreover, if f belongs to $C_{\mathcal{K}}(\Gamma)$, $f_k(\hbar, A, 1) = f(\hbar, A, l)\, \mathcal{F}F_k(1)$ if $\mathcal{F}F_k$ is the Fourier series of F_k. Therefore f_k has a compact support with respect to 1 which depends only upon F_k.

Let now g_j be a continuous function with compact support on $\mathbf{R}_+^N$ such that $0 \leqq g_j(A) \leqq 1$, and $g_j(A) = 1$ for $|A|_\infty \leqq j$. We identify g_j with the kernel $g_j(\hbar, A, l) = g_j(A)\,\delta_{l,0}$. Then again a 3ε argument shows that if f belongs to $C^*(\Gamma)$ $\lim_{j\to\infty} \|f - g_j f\| = 0$. All together, $g_j f_k$ belongs to $C_{\mathcal{K}}(\Gamma)$ and approximate f in the norm of $C^*(\Gamma)$.

We need now to prove the following lemma due to G. Elliott [46].

LEMMA 3.4 (G. Elliott). – *If $f \in C^*(\Gamma)$ is such that $\rho_\hbar(f) = 0$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that if $|\hbar - \hbar'| \leqq \delta$ then $\|\rho_\hbar(f)\| \leqq \varepsilon$.* □

Proof of lemma 3.4. – Using the approximation $g_j f_k$ for f in the C*-algebra and a 3ε argument, we can restrict ourself to the case for which f belongs to $C_{\mathcal{K}}(\Gamma)$. Indeed by (3.16) $\rho_\hbar(f_k) = 0$, and therefore we get $\rho_\hbar(g_j f_k) = \rho_\hbar(g_j)\,\rho_\hbar(f_k) = 0$. Now if f belongs to $C_{\mathcal{K}}(\Gamma)$ the norm of $\rho_\hbar(f_k)$ is bounded from above by $\sup_A \Sigma_l |f(\hbar', A, l)|$ which is a continuous function of $\hbar'$ vanishing whenever $\hbar' = \hbar$ proving the result. □

Proof of theorem 3.1 (*end*). – The end of the proof is now standard: let $f = f^*$ belong to $C^*(\Gamma)$ and let J be a closed interval contained in the

resolvent set of $\rho_\hbar(f)$. We denote by G a continuous function on $\mathbf{R}$ such that $0 \leqq G \leqq 1$, vanishing on the spectrum of $\rho_\hbar(f)$ and taking on the value 1 on J. Thus $\rho_\hbar(G(f))=0$ and applying the lemma 3.3 there is $\delta > 0$ such that if $|\hbar-\hbar'| \leqq \delta$ the norm of $\rho_{\hbar'}(G(f))$ is less than 1/2. For such $\hbar'$, J cannot cut the spectrum of $\rho_{\hbar'}(G(f))$.

Let now O be an open interval cutting the spectrum of $\rho_\hbar(f)$. Let $\hbar_k$ be a sequence converging to $\hbar$ and such that O does not cut the spectrum of $\rho_{\hbar_k}(f)$. Let us assume first that $\hbar \neq 0$. We know that $\pi_{\hbar'}(f)$ acts on the same Hilbert space as long as $\hbar'$ is not zero. Moreover, from (3.8) we see that $\pi_{\hbar'}(f)$ is strongly continuous in $\hbar'$ whenever f has compact support, and by a 3ε argument this is true for f in the C*-algebra. Therefore since the spectrum cannot increase by strong limit the spectrum of $\pi_\hbar(f)$ which coincides with the spectrum of $\rho_\hbar(f)$ cannot cut O, a contradiction. If now $\hbar=0$, let us represent $\pi_{\hbar'}(f)$ from $\hbar' > 0$ in the space $l^2(\mathbf{Z}^N)$ by mean of:

$$\left.\begin{aligned} \langle l | \pi_{\hbar', n\hbar'}(f) | l' \rangle = f(\hbar', (n-l)\hbar', l'-l) \qquad & \text{if} \quad l, l' \leqq n \\ \langle l | \pi_{\hbar', n\hbar'}(f) | l' \rangle = 0 \qquad & \text{otherwise} \end{aligned}\right\} \tag{3.17}$$

This is a representation unitarily equivalent to $\pi_{\hbar'}$. Choosing a sequence n_k in $\mathbf{N}^N$ converging to ∞ such that $n_k\hbar_k$ converges to A, and denoting by π_k the corresponding representation, we see that $\pi_k(f)$ converges strongly to $\pi_{(0, A)}(f)$. Since A can be chosen arbitrarily the same conclusion applies. Therefore in any case, for $\hbar'$ close enough to $\hbar$ the spectrum of $\rho_{\hbar'}(f)$ must cut O.

From these two properties, we conclude easily that the gap boundaries are continuous in $\hbar$. In particular, since for $\hbar \neq 0$ $\rho_\hbar(f)$ is a compact self adjoint operator its spectrum is discrete and each eigenvalue varies continuously with respect to $\hbar$. As $\hbar=0$, $\rho_0(f)$ can be identified with the function $f_{cl}(A, \theta)$ and its spectrum is nothing but the image of f_{cl} namely a closed interval. Therefore the eigenvalues of $\rho_\hbar(f)$ accumulate on that interval as $\hbar \to 0$.

If now f is arbitrary, we have $\|\rho_\hbar(f)\| = \|\rho_\hbar(ff^*)\|^{1/2}$ and therefore the continuity of the norm follows from the continuity of the upper boundary of the spectrum of $\rho_\hbar(ff^*)$. This is just what we need for the definition of a continuous field of C*-algebra (*see* [44]). □

Proof of the Corollary 3.2. – The proposition follows from the estimate. But by (3.4) if $x \in \Gamma^{(0)}$ and $\gamma, \gamma' \in \Gamma(x)$ we get $|f(\gamma'^{-1} \circ \gamma)| \leqq \|\pi_x(f)\| \leqq \|f\|$ leading to the first formula and also to the other one if we replace f by ff^*. □

4. CALCULUS I: AVERAGING AND INTEGRATION

As we already explained the groupoid Γ can be seen as the non commutative analog of the classical phase space. We may wonder whether it is possible to extend to this space the usual rules of Calculus. In this section we intend to explain how to do it concerning two important rules of Classical Mechanics namely the "averaging over angles" and the Liouville density measure.

Averaging is used to provide approximations in presence of adiabatic invariants. The Liouville density measure is an invariant by any hamiltonian motion and is used in statistical mechanics and transport theory. In our language we will see that averaging is nothing but restricting a matrix to its diagonal part, whereas its integral with respect to Liouville's density is nothing but its trace properly normalized as $\hbar \to 0$ in order to recover the usual Lebesgue integration at the classical limit.

In Classical Mechanics, averaging consist in integrating over the angles. Since the quantum phase space is expressed in term of the dual variables "l" we see that it consists in restricting the Fourier transform of a function f at $l=0$. In the quantum case the same definition will be accepted to give:

$$f \in C_{\mathscr{H}}(\Gamma) \quad \Rightarrow \quad \langle f \rangle(\hbar, A, l) = f(\hbar, A, 0)\,\delta_{l,0}. \tag{4.1}$$

Clearly from Cor. 3.2, we get $|\langle f \rangle| \leqq \|f\|$ so that the average extends to any function in $C^*(\Gamma)$. Thanks to formulae 3.7 & 3.10 we see that in the canonical representations $\pi_\hbar$ and $\pi_{(0,A)}$ the average of f is represented by diagonal matrices. In particular let f belong to $C^*(\Gamma)$; diagonalizing it means nothing but finding a *-automorphism β of $C^*(\Gamma)$ and some element f' in $C^*(\Gamma)$ such that $\beta(f) = \langle f' \rangle$. Thus the diagonalization procedure has a purely algebraic interpretation, independent of the specific choice of a representation or of a basis of vectors in this representation. The main property of the average is the following:

$$f \in C^*(\Gamma) \quad \Rightarrow \quad \ll f \gg = \langle f \rangle. \tag{4.2}$$

In the language of C*-algebra the average provides a conditional expectation onto the abelian subalgebra of diagonal elements of $C^*(\Gamma)$ which is nothing but the algebra $\mathscr{C}_0(\Gamma^{(0)})$ of continuous functions vanishing at infinity over the state space $\Gamma^{(0)}$. By analogy we will define the integral of $f \in C_{\mathscr{H}}(\Gamma)$ as follows:

$$f \in C_{\mathscr{H}}(\Gamma), \quad \hbar \neq 0 \quad \Rightarrow \quad \tau_\hbar(f) = \sum_{n \in \mathbf{N}^N} \hbar^N f(\hbar, n\hbar, 0)$$

$$\hbar = 0 \quad \Rightarrow \quad \tau_0(f) = \int_{\mathbf{R}^N} d^N A\, f(0, A, 0) = \int_{\mathbf{R}^N \times \mathbf{T}^N} \frac{d^N A\, d^N \theta}{(2\pi)^N} f_{cl}(A, \theta). \tag{4.3}$$

The main properties of this definition are summarized below:

PROPOSITION 4.1. – (i) *If* $\hbar \neq 0$ $\tau_\hbar$ *is a normalized trace on compact operators namely:*

$$\tau_\hbar(f) = \hbar^N \operatorname{Tr}(\pi_\hbar(f)). \tag{4.4}$$

(ii) *If* $f \in C_\mathscr{K}(\Gamma)$ *the map* $\hbar \in [0, 1] \to \tau_\hbar(f) \in \mathbf{C}$ *is continuous.* □

Proof. – (i) formula (4.4) is obvious in view of the definitions (3.7) & (4.3).

(ii) Since f has a compact support there is $R > 0$ such that whenever $|A|_\infty \geqq R$, $f(\hbar, A, 0) = 0$. In addition, f is continuous. Therefore if $\hbar \neq 0$, the formula (4.3) shows that indeed $\tau_\hbar(f)$ is continuous around $\hbar$. If $\hbar = 0$, the continuity simply means that the discrete sum in (4.3) converges to the integral, a simple property of Riemann's integral. □

PROPOSITION 4.2. – *Let* $\tau(f)$ *denote the map* $\hbar \in [0, 1] \to \tau(f) \in \mathbf{C}$. *If* $f_1, f_2 \in C_\mathscr{K}(\Gamma)$ *then:*

(i) $|\tau(f_1 f_2)| \leqq \tau(|f_1|) \, \|f_2\|$

(ii) $\tau(f_1 f_2) = \tau(f_2 f_1)$

(iii) $\tau(ff^*) \geqq 0$ and $\tau(ff^*) = 0 \Rightarrow f = 0$

(iv) $\tau(\langle f_1 \rangle f_2) = \tau(\langle f_1 \rangle \langle f_2 \rangle)$. □

Since these properties are standard we will leave the proof to the reader.

DEFINITION 4.3. – For $p \geqq 1$, $L^p(\Gamma)$ will denote the completion of $C_\mathscr{K}(\Gamma)$ under the norm

$$\|f\|_{\infty, p} = \sup_{\hbar \in [0, 1]} \tau_\hbar(|f|^p)^{1/p}. \quad \square$$

Remark. – Thanks to the proposition 4.2 (i) this definition makes sense even if p is not an even integer.

5. CALCULUS II: DIFFERENTIAL STRUCTURE

In much the same way we will extend to the quantum phase space the differential structure of the classical one. The easy part consists in defining the derivative with respect to the angles. The difficult part will be to define properly the derivative with respect to actions.

Since differentiating with respect to angles is easily expressed in term of Fourier series in the classical case we are entitled to propose the following generalization in the quantum case:

PROPOSITION 5.1. – *If* $f \in C_\mathscr{K}(\Gamma)$ *the derivative* $\partial f / \partial \theta_\mu$ *is defined by:*

$$\partial f / \partial \theta_\mu(\hbar, A, l) = -i l_\mu f(\hbar, A, l), \qquad \mu = 1, \ldots, N. \tag{5.1}$$

Then they extend as family of commuting unbounded derivations on $C^*(\Gamma)$ *generating a norm pointwise continuous automorphism group.* □

Proof. – The formula (3.15) gives the definition of a norm pointwise continuous automorphism group the generators of which being precisely the partial derivatives defined in (5.1) proving the proposition. However it is interesting to see in detail what makes $\partial/\partial\theta_\mu$ a derivation. Let f, g be two continuous functions with compact support on Γ. Using the definition of the product [eq. (3.2)] we get:

$$\frac{\partial(fg)}{\partial\theta_\mu} = \sum_{l' \in \mathbf{Z}^N} -\mathbf{i}\, l_\mu f(\hbar, A, l') g(\hbar, A, l-l').$$

If we decompose l_μ into $l'_\mu + (l_\mu - l'_\mu)$ we obtain immediately:

$$\partial(fg)/\partial\theta_\mu = \partial f/\partial\theta_\mu . g + f . \partial g/\partial\theta_\mu$$

which is the definition for a derivation. □

It is quite trickier to define the derivative with respect to action variables. The reason comes from the fact that the quantum phase space is discrete at $\hbar \neq 0$. To understand our construction let us consider a simple example namely the definition of a $\mathscr{C}^1$ function on $\mathbf{R}$: we will characterize them from the data of the C*-algebra $\mathscr{C}_0(\mathbf{R})$ only. Let $\mathscr{E}$ be the Banach space $\mathscr{C}_0(\mathbf{R}\times\mathbf{R})$ considered as a bimodule over $\mathscr{C}_0(\mathbf{R})$ as follows:

$$F \in \mathscr{E}, f, g \in \mathscr{C}_0(\mathbf{R}) \quad \Rightarrow \quad f F g(x, x') = f(x) F(x, x') g(x'). \tag{5.2}$$

We will denote by $\| F \|_{\mathscr{E}}$ the sup norm of F in $\mathscr{E}$ and by $\| f \|$ the sup-norm of f in $\mathscr{C}_0(\mathbf{R})$. Let f belong to $\mathscr{C}_0(\mathbf{R})$, we introduce its differential ∂f as a function of two variables namely:

$$\partial f(x, x') = \{f(x) - f(x')\}/\{x - x'\} \quad \text{whenever } x \neq x'. \tag{5.3}$$

It is easy to check that f belongs to the space $\mathscr{C}^1_0(\mathbf{R})$ of continuously differentiable function vanishing together with its derivative at infinity if and only if ∂f coincides on $x \neq x'$ with an element of $\mathscr{E}$. In this case the corresponding element of $\mathscr{E}$ will also be denoted by ∂f and it satisfies the following properties:

(i) if $f, g \in \mathscr{C}^1_0(\mathbf{R})$ then $\partial(fg) = \partial f g + f\, \partial g$.

(ii) $\| \partial f \|_{\mathscr{E}} = \| f' \|$ if f' denotes the derivative of f.

It is interesting to see why (i) holds for indeed:

$$\begin{aligned}\partial(fg)(x, x') &= \{f(x)g(x) - f(x')g(x')\}/\{x - x'\} \\ &= \{f(x) - f(x')\}/\{x - x'\}\, g(x') + f(x)\{g(x) - g(x')\}/\{x - x'\} \\ &= \partial f(x, x') g(x') + f(x)\, \partial g(x, x').\end{aligned}$$

These two properties allows us to define $\mathscr{C}^1_0(\mathbf{R})$ as the subspace of elements of $\mathscr{C}_0(\mathbf{R})$ for which there exists $F \in \mathscr{E}$ such that $F = \partial f$ and the norm will

be:

$$\| f \|_{C^1} = \| f \| + \mathrm{Inf}\{ \| F \|; F = \partial f \}. \tag{5.4}$$

In this approach $\mathbf{R}$ can be replaced by any locally compact subset of $\mathbf{R}$.

We will define partial derivatives on a locally compact subset of $\mathbf{R}^N$ through:

$$\partial_\mu f(x, y) = \{ f(x_1, \ldots, x_{\mu-1}, x_\mu, y_{\mu+1}, \ldots, y_N) - f(x_1, \ldots, x_{\mu-1}, y_\mu, y_{\mu+1}, \ldots, y_N) \} / \{ x_\mu - y_\mu \} \tag{5.5}$$

whenever $x_\mu - y_\mu \neq 0$. This definition is not convenient from the previous point of view for the right hand side does not depend upon $(y_1, \ldots, y_{\mu-1}, x_{\mu+1}, \ldots, x_N)$ and cannot converge to zero at infinity in $\mathbf{R}^{2N}$. However as we will see we will apply it to a case for which the distance between x and y will be controlled in such a way that the formula may be useful.

For each $\mu = 1, \ldots, N$ we define the groupoid $\Gamma \times_\mu \Gamma$ as the set of families $(\hbar, A, B, l)$ such that $0 \leqq \hbar \leqq 1$, $(\hbar, A), (\hbar, B) \in \Gamma^{(0)}$ with $A_\nu = B_\nu$ if $\nu \neq \mu$, and $l \in \mathbf{Z}^N$. The groupoid structure is given by the following rules:

(i) the range of $(\hbar, A, B, l)$ is $(\hbar, A, B)$, its source being $(\hbar, A - l\hbar, B - l\hbar)$.

(ii) the inverse of $(\hbar, A, B, l)$ is $(\hbar, A - l\hbar, B - l\hbar, l)$.

(iii) the composition of $(\hbar, A, B, l)$ and $(\hbar, A - l\hbar, B - l\hbar, l')$ is $(\hbar, A, B, l + l')$.

There are two canonical projections p_1 and p_2 from $\Gamma \times_\mu \Gamma$ onto Γ:

(iv) $p_1(\hbar, A, B, l) = (\hbar, A, l)$ and $p_2(\hbar, A, B, l) = (\hbar, B, l)$. They are obviously continuous groupoid homomorphisms.

The space $C_{\mathscr{K}}(\Gamma \times_\mu \Gamma)$ of continuous functions with compact support on $\Gamma \times_\mu \Gamma$ becomes a $C_{\mathscr{K}}(\Gamma)$ bimodule by mean of the product rule $f \mathrm{F} g = f \circ p_1 \, \mathrm{F} g \circ p_2$. It is indeed easy to check that if f, g belong to $C_{\mathscr{K}}(\Gamma)$ and if F belongs to $C_{\mathscr{K}}(\Gamma \times_\mu \Gamma)$ then $f \mathrm{F} g$ belongs to $C_{\mathscr{K}}(\Gamma \times_\mu \Gamma)$. In much the same way as in section 3 we may define the C*-algebra of $\Gamma \times_\mu \Gamma$ together with its norm.

The derivative with respect to the action will be defined by:

$$\frac{\partial f}{\partial A_\mu}(\hbar, A, B, l) = \frac{f(\hbar, A, l) - f(\hbar, B, l)}{A_\mu - B_\mu} \quad \text{if} \quad A_\mu - B_\mu \neq 0. \tag{5.6}$$

This formula defines a continuous function away from the diagonal $A = B$ of $\Gamma \times_\mu \Gamma$ and vanishing at infinity. The space of such functions is also a $C_{\mathscr{K}}(\Gamma)$ bimodule and (5.6) defines a *-derivation for:

$$\frac{\partial (fg)}{\partial A_\mu} = \frac{\partial f}{\partial A_\mu} g + f \frac{\partial g}{\partial A_\mu}, \qquad \frac{\partial f^*}{\partial A_\mu} = \left\{ \frac{\partial f}{\partial A_\mu} \right\}^*. \tag{5.7}$$

We now remark that if $\partial f/\partial A_\mu$ extends as continuous function on $\Gamma \times_\mu \Gamma$ the extension need not be unique since $\Gamma \times_\mu \Gamma$ is discrete for $\hbar \neq 0$. Nevertheless we propose the following definition [remark that any element of $C_{\mathscr{K}}(\Gamma)$ has a partial derivative with respect to angles in $C_{\mathscr{K}}(\Gamma)$]:

DEFINITION 5.2. – A function f in $C_{\mathscr{K}}(\Gamma)$ will be called differentiable with respect to A_μ whenever there is an element F of $C^*(\Gamma \times_\mu \Gamma)$ which coincides with $\partial f/\partial A_\mu$ away from the diagonal of $\Gamma \times_\mu \Gamma$. It will be called differentiable whenever it is differentiable with respect to A_μ for every μ.

The set of differentiable elements of $C_{\mathscr{K}}(\Gamma)$ will be denoted by $C^1_{\mathscr{K}}(\Gamma)$. $C^1(\Gamma)$ will denote the completion of $C^1_{\mathscr{K}}(\Gamma)$ under the norm:

$$\| f \|_{\mathscr{C}^1} = \| f \| + \sum_{1 \leq \mu \leq N} \{ \| \partial f/\partial \theta_\mu \| + \mathrm{Inf}(\| F \|;\ F = \partial f/\partial A_\mu) \}. \quad \square \quad (5.9)$$

PROPOSITION 5.3. – $C^1(\Gamma)$ *is a normed *-algebra and in particular if* f *and* g *belong to* $C^1(\Gamma)$:

$$\| fg \|_{\mathscr{C}^1} \leq \| f \|_{\mathscr{C}^1} \| g \|_{\mathscr{C}^1}, \qquad \| f^* \|_{\mathscr{C}^1} = \| f \|_{\mathscr{C}^1}. \quad \square \quad (5.10)$$

Since this inequality is essentially obvious we will leave it to the reader.

In Classical Mechanics an extremely important fact concerns the existence of the symplectic structure: any local change of coordinate leaving the symplectic form invariant is called a canonical transformation; it transforms the Hamilton equations into similar equations in the Hamilton form. The symplectic structure can be seen on classical observables through the existence of Poisson's brackets. One of the main contributions of Heisenberg was to provide a correspondence principle between Poisson's brackets in Classical Mechanics and commutators in Quantum Mechanics. Actually the commutator must be divided by $\mathbf{i}\hbar$ if one wants this correspondence principle to be correct at the semiclassical limit.

Accordingly we define the Poisson bracket of two elements f and g of $C^1_{\mathscr{K}}(\Gamma)$ as:

$$\left.\begin{aligned} \hbar \neq 0 \quad &\Rightarrow \quad \{f, g\}(\hbar, A, l) = (fg - gf)(\hbar, A, l)/\mathbf{i}\hbar \\ \hbar = 0 \quad &\Rightarrow \quad \{f, g\}_{cl} = \{f_{cl}, g_{cl}\}. \end{aligned}\right\} \quad (5.11)$$

The main result of this section is given in:

PROPOSITION 5.4. – *If* f *and* g *belong to* $C^1_{\mathscr{K}}(\Gamma)$ *their Poisson bracket* $\{f, g\}$ *defines an element of* $C_{\mathscr{K}}(\Gamma)$ *and satisfies:*

(i) *it is bilinear and continuous for the topologies of* $C_{\mathscr{K}}(\Gamma)$ *and of* $C^1_{\mathscr{K}}(\Gamma)$.

(ii) *it is antisymmetric:* $\{f, g\} = -\{g, f\}$.

(iii) *it satisfies the Jacobi identity:*

$$\{\{f, f'\}, f''\} + \{\{f', f''\}, f\} + \{\{f'', f\}, f'\} = 0.$$

(iv) *it is *-invariant:* $\{f, g\}^* = \{f^*, g^*\}$. □

Proof. – The only non elementary statement in this proposition is the very first one, namely that the $\{f, g\}$ is a continuous function with compact support and in particular that the limit $\hbar \to 0$ matches with the definition (5.11).

To see this let f and g be continuous with compact support on Γ and let $\hbar \neq 0$. Then we can write the Poisson bracket as:

$$\{f, g\}(\hbar, \mathrm{A}, l) = \sum_{l' \in \mathbf{Z}^N} \frac{f(\hbar, \mathrm{A}, l') - f(\hbar, \mathrm{A} - (l-l')\hbar, l')}{\mathrm{i}\hbar} g(\hbar, \mathrm{A}, l-l') - \underset{l' \leftrightarrow l-l'}{f \leftrightarrow g} \quad (5.12)$$

where the double arrow $\leftrightarrow$ is a short-hand notation indicating that the same term with f and g exchanged must be substracted. Using the identity:

$$\begin{aligned} f(\hbar, \mathrm{A}, l) - f(\hbar, \mathrm{B}, l) &= \sum_{1 \leqq \mu \leqq N} \{f(\hbar, \mathrm{A}_1, \ldots, \mathrm{A}_\mu, \mathrm{B}_{\mu+1}, \ldots, \mathrm{B}_N, l) \\ &\quad - f(\hbar, \mathrm{A}_1, \ldots, \mathrm{A}_{\mu-1}, \mathrm{B}_\mu, \ldots, \mathrm{B}_N, l)\} \\ &= \sum_{1 \leqq N} \partial f / \partial \mathrm{A}_\mu (\hbar, \mathrm{A}_1, \ldots, \mathrm{A}_{\mu-1}, \mathrm{A}_\mu, \mathrm{B}_\mu, \mathrm{B}_{\mu+1}, \ldots, \mathrm{B}_N, l) \{\mathrm{A}_\mu - \mathrm{B}_\mu\} \end{aligned}$$

we get:

$$\{f, g\}(\hbar, \mathrm{A}, l) = \sum_{\mu, l'} \frac{\partial f}{\partial \mathrm{A}_\mu} (\hbar, \mathrm{A}_1, \ldots, \mathrm{A}_\mu, \mathrm{A}_\mu - (l-l')_\mu \hbar, \ldots, \mathrm{A}_N - (l-l')_N \hbar, l') \frac{\partial g}{\partial \theta_\mu} (\hbar, \mathrm{A}, l-l') - \underset{l' \leftrightarrow l-l'}{f \leftrightarrow g}.$$

We remark that this expression makes sense for the terms with $l_\mu = l'_\mu$ vanishes anyway. Since both f and g are differentiable, one can extend them by continuity at $\hbar = 0$ and by definition we get the usual derivatives with respect to A. The previous expression is then nothing but the Fourier transform of the usual Poisson bracket of the f_{cl} and g_{cl} according to (5.11). Since both f and g have compact supports there is $\mathrm{R} > 0$ such that whenever $|l'|_1 \geqq \mathrm{R}$ and $|l-l'|_1 \geqq \mathrm{R}$ the corresponding term in the sum vanishes. Thus if $|l|_1 \geqq 2\mathrm{R}$ the left hand side vanishes too. Moreover R can be chosen such that whenever $|\mathrm{A}|_\infty \geqq \mathrm{R}$, $f(\hbar, \mathrm{A}, l)$ and $g(\hbar, \mathrm{A}, l)$ both vanish. Hence whenever $|\mathrm{A}|_\infty \geqq 2\mathrm{R}$ the left hand side also vanishes. Therefore $\{f, g\}$ is continuous with compact support. □

Open Problems:

(i) Can we define the Poisson bracket of two elements of $C^1(\Gamma)$? If yes is the Poisson bracket continuous in norm, namely do we have: $\| \{f, g\} \| \leqq C \| f \|_{\mathscr{C}^1} \| g \|_{\mathscr{C}^1}$ for some $C > 0$?

(ii) We define the Liouville operator L_f associated to f in $C^1_{\mathscr{A}}(\Gamma)$ as $L_f(g) = \{f, g\}$. From Jacobi's identity this is a derivation. If $f = f^*$ this is a *-derivation. What are the necessary and sufficient conditions on $f = f^*$ to make sure that L_f generates an automorphism group ([27], [69])?

6. HOLOMORPHIC FUNCTIONS ON QUANTUM PHASE SPACE

The differential structure defined in the previous section is quite difficult to use in practice, for the differentiability with respect to action variable is not very easy to perform. On the other hand even in Classical Mechanics many technical results such as the Nekhoroshev theorem or the Kolmogorov-Arnold-Moser one, require the use of holomorphic functions on the complexified phase space, at least as an intermediate step. We intend to define in this section a family of dense subalgebras of $C^*(\Gamma)$ generalizing such holomorphic functions. This will provide us with an extremely useful technical tool which will give results similar to those obtained in Classical Mechanics. However the interpretation of these results will be highly non trivial since they will give controls of the semiclassical limit.

Let (η, R, r) be a family of three positive numbers. $D(\eta, R)$ will denote the open subset of the complex space $\mathbf{C}^{N+1}$ defined by (see *fig.* 3):

(i) $|\mathrm{Arg}(\hbar)| < \eta$, $0 < |\hbar| < 1$.

(ii) $|\mathrm{Im}\{e^{-i\,\mathrm{Arg}(\hbar)} A\}|_\infty < R$.

We denote by $\mathscr{D}(\eta, R)$ the space of holomorphic functions on $D(\eta, R)$ continuous, bounded, converging to zero at infinity on the closure of $D(\eta, R)$, equipped with the uniform topology given by the norm:

$$\| f \|_{\eta, R} = \sup_{(\hbar, A) \in D(\eta, R)} | f(\hbar, A) |. \tag{6.1}$$

By construction $D(\eta, R)$ is invariant by the translations $(\hbar, A) \to (\hbar, A - l\hbar)$ ($l \in \mathbf{Z}^N$). We can therefore define a complexified groupoid $\Gamma_{\mathbf{C}}$ the elements of which being the set of $(\hbar, A, l)$ with $(\hbar, A)$ in $D(\eta, R)$ and l in $\mathbf{Z}^N$ with the rules similar to (2.2) & (2.3). $\tilde{\mathscr{A}}(\eta, R, r)$ will denote the space of holomorphic functions f on $\Gamma_{\mathbf{C}}$ continuous on the closure of $\Gamma_{\mathbf{C}}$ such that

$$\| f \|_{\eta, R, r}^{\sim} = \sup_{(\hbar, A) \in D(\eta, R)} \sum_{l \in \mathbf{Z}^N} | f(\hbar, A, l) | e^{r|l|_1} < \infty. \tag{6.2}$$

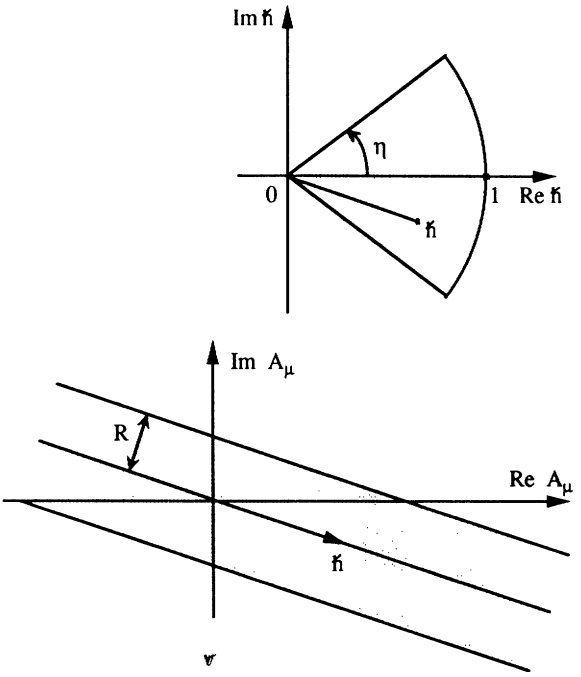

FIG. 3. – The Domain D(η, R).

With the product defined as in (3.2) $\mathscr{A}^{\sim}(\eta, R, r)$ becomes an algebra such that:

PROPOSITION 6.1. – *$\mathscr{A}^{\sim}(\eta, R, r)$ equipped with the norm defined by* (6.2), *is a Banach algebra and:*

$$\| fg \|^{\sim}_{\eta, R, r} \leqq \| f \|^{\sim}_{\eta, R, r} \| g \|^{\sim}_{\eta, R, r}. \quad \square \tag{6.3}$$

Proof. – Let f belong to $\mathscr{A}^{\sim}(\eta, R, r)$. For any $l \in \mathbf{Z}^N$ the function $(\hbar, A) \to f(\hbar, A, l)$ belongs to $\mathscr{H}(\eta, R)$. By Montel's theorem this is a Banach space. Thus any limit of Cauchy sequences in the algebra $\mathscr{A}^{\sim}(\eta, R, r)$ will define a holomorphic function on the complexified groupoid, bounded and continuous on the closure. Moreover the limit will decay in l in such a way that the norm $\| f \|^{\sim}_{\eta, R, r}$ of the limit be finite. Hence $\mathscr{A}^{\sim}(\eta, R, r)$ is a Banach space. Let us now prove (6.3). We get:

$$\| fg \|^{\sim}_{\eta, R, r} \leqq \sup_{D(\eta, R)} \sum_{l, l' \in \mathbf{Z}^N} | f(\hbar, A, l') | | g(\hbar, A - l'\hbar, l - l') | e^{r(|l'|_1 + |l - l'|_1)}$$

for $|l|_1 \leqq |l'|_1 + |l-l'|_1$. Now since $D(\eta, R)$ is translation invariant we have:

$$\sum_{l \in \mathbf{Z}^N} |g(\hbar, A - l'\hbar, l-l')| e^{r|l-l'|_1} \leqq \|g\|^{\sim}_{\eta, R, r}$$

and the estimate can be achieved easily. □

As in section 1 we introduced a cocycle θ as the unique function on the groupoid Γ_C such that:

$$\begin{aligned} &\text{(i)} \quad \gamma \in \Gamma_C \Rightarrow \theta(\gamma) \in \mathbf{C}. \\ &\text{(ii)} \quad \gamma, \gamma' \in \Gamma_C \Rightarrow \theta(\gamma\gamma') = \theta(\gamma)\theta(\gamma'). \\ &\text{(iii)} \quad x \in \Gamma_C^{(0)} \Rightarrow \theta(x) = 1. \\ &\text{(iv)} \quad \theta(\hbar, A, e_\mu) = A_\mu \text{ if } e_\mu = (\delta_{\mu,\nu})_{1 \leqq \nu \leqq N}. \end{aligned} \tag{6.4}$$

Then one gets:

$$\theta(\hbar, A, l) = \prod_{1 \leqq \mu \leqq N} \theta^{(1)}(\hbar, A_\mu, l_\mu)$$

with

$$\theta^{(1)}(\hbar, A_\mu, l_\mu) = A_\mu(A_\mu - \hbar) \ldots (A_\mu - (l_\mu - 1)\hbar) \qquad \text{if} \quad l_\mu > 0 \tag{6.5}$$

and

$$\theta^{(1)}(\hbar, A_\mu, -l_\mu) = \{(A_\mu + \hbar)(A_\mu + 2\hbar) \ldots (A_\mu + l_\mu \hbar)\}^{-1}.$$

In particular θ is a meromorphic function of $D(\eta, R)$.

By analogy with the results of section 1, we define an involution "*" on $\mathscr{A}^{\sim}(\eta, R, r)$ as follows:

$$f^*(\hbar, A, l) = \overline{f(\bar{\hbar}, \bar{A} - \overline{l\hbar}, -l)}\, \theta(\hbar, A, l). \tag{6.6}$$

Because of (6.5) if $f \in \mathscr{A}^{\sim}(\eta, R, r) f^*$ may not belong to $\mathscr{A}^{\sim}(\eta, R, r)$ for if $l_\mu < 0$ $f^*(\hbar, A, l)$ may get a pole at $A_\mu = -k_\mu \hbar$ whenever $1 \leqq k_\mu \leqq l_\mu$. On the other hand it may also be unbounded at infinity. Nevertheless, we get:

$$\begin{aligned} &\text{(i)} \quad \text{the mapping } f \to f^* \text{ is antilinear} \\ &\text{(ii)} \quad \{f^*\}^* = f \\ &\text{(iii)} \quad \{fg\}^* = g^* f^* \end{aligned} \tag{6.7}$$

whenever the right hand side makes sense.

To make sure that this involution is well defined we introduce the subalgebra $\mathscr{A}(\eta, R, r)$ of $\mathscr{A}^{\sim}(\eta, R, r)$ the elements of which having precisely the property that both f and their adjoint f^* belong to $\mathscr{A}^{\sim}(\eta, R, r)$. We endow $\mathscr{A}(\eta, R, r)$ with the norm:

$$\|f\|_{\eta, R, r} = \operatorname{Max}\{\|f\|^{\sim}_{\eta, R, r}, \|f^*\|^{\sim}_{\eta, R, r}\}. \tag{6.8}$$

The main properties of this later algebra are summarized in the following propositions:

PROPOSITION 6.2. – *If f belongs to $\mathscr{A}(\eta, R, r)$ and if $m \in \mathbf{N}^N$, then $f(\hbar, m\hbar, l) = 0$ whenever $\hbar \neq 0$ and $m - l \notin \mathbf{N}^N$.* □

Proof. – We suppose that $m \in \mathbf{N}^N$. Then if $m - l \notin \mathbf{N}^N$ we can find a subset I of [1, N] such that $\mu \in I$ if and only if $l_\mu > m_\mu$. Thus, using (6.5) & (6.6) with $A = m\hbar$ we get:

$$f(\hbar, m\hbar, l) = \overline{f^*(\bar{\hbar}, (m-l)\bar{\hbar}, -l)}\, \theta(\hbar, m\hbar, l).$$

Since f belongs to $\mathscr{A}(\eta, R, r)$, f^* also belongs to $\mathscr{A}(\eta, R, r)$ whereas:

$$\theta(\hbar, m\hbar, l) = \prod_{\nu \notin I} \theta^{(1)}(\hbar, m_\nu \hbar, l_\nu) \prod_{\mu \in I} \{m_\mu (m_\mu - 1) \ldots (m_\mu - l_\mu + 1) \hbar^{l_\mu}\}.$$

Since $l_\mu > m_\mu$ the right hand side vanishes. □

PROPOSITION 6.3. – *If f belongs to $\mathscr{A}(\eta, R, r)$ and if $l \in \mathbf{Z}^N$ is such that whenever μ belongs to the subset I of [1, N], $l_\mu < 0$, then as $|A|_\infty \to \infty$ one gets:*

$$|f(\hbar, A, l)| \leqq \prod_{\mu \in I} \frac{\text{const.}}{(\hbar + |A_\mu|)^{-l_\mu}}. \quad \square \tag{6.9}$$

Proof. – Using (6.5) & (6.6) again the estimate is immediate. □

THEOREM 6.4. – *The algebra $\mathscr{A}(\eta, R, r)$ endowed with the involution defined in* (6.6) *and the norm defined in* (6.8) *is a Banach *-algebra.* □

Proof. – It is enough to show that $\mathscr{A}(\eta, R, r)$ is complete. Let $\{f_n\}_{n \geqq 0}$ be a Cauchy sequence in $\mathscr{A}(\eta, R, r)$. Then both $\{f_n\}_{n \geqq 0}$ and $\{f_n^*\}_{n \geqq 0}$ converge in $\mathscr{A}^\sim(\eta, R, r)$ to limit denoted by f and f^* respectively. In particular we get:

$$f_*(\hbar, A, l) = \lim_{n \to \infty} f_n^*(\hbar, A, l)$$
$$= \lim_{n \to \infty} \overline{f_n(\bar{\hbar}, \bar{A} - \overline{l\hbar}, -l)}\, \theta(\hbar, A, l) = f(\bar{\hbar}, \bar{A} - \overline{l\hbar}, -l)\, \theta(\hbar, A, l)$$

uniformly over Γ_C. In particular $f_* = f^*$ and the sequence converges to f in the norm (6.8). □

THEOREM 6.5. – *If $f \in \mathscr{A}(\eta, R, r)$, $\iota(f)$ is the function over Γ defined by:*

$$\iota(f)(\hbar, A, l) = \{\theta(\hbar, A, l)\}^{-1/2} f(\hbar, A, l) \tag{6.10}$$

*whenever $(\hbar, A, l) \in \Gamma$. Then $\iota(f)$ belongs to $C^*(\Gamma)$ and ι is a *-homomorphism with a dense image. Actually $\iota(f)$ is a C^∞ element in $C^*(\Gamma)$.* □

Proof. – Using (6.4) we get for the product:

$$\iota(fg)(\hbar, m\hbar, l) = \sum_{l' \in \mathbf{Z}^N} f(\hbar, m\hbar, l')\, g(\hbar, (m-l')\hbar, l-l')\, \{\theta(\hbar, m\hbar, l)\}^{-1/2}$$

$$= \sum_{l' \in \mathbf{Z}^N} \frac{f(\hbar, m\hbar, l')}{\sqrt{\theta(\hbar, m\hbar, l')}} \frac{g(\hbar, (m-l)\hbar, l-l')}{\sqrt{\theta(\hbar, (m-l')\hbar, l-l')}}$$

namely:

$$\iota(fg)(\hbar, m\hbar, l) = \sum_{l' \in \mathbf{Z}^N} \iota(f)(\hbar, m\hbar, l')\, \iota(g)(\hbar, (m-l')\hbar, l-l').$$

Using (6.6) we get immediately for $\hbar \neq 0$:

$$\iota(f)(\hbar, m\hbar, l) = \overline{f(\hbar, (m-l)\hbar, -l)}\, \sqrt{\theta(\hbar, m\hbar, l)}.$$

In particular if $m-l$ does not belong to $\mathbf{N}^N$ the right hand side vanishes and thus:

$$\iota(fg) = \iota(f)\, \iota(g).$$

The very same argument holds whenever $\hbar = 0$.

On the other hand using (6.4) again:

$$\iota(f^*)(\hbar, A, l) = \frac{f^*(\hbar, A, l)}{\sqrt{\theta(\hbar, A, l)}} = \overline{f(\hbar, A-l\hbar, -l)}\, \sqrt{\theta(\hbar, A, l)}$$

$$= \frac{\overline{f(\hbar, A-l\hbar, -l)}}{\sqrt{\theta(\hbar, A-l\hbar, -l)}} = \iota(f)^*(\hbar, A, l)$$

showing that

$$\iota(f^*) = \iota(f)^*.$$

Now we have to show that we obtain elements of $C^*(\Gamma)$. If we suppose first $\hbar \neq 0$ and if $\pi_\hbar(\iota(f)) = F$ we have:

$$\langle m | F | m' \rangle = \frac{f(\hbar, m\hbar, m-m')}{\sqrt{\theta(\hbar, m\hbar, m-m')}}$$

$$= \overline{f^*(\hbar, m'\hbar, m'-m)}\, \sqrt{\theta(\hbar, m\hbar, m-m')}.$$

Therefore:

$$|\langle m | F | m' \rangle| \leq \|f\|_{\eta, R, r}\, e^{-r|m-m'|_\infty}$$

$$\operatorname{Min}\{|\theta(\hbar, m\hbar, m-m')|^{1/2}, |\theta(\hbar, m\hbar, m-m')|^{-1/2}\}$$

and since $\operatorname{Min}\{|x|, |x|^{-1}\} \leq 1$ we get:

$$\|F\| \leq \operatorname{Max}\{\sup_m \sum_{m' \in \mathbf{N}^N} |\langle m | F | m' \rangle|, \sup_{m'} \sum_{m \in \mathbf{N}^N} |\langle m | F | m' \rangle|\}$$

$$\leq \|f\|_{\eta, R, r} \sum_{l \in \mathbf{Z}^N} e^{-r|l|_1} \leq \|f\|_{\eta, R, r} \left\{\frac{1+e^{-r}}{1-e^{-r}}\right\}^N.$$

The very same estimate holds for $\hbar = 0$ showing that:

$$\| \iota(f) \| \leqq \left\{ \frac{1+e^{-r}}{1-e^{-r}} \right\}^{N} \| f \|_{\eta, R, r}. \tag{6.11}$$

At last we must show that $\iota(f)$ may be approximated by elements of $C_{\mathscr{H}}(\Gamma)$. We first truncate $\iota(f)$ with respect to the angle variable by setting:

$$\iota(f)_L(\hbar, A, l) = \iota(f)(\hbar, A, l) \qquad \text{whenever } |l|_1 < L$$
$$= 0 \qquad \text{otherwise.}$$

By an argument similar to (6.11) replacing $\iota(f)$ by $\iota(f)_L$ produces an error:

$$\| \iota(f) - \iota(f)_L \| \leqq \| f \|_{\eta, R, r} \frac{2N e^{-rL}}{(1-e^{-r})^N}.$$

Then we truncate with respect to the action variable by choosing a continuous function with compact support χ on $\mathbf{R}^N$ such that $0 \leqq \chi \leqq 1$, and that $\chi(A) = 1$ whenever $|A|_\infty \leqq M + L\hbar$, $\chi(A) = 0$ whenever $|A|_\infty \geqq M + 1 + L\hbar$, and by setting:

$$\iota(f)_{L,M}(\hbar, A, l) = \chi(A)\, \iota(f)_L(\hbar, A, l)\, \chi(A - l\hbar).$$

Since f belongs to the algebra $\mathscr{A}(\eta, R, r)$ it converges to zero at infinity and for any $\varepsilon > 0$ we can find $M > 0$ such that:

$$\sup_{(\hbar, A) \in D(\eta, R);\ |A|_\infty \geqq M} \operatorname{Max} \Big\{ \sum_{l \in \mathbf{Z}^N} |f(\hbar, A, l)| e^{r|l|_1}, \sum_{l \in \mathbf{Z}^N} |f^*(\hbar, A, l)| e^{r|l|_1} \Big\} \leqq \varepsilon.$$

Thus by a similar argument, replacing $\iota(f)_L$ by $\iota(f)_{L,M}$ produces an error:

$$\| \iota(f)_L - \iota(f)_{L,M} \| \leqq \varepsilon \left\{ \frac{1+e^{-r}}{1-e^{-r}} \right\}^{N}.$$

Clearly $\iota(f)_{L,M}$ belongs to $C_{\mathscr{H}}(\Gamma)$ and it converges to $\iota(f)$ in $C^*(\Gamma)$ as L and M goes to infinity. Since $C_{\mathscr{H}}(\Gamma)$ is dense in the C*-algebra the algebra $\iota(\mathscr{A}(\eta, R, r))$ is dense too.

It remains to show that $\iota(f)$ is C^∞. Clearly, since the kernel of f is exponentially decaying in l it gives rise to a C^∞ function of the angles, namely for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$ in $\mathbf{N}^N$ using the estimate $n^\alpha e^{-\delta n} \leqq \{\alpha / e\delta\}^\alpha$ (if $n \geqq 0$) we get:

$$\left\| \prod_{\mu=1}^{N} \frac{\partial^{\alpha_\mu}}{\partial \theta_\mu^{\alpha_\mu}} f \right\|_{\eta, R, r-\delta} \leqq \prod_{\mu=1}^{N} \left(\frac{\alpha_\mu}{e\delta} \right)^{\alpha_\mu} \| f \|_{\eta, R, r}. \tag{6.12}$$

It is a little bit trickier to get smoothness with respect to the action since we did not define derivatives of higher order in the previous sections.

However, we can proceed in the same way, defining the higher derivative by mean of uniform estimates on finite differences. Now if f belongs to $\mathscr{A}(\eta, R, r)$ the first derivative with respect to the action is given by:

$$\frac{\partial f}{\partial A_\mu}(\hbar, A, A'; l)$$
$$= \frac{1}{A_\mu - A'_\mu}\cdot[f(\hbar, A_1, \ldots, A_{\mu-1}, A_\mu, A'_{\mu+1}, \ldots, A'_N; l)$$
$$- f(\hbar, A_1, \ldots, A_{\mu-1}, A'_\mu, A'_{\mu+1}, \ldots, A'_N; l)]$$
$$= \int_0^1 d\sigma \frac{\partial f}{\partial A_\mu}(\hbar, A_1, \ldots, A_{\mu-1}, \sigma A_\mu + (1-\sigma) A'_\mu, A'_{\mu+1}, \ldots, A'_N; l).$$

Since f is analytic with respect to A the right hand side makes sense and defines an holomorphic function on the complexified groupoid $\Gamma_{\mathbf{C}} \times_\mu \Gamma_{\mathbf{C}}$. This formula can be iterated in an obvious way giving rise to a C^∞ function in A too. □

7. A NON COMMUTATIVE CAUCHY-KOWALESKAYA THEOREM

In the 19th century Cauchy and Kowaleskaya [36] gave sufficient conditions on a vector field F on an open subset of $\mathbf{R}^N$ to define locally and for a short time a solution of the differential equation $dx/dt = F(x)$. The method used at that time consisted in expanding F into a power series around the initial point of the trajectory, and to give $x(t)$ as a power series in time. The first step was to prove the existence of a solution as a formal power series in time. The next step was to prove the convergence of that series.

We would like to extend such a theorem in the non commutative context developed before. The differential equation associated to a hamiltonian vector field will be replaced by a differential equation on the time evolution of observables, namely the Heisenberg equation $\partial f/\partial t = \{w, f\} = \mathscr{L}_w(f)$ where the operator $\mathscr{L}_w$ will be called the Liouville operator associated to w. The existence of a solution at short time will be obtained by simply showing that the power series expansion for the exponential $\exp\{t\mathscr{L}_w\}$ is actually convergent for small t's in one of the algebras $\mathscr{A}(\eta, R, r)$ of holomorphic functions on our non commutative quantum phase space.

Let f and g belong to $\mathscr{A}(\eta, R, r)$. We define their Poisson bracket $\{f, g\}$ as in $C^*(\Gamma)$ namely with a suitable choice of variables:

$$\{f, g\}(\hbar, A, l) = \sum_{l' \in \mathbf{Z}^N} \frac{f(\hbar, A, l')\, g(\hbar, A - l'\hbar, l - l') - f(\hbar, A - (l-l')\hbar, l')\, g(\hbar, A, l-l')}{i\hbar}. \quad (7.1)$$

This expression is well defined on the open set $D(\eta, R)$ since $\hbar \neq 0$ there. However it requires some work to show that it defines an element of $\mathscr{A}(\eta, R', r')$ for a suitable choice of R' and r'. As a first remark we get:

PROPOSITION 7.1. – *If f_i belongs to $\mathscr{A}(\eta, R_i, r_i)$ $(i = 1, 2)$ the Poisson bracket $\{f_1, f_2\}$ satisfies*

$$\{f_1, f_2\}^* = \{f_1^*, f_2^*\}. \quad \square \quad (7.2)$$

Proof. – We have $\{f_1, f_2\}(\hbar, A, l) = (i\hbar)^{-1}[fg - gf](\hbar, A, l)$. On the other hand we get:

$$(i[fg - gf])^* = -i[g^* f^* - f^* g^*] = i[f^* g^* - g^* f^*]$$

which implies immediately the result. $\square$

THEOREM 7.2. – *If f_i belongs to $\mathscr{A}(\eta, R_i, r_i)$ $(i = 1, 2)$ the Poisson bracket $\{f_1, f_2\}$ belongs to $\mathscr{A}(\eta, R, r)$ for any $0 < R < \mathrm{Min}\{R_1, R_2\}$ and $0 < r < \mathrm{Min}\{r_1, r_2\}$ and:*

$$\| \{f_1, f_2\} \|_{\eta, R, r} \leqq \frac{1}{e} \left\{ \frac{1}{(r_1 - r)(R_2 - R)} + \frac{1}{(r_2 - r)(R_1 - R)} \right\} \times \| f_1 \|_{\eta, R_1, r_1} \| f_2 \|_{\eta, R_2, r_2}. \quad \square \quad (7.3)$$

Proof. – In order to get such an estimate, let us write (5.12) in the following way:

$$\begin{aligned} &\{f_1, f_2\}(\hbar, A, l) \\ &= \sum_{l' \in \mathbf{Z}^N} \frac{\{f_1(\hbar, A, l') - f_1(\hbar, A - (l-l')\hbar, l')\}}{i\hbar} f_2(\hbar, A, l - l') - f_1 \leftrightarrow f_2 \\ &= \sum_{\mu=1}^{N} \sum_{l' \in \mathbf{Z}^N} \int_0^1 d\sigma \frac{\partial f_1}{\partial A_\mu}(\hbar, A - \sigma(l-l')\hbar, l') \\ &\qquad \times i(l_\mu - l'_\mu) f_2(\hbar, A, l - l') - f_1 \leftrightarrow f_2. \end{aligned} \quad (7.4)$$

This expression shows that each term in the sum is holomorphic in $D(\eta, R)$, bounded, continuous and vanishing at infinity on the closure of $D(\eta, R)$ provided $0 < R < \mathrm{Min}\{R_1, R_2\}$. Indeed the derivative of a

holomorphic function is given by the Cauchy formula namely in our case:

$$\frac{\partial f_1}{\partial A_\mu}(\hbar, A-\sigma(l-l')\hbar, l') = \int_{|z|=\rho} \frac{dz}{2i\pi z^2} f_1(\hbar, A-\sigma(l-l')\hbar+ze_\mu, l') \quad (7.5)$$

where $e_\mu=(\delta_{\mu\nu})_{1\leqq\nu\leqq N}$ is the μ-th vector in the canonical basis of $\mathbf{C}^N$, and $\rho\leqq R_1-R$. Since the expression in the integral is holomorphic in $D(\eta, R)$, bounded, continuous and vanishing at infinity on the closure of $D(\eta, R)$, so does the integral according to the dominated convergence theorem.

Now we may estimate the norm of each term in the left hand side of (7.4) using (7.5). Namely

$$\sum_{l\in\mathbf{Z}^N} |\{f_1, f_2\}(\hbar, A, l)| e^{r|l|_1} \leqq \int_0^1 d\sigma \int_0^{2\pi} \frac{d\theta}{2\pi\rho} \sum_{l', l''\in\mathbf{Z}^N} \sum_{\mu=1}^{N} |f_1(\hbar, A-\sigma l''\hbar+\rho e^{i\theta}e_\mu, l')| \times e^{r_1|l'|_1} |f_2(\hbar, A, l')| |l''_\mu| e^{r_2|l''|_1} e^{-(r_2-r)|l''|_1} + f_1 \leftrightarrow f_2.$$

The sum over l' is dominated by the norm of f_1 in $\mathscr{A}(\eta, R_1, r_1)$ provided $|e^{-i\,\mathrm{Arg}(\hbar)}\,\mathrm{Im}\,A|_\infty < R$ if $R=R_1-\rho$. In the remainder we use again the estimate $\sup_{x>0} xe^{-\delta x} \leqq (e\delta)^{-1}$ with $\delta=r_2-r$. Thus the sum over μ is dominated by $(e\delta)^{-1}$ whereas the remaining sum on l'' can be estimated by the norm of f_2 in $\mathscr{A}(\eta, R_2, r_2)$. This gives (7.2) after remarking that $\{f_1, f_2\}^*=\{f_1^*, f_2^*\}$. □

THEOREM 7.3. — *Let w belong to $\mathscr{A}(\eta, R_0, r_0)$. Then for any pair (R, r) of positive numbers such that $0<R<R_0$, $0<r<r_0$ and any integer $n\geqq 0$, $\mathscr{L}_w^n$ defines a bounded linear operator from the algebra $\mathscr{A}(\eta, R_0, r_0)$ into the algebra $\mathscr{A}(\eta, R, r)$* and:

$$\left\|\frac{\mathscr{L}_w^n}{n!}\right\| \leqq \left\{\frac{2e}{(R_0-R)(r_0-r)}\|w\|_{\eta, R_0, r_0}\right\}^n. \quad (7.6)$$

(ii) *If $t\in\mathbf{C}$ has a small enough modulus then $\exp(t\mathscr{L}_w)$ defines a bounded linear operator from the algebra $\mathscr{A}(\eta, R_0, r_0)$ into the algebra $\mathscr{A}(\eta, R, r)$ such that:*

$$\left.\begin{array}{l}(a)\ \exp(t\mathscr{L}_w)(fg)=\exp(t\mathscr{L}_w)(f)\exp(t\mathscr{L}_w)(g)\\ (b)\ \exp(t\mathscr{L}_w)(f)^*=\exp(t\mathscr{L}_{w^*})(f^*)\end{array}\right\} \quad (7.7)$$

In addition we get:

$$\|\exp\{t\mathscr{L}_w\}\|_{(R_0, r_0)\to(R, r)} \leqq \frac{1}{1-(2e/(R_0-R)(r_0-r))|t|\,\|w\|_{\eta, R_0, r_0}}. \quad \square \quad (7.8)$$

Proof. – For $k=0, 1, \ldots, n$ we will set $R_k = R_0 - \rho k/n$ where $\rho = R_0 - R$ and $r_k = r_0 - \delta k/n$ where $\delta = r_0 - r$. We will also denote by $\mathscr{A}(k)$ the algebra $\mathscr{A}(\eta, R_k, r_k)$. By the theorem 7.2, one can see $\mathscr{L}_w$ as a linear bounded operator from $\mathscr{A}(k-1)$ into $\mathscr{A}(k)$ with a norm:

$$\| \mathscr{L}_w \|_{(k-1)\to k} \leqq \frac{1}{e} \left\{ \frac{1}{(R_0 - R_k)(r_{k-1} - r_k)} + \frac{1}{(R_{k-1} - R_k)(r_0 - r_k)} \right\} \| w \|_{\eta, R_0, r_0}.$$

Using the definition of R_k and r_k we get:

$$\| \mathscr{L}_w \|_{(k-1)\to k} \leqq \frac{2e}{e\delta\rho} \frac{n^2}{k} \| w \|_{\eta, R_0, r_0}.$$

Since $n^n \leqq n!\, e^n$ we get (7.6) immediately. From (7.6), (7.8) follows by summing all terms in the exponential.

The Jacobi identity [Proposition 5.4 (iii)] implies that $\mathscr{L}_w$ is a derivation namely:

$$\mathscr{L}_w(fg) = \mathscr{L}_w(f)\, g + f\, \mathscr{L}_w(g).$$

By recursion on n we obtain the Leibniz rule, namely:

$$\mathscr{L}_w^n(fg) = \sum_{p=0}^{n} \binom{n}{p} \mathscr{L}_w^p(f)\, \mathscr{L}_w^{n-p}(g).$$

Dividing by $n!$ and summing up all terms with respect to n, we get (7.7 *a*) immediately. We also have $\mathscr{L}_w(f)^* = \mathscr{L}_{w^*}(f^*)$ and by recursion $\mathscr{L}_w^n(f)^* = \mathscr{L}_{w^*}^n(f^*)$ for all n leading to (7.7 *b*). □

THEOREM 7.4. – *Let w belong to $\mathscr{A}(\eta, R_0, r_0)$. If $w = w^*$ then $\exp\{t\mathscr{L}_w\}$ $(t \in \mathbf{R})$ defines a one parameter group of *-automorphisms of $C^*(\Gamma)$ which is norm pointwise continuous.* □

Proof. – Let T be positive and small enough to make sure that for some (R, r) [*see* (7.8)]:

$$\| \exp\{t\mathscr{L}_w\} \|_{(R_0, r_0)\to(R, r)} \leqq 2 \qquad \text{for} \quad |t| \leqq T.$$

Now we consider the canonical imbedding ι of $\mathscr{A}(\eta, R, r)$ into $C^*(\Gamma)$ (*see* section 6). For $\hbar \neq 0$ we set $\iota_\hbar(f) = \pi_\hbar(\iota(f))$ where $\pi_\hbar$ is the harmonic representation (*see* (3.8)). Then we get:

$$\iota_\hbar(\mathscr{L}_w(f)) = [\iota_\hbar(w), \iota_\hbar(f)]/i\hbar = L_w(\iota_\hbar(f)) \qquad \text{for} \quad f \in \mathscr{A}(\eta, R_0, r_0).$$

Thus iterating we get for $|t| \leqq T$:

$$\iota_\hbar(\exp\{t\mathscr{L}_w\}(f)) = \exp\{tL_w\}(\iota_\hbar(f)) = e^{-i t\tau_\hbar(w)/\hbar}\, \iota_\hbar(f)\, e^{i t\tau_\hbar(w)/\hbar}.$$

Since $w = w^*$ the operator $\iota_\hbar(w)$ is self adjoint and thus $e^{i\,t\,\iota_\hbar(w)/\hbar}$ is unitary implying:

$$\| \pi_\hbar(\iota(\exp\{t\,\mathscr{L}_w\}(f))) \| \leqq \| \iota_\hbar(f) \| \leqq \| \iota(f) \|. \tag{7.9}$$

If now we assume $\hbar = 0$, we get a canonical *-homomorphism ρ_0 from $C^*(\Gamma)$ into $\mathscr{A}_0$ which will be identified with $\mathscr{C}_0(\mathbf{R}^N_+ \times \mathbf{T}^N)$ (*see theorem* 3.1). Through the mapping $\rho_0 \circ \iota$ an element f in the algebra $\mathscr{A}(\eta, R_0, r_0)$ is transformed into f_{cl} according to (3.12). Moreover the Poisson bracket $\{w, f\}$ is transformed into the classical Poisson bracket $\{w_{cl}, f_{cl}\}$. Therefore $\rho_0 \circ \iota(\exp\{t\,\mathscr{L}_w\}(f))$ is nothing but the solution of the differential equation:

$$\partial F/\partial t = \{w_{cl}, F(t)\} \qquad \text{with} \quad F(0) = f_{cl} \text{ and } F(t) \in \mathscr{C}_0(\mathbf{R}^N_+ \times \mathbf{T}^N).$$

We already know that $F(t)(A, \theta)$ can be expressed as $f_{cl}(A(t), \theta(t))$ where $(A(t), \theta(t))$ is the solution of the Hamilton equation:

$$d\theta_\mu/dt = \partial w_{cl}/\partial A_\mu, \qquad dA_\mu/dt = -\partial w_{cl}/\partial \theta_\mu \quad \text{with } (A(0), \theta(0)) = (A, \theta).$$

Since $w \in \mathscr{A}(\eta, R_0, r_0)$ it is easy to see that w_{cl} is holomorphic with respect to (A, θ) in the domain $|\operatorname{Im} A|_\infty < R_0$, $|\operatorname{Im} \theta|_\infty \leqq r_0$. Since $w = w^*$, w_{cl} is real on the real phase space. Therefore the Hamilton equations admit a unique solution at short time according to (7.8) which is real for real initial conditions. In particular, the uniform norm of $F(t)$ is bounded according to:

$$\| \rho_0 \circ \iota(\exp\{t\,\mathscr{L}_w\}(f)) \| = \| F(t) \| \leqq \| f_{cl} \| \leqq \| f \|. \tag{7.10}$$

Both inequalities (7.9) and (7.10) imply for $|t| \leqq T$ and $f \in \mathscr{A}(\eta, R_0, r_0)$:

$$\| \iota(\exp\{t\,\mathscr{L}_w\}(f)) \| \leqq \| \iota(f) \|.$$

Applying this inequality to $g = \exp\{-t\,\mathscr{L}_w\}(f)$ we get:

$$\| \iota(\exp\{t\,\mathscr{L}_w\}(f)) \| = \| \iota(f) \|. \tag{7.11}$$

Since $\iota(\mathscr{A}(\eta, R_0, r_0))$ is dense in $C^*(\Gamma)$, one can find a linear isometric mapping α_t on $C^*(\Gamma)$ such that:

$$\iota(\exp\{t\,\mathscr{L}_w\}(f)) = \alpha_t(\iota(f)). \tag{7.12}$$

From theorem 7.3, one gets for $|t| + |s| \leqq T$, $\alpha_{t+s} = \alpha_t \circ \alpha_s$, in particular $\alpha_{-t} = \alpha_t^{-1}$. Moreover since $w = w^*$, $\alpha_t(f)^* = \alpha_t(f^*)$, and α_t is a family of *-automorphisms. From (7.12) and the theorem 7.3, $\| \alpha_t(\iota(f)) - \iota(f) \| \to 0$ as $t \to 0$ whenever $f \in \mathscr{A}(\eta, R_0, r_0)$. By density of $\iota(\mathscr{A}(\eta, R_0, r_0))$, this family is norm pointwise continuous. We can now extend it for all time by a standard procedure, namely if $nT \leqq t \leqq (n+1)T$ $(n \in \mathbf{Z})$, we set $\alpha_t = \alpha_{t-nT} \circ \alpha_T^n$. This is the one parameter group of *-automorphism norm pointwise continuous which extends the action of $\exp\{-t\,\mathscr{L}_w\}$ to all time on $C^*(\Gamma)$. □

8. A CONVERGENT PERTURBATION THEORY FOR ONE DEGREE OF FREEDOM

In the first section we restricted ourselves to the study of polynomials in the creation operator a^* and annihilation operator a. Such operators do not belong to the C*-algebra $C^*(\Gamma)$ since they are unbounded. However many of them can be seen as unbounded derivations on $C^*(\Gamma)$ through the Liouville operator they define formally. We will not enter into this question here but rather restrict ourselves to the following simple case, which turns out to be extremely useful in a wide class of problems.

We will consider a diagonal operator H_0 given through an holomorphic kernel $H_0(\hbar, A)\delta_{l,0}$ on the complexified groupoid Γ_C when the number of degrees of freedom is equal to 1. We assume the following properties:

$$\left.\begin{array}{l}\text{(i) } \mathrm{Re}\{\partial H_0/\partial A(\hbar, A)\} \geqq \Omega \quad \text{for some } \Omega > 0 \text{ if } (\hbar, A) \in D(\eta, R) \\ \text{(ii) } H_0 = H_0^* \quad \text{(reality condition).}\end{array}\right\} \tag{8.1}$$

The simplest example of such kernel is provided by the hamiltonian of a perturbed harmonic oscillator namely:

$$H_0(h, A) = A + h(\hbar, A) \tag{8.2}$$

where the function h is holomorphic on the domain $D(\eta, R)$, bounded and continuous on the closure of $D(\eta, R)$ and having a derivative with respect to A uniformly bounded by $1-\Omega$ for some $0 < \Omega < 1$.

Such a kernel can be seen as defining in each representation an unbounded diagonal operator by mean of:

$$\langle m | \pi_\hbar(H_0) | m' \rangle = H_0(\hbar, m\hbar)\delta_{m,m'} \qquad \text{if} \quad \hbar \neq 0 \tag{8.3a}$$

$$(H_0)_{cl}(A, \theta) = H_0(0, A) \qquad \text{if} \quad \hbar = 0. \tag{8.3b}$$

We now consider an element f of $\mathscr{A}(\eta, R, r)$ and we want to form the kernel:

$$H(\hbar, A, l) = H_0(\hbar, A)\delta_{l,0} + \varepsilon . f(\hbar, A, l) \tag{8.4}$$

where ε is a small parameter which will be called a "coupling constant".

The main result of this section is given by the following theorem:

THEOREM 8.1. — *Let* H_0 *and* f *as above. Let* $0 < \rho < R$, $0 < \delta < r$ *and* $0 < \Omega_\infty < \Omega$. *Assume that* d *satisfies the condition:*

$$d \| f \|_{\eta, R, r} \leqq \mathrm{Min}\left\{ \frac{\Omega_\infty \rho \delta}{128\, e}, \frac{\Omega_\infty (\Omega - \Omega_\infty) \rho^2 \delta}{512\, e}, \frac{\Omega_\infty^2 \rho^2 \delta}{512\, e^2} \right\}.$$

(i) *If* $|\varepsilon| \leqq d$ *there is a sequence* $(w_k)_{k \geqq 1}$ *with* $w_k \in \mathscr{A}(\eta, R - 2^{-(k-1)}\rho, r - 2^{-(k-1)}\delta)$ *and* δH *in* $\mathscr{D}(\eta, R-\rho)$ *depending*

analytically upon ε, *such that:*

$$\prod_{k \geq 1} e^{\{\mathscr{L}_{w_k}\}} (H_0 + \varepsilon . f) = H_0 + \delta H = H_\infty \tag{8.5}$$

with

$$\| w_k \|_{\eta, R - 2^{-(k-1)}\rho, r - 2^{-(k-1)}\delta} = O(\varepsilon^{2^{k-1}}) \quad \textit{and} \quad \| \delta H \|_{\eta, R-\rho} = O(\varepsilon) \textit{ as } \varepsilon \approx 0$$

and

$$\mathrm{Re}\{\partial H_\infty / \partial A(\hbar, A)\} \geqq \Omega_\infty \qquad \text{for} \quad (\hbar, A) \in D(\eta, R - \rho).$$

(ii) *If* $\hbar \neq 0$, *the operator* $\pi_\hbar(H_0) + \varepsilon \pi_\hbar(f)$ *can be diagonalized by means of the unitary operator* $\prod_{k \geq 1} e^{-i \pi_\hbar(w_k)/\hbar}$ *and its eigenvalues are*

$$E_n(\hbar) = H_\infty(\hbar, n\hbar), \qquad n \in \mathbf{N}. \tag{8.6}$$

(iii) *The corresponding eigenfunction* ψ_n *is exponentially localized away from the unpertubed one* $|n\rangle$ *namely:*

$$|\langle n' | \psi_n \rangle - \delta_{n, n'}| \leqq |\varepsilon| / (\Omega_\infty \hbar d) e^{1/(\Omega_\infty \hbar)} e^{-(r-\delta)|n-n'|}. \quad \square \tag{8.7}$$

Proof. – (1) In this first step of the proof we will assume that both H_0 and f are holomorphic functions of ε in the disc $\{\varepsilon \in \mathbf{C};\ |\varepsilon| < d\}$. We will use the Schwarz lemma according to which whenever $g(z)$ is holomorphic and bounded in the disc $\{z \in \mathbf{C};\ |z| < d\}$ and satisfies $g(z) = O(z^N)$ around $z = 0$, we get $|g(z)| \leqq \{|z|/d\}^N M$ where $M = \sup_{|z| < d} |g(z)|$.

The first step of the proof consists in diagonalizing $H_0 + f$ to the first order in f. We will assume that $f = O(\varepsilon^N)$. More precisely we look for an element w in $\mathscr{A}(\eta, R', r')$ for some R', r' such that:

$$\exp\{\mathscr{L}_w\}(H_0 + f) = H_0 + \delta H + O(\| f \|^2_{\eta, R, r})$$

where δH is diagonal. Expanding the exponential to the first order we get the linearized equation:

$$\mathscr{L}_w(H_0) + f = \{w, H_0\} + f = \delta H. \tag{8.8}$$

Since H_0 is diagonal taking the average of both sides gives $\langle f \rangle = \delta H$, namely:

$$\delta H(\hbar, A) = f(\hbar, A, 0). \tag{8.9}$$

On the other hand (8.8) gives for the non diagonal part of w:

$$w(\hbar, A, l)\{H_0(\hbar, A - l\hbar) - H_0(\hbar, A)\}/i\hbar + f(\hbar, A, l) = 0 \qquad \text{if} \quad l \neq 0.$$

Since H_0 is holomorphic with respect to A, the solutions is given by:

$$w(\hbar, A, l) = \frac{i f(\hbar, A, l)}{l \int_0^1 d\sigma \, (\partial H_0 / \partial A)(\hbar, A - \sigma l\hbar)} \qquad \text{if} \quad l \neq 0. \tag{8.10}$$

We will choose the solution such that $w(\hbar, A, 0) = 0$.

Because H_0 is real, the adjoint of w is obtained through the same formula replacing f by f^*. Clearly, thanks to (8.1), the right hand side of (8.10) defines an holomorphic kernel in the domain $(\hbar, A) \in D(\eta, R)$ and $|\varepsilon| < d$. Moreover, writing the derivative of H_0 through a Cauchy formula as in (7.5), we can see that w is continuous on the closure of $D(\eta, R')$ for each $R' < R$ and bounded on $D(\eta, R)$. Using (8.1) and remarking that $|l| \geqq 1$, we get an estimate of the norm of w, namely

$$\| w \|_{\eta, R', r} \leqq \{ |\varepsilon| / d \}^N \| f \|_{\eta, R, r} / \Omega. \tag{8.11}$$

This implies $\exp\{\mathscr{L}_w\}(H_0 + f) = H_0 + \langle f \rangle + R_w(f)$ where $R_w(f)$ is given by:

$$R_w(f) = \sum_{n=1}^{\infty} \frac{\mathscr{L}_w^n}{n!} \left(\frac{nf + \langle f \rangle}{n+1} \right). \tag{8.12}$$

The theorem 7.3 shows that $R_w(f)$ belongs to $\mathscr{A}(\eta, R - \rho, r - \delta)$ for any $0 < \rho < R' < R$ and $0 < \delta < r$ and the estimate (7.6) allows to get:

$$\| R_w(f) \|_{\eta, R-\rho, r-\delta} \leqq \left(\frac{\varepsilon}{d} \right)^{2N} \frac{1}{\Omega} \left(\frac{4e}{\rho\delta} \right) \| f \|^2_{\eta, R, r}$$

provided

$$\frac{2e}{\rho\delta} \| w \|_{\eta, R, r} \leqq \left(\frac{\varepsilon}{d} \right)^N \frac{1}{\Omega} \left(\frac{2e}{\rho\delta} \right) \| f \|_{\eta, R, r} \leqq \frac{1}{2}. \tag{8.13}$$

(2) The next step consists in defining a recursion process starting with the original hamiltonian $H_0 + f_0 = H_0 + \varepsilon . f$. At the k-th step we assume that we have obtained an hamiltonian $H_k + f_k$ with H_k diagonal belonging to $\mathscr{D}(\eta, R_k)$, and f_k belonging to $\mathscr{A}(\eta, R_k, r_k)$ both being analytic with respect to ε in the disc $\{\varepsilon \in \mathbf{C};\ |\varepsilon| < d\}$, with the conditions $R_k < R$, $r_k < r$.

We set $\rho_k = R_{k-1} - R_k$, $\delta_k = r_{k-1} - r_k$, together with $\sup_{|\varepsilon| \leqq d} \| f_k \|_k = \varepsilon_k$ where $\| \cdot \|_k$ denotes the norm of the algebra $\mathscr{A}(\eta, R_k, r_k)$. We will assume the following recursion hypothesis:

for $1 \leqq k \leqq n-1$

(*a*) $\mathrm{Re}\, \partial H_k / \partial A(\hbar, A) \geqq \Omega_k \geqq \Omega_\infty > 0$ on $D(\eta, R_k)$.

(*b*) $H_k = H_k^*$ and $f_k = f_k^*$.

(*c*) $f_k = O(\varepsilon^{N_k})$ as $\varepsilon \approx 0$.

(*d*) $\varepsilon_k \leqq (1/4e)\, \Omega_k \rho_k \delta_k$.

To construct H_n and f_n we proceed as follows:

$$H_n = H_{n-1} + \langle f_{n-1} \rangle. \tag{8.14}$$

Then we construct w_n as the solution of $\{w_n, H_{n-1}\} + f_{n-1} = \langle f_{n-1} \rangle$ with zero diagonal elements as in (8.10). This gives rise to:

$$f_n = R_{w_n}(f_{n-1}).$$

Therefore H_n belongs to $\mathscr{D}(\eta, R_n)$ for any $R_n = R_{n-1} - \rho_n < R_{n-1}$ and f_n belongs to $\mathscr{A}(\eta, R_n, r_n)$ for any $r_n = r_{n-1} - \delta_n < r_{n-1}$ because of (*d*). Moreover, using a Cauchy formula to estimate the norm of the derivative of f_{n-1} we get:

$$\mathrm{Re}\,\partial H_n/\partial A(\hbar, A) \geqq \Omega_{n-1} - (1/\rho_n)\,\|f_{n-1}\|_{n-1} = \Omega_n \quad \text{on } D(\eta, R_n). \tag{8.15}$$

From the recursion hypothesis (*b*) and (8.10) and (8.12) it follows that H_n, w_n and f_n are self adjoint too.

From (8.13) we get $f_n = O(\varepsilon^{N_n})$ as $\varepsilon \approx 0$, with $N_n \equiv 2N_{n-1}$. Since $N_0 = 1$ we conclude that $N_n = 2^n$.

Using (8.13) and the recursion hypothesis (*a*) again we also get:

$$\varepsilon_n = \sup_{|\varepsilon| < d} \|f_n\|_n \leqq \frac{1}{\Omega_\infty}\left(\frac{4e}{\rho_n \delta_n}\right)\varepsilon_{n-1}^2. \tag{8.16}$$

If we raise both sides of (8.16) to the power 2^{-n}, we get after iteration:

$$\varepsilon_n^{2^{-n}} \leqq \frac{4e}{\Omega_\infty} \prod_{k=1}^{n} \left(\frac{1}{\rho_k \delta_k}\right)^{2^{-k}} \varepsilon_0. \tag{8.17}$$

It turns out that the infimum of the right hand side over the sequences (ρ_n, δ_n) of non negative numbers such that $\sum_{1 \leqq n \leqq \infty} \rho_n = \rho$ and $\sum_{1 \leqq n \leqq \infty} \delta_n = \delta$ is reached for $\rho_n = \rho\, 2^{-n}$ and $\delta_n = \delta\, 2^{-n}$. In this case we have:

$$\varepsilon_n \leqq \left\{\frac{2^6 e}{\Omega_\infty \rho\delta}\varepsilon_0\right\}^{2^n} \text{ which converges to zero if } \varepsilon_0 < \frac{\Omega_\infty \rho\delta}{64e}.$$

With this choice the condition (*a*) will be satisfied at each step provided $\sum_{n \geqq 1} 2^n \varepsilon_{n-1} \leqq \rho(\Omega - \Omega_\infty)$. Imposing the condition:

$$\varepsilon_0 \leqq \Omega_\infty \rho\delta/128\, e \tag{8.18}$$

a sufficient condition to satisfy (*a*) is therefore:

$$\varepsilon_0 \leqq \Omega_\infty (\Omega - \Omega_\infty)\, \rho^2 \delta/512\, e. \tag{8.19}$$

At last we have to check that (*d*) holds at each step. With our previous choice it is satisfied if:

$$\left\{\frac{64e}{\Omega_\infty \rho\delta}\varepsilon_0\right\}^{2^n} \leqq \frac{\Omega_\infty \rho\delta}{4^{n+1} e} \quad \text{for all } n \geqq 0.$$

A sufficient condition for this condition to hold is then given by:

$$\varepsilon_0 \leqq \mathrm{Min}\left\{\frac{\Omega_\infty \rho\delta}{128\, e}, \frac{\Omega_\infty^2 \rho^2 \delta}{512\, e^2}\right\}. \tag{8.20}$$

Now we have $\varepsilon_0 = d\,\|f\|_{\eta, R, r}$, therefore gluing together (8.18), (8.19) and (8.20) we get the convergence of the recursion process provided there are

$0 < \rho < R$, $0 < \delta < r$ and $0 < \Omega_\infty < \Omega$ for which:

$$d \| f \|_{\eta, R, r} \leqq \text{Min} \left\{ \frac{\Omega_\infty \rho \delta}{128 e}, \frac{\Omega_\infty (\Omega - \Omega_\infty) \rho^2 \delta}{512 e}, \frac{\Omega_\infty^2 \rho^2 \delta}{512 e^2} \right\}. \quad (8.21)$$

Then:

(1) H_n converges to $H_\infty = H_0 + \sum_{n \geqq 0} \langle f_n \rangle = H_0 + \delta H$ in $\mathscr{D}(\eta, R - \rho)$.

(2) $\text{Re} \{ \partial H_\infty / \partial A (\hbar, A) \} \geqq \Omega_\infty$ for $(\hbar, A) \in D(\eta, R - \rho)$.

(3) $\prod_{n \geqq 1} e^{\{\mathscr{L}_{w_n}\}}$ converges as a bounded *-homomorphism from $\mathscr{A}(\eta, R, r)$ into $\mathscr{A}(\eta, R - \rho, r - \delta)$ transforming $H_0 + \varepsilon . f$ into H_∞.

(4) the previous convergences hold uniformly with respect to ε in $\{\varepsilon \in \mathbf{C}; |\varepsilon| < d\}$. In particular both H_∞ and $\prod_{n \geqq 1} e^{\{\mathscr{L}_{w_n}\}}$ are analytic with respect to ε in the disc $\{\varepsilon \in \mathbf{C}; |\varepsilon| < d\}$.

If we represent these kernel by means of the harmonic representation, $\pi_\hbar(H_\infty)$ is diagonal and the corresponding eigenvalues are given by (8.6). On the other hand for each $k \geqq 1$, $e^{\{\mathscr{L}_{w_k}\}}$ is implemented by the unitary operator $e^{-i \pi_\hbar (w_k)/\hbar}$. Thus the eigenfunction of $H_0 + \varepsilon . f$ corresponding to the eigenvalue $H_\infty(\hbar, n\hbar)$ $(n \in \mathbf{N})$ is given by $\psi_n = \prod_{k \geqq 1} e^{-i \pi_\hbar (w_k)/\hbar} | n \rangle$. Thus we get:

$$|\langle n' | \psi_n \rangle - \delta_{n, n'}| = |\langle n' | \prod_{k \geqq 1} e^{-i \pi_\hbar (w_k)/\hbar} - \mathbf{1} | n \rangle|$$
$$\leqq \sum_{k \geqq 1} |\langle n' | \prod_{1 \leqq j < k} e^{-i \pi_\hbar (w_j)/\hbar} \{ e^{-i \pi_\hbar (w_k)/\hbar} - \mathbf{1} \} | n \rangle|.$$

We introduce on the set of matrices $\mathbf{A} = (A_{n, n'})$ the norm:

$$\| \mathbf{A} \|_r = \text{Max} \{ \sup_n \sum_{n'} |A_{n, n'}| e^{r|n - n'|}, \sup_{n'} \sum_n |A_{n, n'}| e^{r|n - n'|} \}.$$

The operator norm $\| \mathbf{A} \|$ is dominated by $\| \mathbf{A} \|_r$ and $\| \cdot \|_r$ is a *-algebraic norm. Moreover we get $\| \pi_\hbar (w) \|_r \leqq \| w \|_{\eta, R, r}$ if $w \in \mathscr{A}(\eta, R, r)$. Thus we get, using $e^x - 1 \leqq x e^x$ for $x \geqq 0$:

$$|\langle n' | \psi_n \rangle - \delta_{n, n'}| \leqq \sum_{k \geqq 1} \prod_{1 \leqq j \leqq k} \exp [\| w_j \|_{j-1} / |\hbar|] \| w_k \|_{k-1} / |\hbar| e^{-(r - \delta)|n - n'|}$$
$$\leqq \exp [\sum_{j \geqq 1} \| w_j \|_{j-1} / |\hbar|] \{ \sum_{k \geqq 1} \| w_k \|_{k-1} / |\hbar| \} e^{-(r - \delta)|n - n'|}.$$

Recursively, (8.11) and (8.18) give the following estimate:

$$\| w_k \|_{k-1} \leqq \frac{1}{\Omega_\infty} \left(\frac{|\varepsilon|}{d} \right)^{2^{k-1}} \| f_{k-1} \|_{k-1}$$
$$\leqq \frac{|\varepsilon|}{\Omega_\infty d} \left(\frac{1}{2} \right)^{2^{k-1}} \leqq \frac{|\varepsilon|}{\Omega_\infty d} \left(\frac{1}{2} \right)^k \quad \text{for} \quad k \geqq 1$$

which implies for $|\varepsilon| \leqq d$:

$$|\langle n' | \psi_n \rangle - \delta_{n,n'}| \leqq |\varepsilon|/(\Omega_\infty \hbar d)\, e^{1/(\Omega_\infty \hbar)}\, e^{-(r-\delta)|n-n'|}. \quad \square \quad (8.22)$$

9. BIRKHOFF'S EXPANSION AND RAYLEIGH-SCHRÖDINGER PERTURBATION SERIES

The first important question addressed by physicists in the field of semi classical analysis is to understand the relationship between the classical and the quantum perturbation expansions. In classical hamiltonian mechanics, investigating the motion around a fixpoint (or a periodic orbit), Birkhoff [22] proved the existence of canonical changes of coordinate in phase space transforming the original hamiltonian into a completely integrable one to arbitrary order in the distance to the fixpoint. This perturbation series was already commonly used since the nineteenth century by the astronomers, and the question of its convergence has been the cornestone of further studies on the existence of instabilities. We know nowadays that for sure such an expansion cannot converge unless the system is completely integrable, because of the existence of resonant regions in the phase space leading to some chaotic motion.

On the contrary the corresponding expansion in quantum mechanics, called the Rayleigh-Schrödinger perturbation series, does converge for each individual eigenvalue. It has been guessed, and proved in some cases that it converges order by order to the Birkhoff expansion as Planck's constant goes to zero [51]. This remark has been widely used among physicists and theoretical chemists to get accurate estimates of eigenvalues for complicate systems as nuclei or molecules, when the methods involving partial differential operators fail to produce reliable results. The most remarkable fact about this method is the amazing accuracy of such a method, even if Planck's constant is not very small in the natural units of the problem.

In this section we intend to give a proof of the previous guess in the framework of the non commutative approach of the semiclassical limit. More precisely we will consider a system with N degrees of freedom (N is an arbitrary integer) which is a perturbation of a non resonant family of coupled harmonic oscillators. Such a system arises locally in phase space around each fixpoint of any hamiltonian system in regions called sometimes "island of stability". Preliminary calculations show that it is actually possible to represent the quantum and the classical motion as well around such a point by mean of such a perturbed harmonic oscillator. For this reason we are entitled to restrict ourselves to such a case without loss of generality. We will prove that at any order in the perturbation parameter

ε it is possible to find a canonical transformation of the quantum phase space which diagonalizes the hamiltonian at this order. We will give a precise estimate of the remainder uniformly in the Planck constant in the next section: we get an error term of the form $O(\exp(-\text{cont. } \varepsilon^{-1/2N}))$ when we stop the expansion at the optimal order, a result similar to the Nekhoroshev estimate in classical mechanics ([66], [67]).

Such an estimate explains why this method can be so successful in practice. Indeed if the parameter ε is chosen so small to produce a remainder negligeable with respect to Planck's constant $\hbar$, say it is of order $\hbar^\beta$ for some $\beta > 1$, we are entitled to expect the expansion to give accurate results. Using the Nekhoroshev-type estimate, this leads to a value $\varepsilon = O(|\mathrm{Ln}(\hbar)|^{-2N})$ which may be macroscopic even if $\hbar$ is extremely small!

Beyond that value of the perturbation parameter, the quantum system starts seeing the classical chaotic regions. However, it is possible to localize the quantum system outside these resonant regions in order to produce better estimates of the K.A.M. type [49]. The net result of this procedure will be to replace the parameter ε by an effective one of order $\varepsilon^2, \varepsilon^4, \varepsilon^8, \varepsilon^{16}, \ldots$ The process will be stopped after n steps, namely whenever this effective parameter is $O(|\mathrm{L}n(\hbar)|^{-2N})$, that is to say when $\varepsilon = O(|\mathrm{L}n(\hbar)|^{-N/2^{n-1}})$, an even bigger value!

Let us start by describing the system we want to study here. The unperturbed one is given by the unbounded hamiltonian of a family of harmonic oscillators. In action-angle variables it is given by the kernel:

$$\mathrm{H}_0(\hbar, \mathrm{A}, l) = \mathrm{A} \cdot \omega \delta_{l,0} \tag{9.1}$$

where $\mathrm{A} \cdot \omega = \mathrm{A}_1 \omega_1 + \mathrm{A}_2 \omega_2 + \ldots + \mathrm{A}_N \omega_N$. We will restrict ourselves to the "non resonant case" namely we will assume that there exists $\sigma > N-1$ and $\Omega > 0$ such that:

$$|\omega \cdot l| \geqq \Omega / |l|_1^\sigma \quad \text{for all } l \in \mathbf{Z}^N \setminus \{0\}. \tag{9.2}$$

We perturb it by an element f of the algebra $\mathscr{A}(\eta, \mathrm{R}, r)$ namely we consider the hamiltonian:

$$\mathrm{H}(\hbar, \mathrm{A}, l) = \mathrm{A} \cdot \omega \delta_{l,0} + \varepsilon . f(\hbar, \mathrm{A}, l). \tag{9.3}$$

Our first result is summarized as follows:

THEOREM 9.1. — *Let* $\mathrm{R} > \mathrm{R}_1 > \ldots > \mathrm{R}_L > 0$ *and* $r > r_1 > \ldots > r_L > 0$ *be given. Then for each* $n \in [1, L]$ *there is* $w_n \in \mathscr{A}(\eta, \mathrm{R}_n, r_n)$ *such that if* $w_{(L)}(\varepsilon) = \varepsilon . w_1 + \ldots + \varepsilon^L . w_L$ *then for* ε *small enough:*

$$e^{\mathscr{L}_{w_{(L)}(\varepsilon)}} \{\mathrm{H}_0 + \varepsilon . f\} = \mathrm{H}_0 + \sum_{n=1}^{L} \varepsilon^n \mathrm{H}_n + \mathscr{R}_L(f) \tag{9.4}$$

where $H_n = \langle H_n \rangle \in \mathscr{D}(\eta, R_n)$ (cf. (6.1)), $\mathscr{R}_L(f) \in \mathscr{A}(\eta, R_L, r_L)$ *and denoting by* $\| \cdot \|_L$ *the norm of* $\mathscr{A}(\eta, R_L, r_L)$, $\| \mathscr{R}_L(f) \|_L = O(\varepsilon^{L+1})$ as $\varepsilon \approx 0$. □

Proof. – We proceed as in classical mechanics, namely we suppose that (9.4) holds, we expand both sides in powers of ε and we identify them term by term. It will give a recursion formula permiting to compute the w_n's. In order to shorten the notations we denote by $\mathscr{L}_n$ the Liouville operator associated to w_n and by $\mathscr{L}(\varepsilon)$ the one associated to $w_{(L)}(\varepsilon)$. We obtain:

$$e^{\mathscr{L}(\varepsilon)} = \mathbf{1} + \sum_{n=1}^{L} \varepsilon^n \sum_{k=1}^{n} \frac{1}{k!} \sum_{n_i \geq 1;\, n_1 + \ldots + n_k = n} \mathscr{L}_{n_1} \ldots \mathscr{L}_{n_k} + O(\varepsilon^{L+1}). \quad (9.5)$$

At the first order (9.4) reads:

$$\mathscr{L}_1(H_0) + f = \{w_1, H_0\} + f = H_1 \quad (9.6)$$

and for $n \geq 2$ it gives:

$$\mathscr{L}_n(H_0) + \sum_{k=2}^{n} \frac{1}{k!} \sum_{n_i \geq 1;\, n_1 + \ldots + n_k = n} \mathscr{L}_{n_1} \ldots \mathscr{L}_{n_k}(H_0) + \sum_{k=1}^{n-1} \frac{1}{k!} \sum_{n_i \geq 1;\, n_1 + \ldots + n_k = n-1} \mathscr{L}_{n_1} \ldots \mathscr{L}_{n_k}(f) = H_n. \quad (9.7)$$

To compute $H_1, \ldots, H_L$ we remark that $\mathscr{L}_n(H_0) = -\mathscr{L}_\omega(w_n)$ where $\mathscr{L}_\omega$ is the operator acting on $\mathscr{A}(\eta, R, r)$ as:

$$\mathscr{L}_\omega(w)(\hbar, A, I) = i\omega . lw(\hbar, A, I). \quad (9.8)$$

In particular $\langle \mathscr{L}_n(H_0) \rangle = 0$. This gives:

$$\langle f \rangle = H_1$$
$$\sum_{k=2}^{n} \frac{1}{k!} \sum_{n_i \geq 1;\, n_1 + \ldots + n_k = n} \langle \mathscr{L}_{n_1} \ldots \mathscr{L}_{n_k}(H_0) \rangle \quad (9.9)$$
$$+ \sum_{k=1}^{n-1} \frac{1}{k!} \sum_{n_i \geq 1;\, n_1 + \ldots + n_k = n-1} \langle \mathscr{L}_{n_1} \ldots \mathscr{L}_{n_k}(f) \rangle = H_n$$

We can actually take advantage of (9.6) and (9.9) to write the solution of (9.7) in a more compact form. Let us define $K(f) = f - \langle f \rangle$ and:

$$G_1^{(k)} = \{kf + \langle f \rangle\}/(k+1), \qquad G_n^{(k)} = -\mathscr{L}_\omega(w_n)/(k+1) \quad \text{for } n \geq 2. \quad (9.10)$$

A tedious computation gives:

$$\mathscr{L}_\omega(w_n) = \sum_{k=1}^{n-1} \frac{1}{k!} \sum_{n_i \geq 1;\, n_0 + \ldots + n_k = n} K\{\mathscr{L}_{n_k} \ldots \mathscr{L}_{n_1}(G_{n_0}^{(k)})\}. \quad (9.11)$$

To compute the w_n's recursively it is necessary and sufficient to solve the equation:

$$\mathscr{L}_\omega(w) = g - \langle g \rangle \quad \Leftrightarrow \quad w = \mathscr{L}_\omega^{-1}(\mathrm{K}\, g). \tag{9.12}$$

Indeed assuming that $\mathscr{L}_\omega^{-1}(\mathrm{K}\, g)$ is well defined, (9.11) permits to compute w_n recursively for the right hand side depends only upon the w_k's with $1 \leqq k \leqq n-1$ and w_1 is given by (9.6).

To solve the linearized equation (9.12) we need the:

LEMMA 9.2. – *Let* $\mathrm{R} > 0$ *and* $r > 0$ *be given. Let* $g \in \mathscr{A}(\eta, \mathrm{R}, r)$ *and let* $\omega \in \mathbf{R}^\mathrm{N}$ *satisfy the diophantine estimate* (9.2). *Then for any* $r' < r$ *there is a unique* w *in* $\mathscr{A}(\eta, \mathrm{R}, r')$ *such that:*

(i) $\langle w \rangle = 0$.

(ii) $\mathscr{L}_\omega(w) = g - \langle g \rangle = \mathrm{K}(g)$.

It obeys to the following estimate, with $\delta \equiv r - r'$*:*

$$\| w \|_{\eta, \mathrm{R}, r-\delta} \leqq \left(\frac{\sigma}{e\delta} \right)^\sigma \frac{\| g \|_{\eta, \mathrm{R}, r}}{\Omega}. \quad \square \tag{9.13}$$

Proof of the lemma 9.2. – From (9.8) the solution is given by:

$$w(\hbar, \mathrm{A}, l) = -\mathbf{i}(\omega . l)^{-1} g(\hbar, \mathrm{A}, l) \qquad \text{if} \quad l \neq 0$$
$$= 0 \qquad \text{if} \quad l = 0$$

We notice the estimate $|l|_1^\sigma e^{-\delta |l|_1} \leqq (\sigma/e\delta)^\sigma$ for all $l \in \mathbf{Z}^\mathrm{N}$ and:

$$\sum_{l \in \mathbf{Z}^\mathrm{N}} |w(\hbar, \mathrm{A}, l)| e^{(r-\delta)|l|_1} \leqq \frac{1}{\Omega} \sum_{l \in \mathbf{Z}^\mathrm{N}} |l|_l^\sigma e^{-\delta |l|_1} |g(\hbar, \mathrm{A}, l)| e^{r|l|_1}$$

giving the result immediately when applied to w and to w^*. $\square$

Proof of theorem 9.1 (*continued*). – To simplify the notations we will set

$$r_{m-1/2} = (r_m + r_{m-1})/2, \qquad \mathrm{R}_{m-1/2} = \mathrm{R}_{m-1} \quad \text{if } m \in \mathbf{Z}$$
$$\mathscr{A}(\eta, \mathrm{R}_{m-1}, r_{m-1}) = \mathscr{A}(m) \qquad \text{if} \quad 2m \in \mathbf{Z}.$$

Then the sequences R_m and r_m are decreasing. Let us assume that for $1 \leqq m \leqq n-1$ $\mathscr{L}_\omega(w_m)$ belongs to $\mathscr{A}(m)$. From (9.6) this is already true for $m = 1$. Then from the lemma 9.2, w_m belongs to $\mathscr{A}(m + 1/2)$.

Now if m and m' are positive integers, by the theorem 7.2 it follows that $\{w_m, f\}$ belongs to $\mathscr{A}(m + m')$ whenever $f \in \mathscr{A}(m')$. For indeed R_{m-1}, $\mathrm{R}_{m'-1} > \mathrm{R}_{m+m'-1}$ and in much the same way $r_{m-1/2}$, $r_{m'-1} > r_{m+m'-1/2}$. Applying this remark to the right hand side of (9.11), we get for each term:

(i) $\mathrm{G}_m^{(k)}$ belongs to $\mathscr{A}(m)$ because of (9.10).

(ii) $\mathscr{L}_{m'}$ is a linear bounded operator from $\mathscr{A}(m)$ into $\mathscr{A}(m + m')$ if $m \geqq 1$.

(iii) K is a contraction from $\mathscr{A}(m)$ into itself.

Therefore each term indexed by $(n_0, n_1, \ldots, n_k)$ such that $n_0+n_1+\ldots+n_k=n$ in the right hand side of (9.11) belongs to $\mathscr{A}(n)$. So does $\mathscr{L}_\omega(w_n)$.

Thus the kernel $w_{(L)}(\varepsilon)=\varepsilon.w_1+\ldots+\varepsilon^L.w_L$ belongs to $\mathscr{A}(L+1/2)$ for any L, and its norm goes to zero as $\varepsilon\to 0$. From the theorem 7.3, for ε small enough, $\exp\{\mathscr{L}(\varepsilon)\}$ is linear and bounded from $\mathscr{A}(0)$ into $\mathscr{A}(L+1)$. Thus $\exp\{\mathscr{L}(\varepsilon)\}(f)$ belongs to $\mathscr{A}(L+1)$. On the other hand expanding the exponential in power series we get

$$\exp\{\mathscr{L}(\varepsilon)\}(H_0)=H_0-\sum_{k\geqq 0}\mathscr{L}(\varepsilon)^k\{\mathscr{L}_\omega(w(\varepsilon))\}/(k+1)!$$

We have already shown that $\mathscr{L}_\omega(w(\varepsilon))$ belongs to $\mathscr{A}(L)$ and again by the theorem 7.3 it follows that $\mathscr{L}(\varepsilon)^k\{\mathscr{L}_\omega(w(\varepsilon))\}/(k+1)!$ belongs to $\mathscr{A}(L+1)$ with a norm of $O(\varepsilon^k)$. Therefore the right hand side $\exp\{\mathscr{L}(\varepsilon)\}(H_0)-H_0$ also belongs to $\mathscr{A}(L+1)$.

Since for ε small enough the expansion $\exp\{\mathscr{L}(\varepsilon)\}(H_0+\varepsilon f)$ in powers of ε does converges, the very definition of $\mathscr{L}(\varepsilon)$ implies that the terms of order ε^k with $1\leqq k\leqq L$ are diagonal, and that the remainder has a norm in $\mathscr{A}(L+1)=\mathscr{A}(\eta, R_L, r_L)$ of order ε^{L+1}. □

THEOREM 9.3. – *Order by order in* ε, *the quantum perturbation expansion for the eigenvalues of the hamiltonian* $H(\hbar)=\pi_\hbar(H_0+\varepsilon.f)$ *converges in the classical limit to the integrable form of the classical hamiltonian given by Birkhoff's expansion. Moreover, order by order in* ε, *the classical canonical transformation which produces Birkhoff's expansion is the semiclassical limit of the family of quantum automorphisms defined by the unitary transformation diagonalizing* $H(\hbar)$. □

Proof. – The quantum perturbation expansion for an individual eigenvalue labelled by $m\in\mathbf{Z}^N$ consists in finding an eigenvector and the corresponding eigenvalue in the form:

$$\left.\begin{aligned}\psi_m(\varepsilon)&=|m\rangle+\varepsilon.\psi_m^{(1)}+\varepsilon^2\psi_m^{(2)}+\ldots\\ E_m(\varepsilon)&=H_0(\hbar, m\hbar)+\varepsilon.H_1(\hbar, m\hbar)+\varepsilon^2 H_2(\hbar, m\hbar)+\ldots\end{aligned}\right\}\tag{9.14}$$

satisfying the eigenvalue equation:

$$(H_0+\varepsilon.f)\psi_m(\varepsilon)=E_m(\varepsilon)\psi_m(\varepsilon)\tag{9.15}$$

and the orthogonality condition:

$$\langle\psi_m(\varepsilon)|\psi_{m'}(\varepsilon)\rangle=\delta_{m,m'}.$$

This means that the new basis $\{\psi_m(\varepsilon)\}_{m\in\mathbf{Z}^N}$ is obtained from the canonical one by a unitary operator $U(\varepsilon)$ which can be expanded in perturbation expansion. A convenient way of writting it is to introduce a self adjoint operator $W_\hbar(\varepsilon)$ such that:

$$U(\varepsilon)=\exp\{\mathbf{i}/\hbar\, W_\hbar(\varepsilon)\}=\exp\{\mathbf{i}/\hbar(\varepsilon.W_\hbar^{(1)}+\varepsilon^2 W_\hbar^{(2)}+\ldots)\}.$$

The equation (9.15) becomes equivalent to the equation:

$$\exp\{-\mathbf{i}/\hbar\, W_\hbar(\varepsilon)\}\,\pi_\hbar(H_0+\varepsilon.f)\exp\{\mathbf{i}/\hbar\, W_\hbar(\varepsilon)\} = \pi_\hbar(H_0+\varepsilon.H_1+\varepsilon^2 H_2+\ldots). \quad (9.16)$$

If $w(\varepsilon)$ is defined through $\pi_\hbar(w(\varepsilon))=W_\hbar(\varepsilon)$, (9.16) is nothing but the equation (9.4) written in the representation $\pi_\hbar$.

Since by theorem 9.1 there is a unique solution as a formal power series expansion in ε belonging to the algebras $\mathscr{A}(\eta, R', r')$ for $R' < R$, $r' < r$, the expansion converges term by term as $\hbar \to 0$ to the expansion we get by taking $\hbar = 0$. But in this latter case, using the results of the sections 3 and 6, at $\hbar = 0$, we get nothing but the standard perturbation expansion of Birkhoff, namely $\mathscr{L}(\varepsilon)$ is the Poisson bracket by $w_{cl}(\varepsilon)$, which is analytic with respect to the actions and angles variables order by order in ε, and so its exponential is well defined giving rise to a canonical transformation. In the new variables (A', θ'), the classical hamiltonian $H(A, \theta)=H_0(A)+\varepsilon.f_{cl}(A, \theta)$ becomes independent of the angles namely $H(A', \theta')=H_0(A')+\varepsilon.H_1(A')+\varepsilon^2.H_2(A')+\ldots$, where

$$H_k(A)=\lim_{\hbar\to 0,\, m\hbar\to A} H_k(\hbar, m\hbar). \quad \square$$

10. A NEKHOROSHEV-TYPE ESTIMATE

In the previous section we have derived a perturbation theory valid in the sense of formal power series expansion. Actually we will see now that it is possible to get a better estimate firstly by estimating the terms of that series and then by getting an optimal estimate on the remainder. The method we use here has been introduced already in classical mechanics by several authors ([43], [59], [77]). Only the framework of noncommutative geometry is perfectly fitted with extending them to the quantum case. However, as discussed in the last section, the interpretation of the results in quantum mechanics is very non trivial. For indeed, by stopping the perturbation expansion at the order which minimizes the error term produce an error of the order of $O(\exp\{-\text{const}.\varepsilon^{-1/2\alpha}\})$ for some α which is approximately equal to N the number of degrees of freedom. In particular, whenever this error term is negligeable with respect to $\hbar$, the quantum eigenvalues will be known through this formula with a sufficient accuracy. Even if we allow this error to be of order $\hbar^\beta$ for some $\beta > 1$, this gives a bound $O(|\operatorname{Ln}(\hbar)|^{-2\alpha})$ for ε, namely quite a large quantity even if the effective Planck constant is very small! This probably explains why such a method of computing eigenvalues for complicated systems produces

results so amazingly accurate ([43], [59], [77]). More precisely:

THEOREM 10.1. – *Let f belong to $\mathscr{A}(\eta, R, r)$ with $\|f\|_{\eta, R, r}=1$. H_0 is defined by* (9.1) *and* (9.2).

For any $\beta > 1$, $\sigma > N-1$, we set $\alpha=\beta(\sigma+2)+1$, and

$$C_0=\frac{1}{12\zeta(\beta)^2}\left(\frac{e}{2\sigma}\right)^{\sigma}, \qquad \zeta(\beta)=\sum_{n=1}^{\infty}\frac{1}{n^{\beta}}.$$

Let us choose R_∞, r_∞ such that $0 < R_\infty < R$, $0 < r_\infty < r$. Then if:

$$\varepsilon \leqq C_0\Omega(R-R_\infty)(r-r_\infty)^{\sigma+1}e^{-2\alpha}=\varepsilon_c \tag{10.1 a}$$

there is an $L \geqq 2$ such that if $w_{(L)}(\varepsilon)$ is given by the theorem 9.1, *then:*

$$e^{\mathscr{L}_{w_{(L)}(\varepsilon)}}\{H_0+\varepsilon.f\}=H_0+\sum_{n=1}^{L}\varepsilon^n H_n+\mathscr{R}_L(f)$$

with $\mathscr{R}_L(f)\in\mathscr{A}(\eta, R_\infty, r_\infty)$ and

$$\|\mathscr{R}_L(f)\|_{\eta, R_\infty, r_\infty} \leqq 2e^{2\alpha}\varepsilon_c \exp\left\{-(2\alpha-1)\left(\frac{\varepsilon_c}{\varepsilon}\right)^{1/2\alpha}\right\}. \quad \square \tag{10.1 b}$$

Proof. – We introduce some notations. Starting from the pair $(R_0, r_0)=(R, r)$ of positive numbers, we define the sequence (R_n, r_n) by mean of:

$$R_n=R_{n-1}-\rho/n^{\beta}, \qquad r_n=r_{n-1}-\delta/n^{\beta} \quad \text{for } n \geqq 1, \beta > 1$$

where $\sum_{n \geqq 1}\rho/n^{\beta}=R_0-R_\infty$ and $\sum_{n \geqq 1}\delta/n^{\beta}=r_0-r_\infty$. We will set $r_{n-1/2}=(r_n+r_{n-1})/2$. We also denote by $\mathscr{A}(n)$ the algebra $\mathscr{A}(\eta, R_n, r_n)$ and by $\mathscr{A}(n+1/2)$ the algebra $\mathscr{A}(\eta, R_n, r_{n+1/2})$. At last we will set $Y_n=\|\mathscr{L}_\omega(w_n)\|_{n-1}$, where $\|\,.\,\|_n$ denotes the norm in $\mathscr{A}(\eta, R_n, r_n)$.

LEMMA 10.2. – *For any $n \geqq 1$ Y_n satisfies the estimate:*

$$Y_n \leqq C_1^{n-1}\|f\|_{\eta, R, r}^{n}\, n^{\alpha n} \tag{10.2}$$

with

$$C_1=\frac{6}{\Omega\rho\delta^{1+\sigma}}\left(\frac{2\sigma}{e}\right)^{\sigma}. \quad \square$$

Proof. – Our first estimate uses the inequality (9.13) (lemma 9.2) to get:

$$\|w_n\|_{n-1/2} \leqq \frac{1}{\Omega}\left(\frac{2\sigma}{e\delta}\right)^{\sigma} n^{\sigma\beta}\, Y_n. \tag{10.3}$$

Now from the equations (9.6) and (9.9) we see that $Y_1 \leqq \|f\|_0$. We will assume by recursion that

$$Y_m \leqq CD^m m^{\alpha m}, \qquad \alpha=(\sigma+2)\beta+1, \qquad 1 \leqq m \leqq n-1. \tag{10.4}$$

To estimate Y_n we now use the formula (9.11) in which we see $\mathscr{L}_m$ acting on $\mathscr{A}(m'-1)$ as a bounded operator with values in $\mathscr{A}(m'+m)$. By theorem 7.2, we can estimate its bound: since w_m belongs to $\mathscr{A}(m-1/2)=\mathscr{A}(\eta, R_{m-1}, r_{m-1/2})$ and since

$$\mathscr{A}(m'-1)=\mathscr{A}(\eta, R_{m'-1}, r_{m'-1})$$

we get

$$\begin{aligned} R_{m-1}-R_{m+m'-1} &\geqq \rho/m^{\beta}, \qquad R_{m'-1}-R_{m+m'-1} \geqq \rho/m'^{\beta} \\ r_{m'-1}-r_{m+m'-1} &\geqq \delta/m'^{\beta}, \qquad r_{m-1/2}-r_{m+m'-1} \geqq \delta/2m^{\beta} \end{aligned}$$

and

$$\| \mathscr{L}_m \|_{(m'-1)\to(m+m'-1)} \leqq (3/e\,\rho\delta)\, m^{\beta} m'^{\beta} \| w_m \|_{m-1/2}.$$

From the formula (9.11) we obtain:

$$Y_n \leqq \sum_{k=1}^{n-1} \frac{1}{k!} \left\{ \frac{3.2^{\sigma}\sigma^{\sigma}}{e^{1+\sigma}\Omega\,\rho\delta^{1+\sigma}} \right\}^k n^{k\beta} \sum_{n_i \geqq 1;\ n_0+\ldots+n_k=n} \prod_{j=1}^{k} n_j^{(\sigma+1)\beta} \prod_{i=0}^{k} Y_{n_i}.$$

Now we use the following inequalities

$$n_1 n_2 \ldots n_k \leqq \{(n_1+n_2+\ldots+n_k)/k\}^k$$

and $k! \geqq k^k e^{-k}$ to get:

$$Y_n \leqq \sum_{k=1}^{n-1} \xi^k \frac{n^{\beta(\sigma+2)k}}{k^{\{\beta(\sigma+1)+1\}k}} \sum_{n_i \geqq 1;\ n_0+\ldots+n_k=n} Y_{n_0} \ldots Y_{n_k} \tag{10.5}$$

with

$$\xi = \frac{3}{\Omega\,\rho\delta^{\sigma+1}} \left(\frac{2\sigma}{e}\right)^{\sigma}.$$

Using (10.4) we may estimate the product of the Y_m's. To this purpose we remark that the function

$$\begin{aligned} G(n_1, n_2, \ldots, n_k) = {} & n_1 L n(n_1) + \ldots + n_k L n(n_k) \\ & + \{n-(n_1+n_2+\ldots+n_k)\} L n \{n-(n_1+n_2+\ldots+n_k)\} \end{aligned}$$

defined on the convex domain $\Delta_{n,k}=\{1 \leqq n_i, (n_1+n_2+\ldots+n_k) \leqq n\}$ is convex, and that the extremal points of $\Delta_{n,k}$ are the vectors $\mathbf{e}_0=(1, 1, \ldots, 1)$ and $\mathbf{e}_i=(1, \ldots, n-k+1, \ldots, 1)$. Consequently, G takes on its maximal value on one of these extremal points namely $G(n_1, n_2, \ldots, n_k) \leqq (n-k) L n(n-k)$. Thus:

$$Y_{n_0} \ldots Y_{n_k} \leqq C^k D^n (n-k)^{\alpha(n-k)} \leqq C^k D^n n^{\alpha n} \frac{1}{n^{\alpha k}}.$$

Using (10.4), if $C = 1/(2\xi)$, this estimate gives:

$$Y_n \leqq CD^n n^{\alpha n} \sum_{k=1}^{n-1} \left(\frac{C\xi}{n}\right)^k \binom{n-1}{k} \leqq CD^n n^{\alpha n} \left\{\left(1+\frac{C\xi}{n}\right)^{n-1} - 1\right\}$$
$$\leqq CD^n n^{\alpha n} C\xi e^{C\xi} \leqq CD^n n^{\alpha n}.$$

To finish the recursive proof it is sufficient to check this estimate for $n=1$: if $C_1 = 2\xi$ we get easily $\| f \|_{\eta, R, r} = CD$ namely $D = C_1 \| f \|_{\eta, R, r}$ and $C = (C_1)^{-1}$, proving the lemma. □

Proof of theorem 10.1 (*continued*). – Using the definition of the remainder we get for any L the following expansion:

$$\mathscr{R}_L(f) = \sum_{n=L+1}^{\infty} \varepsilon^n \sum_{k \geqq (n/L)-1}^{n-1} \frac{1}{k!} \sum_{\substack{n_0 + \ldots + n_k = n \\ 1 \leqq n_i \leqq L}} \mathscr{L}_{n_k} \ldots \mathscr{L}_{n_1} (G_{n_0}^{(k)}). \quad (10.6)$$

For $k \in \mathbf{N}$ given, let us introduce

$$R(j) = R_{L-1}(1 - j/k) + R_L j/k = R_{L-1} - \rho_L j/k$$

and similarly we define $r(j) = r_{L-1/2}(1-j/k) + r_L j/k = r_{L-1/2} - \delta_L j/2k$. Since for $1 \leqq m \leqq L$, w_m belongs to $\mathscr{A}(\eta, R_{L-1}, r_{L-1})$, one can see $\mathscr{L}_m$ as a linear and bounded operator from $\mathscr{A}(\eta, R(j-1), r(j-1))$ into $\mathscr{A}(\eta, R(j), r(j))$ with a bound given by the theorem 7.2, namely [by (10.3)]:

$$\| \mathscr{L}_m \|_{(j-1) \to j} \leqq \frac{4}{e} \frac{k^2}{j} \frac{L^{2\beta}}{\rho\delta} \| w_m \|_m \leqq \frac{4}{\Omega \rho \delta^{\sigma+1}} \left(\frac{2\sigma}{e}\right)^{\sigma} L^{\beta(\sigma+2)} Y_m \frac{k^2}{ej}.$$

Using (10.6), the inequality $m^{\alpha m} \leqq L^{\alpha m}$ for $1 \leqq m \leqq L$ and remarking that there are at most L^{k+1} terms in the sum over $(n_0, \ldots, n_k)$ we get (since $\| f \|_0 = 1$):

$$\| \mathscr{R}_L(f) \|_L \leqq \frac{L}{C_1} \sum_{n=L+1}^{\infty} \{\varepsilon C_1 L^{\alpha}\}^n \sum_{k \geqq (n-1)/L}^{n-1} \left\{\frac{2eL^{\alpha}}{3}\right\}^k.$$

If $L \geqq 2$ the sum over k is bounded by $\{2eL^{\alpha}/3\}^n$. In much the same way remarking that $2e/2 \leqq 3$ if $\varepsilon C_1 L^{2\alpha} \leqq 1/4$ the sum over n is bounded by:

$$\| \mathscr{R}_L(f) \|_L \leqq L/C_1 \{2\varepsilon C_1 L^{2\alpha}\}^L.$$

The minimum of the right hand side of (10.7) is reached for $\varepsilon C_1 L^{2\alpha} = e^{-2\alpha}$ which is smaller than or equal to 1/4 for $\alpha > 1$. Using $Le^{-L} \leqq 1$, the relation (10.2), the definition of ρ and δ, we get the inequalities (10.1 *a* and *b*), and the theorem is proved. □

11. ANOTHER BIRKHOFF EXPANSION

We start again with the perturbed harmonic oscillator described in (9.1)-(9.3). But now we denote by $T(\varepsilon)$ the automorphism $\exp\{\mathscr{L}_{w(\varepsilon)}\}$ and by M the averaging operator $\langle \ldots \rangle$ introduced in (4.1):

$$M(f) \equiv \langle f \rangle.$$

So we are looking for a *-automorphism $T(\varepsilon)$ such that for any "small" ε:

$$T(\varepsilon).(H_0+\varepsilon.f)=M.T(\varepsilon).(H_0+\varepsilon.f). \tag{11.1}$$

Here again, we assume that $T(\varepsilon)$ may be expanded as a formal power series in ε:

$$T(\varepsilon)=\sum_{n=0}^{\infty} \varepsilon^n.T_n \tag{11.2}$$

where the coefficients T_n are operators. For instance we get, when $\varepsilon=0$:

$$T_0=T(0)=1$$

(since H_0 is integrable: $H_0=M.H_0$). We follow the method exposed in [80].

In order to compute $T(\varepsilon)$ our first result will explicit the "Hamilton-Jacobi" equation (11.1):

PROPOSITION 11.1. – *The "Hamilton-Jacobi" equation* (11.1) *for the formal series* $\varepsilon \to T(\varepsilon)$ *is equivalent to the following integral equation:*

$$T(\varepsilon).(H_0+\varepsilon.f)=H_0+M.\left[\int_0^{\varepsilon} ds.T(s)\right].f. \quad \square \tag{11.3}$$

Since the right hand side of (11.3) is the sum of H_0 and the *average* of some hamiltonian, clearly the transformed hamiltonian $T(\varepsilon).(H_0+\varepsilon.f)$ is integrable, so that (11.3) implies (11.1).

The linear operator in the bracket (which applies to the perturbation f) is nothing else than the sum of the automorphisms $T(s)$ when the parameter s varies from 0 to ε. Let us note that this sum is approximately equal to $\varepsilon.1$, (*i.e.* proportional to the identity operator), since $T(0)=1$.

By definition, the "normal form" of the perturbed hamiltonian $H_0+\varepsilon.f$ is the left hand side of (11.3), *i.e.* is equal to this hamiltonian expressed in its "adapted coordinates". Hence we deduce from (11.3) that the normal form of $H_0+\varepsilon.f$ is approximately $H_0+\varepsilon.M.f$.

Here we shall give a formal solution by computing the coefficients T_n by some recurrence formulas.

Proof of proposition 11.1. – To prove that (11.1) implies (11.3) we first set:

$$H(\varepsilon) \equiv T(\varepsilon) . (H_0 + \varepsilon . f). \tag{11.4}$$

By assumption (11.2), $H(\varepsilon)$ is a formal power series in ε:

$$H(\varepsilon) = H_0 + \sum_{n=1}^{\infty} \varepsilon^n . (T_n . H_0 + T_{n-1} . f)$$

and by assumption (11.1), $H(\varepsilon) = M . H(\varepsilon)$. Hence, deriving formally with respect to ε:

$$(d/d\varepsilon) H(\varepsilon) = M . [(d/d\varepsilon) H(\varepsilon)] \tag{11.5}$$

where

$$(d/d\varepsilon) H(\varepsilon) = \sum_{n=1}^{\infty} n . \varepsilon^{n-1} . (T_n . H_0 + T_{n-1} . f).$$

By definition (11.4):

$$(d/d\varepsilon) H(\varepsilon) = [(d/d\varepsilon) T(\varepsilon)] . (H_0 + \varepsilon . f) + T(\varepsilon) . f \tag{11.6}$$

where

$$(d/d\varepsilon) T(\varepsilon) = \sum_{n=0}^{\infty} n . \varepsilon^{n-1} . T_n.$$

We will now prove that:

$$M . [(d/d\varepsilon) T(\varepsilon)] . (H_0 + \varepsilon . f) = 0. \tag{11.7}$$

This completes the proof of proposition 11.1, since (11.6) plugged into the r. h. s. of (11.5) yields:

$$\begin{aligned} (d/d\varepsilon) H(\varepsilon) &= M . [[(d/d\varepsilon) T(\varepsilon)] . (H_0 + \varepsilon . f) + T(\varepsilon) . f] \\ &= M . T(\varepsilon) . f \end{aligned} \tag{11.8}$$

so that, by integration:

$$H(\varepsilon) - H(0) = \int_0^{\varepsilon} ds . M . T(s) . f$$

which is (11.3) since $H(0) = H_0$.

To prove (11.7) we just have to compute $(d/d\varepsilon)\,T(\varepsilon)$:

$$\begin{aligned}(d/d\varepsilon)\,T(\varepsilon) &\equiv (d/d\varepsilon)\,(e^{\mathscr{L}_W}) \\ &= \int_0^1 dt\,.\,e^{t\,\mathscr{L}_W}\,.\,[(d/d\varepsilon)\,\mathscr{L}_W]\,.\,e^{(1-t)\,\mathscr{L}_W} \\ &= \int_0^1 dt\,.\,[e^{t\,\mathscr{L}_W}\,.\,[(d/d\varepsilon)\,\mathscr{L}_W]\,.\,e^{-t\,\mathscr{L}_W}]\,.\,e^{\mathscr{L}_W} \\ &= \mathscr{L}_{\left(\int_0^1 dt\,.\,e^{t\,\mathscr{L}_W}\,.\,(d/d\varepsilon)\,W\right)}\,.\,e^{\mathscr{L}_W} \\ &\equiv \mathscr{L}_{F(\varepsilon)}\,.\,T(\varepsilon) \qquad (11.9)\end{aligned}$$

where $F(\varepsilon)$ is some formal series:

$$F(\varepsilon) \equiv \int_0^1 dt\,.\exp\{t\,.\,\mathscr{L}_{W(\varepsilon)}\}\,.\,(d/d\varepsilon)\,W(\varepsilon).$$

From now on, we will forget $W(\varepsilon)$. In order to compute $T(\varepsilon)$, (11.9) and $F(\varepsilon)$ suffice. Therefore (11.7) rewrites:

$$M\,.\,[\mathscr{L}_{F(\varepsilon)}\,.\,T(\varepsilon)]\,.\,(H_0+\varepsilon\,.f)=0$$

or by definition (11.4):

$$M\,.\,\mathscr{L}_{F(\varepsilon)}\,.\,H(\varepsilon)=0$$

or even, using the requirement (11.1):

$$M\,.\,\mathscr{L}_{F(\varepsilon)}\,.\,M\,.\,H(\varepsilon)=0.$$

And it is obvious that for any hamiltonian F, the following operator is identically zero:

$$M\,.\,\mathscr{L}_F\,.\,M=\mathscr{L}_{M.F}\,.\,M=0. \quad \square \qquad (11.10)$$

Let us remark that the ε^n-term of (11.3) is (for any $n \geqq 1$):

$$T_n\,.\,H_0+T_{n-1}\,.f=M\,.\,T_{n-1}\,.f/n.$$

This relation is not quite useful to compute T_n.

We now give a modified version of the equation (11.3), more suitable to derive recurrence formulas:

PROPOSITION 11.2. – *The "Hamilton-Jacobi" equation* (11.1) *for the formal series* $\varepsilon \to T(\varepsilon)$ *is equivalent to the following integral equation:*

$$\mathscr{L}_\omega\,.\,F(\varepsilon)=K\,.\left[T(\varepsilon)+\mathscr{L}_{F(\varepsilon)}\,.\,M\int_0^\varepsilon ds\,.\,T(s)\right].f \qquad (11.11)$$

where $\mathscr{L}_\omega \equiv \mathscr{L}_{H_0}$, $K \equiv 1-M$ *and* $F(\varepsilon)$ *generates* $T(\varepsilon)$ *by:*

$$(d/d\varepsilon)\,T(\varepsilon)=\mathscr{L}_{F(\varepsilon)}\,.\,T(\varepsilon). \qquad (11.12)$$

So if we define the coefficients F_n, $\mathscr{L}_n$ *of the formal series in* ε:

$$F(\varepsilon) \equiv \sum_{n=0}^{\infty} \varepsilon^n . F_n, \qquad \mathscr{L}_{F(\varepsilon)} \equiv \sum_{n=0}^{\infty} \varepsilon^n . \mathscr{L}_n$$

(i.e. $\mathscr{L}_n \equiv \mathscr{L}_{F_n}$) *then* (11.12) *rewrites, for any* $N \geqq 1$:

$$T_N = \frac{1}{N} \sum_{n=0}^{N-1} \mathscr{L}_n . T_{N-1-n} \tag{11.13}$$

(*and* $T_0 = 1$). *Likewise* (11.11) *rewrites, for any* $N \geqq 1$:

$$\mathscr{L}_\omega . F_N = K . \left[T_N + \sum_{n=0}^{N-1} \mathscr{L}_n . \frac{M}{N-n} . T_{N-1-n} \right] . f$$

(*and* $\mathscr{L}_\omega . F_0 = K . f$) i.e. *using* (11.13):

$$\mathscr{L}_\omega . F_N = K . \sum_{n=0}^{N-1} \mathscr{L}_n . \left[\frac{1}{N} + \frac{M}{N-n} \right] . T_{N-1-n} . f. \quad \square \tag{11.14}$$

Now if $\omega \in \mathbf{R}^N$ verifies the diophantine estimate (9.2) the lemma 9.2 shows that for any g belonging to some algebra $\mathscr{A}(\eta, R, r)$, the equation $\mathscr{L}_\omega . F = K . g$ has a solution in $\mathscr{A}(\eta, R, r')$ $(\forall r' < r)$. Proposition 11.3, below, precises this.

But at a formal level (11.13)-(11.14) allow us to compute the coefficients T_N and F_N. The algorithm is the following:

$T_0 = 1, \quad F_0 = \mathscr{L}_\omega^{-1} . K . f \Rightarrow \mathscr{L}_0 = \mathscr{L}_{F_0}$

$\Rightarrow T_1 = \mathscr{L}_0,$

$\Rightarrow F_1 = \mathscr{L}_\omega^{-1} . K . \mathscr{L}_0 . (1+M) . f \; (= \mathscr{L}_\omega^{-1} . K . \{(1+M) . f, \mathscr{L}_\omega^{-1} . K . f\})$

$\Rightarrow \mathscr{L}_1 = \mathscr{L}_{F_1}$

$\Rightarrow T_2 = [(\mathscr{L}_0)^2 + \mathscr{L}_1]/2,$

$F_2 = \mathscr{L}_\omega^{-1} . K [\mathscr{L}_0 . (1+M) . \mathscr{L}_0 + \mathscr{L}_1 . (1+2M)] . f/2 \Rightarrow \mathscr{L}_2 = \mathscr{L}_{F_2} \ldots$

Proof of proposition 11.2. – Let us rewrite (11.8) using the generating function $F(\varepsilon)$ of $T(\varepsilon)$:

$$[(d/d\varepsilon) T(\varepsilon)] . (H_0 + \varepsilon . f) + T(\varepsilon) . f \equiv (d/d\varepsilon) H(\varepsilon) = M . T(\varepsilon) . f$$

or

$$[(d/d\varepsilon) T(\varepsilon)] . (H_0 + \varepsilon . f) = -K . T(\varepsilon) . f.$$

The l.h.s. is $[\mathscr{L}_{F(\varepsilon)} . T(\varepsilon)] . (H_0 + \varepsilon . f)$ [*cf.* (11.12)]; hence using (11.3) the above equality becomes:

$$\mathscr{L}_{F(\varepsilon)} . \left[H_0 + M . \int_0^\varepsilon ds . T(s) . f \right] = -K . T(\varepsilon) . f$$

or

$$\mathscr{L}_{F(\varepsilon)} . H_0 = -\left[K . T(\varepsilon) + \mathscr{L}_{F(\varepsilon)} . M . \int_0^\varepsilon ds . T(s) \right] . f.$$

By (11.10) we can factorize K in front of the bracket. And by the antisymmetry of the Poisson bracket: $\mathscr{L}_{F(\varepsilon)} . H_0 = -\mathscr{L}_{H_0} . F(\varepsilon)$. So we have proven that (11.3) [*i.e.* (11.1)] implies (11.11).

To prove that (11.11) implies (11.3), we will show that the formal series $R(\varepsilon)$ defined by

$$R(\varepsilon) \equiv T(\varepsilon) . (H_0 + \varepsilon . f) - H_0 - M . \int_0^\varepsilon ds . T(s) . f$$

vanishes identically when (11.11) holds. Indeed, if $L(\varepsilon) \equiv \mathscr{L}_{F(\varepsilon)}$ we get from (11.12):

$$\begin{aligned}
(d/d\varepsilon) R(\varepsilon) &\equiv [(d/d\varepsilon) T(\varepsilon)] . (H_0 + \varepsilon . f) + T(\varepsilon) . f - M . T(\varepsilon) . f \\
&= [L(\varepsilon) . T(\varepsilon)] . (H_0 + \varepsilon . f) + [K . T(\varepsilon) . f] \\
&= [L(\varepsilon) . T(\varepsilon)] . (H_0 + \varepsilon . f) + \left[\mathscr{L}_\omega . F(\varepsilon) - \mathscr{L}_{F(\varepsilon)} . M \int_0^\varepsilon ds . T(s) \right] . f \\
&= L(\varepsilon) . \left[T(\varepsilon) . (H_0 + \varepsilon . f) - H_0 - M . \int_0^\varepsilon ds . T(s) . f \right] \\
&\equiv L(\varepsilon) . R(\varepsilon).
\end{aligned}$$

Since $R(\varepsilon)$ is by definition a formal series: $R(\varepsilon) \equiv \sum_{n=0}^{\infty} \varepsilon^n . R_n$, the preceding relation yields:

$$R_N = \frac{1}{N} \sum_{n=0}^{N-1} \mathscr{L}_n . R_{N-1-n}.$$

Finally $R_0 = R(0) = 0$ and therefore $R_N = 0$ for any N. □

For the moment, the meaning of the inverse $\mathscr{L}_\omega^{-1}$ of the operator $\mathscr{L}_\omega$ is purely formal. To go further, we need some assumption on the smoothness of f:

PROPOSITION 11.3. — *Let f be in some algebra $\mathscr{A}_0 \equiv \mathscr{A}(\eta, R, r)$ and q be the function:*

$$q(x) \equiv 1 - \gamma + [\gamma/(1 + 2 . x)^b] \text{ for some } \gamma \in]0; 1[\text{ and } b \in]0; 1/12].$$

For any $n \geqq 0$ we set:

$$\mathscr{A}_n \equiv \mathscr{A}(\eta, R . q(n), r . q(n))$$
$$\| \ldots \|_n \text{ is the norm in } \mathscr{A}_n$$
$$\| \ldots \|_{m \leftarrow n} \text{ is the norm for operators between } \mathscr{A}_n \text{ and } \mathscr{A}_m.$$

To avoid any ambiguity between ε *and* f, *we may assume* $\| f \|_0 = 1$.

If $\omega \in \mathbf{R}^N$ *satisfies, for some* $\sigma > N-1$, $\Omega > 0$, *the diophantine estimate:*

$$|\omega . l| \geqq \Omega / |l|^{\sigma}, \qquad \forall l \in \mathbf{Z}^N - \{0\} \tag{11.15}$$

then for any $n \geqq 0$: $F_n \in \mathscr{A}_{n+1/2}$ (11.16)

T_n *is a bounded operator between* $\mathscr{A}_0$ *and* $\mathscr{A}_n$. (11.17)

Furthermore estimates on the norms are:

$$\| F_n \|_{n+1/2} \leqq B . C^n . n^{a . n} . (n+1)^{1+(b+1)\sigma} \tag{11.18}$$

$$\| T_n \|_{n \leftarrow 0} \leqq C^n . n^{a . n} \tag{11.19}$$

where $a \equiv (b+1)(\sigma+2)$, $B \equiv (\sigma / r . b . \gamma)^{\sigma} / \Omega$, $C \equiv 4 . B/R . r . b^2 . \gamma^2$. □

Proof of proposition 11.3. – For $n=0$, (11.17) is trivial ($T_0 = 1$) and $F_0 \equiv \mathscr{L}_{\omega}^{-1} . K . f \in \mathscr{A}_{1/2}$ by applying the lemma 9.2 with $r' = r . q(1/2) < r$.

Let us assume that for a given $N \geqq 1$, (11.16)-(11.19) hold for $n = 0, \ldots, N-1$, and consider any term of the sum (11.13).

By (11.17) T_{N-n-1} is a bounded operator between $\mathscr{A}_0$ and $\mathscr{A}_{N-n-1}$ and by (11.16) $F_n \in \mathscr{A}_{n+1/2}$. Since $q(N) < \min \{ q(n+1/2); q(N-n-1) \}$, the theorem 7.2 yields that $\mathscr{L}_n \equiv \mathscr{F}_{F_n}$ is bounded between $\mathscr{A}_{N-n-1}$ and $\mathscr{A}_N$. And (11.17) holds also for $n = N$.

Furthermore:

$$\begin{aligned} \| \mathscr{L}_n \|_{N \leftarrow N-n-1} &\leqq (2/R . r . e) . \| F_n \|_{n+1/2} . [q(N-n-1) - q(N)]^{-1} . [q(n+1/2) - q(N)]^{-1} \\ &\leqq (2/R . r . e) . B . C^n . n^{a . n} . (n+1)^{1+(b+1)\sigma} . [(n+1) . |q'(N)|]^{-1} \\ &\quad \times [(N-n-1/2) . |q'(N)|]^{-1}. \end{aligned}$$

Indeed

$$\forall x < y : q(x) - q(y) \geqq |q'(y)| . (y-x)$$

$$|q'(N)| \geqq 2\gamma . b / (3N)^{b+1} \qquad \text{and} \qquad (N-n-1/2) \geqq 1/2.$$

Hence:

$$\begin{aligned} \| T_N \|_{N \leftarrow 0} &\leqq \frac{1}{N} \sum_{n=0}^{N-1} \| \mathscr{L}_n \|_{N \leftarrow N-n-1} \cdot \| T_{N-1-n} \|_{N-1-n \leftarrow 0} \\ &\leqq \frac{1}{N} \sum_{n=0}^{N-1} [9^{b+1} . B/R . r . e . b^2 . \gamma^2] . C^n . n^{a . n} \\ &\quad \times (n+1)^{(b+1)\sigma} . N^{2(b+1)} . C^{N-1-n} . (N-1-n)^{a . (N-1-n)} \\ &\leqq \frac{1}{N} \sum_{n=0}^{N-1} [9^{b+1} B/R \, reb^2 \gamma^2] C^{N-1} N^a [n^n (N-1-n)^{(N-1-n)}]^a. \end{aligned}$$

The first bracket is $\leqq C$ (since $b < 1/12$) and the last one is maximized when n is extremal, *i.e.* when $n=0$ (or $n=N-1$). And its value is $\leqq N^{a . (N-1)}$. So:

$$\| T_N \|_{N \leftarrow 0} \leqq C^N . N^{a . N}.$$

Likewise let us consider any term of the sum (11.14).

The operator $\left[\frac{1}{N}+\frac{M}{N-n}\right]$ does not decrease the analyticity properties of its argument.

Hence $\mathscr{L}_\omega . F_N \in \mathscr{A}_N$ by the same argument as above, and:

$$\| \mathscr{L}_\omega . F_N \|_N \leqq \sum_{n=0}^{N-1} \| \mathscr{L}_n \|_{N \leftarrow N-n-1} \cdot \left[\frac{1}{N}+\frac{1}{N-n}\right] \cdot \| T_{N-1-n} \|_{N-1-n \leftarrow 0} \cdot \| f \|_0$$

$$\leqq \left[\frac{1}{N}+1\right] . N . (C^N . N^{a.N}).$$

Applying once more the lemma 9.2, we verify that (11.16) holds also for $n=N$, and that:

$$\| F_N \|_{N+1/2} \leqq [(\sigma/r.e)^\sigma/\Omega] . [q(N)-q(N+1/2)]^{-\sigma} . (N+1) . C^N . N^{a.N}$$

$$\leqq B . C^N . N^{a.N} . (N+1)^{1+(b+1)\sigma}. \quad \square$$

Finally we give another version of the theorem 10.1:

THEOREM 11.4. – *Let f be in some algebra $\mathscr{A}(\eta, R, r)$ with $\| f \|_{\eta, R, r}=1$, $b \in]0; 1/12]$. We set $R_\infty \equiv R.(1-\gamma)$, $r_\infty \equiv r.(1-\gamma)$ (for some $\gamma \in]0; 1[$), and define H_0 by (9.1) with $\omega \in \mathbf{R}^N$ satisfying the diophantine estimate (11.15) for some $\sigma > N-1$, $\Omega > 0$. We also set $\alpha \equiv (b+1)(\sigma+2)+1$.*

If

$$\varepsilon \leqq \varepsilon_c \equiv \Omega . R . r . b^2 . \gamma^2 . (r . b . \gamma/\sigma)^\sigma/64 . e^\alpha \tag{11.20}$$

*then there is some integer $L \geqq 1$, and a *-automorphism*

$$T^{(L)}(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n . T_n^{(L)} \tag{11.21}$$

such that

$$T^{(L)}(\varepsilon) . (H_0 + \varepsilon . f) = H_0 + \sum_{n=1}^{L} \varepsilon^n . H_n + \mathscr{R}_L(f) \tag{11.22}$$

where

$$H_n = M . T_{n-1}^{(L)} . f/n \in \mathscr{D}(\eta, R_\infty) \tag{11.23}$$

and

$$\mathscr{R}_L(f) \equiv \sum_{n=L+1}^{\infty} \varepsilon^n . [T_n^{(L)} . H_0 + T_{n-1}^{(L)} . f] \in \mathscr{A}(\eta, R_\infty, r_\infty) \tag{11.24}$$

$$\| \mathscr{R}_L(f) \|_{\eta, R_\infty, r_\infty} \leqq 18 . \varepsilon_c . (\varepsilon/\varepsilon_c)^{1-(1/\alpha)} . \exp -\alpha . [(\varepsilon_c/\varepsilon)^{1/\alpha}]. \quad \square (11.25)$$

Proof of theorem 11.4. – The construction of $T^{(L)}(\varepsilon)$ is the same as in chapters 9 and 10, *i.e.* we first define $T_0^{(L)}=1$, $F_0^{(L)}=\mathscr{L}_\omega^{-1} . K . f$,

$T_1^{(L)}$, $F_1^{(L)}$, ... as in proposition 11.2, with the exception that we set:

$$F_N^{(L)} \equiv 0 \qquad \text{for} \quad N \geqq L.$$

So, for $N \leqq L-1$, $F_N^{(L)} = F_N$ and for $N \leqq L$, $T_N^{(L)} = T_N$.

In particular this explains (11.23), since we have already seen that:

$$T_n . H_0 + T_{n-1} . f = M . T_{n-1} . f/n.$$

Now we rewrite $T_N^{(L)}$ under an expanded form, obtained by iterating (11.13):

$$T_N^{(L)} = \sum_{k=1}^{N} \sum_{n_1=0}^{(N \wedge L)-1} \cdots \sum_{n_k=0}^{(N \wedge L)-1} \frac{\mathscr{L}_{n_k}}{|n|_k + k} \cdots \frac{\mathscr{L}_{n_1}}{|n|_1 + 1} \tag{11.26}$$

where:

(*) the sum is restricted by the condition $n_1 + \ldots + n_k = N - k$

(*) $N \wedge L \equiv \min(N; L)$

(*) and for any m: $|n|_m \equiv \sum_{i=1}^{m} n_i$.

Obviously (11.26) is also true when $L = \infty$ (which is the case of the proposition 11.2). The advantage of this truncation is seen in the following lemma:

LEMMA 11.5. – *Under the same assumptions as in proposition* 11.3, *we get for any* $L \geqq 1$, $N \geqq 0$:

$$T_N^{(L)} \text{ is a bounded operator between } \mathscr{A}_0 \text{ and } \mathscr{A}_L \tag{11.27}$$

and

$$\| T_N^{(L)} \|_{L \leftarrow 0} \leqq (16 . C)^N . (N \wedge L)^{\alpha . N} \tag{11.28}$$

where

$$\alpha \equiv (b+1)(\sigma+2)+1, \qquad B \equiv (\sigma/r . b . \gamma)^\sigma/\Omega, \qquad C \equiv 4 . B/R . r . b^2 . \gamma^2.$$

And if

$$\varepsilon \leqq 1/32 . C . L^\alpha \tag{11.29}$$

then

$$\mathscr{R}_L(f) \equiv \sum_{N=L+1}^{\infty} \varepsilon^N . [T_N^{(L)} . H_0 + T_{N-1}^{(L)} . f] \in \mathscr{A}_L \tag{11.30}$$

and

$$\| \mathscr{R}_L(f) \|_L \leqq 18 . \varepsilon . L . (16 . C . L^\alpha . \varepsilon)^L. \quad \square \tag{11.31}$$

Proof of lemma 11.5. – First of all, (11.27)-(11.28) are already established in proposition 11.3 when $N \leqq L$. So we investigate the case $N > L$.

Let us consider any term of (11.26) as follow:

$$\mathscr{A}_L = \mathscr{A}_{L-(k-k)/2k} \overset{\mathscr{L}_{n_k}}{\leftarrow} \dots \overset{\mathscr{L}_{n_3}}{\leftarrow} \mathscr{A}_{L-(k-2)/2k} \overset{\mathscr{L}_{n_2}}{\leftarrow} \mathscr{A}_{L-(k-1)/2k} \overset{\mathscr{L}_{n_1}}{\leftarrow} \mathscr{A}_0.$$

Using (11.18), the norm of a generic factor is bounded by:

$$\begin{aligned}
&\| \mathscr{L}_{n_i} \|_{L-(k-i)/2k \leftarrow L-(k-i+1)/2k} \\
&\quad \leqq \| F_{n_i} \|_{n_i+1/2} \cdot (2/R \cdot r \cdot e) \cdot [q(L-(k-i+1)/2k) - q(L-(k-i)/2k)]^{-1} \\
&\qquad \times [q(n_i+1/2) - q(L-(k-i)/2k)]^{-1} \\
&\qquad \leqq [B \cdot C^{n_i} \cdot n_i^{a \cdot n_i} \cdot (n_i+1)^{1+(b+1)\sigma}] \cdot (2/R \cdot r \cdot e) \cdot (1/2k)^{-1} \\
&\qquad\qquad \times |q'(L-(k-i)/2k)|^{-2} \cdot (L-1-n_i+(i/2k))^{-1}.
\end{aligned}$$

The last factor is maximal when $n_i = L-1$, and the argument of the q' function may be replaced by L. Furthermore in (11.26), this factor is associated to a denominator $|n|_i + i$, which may be lower bounded by i. Hence:

$$\| T_N^{(L)} \|_{L \leftarrow 0} \leqq \sum_{k=1}^{N} \sum_{n_1=0}^{L-1} \dots \sum_{n_k=0}^{L-1} \prod_{i=1}^{k}$$

$$\times \frac{1}{i} [B \cdot C^{n_i} \cdot L^{a \cdot n_i} \cdot L^{1+(b+1)\sigma} \cdot (2/R\, re \cdot b^2 \cdot \gamma^2)(k^2/i)(2L+1)^{2(b+1)}] \quad (11.32)$$

where the sum is restricted by the condition $n_1 + \dots + n_k = N-k$.

Now we replace $2L+1$ by $3L$ since $L \geqq 1$, and by the Stirling formula we get an upper bound

$$\prod_{i=1}^{k} (k/i)^2 = (k^k/k!)^2 \leqq e^{2k}.$$

Let us also notice that

$$\prod_{i=1}^{k} C^{n_i} \cdot L^{a \cdot n_i} = (C \cdot L^a)^{n_1 + \dots + n_k} = (C \cdot L^a)^{N-k}$$

and that there are $\binom{N-1}{k-1}$ k-uple $(n_1, \dots, n_k)$ such that

$$n_1 + \dots + n_k = N-k.$$

Therefore, using the value of a:

$$\begin{aligned}
\| T_N^{(L)} \|_{L \leftarrow 0} &\leqq \sum_{k=1}^{N} \binom{N-1}{k-1} \cdot [9^{b+1} \cdot e \cdot L/2]^k \cdot (C \cdot L^a)^N \\
&\leqq 15 \cdot L \cdot [1+15 \cdot L]^{N-1} \cdot (C \cdot L^a)^N \quad (11.33)
\end{aligned}$$

since $9^{b+1} \cdot e/2 \leqq 15$ ($b \leqq 1/12$). Finally $1+15 \cdot L \leqq 16 \cdot L$, and $a+1 \equiv \alpha$.

This proves (11.28), *i.e.* that $\| T_N^{(L)} \|_{L \leftarrow 0} \leqq (16 \cdot C \cdot L^\alpha)^N$.

So (11.30) is a convergent geometric series when (11.29) holds.

Let us now turn to (11.30).

Estimating $\| T_{N-1}^{(L)} . f \|_L$ is easy (using $\| f \|_0 = 1$):

$$\| T_{N-1}^{(L)} . f \|_L \leqq (16 . C . L^{\alpha})^{N-1}. \tag{11.34}$$

For what concerns $\| T_N^{(L)} . H_0 \|_L$ we replace the last factor in (11.26) $\mathscr{L}_{n_1} . H_0$ by $-\mathscr{L}_{\omega} . F_{n_1}$ and:

$$\begin{aligned} \| \mathscr{L}_{\omega} . F_{n_1} \|_{L-(k-1)/2k} &\leqq \| \mathscr{L}_{\omega} . F_{n_1} \|_{L-1} \\ &\leqq \| \mathscr{L}_{\omega} . F_{n_1} \|_{n_1} \\ &\leqq (n_1 + 1) . C^{n_1} . n_1^{a . n_1} \end{aligned}$$

(*cf.* the end of the proof of proposition 11.3). So this r.h.s. replaces the last factor $\| \mathscr{L}_{n_1} \|_{L-(k-1)/2k \leftarrow 0}$ of the product in (11.32):

$$\| T_N^{(L)} . H_0 \|_L \leqq (16 . C L^{\alpha})^N / 2 C L^{\alpha - 1} = (16 . C L^{\alpha})^{N-1} . 8 L. \tag{11.35}$$

Plugging (11.34)-(11.35) into the definition (11.30) of $\mathscr{R}_L(f)$ we get:

$$\begin{aligned} \| \mathscr{R}_L(f) \|_L &\leqq \sum_{N=L+1}^{\infty} (16 . C . L^{\alpha} . \varepsilon)^{N-1} . (8 L + 1) . \varepsilon \\ &= (16 . C . L^{\alpha} . \varepsilon)^L . (8 L + 1) . \varepsilon / [1 - 16 . C . L^{\alpha} . \varepsilon] \\ &\leqq 18 . \varepsilon . L . (16 . C . L^{\alpha} . \varepsilon)^L \end{aligned}$$

(because $\varepsilon \leqq 1/32 . C . L^{\alpha}$, and $L \geqq 1 : 8 L + 1 \leqq 9 L$). □

Let us complete the proof of the theorem 11.4: since $q(L) \geqq 1 - \gamma$, $\mathscr{A}_L \subseteqq \mathscr{A}(\eta, R_{\infty}, r_{\infty})$.

There remains to choose L. We will take a value which (nearly) minimizes the bound (11.31), *i.e.*:

$$L = [(\varepsilon_c / \varepsilon)^{1/\alpha}]$$

with

$$\varepsilon_c \equiv 1/16 . C . e^{\alpha} \tag{11.36}$$

(the integer part is denoted by [. . .]).

Finally condition (11.20) insures that this value of L is $\geqq 1$, and it also implies condition (11.29) of lemma 11.5 (since $e^{\alpha} \geqq 2$). □

REFERENCES

[1] N. ASHCROFT and D. MERMIN, *Solid state physics*, Saunders, Philadelphia, Tokyo, 1976.

[2] J. AVRON and R. SEILER, Quantization of the Hall Conductance for General Multiparticule Schrödinger Hamiltonians, *Phys. Rev. Lett.*, Vol. **54**, 1985, pp. 259-262.

[3] R. BALIAN and C. BLOCH, Distribution of Eigenfrequencies for the Wave Equation in a Finite Domain: I-Three-Dimensional Problem with Smooth Boundary Surface, *Ann. Phys.*, Vol. **60**, 1970, pp. 401-447.

[4] R. BALIAN and C. BLOCH, Distribution of Eigenfrequencies for the Wave Equation in a Finite Domain: II-Electromagnetic Field, Riemannian Spaces, *Ann. Phys.*, Vol. **64**, 1971, pp. 271-307.

[5] R. BALIAN and C. BLOCH, Distribution of Eigenfrequencies for the Wave Equation in a Finite Domain: III-Eigenfrequency Density Oscillations, *Ann. Phys.*, Vol. **69**, 1972, pp. 76-160.

[6] R. BALIAN and C. BLOCH, Solution of the Schrödinger Equation in Terms of Classical Paths, *Ann. Phys.*, Vol. **85**, 1974, pp. 514-545.

[7] J. BAYFIELD and P. KOCH, Multiphotonic Ionization of Highly Excited Hydrogen Atoms, *Phys. Rev.*, Vol. **33**, 1974, p. 258.

[8] J. BAYFIELD, Experiment and Theory for the Classically Chaotic Motion of the Driven Bound Electron, in *Non linear evolution and chaotic phenomena*, G. GALLAVOTTI and P. F. ZWEIFEL Eds., Plenum, New York, 1988.

[9] J. BAYFIELD and D. W. SOKOL, Excited Atoms in Strong Microwaves: Classical Resonances and Localization in Experimental Final States Distributions, *Phys. Rev. Lett.*, Vol. **61**, 1988, p. 2007.

[10] J. BAYFIELD, G. CASATI, I. GUARNERI and D. W. SOKOL, Localization of Classically Chaotic Diffusion for Hydrogen Atoms in Microwave Fields, submitted to *Phys. Rev. Lett.*

[11] J. BELLISSARD, Stability and Instability in Quantum Mechanics, in *Trends and developments in the eighties*, S. ALBEVERIO and P. BLANCHARD Eds., World Scientific, Singapore, 1985.

[12] J. BELLISSARD, K-Theory of C*-Algebras in Solid State Physics, in *Statistical mechanics and field theory, mathematical aspects*, T. C. DORLAS, M. N. HUGENHOLTZ and M. WINNINK Eds., *Lect. Notes Phys.*, Vol. **257**, Springer, Berlin, 1986.

[13] J. BELLISSARD, Ordinary Quantum Hall Effect and Non Commutative Cohomology, in *Bad Schandau conference on localization*, W. WELLER and P. ZIESCHE Eds., Teubner, Leipzig, 1988.

[14] J. BELLISSARD, C*-Algebras in Solid State Physics: 2D Electrons in a Uniform Magnetic Field, in *Operators Algebras and Applications*, Vol. II, E. V. EVANS and M. TAKESAKI Eds., Cambridge University Press, Cambridge, 1988.

[15] J. BELLISSARD, Almost Periodicity in Solid State Physics and C*-Algebras, in *The Harald Bohr Centennary*, C. BERG and F. FLUGEDE Eds., Royal Danish Acad. Science, Copenhagen, 1989.

[16] M. V. BERRY and M. TABOR, Level Clustering in the Regular Spectrum, *Proc. R. Soc. London*, Vol. **A356**, 1977, pp. 375-394.

[17] M. V. BERRY, Semiclassical Mechanics of Regular and Irregular Motion, in *Chaotic Behavior of Deterministic Systems*, G. IOOSS, R. H. G. HELLEMAN and R. STORA Eds., North-Holland, Amsterdam, 1983.

[18] M. V. BERRY, Quantal Phase Factors Accompanying Adiabatic Changes, *Proc. R. Soc. London*, Vol. **A392**, 1984, pp. 45-57.

[19] M. V. BERRY, Semiclassical Theory of Spectral Rigidity, *Proc. R. Soc. London*, Vol. **A400**, 1985, pp. 229-251.

[20] M. V. BERRY and M. ROBNIK, Statistics of Energy Levels Without Time-Reversal Symmetry, Aharonov-Bohm Chaotic Billards, *J. Phys.*, Vol. **A19**, 1986, pp. 649-668.

[21] M. V. BERRY, Semiclassical Formula for the Number Variance of the Riemann Zeros, *Nonlinearity*, Vol. **1**, 1988, pp. 399-407.

[22] G. D. BIRKHOFF, *Dynamical systems*, A.M.S. Coll. Pub., Vol. **9**, AMS, Providence, Rhode Island, 1927.

[23] O. BOHIGAS, R. U. HAQ and A. PANDEY, Fluctuation Properties of Nuclear Energy Levels and widths: Comparison of Theory with Experiment, in *Nuclear data for science and technology*, K. H. BÖCKHOFF Ed., ECSC, EEC, EAEC, Brussels and Luxembourg, 1983.

[24] O. BOHIGAS and M. J. GIANNONI, Chaotic Motion and Random Matrix Theories, in *Mathematical and computational methods in nuclear physics*, J. S. DEHESA, J. M. G. GOMEZ and A. POLLS Eds., *Lect. Notes Phys.*, Vol. **209**, Springer, Berlin, 1984.

[25] O. BOHIGAS, M. J. GIANNONI and C. SCHMIT, Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws, *Phys. Rev. Lett.*, Vol. **52**, 1984, pp. 1-4.

[26] O. BOHIGAS, M. J. GIANNONI and C. SCHMIT, Spectral Fluctuations of Classically Chaotic Quantum Systems, in *Quantum chaos and statistical nuclear physics*, T. H. SELIGMAN and H. NISHIOKA Eds., *Lect. Notes Phys.*, Vol. **263**, Springer, Berlin, 1986.

[27] O. BRATTELI and D. W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics, I, II*, Springer, Berlin, 1979-1981.

[28] P. BRIET, J. M. COMBES and P. DUCLOS, On the location of resonances for Schrödinger operators in the semiclassical limit: II, Comm. P.D.E., Vol. **12**, 1987, pp. 201-222.

[29] P. BRIET, J. M. COMBES and P. DUCLOS, Spectral Stability Under Tunneling, *Comm. Math. Phys.*, 1989 (to appear).

[30] L. BRILLOUIN, *J. Phys. Radium*, Vol. **7**, 1926, pp. 353-368.

[31] T. A. BRODY, J. FLORES, J. B. FRENCH, P. A. MELLO, A. PANDEY and S. S. M. WONG, Random-Matrix Physics: Spectrum and Strenght Fluctuations, *Rev. Mod. Phys.*, Vol. **53**, 1981, pp. 385-479.

[32] G. CASATI, B. V. CHIRIKOV, J. FORD and F. M. IZRAELEV, Stochastic Behavior of a Quantum Pendulum Under a Periodic Perturbation, in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems*, G. CASATI and J. FORD Eds., *Lect. Notes Phys.*, Vol. **93**, Springer, Berlin, 1979.

[33] G. CASATI, B. V. CHIRIKOV, I. GUARNERI and D. L. SHEPELYANSKY, Dynamical Stability of Quantum Chaotic Motion in a Hydrogen Atom, *Phys. Rev. Lett.*, Vol. **56**, 1986, p. 2437.

[34] G. CASATI, B. V. CHIRIKOV, I. GUARNERI and D. L. SHEPELYANSKY, New Photoelectric Ionization Peak in the Hydrogen Atom, *Phys. Rev. Lett.*, Vol. **57**, 1986, p. 823.

[35] G. CASATI, B. V. CHIRIKOV, I. GUARNERI and D. L. SHEPELYANSKY, Localization of Diffusive Excitation in the Two-Dimensional Hydrogen Atom in a Monochromatic Field, *Phys. Rev. Lett.*, Vol. **59**, 1987, p. 2927.

[36] A. L. CAUCHY, Résumé des leçons données à l'École Royale Polytechnique sur le calcul infinitésimal, Paris (1823) (repr. in *Œuvres complètes IV*, Gauthier-Villars, Paris, 1899).

[37] J. CHAZARAIN, Spectre d'un hamiltonien quantique et mécanique classique, *Comm. P.D.E.*, Vol. **5**, 1980, pp. 595-644.

[38] B. V. CHIRIKOV, A Universal Instability of Many Dimensional Oscillator Systems, *Phys. Rep.*, Vol. **52**, 1979, pp. 263.

[39] Y. COLIN DE VERDIERE, *Compos. Math.*, Vol. **27**, 1973, p. 83, Vol. **27**, p. 159.

[40] Y. COLIN DE VERDIERE, Quasi-modes sur les variétés Riemanniennes, *Inv. Math.*, 1977, Vol. **43**, pp. 15-52.

[41] A. CONNES, Non Commutative Differential Geometry, *Pub. I.H.E.S.*, 1986, Vol. **62**, pp. 43-144.

[42] W. J. DE HAAS and P. M. VAN ALPHEN, *Proc. Acad. Sci.* (Amsterdam), Vol. **36**, 1933, p. 262.

[43] J. B. DELOS, S. K. KNUDSON and D. W. NOID, High Rydberg States of an Atom in a Strong Magnetic Field, *Phys. Rev. Lett.*, Vol. **50**, 1983, pp. 579-583.

[44] J. DIXMIER, *Les C*-algèbres et leurs représentations*, Gauthiers-Villars, Paris, 1969.

[45] A. EINSTEIN, Zum Quantensatz von Sommerfeld und Epstein, *Verhandl. Deutsch. Phys. Ges.*, Vol. **19**, 1917, p. 82-92.

[46] G. A. ELLIOT, Gaps in the Spectrum of an Almost Periodic Schrödinger Operator, *C.R. Acad. Sci.*, Royal Soc. of Canada, Vol. **IV**, 1982, p. 255-259.

[47] S. FISHMAN, D. R. GREMPEL and R. E. PRANGE, Chaos, Quantum Recurrence and Anderson Localization, *Phys. Rev. Lett.*, Vol. **49**, 1982, pp. 509-512.

[48] S. FISHMAN, D. R. GREMPEL and R. E. PRANGE, Quantum Dynamics of a Non Integrable System, *Phys. Rev.*, Vol. **A29**, 1984, pp. 1639-1647.

[49] G. GALLAVOTTI, Quasi-Integrable Mechanical Systems, in *Critical Phenomena, Random Systems, Gauge Theories*, K. OSTERWALDER and R. STORA Eds., North Holland, 1986.

[50] T. GEISEL, G. RADONS and J. RUBNER, Kolmogorov-Arnol'd-Moser Barriers in the Quantum Dynamics of Chaotic Systems, *Phys. Rev. Lett.*, Vol. **57**, 1986, pp. 2883-2886.

[51] S. GRAFFI and T. PAUL, The Schrödinger Equation and Canonical Perturbation Theory, *Comm. Math. Phys.*, Vol. **108**, 1987, pp. 25-40.

[52] J. M. GREENE, A Method for Determining a Stochastic Transition, *J. Math. Phys.*, Vol. **20**, 1979, pp. 1183-1201.

[53] V. GUILLEMIN and S. STERNBERG, The Metaplectic Representation, Weyl Operators and Spectral Theory, in *Differential Geometric Methods in Mathematical Physics*, P. L. GARCIA, A. PÉREZ-RENDÓN and J. M. SOURIAU Eds., *Lect. Notes Math.*, Vol. **836**, Springer, Berlin, 1979.

[54] M. C. GUTZWILLER, Energy spectrum According to Classical Mechanics, *J. Math. Phys.*, Vol. **11**, 1970, pp. 1791-1806.

[55] M. C. GUTZWILLER, Periodic Orbits and Classical Quantization Conditions, *J. Math. Phys.*, Vol. **12**, 1971, pp. 343-358.

[56] B. HELFFER and J. SJÖSTRAND, Analyse semi-classique pour l'équation de Harper (avec application à l'étude de l'équation de Schrödinger avec champ magnétique) I, II, III, Preprint Univ. Orsay, *Bull. Soc. Math. France*, 1988 (submitted).

[57] E. J. HELLER, Bound-State Eigenfunctions of Classically Chaotic Hamiltonian Systems: Scars of Periodic Orbits, *Phys. Rev. Lett.*, Vol. **53**, 1984, pp. 1515-1518.

[58] E. J. HELLER and R. L. SUNDBERG, Quantum Ergodicity and Intensity Fluctuations, in *Chaotic behavior in quantum systems*, G. CASATI, Ed., NATO ASI, Vol. **B120**, Plenum, New York, 1985.

[59] C. JAFFE and W. P. REINHARDT, Uniform Semiclassical Quantization of Regular and Chaotic Classical Dynamics on the Hénon-Heiles Surface, *J. Chem. Phys.*, Vol. **77**, 1982, pp. 5191-5203.

[60] Y. KATZNELSON, *An introduction to harmonic analysis*, Wiley, New York, 1968.

[61] J. B. KELLER, Corrected Bohr-Sommerfeld Quantum Conditions for Nonseparable Systems, *Ann. Phys.*, Vol. **4**, 1958, pp. 180-188.

[62] H. A. KRAMERS, *Z. Physik*, Vol. **39**, 1926, pp. 828-840.

[63] H. KUNZ, The Quantum Hall Effect for Electron in a Random Potential, *Comm. Math. Phys.*, Vol. **112**, 1987, pp. 121-145.

[64] R. A. MARCUS, Aspects of Intramolecular Dynamics in Chemistry, in *Chaotic behavior in quantum systems*, G. CASATI Ed., NATO ASI, Vol. **B120**, Plenum, New York, 1985.

[65] H. P. MCKEAN, Selberg's Trace Formula as Applied to a Compact Riemann Surface, *Comm. Pure Appl. Math.*, Vol. **25**, 1972, pp. 225-246.

[66] N. N. NEKHOROSHEV, The Behavior of Hamiltonian Systems that Are Close to Integrable Ones, *Funct. Anal. Appl.*, Vol. **5**, 1971, pp. 338-339.

[67] N. N. NEKHOROSHEV, Exponential Estimates of the Time of Stability for Nearly Integrable Hamiltonians, *Russ. Math. Surveys*, Vol. **32**, 1977, pp. 1-63.

[68] L. ONSAGER, Interpretation of the Haas-van Alphen Effect, *Phil. Mag.*, Vol. **43**, 1952, pp. 1006-1008.

[69] G. PEDERSEN, *C*-Algebras and their Automorphisms Groups*, Academic, New York, 1979.

[70] I. C. PERCIVAL, Regular and Irregular Spectra, *J. Phys.*, Vol. **B6L**, 1973, pp. 229-232.

[71] I. C. PERCIVAL, Regular and Irregular Spectra in Molecules, in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems*, G. CASATI and J. FORD Eds., *Lect. Notes Phys.*, Vol. **93**, Springer, Berlin, 1979.

[72] N. POMPHREY, Numerical Identification of Regular and Irregular Spectra, *J. Phys.*, Vol. **B7**, 1974, pp. 1909-1915.

[73] R. RAMMAL, Landau Level Spectrum of Bloch Electron in a Honeycomb Lattice, *J. Phys. France*, Vol. **46**, 1985, pp. 1345-1354.

[73*b*] Y. Y. WANG, PANNETIER and R. RAMMAL, Quasi Classical Approximations for Almost-Mathieu Equation, *J. Phys. France*, Vol. **48**, 1987, pp. 2067-2079.

[74] J. RENAULT, *A Groupoïd Approach to C*-Algebras; Lect. Notes Math.*, Vol. **793**, Springer, Berlin, 1980.

[75] M. ROBNIK and M. V. BERRY, False Time-Reversal Violation and Energy Level Statistics: the Role of Anti-Unitary Symmetry, *J. Phys.*, Vol. **A19**, 1986, pp. 669-682.

[76] A. SELBERG, Harmonic Analysis and Discontinuous Groups in Weakly Symmetric Riemannian Spaces with Applications to Dirichlet Series, *J. Indian. Math. Soc.*, Vol. **20**, 1956, pp. 47-87.

[77] R. B. SHIRTS and W. P. REINHARDT, Approximate Constants of Motion for Classically Chaotic Vibrational Dynamics: Vague Tori, Semiclassical Quantization, and Classical Intramolecular Energy Flow, *J. Chem. Phys.*, Vol. **77**, 1982, pp. 5204-5217.

[78] J. B. TAYLOR, Unpublished (1968) (quoted in [52]).

[79] D. J. THOULESS, M. KOHMOTO, M. P. NIGHTINGALE and M. DEN NIJS, Quantized Hall Conductance in a Two Dimensional Periodic Potential, *Phys. Rev. Lett.*, Vol. **49**, 1982, pp. 405-408.

[80] M. VITTOT, A Simple and Compact Presentation of Birkhoff Series, in *Non Linear Evolution and Chaotic Phenomena*, G. GALLAVOTTI and P. F. ZWEIFEL Eds., Plenum, New York, 1987.

[81] A. VOROS, Asymptotic $\hbar$-Expansions of Stationnary Quantum States, *Ann. I.H.P.*, Vol. **A26**, 1977, pp. 343-403.

[82] M. WILKINSON, Critical Properties of Electrons Eigenstates in Incommensurate Systems, *Proc. R. Soc. London*, Vol. **A391**, 1984, pp. 305-350.

[83] M. WILKINSON, Von Neumann Lattices of Wannier Functions for Bloch Electrons in a Magnetic Field, *Proc. R. Soc. London*, Vol. **A403**, 1986, pp. 135-166.

[84] M. WILKINSON, An Example of Phase Holonomy and WKB Theory, *J. Phys.*, Vol. **A17**, 1984, pp. 3459-3476.

(*Manuscript received May* 2, 1989.)