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Inverse scattering problem for the Maxwell equations outside moving body

by

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ABSTRACT. — We prove the existence of solutions to the Maxwell system for moving obstacles. Moreover, we obtain a representation of the scattering matrix and study the singularities of the matrix. Finally, the results are applied to the inverse scattering problem connected with the recovering of the convex hull of the obstacle.

Key-words: Maxwell, Scattering theory, Inverse problem.

RÉSUMÉ. — On prouve l'existence des solutions pour le systeme de Maxwell dans le cas d'un obstacle mouvant. On obtient une représentation de la matrice de diffusion et on examine la singularité principale de cette matrice. Finalement, on applique ces résultats pour le problème inverse de diffusion lié avec la détermination de l'enveloppe convexe de l'obstacle.

0. INTRODUCTION

Cooper and Strauss examined in [7] the leading singularity of the scattering kernel. In particular, they proved that the convex hull of the obstacle can be recovered from the scattering data.

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In this work we study the same problem for the Maxwell equations outside moving obstacles. Especially, we deal with the following three points:

1. The existence of solutions to the mixed problem with initial data distributions (disturbed plane waves).
2. Representation of the scattering matrix $S^{\#}$ involving disturbed plane waves.
3. Examination of the leading singularity of the kernel $S^{\#}(s', \omega', s, \omega)$ of $S^{\#}$ in the case of back-scattering $\omega' = -\omega$.

Our results combined with the approach of Cooper and Strauss [7] allow us to determine the convex hull of the obstacle.

The first point for the case of the Maxwell equations leads to some difficulties since the moving boundary could be nonuniformly characteristic. For this reason Cooper and Strauss [8] used the symmetry of the Maxwell equations and the results of Kato [15] in order to prove the existence of solutions to corresponding mixed problem with initial data in the energy space. To obtain an existence theorem for initial data distributions we transform the Maxwell equations so that the results in [17], [18], [26], [27] for uniformly characteristic boundaries can be applied.

We obtain a representation of the scattering operator in the spirit of [19], [20], [21], [23] involving a solution to a mixed problem with initial data $\delta(t + s - \langle x, \omega \rangle)$ for large negative time t. This gives us a basis to attack the inverse scattering problem connected with the leading singularity of the scattering kernel. In this problem the construction of microlocal parametrix of the mixed problem plays a crucial role. The fact that the boundary could be nonuniformly characteristic makes some essential difficulties for the construction of parametrix. In fact, denoting by $u = {}^t(E, H)$ the couple of the electric and magnetic fields E and H, we see the Maxwell equations in vacuum has the form

$$\partial_t u = A(\nabla)u, \qquad P(\nabla)u = 0, \tag{0.1}$$

where the differential operators $A(\nabla)$ and $P(\nabla)$ are defined by

$$A(\nabla) = \begin{pmatrix} 0 & \text{rot} \\ -\text{rot} & 0 \end{pmatrix}, \qquad P(\nabla) = \begin{pmatrix} \text{div} & 0 \\ 0 & \text{div} \end{pmatrix}.$$

Here and in the following we use units, in which the light velocity, the magnetic and electric permeabilities in the vacuum are 1. If an electromagnetic field propagates ouside a stationary body, which is a perfect conductor, the vectors E and H can be separated and we obtain two different mixed problems for the vector wave equation (see [20]). Since we study the case, when the obstacle could change its shape and position, the magnetic and electric fields interact on the boundary and they can not be decoupled in the boundary condition. The difficulty is connected with the fact the boundary is not uniformly characteristic and the system (0.1) can not be microlocally diagonalized near the boundary. To overcome this difficulty we propose a suitable microlocal transformation of the Maxwell equations (0.1). Namely, we replace the first equation in (0.1) by

$$\partial_t u - \mathrm{A}(\nabla)u - \mathrm{L}(t, x)\mathrm{P}(\nabla)u = 0,$$

where the matrix $\mathrm{L}(t, x)$ will be chosen appropriately. Following this way we reduce the Maxwell equations to a system with noncharacteristic boundary and this simplifies considerably the construction of the microlocal parametrix near the points of the first reflection of the incoming wave.

There is a lot of papers in the physical literature treating inverse problems for the Maxwell equations outside moving bodies (see [*33*] for references). Nevertheless, it seems that the inverse problem for electromagnetic waves reflecting from moving boundaries has not been treated rigorously.

The plan of the work is the following. In section 2 we construct a translation representation of the Maxwell equations. The existence and uniqueness of solutions to mixed problem associated with the Maxwell equations are discussed in section 3. In section 4 we obtain a representation of the scattering (echo) kernel. Localization of the singularities in the spirit of [*7*], [*19*], [*24*] is done in section 5. Section 6 is devoted to the transformation of the Maxwell equations and the construction of a microlocal parametrix to the mixed problem for the Maxwell system. Finally, in section 7 we obtain the leading term of the scattering amplitude and prove theorems 1, 2 and corollary 3.

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1. ASSUMPTIONS AND MAIN RESULTS

We assume the electromagnetic wave propagates in an exterior space time region $\mathrm{Q} \subset \mathbb{R}^4$ with smooth boundary $\partial\mathrm{Q}$. Denoting by $\nu = (\nu_t, \nu_x)$ the unit spacetime normal to $\partial\mathrm{Q}$ pointed into Q, we make the assumption

(H_1) The normal ν at $\partial\mathrm{Q}$ is spacelike, i. e. $|\nu_t| < |\nu_x|$ on $\partial\mathrm{Q}$.

This assumption means the boundary moves slower than the wave. The body at time t is the set $\mathrm{K}(t) = \{ x; (t, x) \notin \mathrm{Q} \}$. We shall assume that there exists a ball of radius $\rho > 0$ covering the body at any time, i. e.

$$(\mathrm{H}_2) \qquad \mathrm{K}(t) \subseteq \{ x; |x| < \rho \} \quad \text{for} \quad t \in \mathbb{R}.$$

If the surface of the obstacle is a perfect conductor, the electromagnetic wave is connected with the following mixed problem (see [8]).

$$(1.1)\qquad \begin{cases} \partial_t u = A(\nabla)u, & P(\nabla)u = 0 \quad \text{in} \quad Q, \\ \nu_t H + \nu_x \times E = 0, & \langle \nu_x, H \rangle = 0 \quad \text{on} \quad \partial Q, \\ u = f \quad \text{for} \quad t = t_1. \end{cases}$$

One can use the results in [8] (see section 3 too) and define a family of Hilbert spaces $H(t)$, $t \in \mathbb{R}$, and a two parameter group $V(t_2, t_1)$ of operators acting from $H(t_1)$ into $H(t_2)$, such that the solution to (1.1) can be represented in the form $u = V(t, t_1)f$ provided $f \in H(t_1)$. Indeed, given any open subset $U \subset \mathbb{R}^3$, we denote by $L_2(U; \mathbb{R}^k)$ (resp. $C_0^\infty(U; \mathbb{R}^k)$) the spaces of functions $f(x) = {}^t(f_1(x), \ldots, f_k(x))$, such that $f_j(x) \in L_2(U)$, ($f_j(x) \in C_0^\infty(U)$), $j = 1, \ldots, k$. Setting $\Omega(t) = \{ x ; (t, x) \in Q \}$ we define the Hilbert space $H(t)$ as the closure of functions $f = (f_1, f_2)$, $f_j \in C_0^\infty(\Omega(t); \mathbb{R}^3)$, $j = 1, 2$, satisfying the boundary condition

$$\nu_t f_2 + \nu_x \times f_1 = 0, \qquad \langle \nu_x, f_2 \rangle = 0 \quad \text{on} \quad \partial\Omega(t)$$

and $P(\nabla)f = 0$ in the sense of distributions on $\Omega(t)$ with respect to the L_2-norm in $\Omega(t)$.

The wave operators connect the solution to the mixed problem (1.1) with a solution to the free Maxwell equations

$$(1.2)\qquad \begin{cases} \partial_t u = A(\nabla)u, \quad P(\nabla)u = 0 \quad \text{in} \quad \mathbb{R}^4, \\ u(t_1, x) = g(x), \end{cases}$$

where $g(x) \in H_0$ and H_0 is the Hilbert space of functions $g = (g_1, g_2)$, $g_j \in L_2(\mathbb{R}^3; \mathbb{R}^3)$, $j = 1, 2$, satisfying $P(\nabla)g = 0$ in the sense of distributions on $\mathbb{R}^3$. The unperturbed Hilbert space H_0 can also be defined as the absolutely continuous Hilbert subspace $H_{ac}(A_0)$ of $L_2(\mathbb{R}^3; \mathbb{R}^6)$ with respect to the self-adjoint operator $A_0 = -iA(\nabla)$. The solution to (1.2) can be represented as $u = U_0(t - t_1)g$, where $U_0(t)$ is a unitary group of operators acting in H_0. Then the scattering operator can be defined by (see [16], [24])

$$(1.3)\qquad S(g) = \lim_{t \to +\infty} U_0(-t)H^*(t)V(t, -t)H(-t)U_0(-t)g,$$

where $g \in H_0$. For simplicity of symbols $H(t)$ will denote the Hilbert space introduced above or the orthogonal projection from H_0 on the Hilbert space $H(t)$. If $f \in H_0$ then $H(t)f$ is the restriction of f onto $\Omega(t) = \{ x ; (t, x) \in Q \}$. Moreover $H^*(t)$ is the operator adjoint to $H(t)$.

Our last assumption is

(H_3) { The scattering operator S exists and S is a bounded operator in H_0.

The above assumption can be weaken if the motion of the obstacle is periodic (see [5], [6]). Moreover, by using the approach in [1], [2] one can prove the above assumption provided the energy of the perturbed system is uniformly bounded. Since our main goal is the leading term of the scattering amplitude we shall not search for sufficient conditions for (H_3).

To state our main results introduce the vector bundle N over the unit sphere $\mathbb{S}^2$, such that any fiber $N(\omega)$ over $\omega \in \mathbb{S}^2$ consists of vectors $k = {}^t(k_1, k_2)$, $k_1, k_2 \in \mathbb{R}^3$, such that $k_1 \perp \omega$ and $k_2 = \omega \times k_1$. Any section of the bundle N is a function $k(\omega)$ from $\mathbb{S}^2$ into $\mathbb{R}^6$. Then the translation representation, constructed in section 2 is a unitary operator $\mathcal{R}$ from H_0 onto $L_2(\mathbb{R}, L_2(N))$, where $L_2(N)$ is the Hilbert space of square integrable sections of the fibre bundle N. By using the Schwartz kernel theorem and the assumption (H_3) one can find the kernel of the operator $\mathcal{R} \circ S \circ \mathcal{R}^{-1}$ and connect this kernel with the scattering (echo) kernel $S^{\#}(s', \omega', s, \omega)$, which is a (6×6) matrix valued distribution for $(s', \omega', s, \omega) \in \mathbb{R} \times \mathbb{S}^2 \times \mathbb{R} \times \mathbb{S}^2$ (see section 4). An important role in the investigation of the scattering amplitude is played by the arrival surface

$$\Gamma(s, \omega) = \{ x \in \mathbb{R}^3 ; (\langle x, \omega \rangle - s, x) \in \partial Q \}, \tag{1.4}$$

which is the projection of the intersection of the arrival plane $\langle x, \omega \rangle = t + s$ and the boundary ∂Q onto the space of variables x. As in the case of the wave equation the singularities of the scattering kernel $S^{\#}(s', \omega', s, \omega)$ with respect to s' are closely connected with the quantity

$$h(s, \omega) = -2 \min \{ \langle x, \omega \rangle ; x \in \Gamma(s, \omega) \}. \tag{1.5}$$

The first result in the work is

THEOREM 1. — Let $s, \omega, \omega' = -\omega$ be fixed. Then we have the properties:

a) $\max \operatorname{supp}_{s'} S^{\#} \leq s + h(s, \omega)$,

b) $\max \operatorname{sing\,supp}_{s'} S^{\#} = s + h(s, \omega)$.

In order to study the singularities of the scattering kernel near $s' = s + h(s, \omega)$ introduce the filterred scattering amplitude (see [*20*]).

$$a(\lambda) = \int S^{\#}(s', -\omega, s, \omega)\varphi(s') \exp(-i\lambda s')ds', \tag{1.6}$$

where $\varphi(s')$ is a smooth cut-off function, which is 1 near $s' = s + h$ and the integral is taken in the sense of distributions.

In order to state our main result set

$$R(s, \omega) = \{ x \in \Gamma(s, \omega); \langle x, \omega \rangle = -h(s, \omega)/2 \}.$$

Given any $(t, x) \in \partial Q$ we denote by $\gamma(t, x)$ the surface velocity at (t, x), i. e.

$$\gamma(t, x) = v_t(t, x)/|v_x(t, x)|.$$

If $x \in \Gamma(s, \omega)$ we set

$$\gamma(x) = \gamma(\langle x, \omega \rangle - s, x).$$

The second result is

THEOREM 2. — There exists a dense subset $T \subseteq \mathbb{R} \times \mathbb{S}^2$, such that given any $(s, \omega) \in T$ the filterred scattering amplitude (1.6) has the following asymptotic expansion near $s' = s + h(s, \omega)$ as $\lambda \to +\infty$

$$(1.7) \quad a(\lambda) = -i\lambda(4\pi)^{-1}b(\omega)\exp(-i\lambda s')\sum_{j=1}^{k}\frac{(1+\gamma_j)}{(1-\gamma_j)}K(x_j)^{-1/2}+0(1),$$

where $x_1, x_2, \ldots, x_k$ are discrete points forming the set $R(s, \omega)$, $K(x_j)$ is the Gauss curvature at $x_j \in \Gamma(s, \omega)$, $\gamma_j = \gamma(x_j)$ and $b(\omega)$ is the linear operator acting on the fibres $N(\omega)$ of the bundle N as the (6×6) matrix

$$\begin{pmatrix} -I_3 & 0 \\ 0 & I_3 \end{pmatrix}.$$

REMARK 1. — The dense subset $T \subset \mathbb{R} \times \mathbb{S}^2$ is chosen so that the set $R(s, \omega) = \{ x \in \Gamma(s, \omega) ; -2\langle x, \omega \rangle = h(s, \omega) \}$ consists of finite number of isolated points $x_1, \ldots, x_k$. The choice of T is done by using Sard's theorem (see lemma 7.5).

REMARK 2. — We see that the leading term in the asymptotic expansion (1.7) is similar to the leading term obtained in [*20*]. The influence of the moving body is connected with the Dopler factor $(1 + \gamma_j)/(1 - \gamma_j)$. Since the Gauss curvature $K(x_j)$ with respect to $\Gamma(s, \omega)$ can be represented by (see [*7*])

$$K(x_j) = (1 - \gamma_j)^{-2}K'(x_j).$$

$K'(x_j)$ is the Gauss curvature with respect to the boundary $\partial\Omega(t)$ of $\Omega(t)$, we can rewrite (1.7) in the form

$$(1.8) \quad a(\lambda) = -i\lambda(4\pi)^{-1}b(\omega)\exp(-i\lambda s')\sum_{j=1}^{k}K'(x_j)^{-1/2}(1+\gamma_j) + 0(1).$$

REMARK 3. — The above theorem shows that the leading singularity of the scattering matrix $S(s', -\omega, s, \omega)$ is

$$\delta'(s' - s - h(s, \omega))(4\pi)^{-1}b(\omega)\sum_{j=1}^{k}K(x_j)^{-1/2}(1 + \gamma_j)/(1 - \gamma_j).$$

Following the approach in [*7*], [*19*] one obtains

COROLLARY 3. — The convex hull of the obstacle $\Omega(t) = \{ x ; (x, t) \in Q \}$ at any instant t can be recovered from the support of the back-scattering data of the scattering (echo) kernel $S^{\#}$.

2. TRANSLATION REPRESENTATION FOR THE FREE MAXWELL EQUATIONS

Translation representation for the free Maxwell equations was obtained in [*6*], [*28*]. We construct this representation in a form suitable for our investigations and closely connected with the Radon transform

$$R(f)(s, \omega) = \int_{\langle x, \omega \rangle = s} f(x) dS_x .$$

Here $\langle , \rangle$ denotes the inner product in $\mathbb{R}^3$, $s \in \mathbb{R}$, ω is unit vector in $\mathbb{R}^3$, while dS_x is the surface measure on the plane $\langle x, \omega \rangle = s$. Moreover $f(x) \in S(\mathbb{R}^3)$-the space of smooth functions tending to 0 at infinity faster than any polynomial of $1/|x|$. Consider the matrix

$$A(\xi) = \begin{pmatrix} 0 & D(\xi) \\ -D(\xi) & 0 \end{pmatrix}, \qquad \xi \in \mathbb{R}^3 - \{0\},$$

where $D(\xi)$ is the linear operator in $\mathbb{R}^3$ defined by $D(\xi) = \xi \times v$, the vector product of the vectors ξ and v in $\mathbb{R}^3$. The matrix $A(\xi)$ is the symbol of the operator $1/i\begin{pmatrix} 0 & \text{rot} \\ -\text{rot} & 0 \end{pmatrix}$ of the Maxwell equations. The eigenvalues of the matrix $A(\xi)$ are $-|\xi|$, 0 and $|\xi|$. They have constant multiplicity 2 for ξ nonzero. The corresponding projectors are

$$\pi_\sigma(\xi) = 1/2\begin{pmatrix} \pi(\xi) & \sigma D(\xi) \\ -\sigma D(\xi) & \pi(\xi) \end{pmatrix}, \quad \sigma = \pm, \quad \pi_0(\xi) = \begin{pmatrix} I - \pi(\xi) & 0 \\ 0 & I - \pi(\xi) \end{pmatrix},$$

where $\pi(\omega) = -D(\omega)D(\omega)$ is the projection in $\mathbb{R}^3$ on the plane orthogonal to the vector $\omega \in \mathbb{S}^2$. The projections satisfy the properties

$$(2.1) \qquad \begin{aligned} &a) \quad A(\omega)\pi_\pm(\omega) = \pm \pi_\pm(\omega), \qquad A(\omega)\pi_0(\omega) = 0, \\ &b) \quad \pi_+(\omega)\pi_-(\omega) = \pi_0(\omega)\pi_0(\omega) = 0, \qquad \sigma = \pm, \\ &c) \quad \pi_+(\omega), \pi_-(\omega) \quad \text{and} \quad \pi_0(\omega) \text{ commute}, \\ &d) \quad \pi_0(\omega) = \pi_{-\sigma}(-\omega), \qquad \sigma = \pm, \\ &e) \quad \pi_0(\omega) = \pi_0(-\omega), \end{aligned}$$

which follows directly from the definition of the projections.

The main goal of this section is to construct a Hilbert space $\mathbb{N}$ and a unitary operator

$$\mathcal{R} : H_{ac}(A_0) \to L_2(\mathbb{R}, \mathbb{N}),$$

such that the unperturbed group $U_0(t)$, introduced in section 1 can be represented by $U_0(t) = \mathscr{R}^{-1}T_t\mathscr{R}$, with T_t the operator of the right translation in $L_2(\mathbb{R}, \mathbb{N})$. For this purpose denote by $\mathbb{N}$ the Hilbert space

$$\mathbb{N} =. \{ k(\omega) \in L_2(\mathbb{S}^2, \mathbb{R}^6) ; \pi_-(\omega)k(\omega) = k(\omega) \}.$$

It is easy to see that any $k \in \mathbb{N}$ has the form ${}^t(k_1, k_2)$ with

$$(2.2) \qquad k_1(\omega) \perp \omega, \qquad k_2(\omega) = \omega \times k_1(\omega) \quad \text{in} \quad \mathbb{R}^3,$$

The properties (2.2) were used in [6] to characterize the translation representation as an asymptotic wave profil. We introduce the translation representation by the equality

$$(2.3) \qquad \mathscr{R}_{tr} = c\pi_-(\omega)\partial_s R,$$

where

$$(2.4) \qquad c = 1/(2\pi).$$

THEOREM 1. — The operator $\mathscr{R}_{tr} = c\pi_-(\omega)\partial_s R$, $c=(2\pi)^{-1}$, can be extended as a unitary operator from $H_{ac}(A_0)$ onto $L_2(\mathbb{R}, \mathbb{N})$, such that $U_0(t) = \mathscr{R}_{tr}^{-1}T_t\mathscr{R}_{tr}$ and the equality

$$(2.5) \qquad \mathscr{R}_{tr}^{-i}(\pi_-(\omega)k(s, \omega)) = - c \int_{|\omega|=1} \pi_-(\omega)\partial_s k(\langle x, \omega \rangle, \omega)d\omega$$

holds for any $k \in S(\mathbb{R} \times \mathbb{S}^2, \mathbb{R}^6)$.

Proof. — By the aid of the Fourier transform

$$\hat{f}(\xi) = \int f(x) \exp(- i \langle x, \xi \rangle)dx, \qquad f \in S(\mathbb{R}^3, \mathbb{R}^6)$$

it is easy to see that given any $f \in S(\mathbb{R}^3, \mathbb{R}^6)$ the properties

$$(2.6) \qquad a) \quad f \in H_{ac}(A_0),$$

$$(2.6) \qquad b) \quad f(\xi) \perp \operatorname{Ker} A(\xi) = \operatorname{Im} \pi_0(\xi) \text{ for } \xi \text{ nonzero}$$

are equivalent. The equality (see [11])

$$R(f)(s, \omega) = 1/(2\pi) \int_{-\infty}^{+\infty} \hat{f}(\sigma\omega) \exp(- is\sigma)d\sigma$$

together with the equivalence of the properties (2.6) imply

$$(2.7) \qquad a) \quad H_{ac}(A_0) \cap S(\mathbb{R}^3, \mathbb{R}^6) \quad \text{is dense in } H_{ac}(A_0),$$

$$(2.7) \qquad b) \quad R(f)(s, \omega) = \pi_+(\omega)R(f) + \pi_-(\omega)R(f).$$

Now we use the following identity (see [11])

$$\int_{\mathbb{R}} \int_{\mathbb{S}^2} | \partial_s R(f) |^2 d\omega ds = 2(2\pi)^2 \int_{\mathbb{R}^3} | f(x) |^2 dx$$

and obtain the equality

$$\int_{\mathbb{R}^3} | f(x) |^2 dx = 1/(8\pi^2)(I_+ + I_-),$$

where
$$I_\sigma = \int_{\mathbb{R}} \int_{\mathbb{S}^2} | \pi_\sigma(\omega)\partial_s R(f) |^2 d\omega\, ds, \quad \sigma = \pm .$$

The function $\partial_s R(f)(s, \omega)$ is an odd one. This fact and the property (2.1)d) enable one to change the variables $s' = -s$, $\omega' = -\omega$ in I_+ and derive that $I_+ = I_-$. Now we are going to

$$\int_{\mathbb{R}^3} | f(x) |^2 dx = \int_{\mathbb{R}} \int_{\mathbb{S}^2} | \mathscr{R}_{tr}(f)(s, \omega) |^2 d\omega ds$$

with $\mathscr{R}_{tr} = c\pi_-(\omega)\partial_s R$. The density property in (2.7) shows $\mathscr{R}_{tr}$ can be extended as an isometry.

In order to recover $f(x)$ from $\pi_-(\omega)k(s, \omega) = \mathscr{R}(f)(s, \omega)$ we use the identity (see [*10*])

$$f(x) = -1/(8\pi^2) \int_{\mathbb{S}^2} \partial_s^2 R(f)(\langle x, \omega \rangle, \omega) d\omega \tag{2.8}$$

provided $f \in S(\mathbb{R}^3, \mathbb{R}^6)$. Taking advantage of (2.7) and (2.8) we get $-8\pi^2 f(x) = J_+ + J_-$, where

$$J_\sigma = \int_{\mathbb{S}^2} \pi_\sigma(\omega)\partial_s^2 R(f)(\langle x, \omega \rangle, \omega) d\omega, \qquad \sigma = \pm .$$

Making the change of variables $\omega' = -\omega$ and by using the properties (2.1) together with the fact that $\partial_s^2 R(f)(s, \omega)$ is an even function we get $J_+ = J_-$. This observation leads to the equality

$$-4\pi^2 f(x) = J_-,$$

from which we obtain (2.5).

Finally, we shall verify the equality $T_t = \mathscr{R}_{tr}^{-i} U_0(t)\mathscr{R}_{tr}$. Applying the transform $\mathscr{R}_{tr}$ to the Maxwell equations and using the properties (2.1) of the projection π_- we obtain the equations

$$\partial_t m = -\partial_s m, \qquad m(s, \omega, 0) = k(s, \omega),$$

where

$$m(s, \omega, t) = \mathscr{R}_{tr}\begin{pmatrix} E(t, x) \\ H(t, x) \end{pmatrix}, \qquad k(s, \omega) = m(s, \omega, 0).$$

The Cauchy problem for m can be resolved straightforward. We find $h(s, \omega, t) = k(s - t, \omega)$. This proves the translation property $RU_0(t)\mathscr{R}^{-i} = T_t$ and completes the proof of the theorem.

COROLLARY 2.2. — Suppose $f \in H_{ac}(A_0) \cap S(\mathbb{R}^3, \mathbb{R}^6)$ and $k(s,\omega)=\mathscr{R}_{tr}(f)$ is the translation representation of f. Then we have

$$U_0(t)f = -c \int_{\mathbb{S}^2} \partial_s k(\langle x, \omega \rangle - t, \omega) d\omega .$$

Proof. — It is sufficient to exploit the translation property

$$U_0(t) = \mathscr{R}_{tr}^{-i} T_t \mathscr{R}_{tr}$$

together with the inverse formula (2.5). This completes the proof.

Next we introduce the deparing and entering spaces D^ρ and E^ρ (see [6]).

DEFINITION 2.1. — Given any $\rho \in \mathbb{R}$, the deparing and entering spaces are defined by

$$D^\rho = \{ f \in H_{ac}(A_0);\ \text{supp}_s\, \mathscr{R}_{tr}(f)(s, \omega) \subset [\rho, +\infty) \},$$
$$E^\rho = \{ f \in H_{ac}(A_0);\ \text{supp}_s\, \mathscr{R}_{tr}(f)(s, \omega) \subset (-\infty, \rho] \}.$$

We close this section with the following result obtained by Cooper and Strauss in [6]

PROPOSITION 2.3 (see [6]). — We have the equalities

$$D^\rho = \{ f \in H_{ac}(A_0);\ U_0(t)f = 0 \quad \text{for} \quad |x| < \rho + t, \quad t > -\rho \},$$
$$E^\rho = \{ f \in H_{ac}(A_0);\ U_0(t)f = 0 \quad \text{for} \quad |x| < -\rho - t, \quad t < -\rho \}.$$

3. SOLUTIONS TO THE PERTURBED SYSTEM

An important role for the representation of the scattering matrix will play the solution $G = (G', G'')$ to the mixed problem

$$(3.1) \qquad \begin{cases} \partial_t G = A(\nabla)G, \qquad P(\nabla)G = 0 \quad \text{in} \quad Q, \\ \Lambda(G + f) = 0 \quad \text{on} \quad \partial Q, \\ G(t, x) = 0 \quad \text{for} \quad t = t_1 \ll 0, \end{cases}$$

where $f = \delta'(t + s - \langle x, \omega \rangle)v$ and $v \in \mathbb{R}^6$ is any vector in the fibre $N(\omega)= \{ v=(v', v''); v' \perp \omega, v''=\omega \times v' \}$. The matrix Λ of the boundary condition is determined by (1.1), i.e. given any vector $w=(E, H) \in \mathbb{R}^3 \times \mathbb{R}^3$ we have $\Lambda(t, x)w=(v_t H + v_x \times E, \langle v_x, H \rangle)$. Setting $G= -\delta' t + s - \langle x, \omega \rangle)v + \partial_s^4 u$ one can reduce the mixed problem (3.1) to the following one for u

$$(3.2) \qquad \begin{cases} \partial_t u = A(\nabla)u, \qquad P(\nabla)u = 0 \quad \text{in} \quad Q, \\ \Lambda(u) = 0 \quad \text{on} \quad \partial Q, \\ u(t, x) = g \quad \text{for} \quad t = t_1 \ll 0, \end{cases}$$

where $g(x) = d(t_1 + s - \langle x, \omega \rangle)v$ and the function $d(s)$ is 0 for $s < 0$ and $d(s) = s^2/2$ for $s \geq 0$. We shall use the notations

$$Q(t_2, t_1) = \{ (t, x) \in Q ; t_1 < t < t_2 \},$$
$$\partial Q(t_2, t_1) = \{ (t, x) \in \partial Q ; t_1 < t < t_2 \},$$
$$\Omega(t) = \{ x ; (t, x) \in Q \}.$$

Introduce the local space $H_{loc}(t)$ formed by $f = (f^1, f^2)$, $f^j(x) \in L^2_{loc}(\Omega(t); R^3)$, such that $\varphi(x) f \in H(t)$ for any $\varphi(x) \in C^\infty_{(0)}(\Omega(t))$. It is easy to see that $g \in H_{loc}(t_1)$. Following [18], [26], [27] introduce

DEFINITION 3.1. — A function $u(t, x) \in L^2_{loc}(Q(t_2, t_1); \mathbb{R}^6)$ is a (strong) solution to (3.2) if there exists a sequence of functions $u_k(t, x) \in C^\infty(\overline{Q(t_2, t_1)})$, such that $\Lambda(u_k) = 0$ on $\partial Q(t_2, t_1)$ and the following properties are fulfilled

$$(3.3) \quad \begin{cases} a) & \partial_t u_k - A(\nabla) u_k \text{ tends to 0 in } L^2_{loc}(Q(t_2, t_1); \mathbb{R}^6), \\ b) & P(\nabla) u_k \text{ tends to 0 in } L^2_{loc}(Q(t_2, t_1); \mathbb{R}^6), \\ c) & u_k(t_1, x) \text{ tends to } g(x) \text{ in } L^2_{loc}(\Omega(t); \mathbb{R}^6). \end{cases}$$

REMARK 3.2. — In the case, when $g(x) \in H(t)$, the Hilbert space introduced in section 1, one can replace L^2_{loc} by L^2 and C^∞ by $C^\infty_{(0)}$ in the above definition.

THEOREM 3.3. — Suppose $g \in H_{loc}(t_1)$. Then there exists a unique strong solution to the mixed problem (3.2) for the Maxwell equations.

Proof. — By using the principle of causality (see [8]) and the assumption (H_2) that the obstacle lies in a fixed ball one can reduce the mixed problem to a similar problem where $Q(t_2, t_1)$ and $\partial Q(t_2, t_1)$ are replaced by $Q \cap V$ and $\partial Q \cap V$, where V is a sufficiently small neighbourhood in $\mathbb{R}^4$ intersecting ∂Q and the plane $t = t_1$. The solution $u = (E, H)$ describes an electromagnetic wave, for which one can define the field strength tensor F_{jk} in the coordinate system (t, x) by

$$(3.4) \quad \begin{cases} F_{01} = -E_1, \quad F_{02} = -E_2, \quad F_{03} = -E_3, \\ F_{12} = H_3, \quad F_{31} = H_2, \quad F_{23} = H_1. \end{cases}$$

The tensor F_{jk} is antisymmetric, i. e. $F_{jk} = -F_{kj}$, and hence (3.4) defines correctly this tensor. Since we shall use the transformation laws of tensors under coordinate transformations, we shall denote the coordinates by upper indices, $x^0 = t, x^1, x^2, x^3$ and shall use the rule of changing the upper and lower indices by means of the metric tensor, which in the coordinate system x^j has the form $\eta_{jk} = \text{diag}(1, -1, -1, -1)$. For instance $F^{jk} = \eta^{js}\eta^{kr}F_{sr}$, where $\{ \eta^{jk} \}$ is the inverse matrix to $\{ \eta_{jk} \}$. Notice that here and below in this section we use the summation convention for repeated upper and lower indices running from 0 to 3. Then the mixed problem (3.2) locally in V reads (see [32])

$$(3.5)\quad \begin{cases} \partial_j F^{jk} = 0, \quad \partial_j F_{kr} + \partial_r F_{jk} + \partial_k F_{rj} = 0 \quad \text{in} \quad Q \cap V, \\ \nu_j(^*F^{jk}) = 0 \quad \text{on} \quad \partial Q \cap V, \\ F^{jk} = \overset{\circ}{F}{}^{jk} \quad \text{for} \quad x^0 = t_1 \ll 0, \end{cases}$$

where $\partial_j = \partial_{x^j}$, $* F^{jk} = 1/2\, \varepsilon^{jkmn} F_{mn}$ is the tensor dual to F_{mn}, ε^{jkmn} are the Levi-Civita symbols and $\overset{\circ}{F}{}^{jk}$ is determined by (3.4) and the initial data in (3.2). Let the boundary ∂Q be determined by $x^3 = \mu(x^0, x^1, x^2)$ locally in V, while the domain Q is defined by $x^3 < \mu$. Changing locally the variables

$$(3.6)\quad y^0 = x^0,\ y^1 = x^1,\ y^2 = x^2,\ y^3 = x^3 - \mu(x^0,\ x^1,\ x^2),$$

we see that the metric tensor in the new variables y^j becomes $g^{jk} = \frac{\partial y^j}{\partial x^s}\frac{\partial y^k}{\partial x^r}\eta^{sr}$. Similarly $f^{jk} = \frac{\partial y^j}{\partial x^s}\frac{\partial y^k}{\partial x^r} F^{sr}$ represents the strength tensor in the new coordinates y^j. Finally, F_{jk} in the new coordinates becomes $f_{sr} = \frac{\partial y^j}{\partial x^s}\frac{\partial y^k}{\partial x^r} F_{jk}$. Moreover, the Jacobian of the transformation (3.6) is 1 and the metric volume in the new coordinates is $g = \det(g^{jk}) = -1 = \det(\eta^{jk})$. Therefore, the Maxwell equations in the new coordinate system y^j can be written in the form (see [32]).

$$\partial'_j f^{jk} = 0, \qquad \partial'_j f_{kr} + \partial'_r f_{jk} + \partial'_k f_{rj} = 0,$$

where $\partial'_j = \partial_{y^j}$. Setting $\phi(x) = x^3 - \mu(x^0, x^1, x^2)$ we see that the boundary condition in (3.5) is $(* F^{jk})\partial_j \phi = 0$ on ∂Q. In the new coordinates y^j it becomes $\varepsilon^{3kmn} f_{mn} = 0$. The initial data are determined by $f^{jk} = a^j_s a^k_m \overset{\circ}{F}{}^{sm}$ at $y^0 = t_1$, where $a^j_s = (\partial y^j)/(\partial x^s)$ at $y^0 = t_1$. Thus we are going to the mixed problem

$$(3.7)\quad \begin{cases} \partial'_j f^{jk} = 0, \partial'_j f_{kr} + \partial'_r f_{jk} + \partial'_k f_{rj} = 0 \quad \text{on} \quad \{ y^3 < 0 \} \cap V^*, \\ \varepsilon^{3kmn} f_{mn} = 0 \quad \text{on} \quad \{ y^3 = 0 \} \cap V^*, \\ f^{jk} = a^j_s a^k_m \overset{\circ}{F}{}^{sm} \quad \text{at} \quad y^0 = t_1. \end{cases}$$

Here V* is the image of V under the transformation (3.6). As usual (see [8], [32]), one can introduce the vectors e, b, d, h called respectively electric field, magnetic induction, deplacement and magnetic field by the equalities

$$(3.8)\quad \begin{cases} e_j = f_{j0}, & b_1 = f_{23}, & b_2 = f_{31}, & b_3 = f_{12}, \\ d_j = f^{0j}, & h_1 = f^{23}, & h_2 = f^{31}, & h_3 = f^{12}. \end{cases}$$

Then the mixed problem (3.7) becomes

$$(3.9)\quad \begin{cases} \partial_0' d - \operatorname{rot}' h = 0, & \operatorname{div}' d = 0 \quad \text{on} \quad \{ y^3 < 0 \} \cap V^*, \\ \partial_0' b + \operatorname{rot}' e = 0, & \operatorname{div}' b = 0 \quad \text{on} \quad \{ y^3 < 0 \} \cap V^*, \\ e_1 = e_2 = b_3 = 0 \quad \text{on} \quad \{ y^3 = 0 \} \cap V^*, \\ d = d_0, \qquad h = h_0 \quad \text{for} \quad y^0 = t_1 . \end{cases}$$

Next by using the assumption (H_1) together with the results in [8] one can prove that ${}^t(d, b) = M(t, x)^t(e, h)$ for some positive symmetric (6×6) matrix depending only on the function μ. Since the boundary conditions are fulfilled, for $y^0 = 0$ we have

$$(3.10)\qquad b_3 = 0 \quad \text{on} \quad \{ y^3 = 0 \} \cap \{ y^0 = 0 \} \cap V^*.$$

To prove the existence and uniqueness of a strong solution to (3.9) consider the mixed problem

$$(3.11)\quad \begin{cases} \partial_0' d - \operatorname{rot}' h = 0 \quad \text{on} \quad \{ y^3 < 0 \} \cap V^*, \\ \partial_0' b + \operatorname{rot}' e = 0 \quad \text{on} \quad \{ y^3 < 0 \} \cap V^*, \\ e_1 = e_2 = 0 \quad \text{on} \quad \{ y^3 = 0 \} \cap V^*, \\ d = d_0, \qquad h = h_0 \quad \text{for} \quad y^0 = t_1 . \end{cases}$$

By using the results in [18], [27] one can prove the existence and uniqueness of a strong solution to the above problem since the boundary is uniformly characteristic and the boundary condition is conservative one. To approximate any solution to (3.11) by smooth functions one can use Friedrich's type molifiers. More precisely, choose $j \in C_0^\infty(\mathbb{R}^3)$, such that $\operatorname{supp} j \subseteq \{ y^0 > 0 \}$ and $\int j(y')dy' = 1$, $y' = (y^0, y^1, y^2)$. Set

$$e^\varepsilon = \int e(y' + \varepsilon z', y^3) j(z')dz', \; z' = (z^0, z^1, z^2),$$

$$h^\varepsilon = \int h(y' + \varepsilon z', y^3) j(z')dz', \; {}^t(d^\varepsilon, b^\varepsilon) = M^t(e^\varepsilon, h^\varepsilon).$$

Then e^ε, h^ε, b^ε, d^ε are smooth with respect to y' and satisfy

$$(3.12)\quad \partial_0' d^\varepsilon - \operatorname{rot}' h^\varepsilon = \partial_0' b + \operatorname{rot}' e^\varepsilon = 0 \quad \text{in} \quad \{ y^3 < 0 \} \cap V^* .$$

On the other hand, in the coordinates x^k the electric and magnetic fields satisfy for $x^0 = t_1$ the equalities

$$(3.13)\qquad \operatorname{div} E = \operatorname{div} H = 0$$

in the sense of distributions in $\Omega(t_1)$. From (3.12), (3.13) and the transformation laws of f^{jk} and f_{jk} one obtains $\operatorname{div}' \mathring{d}^\varepsilon = \operatorname{div}' \mathring{b}^\varepsilon = 0$, where $\mathring{d}^\varepsilon$ and $\mathring{b}^\varepsilon$ are the traces of d^ε and b^ε on $y^3 = 0$. By using once more (3.12) we get $\mathrm{div}'\, d^\varepsilon = \mathrm{div}'\, b^\varepsilon = 0$ in $\{ y^3 < 0 \} \cap V^*$ and conclude that (3.13) together with the above equalities can be written in the form

$$[A_3(y)\partial'_3 + B(y, \partial_{y'})]^t(e^\varepsilon, h^\varepsilon) = 0,$$

where $A_3(y)$ is an invertible (6×6) matrix. Therefore $e^\varepsilon, h^\varepsilon, d^\varepsilon, b^\varepsilon$ are smooth functions of $y \in \{ y^3 \leq 0 \} \cap V^*$. From the Maxwell equation (3.12) we obtain $\partial'_0 b^\varepsilon_3 + \partial'_1 e^\varepsilon_2 - \partial'_2 e^\varepsilon_1 = 0$. Choosing $y^3 = 0$ and using the boundary condition in (3.11) we get $0 = e^\varepsilon_1 = e^\varepsilon_2 = \partial'_0 b^\varepsilon_3$ on $y^3 = 0$. Now the property (3.10) implies that $b^\varepsilon_3 = 0$ on $y^3 = 0$. Therefore, (3.9) holds with e, h, d, b replaced by $e^\varepsilon, h^\varepsilon, d^\varepsilon, b^\varepsilon$.

This completes the proof of the Theorem.

REMARKS 3.4. — *a*) Replacing L^2_{loc} by L^2 and C^∞ by C^∞_0 in definition 3.1, one can prove the existence and uniqueness of the solutions to the mixed problems with the initial data in $H(t)$. Therefore, one obtains a result similar to the one obtained in [*8*] and can construct the operator $V(t, s)$ acting from $H(s)$ into $H(t)$.

b) By using the techniques developed in [*26*], [*27*] one can prove H^s estimates assuming the initial data in (3.2) satisfies certain compatibility conditions.

c) Extending $b^\varepsilon, h^\varepsilon, e^\varepsilon, d^\varepsilon$ as 0 for $y^3 \geq 0$ and using the inverse transformation to (3.6), one can prove the injection $H(t) \subset H_0$.

d) The above Theorem shows that the mixed problem (3.2) for $u = {}^t(E, H)$ can be transformed locally in V under the transform (3.6) and $w(y) = N(y)u$ into the following mixed problem for w

$$\begin{aligned} &\partial'_0(M(y)w) = A(\nabla')w, \qquad P(\nabla')(Mw) = 0 \quad \text{in} \quad \{ y^3 < 0 \} \cap V^*, \\ &\Lambda(y)(N^{-1}w) = \text{on} \quad \{ y^3 = 0 \} \cap V^*, \\ &w = Ng \quad \text{for} \quad y^0 = t_1 \ll 0, \end{aligned}$$

where $M(y)$ is a positive (6×6) matrix defined by the equation

$$\begin{pmatrix} d \\ b \end{pmatrix} = M \begin{pmatrix} e \\ h \end{pmatrix},$$

while $N(y)$ is an invertible (6×6) matrix defined by

$$\begin{pmatrix} e \\ h \end{pmatrix} = N \begin{pmatrix} E \\ H \end{pmatrix}.$$

We close this section by a verification of a representation formula for the solution $u(t, x)$ to (3.2) with initial data $u(t_1, x) = \alpha(t_1, x)$.

THEOREM 3.5. — Suppose $\alpha(t, x) = U_0(t)f$, $f = \mathcal{R}^{-1}(\pi_-(\omega)k(s, \omega))$, where $\mathcal{R}$ is the translation representation constructed in section 2, while

$k(s, \omega) \in C_0^\infty(\mathbb{R} \times \mathbb{S}^2)$. Then the solution $u(t, x)$ to (3.2) with initial data $u(t_1, x) = \alpha(t_1, x)$ can be represented by

$$(3.14) \qquad u(t, x) = \alpha(t, x) + c \int_{\mathbb{R}} \int_{\mathbb{S}^2} G(t, x, s, \omega) k(s, \omega) d\omega ds ,$$

where G is the solution to (3.1).

Proof. — We know that $G = - \delta'(t + s - \langle x, \omega \rangle) v + \partial_s^4 w$, where w is a solution to (3.2) with initial data $w(t_1, x) = d(t_1 + s - \langle x, \omega \rangle) v$. Thus the right-hand side of (3.14) is

$$\alpha(t, x) + c \int_{\mathbb{R}} \int_{\mathbb{S}^2} w(t, x, s, \omega) \partial_s^4 k(s, \omega) d\omega ds + c \int_{\mathbb{S}^2} \partial_s k(\langle x, \omega \rangle - t, \omega) d\omega .$$

Now the Corollary 2.2 implies that the right hand side of (3.14) is $c \int_{\mathbb{R}} \int_{\mathbb{S}^2} w(t, x, s, \omega) \partial_s^4 k(s, \omega) d\omega ds$. This function is also a solution to (3.2) with initial data $\alpha(t_1, x)$, according to Corollary 2.2. Applying the uniqueness of the solution to the mixed problem (3.2) we obtain (3.14).

This completes the proof of the Theorem.

4. REPRESENTATION OF THE SCATTERING KERNEL

The scattering operator is defined by

(4.1) $\quad S(f) = \lim_{t \to +\infty} U_0(-t) H^*(t) V(t, -t) H(-t) U_0(-t) f$

for $f \in H_{ac}(A_0)$. Our goal is to represent the operator $S^* = \mathcal{R}_{tr} S \mathcal{R}_{tr}^{-1}$: $L_2(\mathbb{R}, \mathbb{N}) \to L_2(\mathbb{R}, \mathbb{N})$, where $\mathcal{R}_{tr}$ is the translation representation constructed in section 2 and $\mathbb{N} = \{ k(\omega) \in L_2(\mathbb{S}^2, \mathbb{C}^6) ; \pi_-(\omega) k = k \}$. The definition of the Hilbert space $\mathbb{N}$ together with the properties (2.1) of the projection $\pi_-(\omega)$ imply the equality $S^* = \pi_- S^* \pi_-$ on $L_2(\mathbb{R}, \mathbb{N})$. The operator in the right hand side of this equality can be extended as an operator on $L_2(\mathbb{R} \times \mathbb{S}^2, \mathbb{R}^6)$. The assumption ($H_3$), concerning the existence of the scattering operator S, guarantees that this operator is bounded. The Schwartz kernel theorem enables one to find a kernel $\mathbb{S}^\#$ of the operator $\pi_-(S^* - id)\pi_-$, which is (6×6) matrix-valued distribution

$$S^\#(s', \omega', s, \omega) = (S^\#_{jk}(s', \omega', s, \omega))_{j,k=1,\ldots,6} ,$$

where $S^\#_{jk}$ are distributions on $\mathbb{R} \times \mathbb{S}^2 \times \mathbb{R} \times \mathbb{S}^2$, s and s' are real numbers, while ω and ω' are unit vectors on $\mathbb{S}^2$. The Schwartz kernel theorem asserts that the equality

$$(4.2) \quad (S^* \pi_- k, \pi_- \imath)_{L_2(\mathbb{R} \times \mathbb{S}^2, \mathbb{R}^6)} - (\pi_- k, \pi_- \imath)_{L_2(\mathbb{R} \times \mathbb{S}^2, \mathbb{R}^6)}$$
$$= \iiiint \langle S^\#(s', \omega', s, \omega) k(s, \omega), \imath(s', \omega') \rangle d\omega' ds' d\omega ds$$

holds. Here $\langle\, ,\, \rangle_K$ denotes the inner product in the Hilbert space K, $k(s, \omega)$, $\imath(s', \omega') \in C_0^\infty(\mathbb{R} \times \mathbb{S}^2, \mathbb{R}^6)$ and the integral in the right hand side of (4.2) is taken in the sense of distributions over $\mathbb{R} \times \mathbb{S}^2 \times \mathbb{R} \times \mathbb{S}^2$.

The matrix-valued distribution $S^\#$ is called scattering (echo) kernel (see [*7*], [*24*]). The main goal of this section is to represent the scattering (echo) kernel $S^\#$.

For the purpose choose $k(s, \omega)$, $\imath(s, \omega)$ to be smooth vector-valued in $\mathbb{C}^6$ functions, such that

$$(4.3) \qquad k(s, \omega) = \imath(s, \omega) = 0 \qquad \text{for} \qquad |s| > a > 0.$$

Set

$$(4.4) \quad f = \mathcal{R}_{tr}^{-i}(k), \quad g = \mathcal{R}_{tr}^{-i}(\imath), \quad \alpha(t, x) = U_0(t)f, \quad \beta(t, x) = U_0(t)g.$$

Since $f, g \in S(\mathbb{R}^3, \mathbb{C}^6)$, the functions α and β are smooth for $(t, x) \in \mathbb{R}^4$. Moreover, the property $\pi_-(\omega)k \in L_2(\mathbb{R}, \mathbb{N})$ together with the fact that $\mathcal{R}_{tr}^{-i}$ maps $L_2(\mathbb{R}, \mathbb{N})$ onto $H_{ac}(A_0)$ yield $f \in H_{ac}(A_0)$. Similarly, $g \in H_{ac}(A_0)$.

On the other hand, combining (4.3), (4.4) with the Definition 2.1 of the entering and deparing space E^ρ, D^ρ and Proposition 2.3, we obtain the properties

$$(4.5)\ a) \qquad \alpha(t, x) = 0 \qquad \text{for} \qquad |x| < -a - t, t < -a,$$

$$(4.5)\ b) \qquad \beta(t, x) = 0 \qquad \text{for} \qquad |x| \langle -a + t, t \rangle a,$$

$$(4.5)\ c) \quad U_0(-t)f \in E^{-\rho} \subset H(-t) \qquad \text{for} \qquad t > \rho_a = \rho + a,$$

$$(4.5)\ d) \qquad U_0(t)g \in D^\rho \subset H(t) \qquad \text{for} \qquad t > \rho_a = \rho + a.$$

For completeness we shall verify (4.5) *d*) since the proofs of the other properties are similar. To prove the mentioned property we start with the identity $\mathcal{R}_{tr}(U_0(t)g) = \pi_-(\omega)\imath(s - t, \omega)$. Assuming $s < \rho$ and $t > a + \rho$, we derive $s - t < -a$ and the property (4.3) shows that we have the property $\mathcal{R}_{tr}(U_0(t)g)(s, \omega) = 0$ for $s < \rho$. Hence, $U_0(t)g \in D^\rho$. The inclusion $D^\rho \subset H(t)$ follows from proposition 2.3. This proves the property (4.5).

The definition (4.1) of the scattering operator suggests us to consider the function $u(t, x, t') = V(t, t')H(t')U_0(t')f$, $t' < -\rho_a$. Then $u(t, x, t')$ is a strong solution to the mixed problem

$$\partial_t u = A(\nabla)u \quad \text{in} \quad Q, \qquad \Lambda(t, x)u = 0 \quad \text{on} \quad \partial Q,$$
$$u(t, x) = \alpha(t, x) \qquad \text{for} \qquad t = t'.$$

Using the uniqueness of the strong solution and the inclusion

$$\alpha(t, x) = U_0(t)f \in E^{-\rho} \subset \{\alpha; \alpha(t, x) = 0 \qquad \text{for} \qquad |x| < \rho, t < -\rho_a\},$$

we obtain the equality $u(t, x, t') = \alpha(t, x)$ for $t < -\rho_a$. Hence $u(t, x, t')$ is independent of t' and solves the mixed problem

$$(4.6) \qquad \partial_t u = A(\nabla)u \quad \text{in} \quad Q, \qquad \Lambda(t, x)u = 0 \quad \text{on} \quad \partial Q,$$
$$u(t, x) = \alpha(t, x) \qquad \text{for} \qquad t < -\rho + a.$$

The results in [8] (section 3 too) show that $u(t, x)$ is smooth up to the boundary ∂Q since α is a smooth function. Moreover, finite dependence domain argument yields

$$(4.7) \quad u(t, x) = \alpha(t, x) \qquad \text{for} \qquad |x| > \rho_a + \rho + t = 2\rho + a + t .$$

After this preparation, we turn to the representation of the scattering kernel. Setting

$$(4.8) \qquad h = S(f), \qquad m = \mathcal{R}_{tr}(h), \qquad \gamma(t, x) = U_0(t)h ,$$

we see the existence of the scattering operator leads to the property

$$\lim_{t \to +\infty} \| h(x) - P_{ac}(A_0)U_0(-t)H^*(t)u(t, x) \|_{L_2(\mathbb{R}^3, \mathbb{R}^6)} = 0 .$$

Since α and β are elements of $H_{ac}(A_0)$, we have the equality

$$(h - f, g)_{L_2(\mathbb{R}^3, \mathbb{R}^6)} = \lim_{t \to +\infty} (H^*(t)u - \alpha(t, x), \beta(t, x))_{L_2(\mathbb{R}^3, \mathbb{R}^6)} .$$

Now using (4.5), (4.7) we find the equality

$$(S^*\pi_- k - \pi_- k, \iota)_{L_2(\mathbb{R}^3, \mathbb{R}^6)} = \lim_{t \to +\infty} \int_{\Omega(t)} \langle u(t, x) - \alpha(t, x), \beta(t, x) \rangle \, dx .$$

Next we need the relation

$$(4.9) \quad \left\{ \begin{aligned} &\int_{\Omega(t)} \langle u(T, x) - \alpha(T, x), \beta(T, x) \rangle \, dx = \\ &\qquad \int_{\Sigma(T)} \langle (A(\nu_x) - \nu_t I)(u - \alpha), \beta \rangle \, dS_{x,t} , \end{aligned} \right.$$

where $T > \rho + a$, $\Sigma(T) = \{ (x, t) \in \partial Q ; t < T \}$. To prove (4.9) we exploit the smoothness of u, α, β up to the boundary ∂Q and the identities

$$\begin{aligned} 0 &= \langle u - \alpha, [\partial_t I - A(\nabla)]\beta \rangle + \langle [\partial_t I - A(\nabla)](u - \alpha), \beta \rangle \\ &= \partial_t(\langle u - \alpha, \beta \rangle) - \Sigma_{j=1}^{3} \partial_{x_j}(\langle A^j(u - \alpha), \beta \rangle) . \end{aligned}$$

Integrating the above identities into domain $Q(T) = \{ (x, t) \in Q, t < T \}$ using (4.5), (4.7) and the property $u(t, x) - \alpha(t, x) = 0$ for $t < \rho + a$, we obtain (4.9). Thus we derive the equality

$$(4.10) \quad \left\{ \begin{aligned} &(S^*\pi_- k - \pi_- k, \pi_- \iota)_{L_2(\mathbb{R}^3, \mathbb{C}^6)} = \\ &\qquad = \lim_{T \to +\infty} \int_{\Sigma(T)} \langle (A(\nu_x) - \nu_t I)(u - \alpha), \beta \rangle \, dS_{x,t} , \end{aligned} \right.$$

For $u - \alpha$ one can use the representation formula

$$u - \alpha = c \int_{\mathbb{R}} \int_{\mathbb{S}^2} G(t, x, s, \omega) \partial_s^4 k(s, \omega) \, d\omega \, ds ,$$

obtained in Theorem 3.5, while for β Corollary 2.2 yields

$$\beta = c \int_{\mathbb{R}} \int_{\mathbb{S}^2} d(t + s' - \langle x, \omega \rangle) \pi_-(\omega') \partial_{s'}^4 \iota(s', \omega') d\omega' ds' .$$

Therefore the equality (4.10) leads to

THEOREM 4.1. — The scattering (echo) kernel is

$$S^{\#}(s', \omega', s, \omega) = \partial_s^4 \partial_{s'}^4 L^{\#}(s', \omega', s, \omega,),$$

where

$$L^{\#}(s', \omega', s, \omega) = (2\pi)^{-2} \int_{\partial Q} d(t + s' - \langle x, \omega \rangle) \pi_-(\omega') [A(\nu_x) - \nu_t I] G d S_{x,t},$$

$$d(\sigma) = \sigma^2/2 \quad \text{for} \quad \sigma > 0, \qquad d(\sigma) = 0 \quad \text{for} \quad \sigma < 0$$

and the function

$$G = G(t, x, s, \omega) \in C(\mathbb{R} \times \mathbb{S}^2 ; L_2(Q(T) ; \mathbb{R}^{36})),$$
$$Q(T) = \{ (x, t) \in Q ; t < T \},$$

is the unique strong solution to the mixed problem

$$(4.11) \quad \begin{cases} (\partial_t - A(\nabla))G = 0 \text{ in } Q, \ \Lambda(t, x)[G + d(t + s - \langle x, \omega \rangle)\pi_-] = 0 \text{ on } \Sigma, \\ G(t, x, s, \omega) = 0 \qquad \text{for} \quad t \ll 0, \end{cases}$$

By using H^s-estimates with $s < 0$ (see section 3) one can derive the property

$$G(t, x, s, \omega) \in C^r(\mathbb{R} \times \mathbb{S}^2 ; H^{-r}(Q(T), \mathbb{R}^{36}))$$

for any integers $r > 0$ and $T > 0$. This fact leads to

COROLLARY 4.2. — $L^{\#}((s', \omega', s, \omega)$ and $S^{\#}(s', \omega', s, \omega)$ are smooth functions of $(s, \omega, \omega') \in \mathbb{R} \times \mathbb{S}^2 \times \mathbb{S}^2$ with values in the space of distributions of $s' \in \mathbb{R}$.

5. LOCALIZATION OF SINGULARITIES

An important role in the investigation of singularities of the scattering kernel is played by the arrival surface

$$\Gamma(s, \omega) = \{ x \in \mathbb{R}^3 ; (\langle x, \omega \rangle - s, x) \in \partial Q \}, \qquad \text{(see [7])}.$$

Throughout this section suppose $\omega \neq \omega'$, $(\omega, \omega') \in \mathbb{S}^2 \times \mathbb{S}^2$ fixed. Set

$$h(s) = \max \{ \langle x, \omega' - \omega \rangle ; x \in \Gamma(s, \omega) \},$$
$$\Sigma(s, \omega) = \{ (t, x) \in \partial Q ; t = \{ x, \omega \rangle - s \}.$$

The following two lemmas due to Cooper and Strauss [9], will play an important role in our investigation.

LEMMA 5.1 (see [9], [24]). — Let $(t^*, x^*) \in \Sigma(s^*, \omega)$. Given any $s > s^*$, there exists $(t, x) \in \Sigma(s, \omega)$, such that

$$|x - x^*| < t^* - t. \tag{5.1}$$

LEMMA 5.2 (see [9], [24]). — Given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that for $0 < \delta < \delta(\varepsilon)$ we have

$$\partial Q \cap \{ (t, x); \quad t + s + h(s) \leq \langle x, \omega' \rangle + \delta \} \subset \{ (t, x); \\ t + s \leq \langle x, \omega \rangle + \varepsilon R \}.$$

Given any smooth function $\psi(x) \in C_0^\infty(\mathbb{R}^3)$, denote by $u_\psi(t, x)$ the solution to the mixed problem

$$\left\{ \begin{array}{lll} \partial_t u = A(\nabla) u \quad \text{in} \quad Q, & \Lambda(t, x)(u + f_\psi) = 0 \quad \text{on} \quad \partial Q, \\ & u(t, x) = 0 \quad \text{for} \quad t \ll 0, \end{array} \right. \tag{5.2}$$

where $\Lambda(t, x)$ is the operator of the boundary condition and

$$f_\psi = \psi(x)\pi_-(\omega)\delta'(t + s - \langle x, \omega \rangle)|_{\partial Q}. \tag{5.3}$$

Introduce the cut-off function

$$\varphi_\delta(t, x) = \varphi\left(\frac{\langle x, \omega' \rangle - t - s - h(s)}{\delta}\right),$$

where $\delta > 0$, $\varphi(\sigma)$ is a smooth function with support in the segment $(-1, 1)$ and $\varphi(\sigma) = 1$ for $|\sigma| \leq 1/2$. The main result of this section is

PROPOSITION 5.3. — Suppose $\operatorname{supp} \psi \cap \Gamma(s, \omega) \neq \emptyset$. Given any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ with the property

(5.4) Given any $(t, y) \in \partial Q \cap \operatorname{sing\,supp} (\varphi_\delta u_\psi |_{\partial Q})$ there exists $x \in \operatorname{supp} \psi$, such that the inequalities

$$|\langle y, \omega' - \omega \rangle - h(s)| + |\langle x, \omega' - \omega \rangle - h(s)| + |t + s - \langle x, \omega \rangle| \leq \varepsilon,$$

$|t - T_{\min}| \leq \varepsilon$ hold for $\operatorname{diam} (\operatorname{supp} \psi) \leq \delta \leq \delta(\varepsilon)$.

Here $\operatorname{diam} (\operatorname{supp} \psi)$ is the diameter of $\operatorname{supp} \psi$ and

$$T_{\min} = \min \{ \langle x', \omega' \rangle - s; x' \in \operatorname{supp} \psi \}.$$

Proof. — Let $(t, y) \in \partial Q \cap \operatorname{sing\,supp} (\varphi_\delta u_\psi |_{\partial Q})$. The first step the proof is to choose $x \in \operatorname{supp} \psi$, such that $|y - x| < t - T_{\min}$.

To prove this we need the following

LEMMA 5.4. — Given any $(t, y) \in \partial Q \cap \operatorname{sing\,supp} (u_\psi |_{\partial Q})$, the distance between y and $\operatorname{supp} \psi$ satisfies the inequality

$$\operatorname{dist} (y, \operatorname{supp} \psi) < t - T_{\min}. \tag{5.5}$$

Proof of Lemma 5.4. — We shall follow closely the proof in [*24*]. Denote by Γ_ψ the solution to the mixed problem

$$\begin{cases} \partial_t\Gamma + A(\nabla)\Gamma = 0 \quad \text{on} \quad Q, & \Lambda(t,x)(\Gamma + \phi_\psi) = 0 \quad \text{on} \quad \partial Q, \\ \Gamma(t,x) = 0 & \text{for} \quad t \ll 0, \end{cases}$$

where $\phi_\psi(t,x) = \psi(x)\pi_-(\omega)d(t + s - \langle x, \omega \rangle)$, $d(\sigma) = \sigma^2/2$ for $\sigma \geq 0$ and $d(\sigma) = 0$ for $\sigma \leq 0$. Since $\partial_\sigma^4 d = \delta'$ the uniqueness of the solution to (5.2) yields

$$u_\psi = \partial_s^4\Gamma_\psi. \tag{5.6}$$

If the property (5.5) is not valid, we have the inequality

$$\text{dist}(y, \text{supp}\,\psi) > t - T_{\min} + 2\varepsilon_i$$

for some $\varepsilon_i > 0$. Denote by $C_{y,t}$ the cone

$$C_{y,t} = \{ (x, \tau) \in \overline{Q}\,; \,|\, y - x \,|\, \leq t - \tau, \tau \leq t \}.$$

Therefore, we are going to

$$\partial Q \cap C_{y,t+\varepsilon_i} \cap \text{supp}\,(\psi(x)d(t + s - \langle x, \omega \rangle)) = \emptyset.$$

Now the principle of causality (see [*8*]) for the solution Γ_ψ implies that $\Gamma_\psi(t,x) = 0$ in $C_{y,t+\varepsilon_i}$, which contradicts the assumption $(t, y) \in \text{sing supp}\,(u_\psi|_\Sigma)$. This proves the lemma.

Turning again to the proof of Proposition 5.3, one can apply the above lemma and find $x \in \text{supp}\,\psi$, such that

$$|\, y - x \,| < t - T_{\min}. \tag{5.7}$$

On the other hand, we have the inequality

$$0 \leq \langle x, \omega \rangle - s - T_{\min} \leq \text{diam}\,(\text{supp}\,\psi) \tag{5.8}$$

since $x \in \text{supp}\,\psi$. Consider the identity (see [*24*])

$$t - \langle y, \omega' \rangle + s + h(s) = D_1 + D_2 + D_3, \tag{5.9}$$

where

$$D_1 = t - [\langle x, \omega \rangle - s] + \langle x - y, \omega' - \omega \rangle/2,$$

$$D_2 = [h(s) - \langle x, \omega' - \omega \rangle]/2, \qquad D_3 = [h(s) - \langle y, \omega' - \omega \rangle]/2.$$

Each of the terms D_1, D_2, D_3 has a small lower bound. Indeed,

$$D_1 \geq -\,\text{diam}\,(\text{supp}\,\psi), \tag{5.10}$$

in view of (5.7) and (5.8). By using the assumption

$$\text{supp}\,\psi \cap \Gamma(s, \omega) \neq \emptyset$$

together with the estimate diam (supp ψ) $< \delta$, we can arrange the inequality

$$D_2 \geq -\,\varepsilon/12. \tag{5.11}$$

Finally, Lemma 5.2 guarantees that $t + s \leq \langle y, \omega \rangle + \varepsilon/6$ provided $(t, y) \in \operatorname{supp} \varphi_\delta$. Since $| t + s - \langle y, \omega' \rangle + h(s) | < \delta$, we obtain

$$(5.12) \quad D_3 = [t+s-\langle y,\omega' \rangle + h(s) + \langle y, \omega \rangle - t - s]/2 \geq -\delta/2 - \varepsilon/12 .$$

Since the left hand side of (5.9) satisfies the estimate $| t+s-\langle y,\omega' \rangle + h(s) | < \delta$, combining (5.10)-(5.12), we get

$$| h(s) - \langle x, \omega' - \omega \rangle | \leq 5\delta + \varepsilon/6, | h(s) - \langle y, \omega' - \omega \rangle | \leq 4\delta + \varepsilon/6 ,$$
$$| \langle x - y, \omega' - \omega \rangle | \leq 9\delta + \varepsilon/3$$

provided diam (supp ψ) $\leq \delta$. From (5.8) and (5.10) we get

$$| t - T_{\min} | = | t - T_{\min} + \langle x, \omega \rangle - s - \langle x, \omega \rangle + s |$$
$$\leq \delta + D_1 + | \langle x - y, \omega' - \omega \rangle | \leq 2\delta - D_2 - D_3 + 9\delta + \varepsilon/3 \leq 12\delta + \varepsilon/2 .$$

Finally, the equality

$$t - \langle y, \omega' \rangle + s + h(s) = t + s - \langle y, \omega \rangle + h(s) - \langle y, \omega' - \omega \rangle$$

leads to

$$| t+s-\langle y, \omega \rangle | \leq \delta + | h(s) - \langle y, \omega' - \omega \rangle | \leq 5\delta + \varepsilon/6 ,$$

The above estimates imply that

$$| h(s) - \langle x, \omega' - \omega \rangle | + | h(s) - \langle y, \omega' - \omega \rangle | + | t+s-\langle y, \omega \rangle | \leq 14\delta + \varepsilon/2,$$
$$| t - T_{\min} | \leq 12\delta + \varepsilon/2 .$$

provided diam (supp ψ) $\leq \delta$. Choosing $\delta \leq \delta(\varepsilon)$ and $\delta(\varepsilon)$ small enough (say $\delta(\varepsilon) \leq \varepsilon/28$), we obtain the property (5.4).

This proves the Proposition.

6. TRANSFORMATION OF SOLUTIONS AND CONSTRUCTION OF MICROLOCAL PARAMETRIX TO THE MAXWELL EQUATIONS IN THE PRESENCE OF MOVING OBSTACLES

In this section we shall construct suitable local transformations of the variables (t, x) and the solution

$$u(t, x) = (E(t, x), H(t, x))$$

to the Maxwell equations near a fixed point

$$(t^*, x^*) \in \Sigma(s, \omega) = \{ (t, x) \in \partial Q ; t = \langle x, \omega \rangle - s \} .$$

The localization of the previous section suggests us to study only the case, when $x^* \in \Gamma(s, \omega) = \{ x \in \mathbb{R}^3 ; (\langle x, \omega \rangle - s, x) \in \partial Q \}$ is an isolated point on $\Gamma(s, \omega)$ with the property

$$\langle x^*, \omega \rangle = \min \{ \langle x, \omega \rangle ; x \in \Gamma(s, \omega) \} .$$

Choose $\mathscr{V} \subset \mathbb{R}^4$ as a small neighbourhood of (t^*, x^*) and the boundary ∂Q is defined locally in $\mathscr{V}$ by the equation $x_3 = \mu(t, x')$, $x' = (x_1, x_2)$, while the domain Q is determined in $\mathscr{V}$ by the inequality $x_3 < \mu(t, x')$.

The mixed problem for the Maxwell equations, needed for a representation of the scattering matrix has the form

$$(6.1) \qquad \begin{cases} \partial_t u = A(\nabla)u, \quad P(\nabla)u = 0 & \text{in } Q \cap \mathscr{V}, \\ \Lambda(u + f) = 0 & \text{on } \partial Q \cap \mathscr{V}, \\ u(t, x) = 0 \quad \text{for} \quad t \leq t^* - \delta, \end{cases}$$

where $\Lambda = \Lambda(t, x)$ is the matrix of the boundary condition in (1.1).

The purpose of this section is to study the local transformation of the mixed problem (6.1) under the following local coordinate transformation

$$(6.2) \quad y_0 = t, \qquad y_1 = x_1, \qquad y_2 = x_2, \qquad y_3 = x_3 - \mu(t, x').$$

The rotation and translation invariance of the Maxwell equations show, that without loss of generality we can assume

$$(6.3) \qquad x^* = (x^{*\prime}, 0) \qquad \text{and} \qquad \nabla_{x'}\mu(t^*, x^{*\prime}) = 0.$$

Denote by $\mathscr{V}^*$, y^*, Q^* the images of $\mathscr{V}$, (t^*, x^*) and Q respectively under the coordinate transform (6.2). Then Q^* locally is defined by $y_3 < 0$. Moreover the boundary ∂Q^* is flat and defined by $y_3 = 0$.

One natural idea is to use the transform of section 3 based on the tensor properties of the Maxwell equations. The remark (3.4)d) shows that we have to solve the equations

$$\begin{pmatrix} e \\ h \end{pmatrix} = N(y)\begin{pmatrix} E \\ H \end{pmatrix}, \qquad \begin{pmatrix} d \\ b \end{pmatrix} = M(y)\begin{pmatrix} e \\ h \end{pmatrix}$$

with respect to the unknown matrices N, M. Then the substitution

$$w = N(y)u$$

will allow us to construct a parametrix of the Maxwell system. The direct calculations of $N(y)$ and $M(y)$ are too complicated. Therefore, we propose another substitution

$$(6.4) \qquad w = [I + \gamma A(i_3)]u,$$

where

$$(6.5) \quad i_3(y) = n(y)/|n(y)|, n(y) = (-\partial_{y_1}\mu, -\partial_{y_2}\mu, 1), \qquad \gamma = v_t/|v_x|.$$

This substitution is similar to the one obtained in section 3. Namely, one can verify the property

$$N(y^*) = [I + \gamma A(i_3)] \quad \text{at} \quad y = y^*.$$

The main advantage of the new substitution is the simplification of the exact computation of the leading singularity of the scattering kernel.

The first result of this section is

PROPOSITION 6.1. — Suppose $u(t, x)$ is a solution to the mixed problem (6.1). Then $w(y) = [I + \gamma A(e_3)]u$ is a solution in the new variables y, defined by (6.4), to the mixed problem

$$(6.6)\quad \begin{cases} A(n)\partial_{y_3}w + A(\nabla_{y'})w \\ \qquad - (\partial_{y_0}+\gamma \langle e_3, \nabla_{y'} \rangle)[I+\gamma A(i_3)]^{-1}w = Dw \quad \text{in} \quad Q^* \cap \mathscr{V}^*, \\ \Lambda^*(w + [I + \gamma A(i_3)]f) = 0 \quad \text{on} \quad \partial Q^* \cap \mathscr{V}^*, \\ \qquad w(y) = 0 \qquad \text{for} \qquad y_0 < t^* - \delta, \end{cases}$$

where $\Lambda^* = \Lambda[I + \gamma A(i_3)]^{-i}$, $\nabla_{y'} = (\partial_{y_1}, \partial_{y_2}, 0)$, i_3, n, γ are defined according to (6.5) and D is a differential operator of order 0.

Proof. — The eigenvalues of the matrix $A(i_3)$ are $0, \pm 1$. Therefore the matrix $[I + \gamma A(i_3)]$ is an invertible one. Comparing the boundary conditions in (6.1) and (6.6) we see they are equivalent. To obtain the first equation in (6.6), we use the fact that the solution to (6.1) satisfies also the equation

$$(6.7)\qquad \partial_t u - A(\nabla)u - L(t, x)P(\nabla)u = 0 \quad \text{in} \quad Q \cap \mathscr{V},$$

where $L(t, x)$ is an arbitrary (6×2) matrix. Next, we use the equalities $\nabla_x = \nabla_{y'} + n(y)\partial_{y_3}$, $\partial_t = \partial_{y_0} - \partial_{y_0}\mu\partial_{y_3}$ and see that (6.7) in the new variables y takes the form

$$(6.8)\quad \partial_{y_0}u - A(\nabla_{y'})u - LP(\nabla_{y'})u + [- LP(n) - \partial_{y_0}\mu - A(n)]\partial_{y_3}u = 0.$$

The vector i_3 enables one to introduce the orthogonal projection $\pi_{\text{norm}}(y) = \pi_0(i_3(y))$ onto the linear space spanned by $f_3 = {}^t(i_3, i_3) \in \mathbb{R}^6$ and the complementary projection $\pi_{\tan}(y) = I - \pi_{\text{norm}}(y) = \pi_+(i_3) + \pi_-(i_3)$. Then the solution u can be represented in the form $u = \pi_{\text{norm}}(u) + \pi_{\tan}(u)$. We shall need the following properties of these projectors

$$(6.9)\quad \begin{cases} a) \quad A(i_3)\pi_{\text{norm}} = 0, \\ b) \quad \pi_{\tan}A(i_3)\pi_{\tan} \quad \text{is an invertible operator on Im } \pi_{\tan}(y) \text{ for } y \in \mathscr{V}^*. \end{cases}$$

We choose the linear operator $L(y) : \mathbb{R}^2 \to \mathbb{R}^6$ so that the term in (6.8) involving the normal derivative $\partial_{y_3}\pi_{\text{norm}}(u)$ vanishes in $\mathscr{V}^*$. Therefore we are going to the equation

$$(6.10)\qquad \gamma I = - LP(i_3) \qquad \text{on} \qquad \text{Im } \pi_{\text{norm}}(y).$$

An operator $L(y)$ with the above property can be found directly

$$(6.11)\quad L(y)m = {}^t(-\gamma m_1 i_3(y), -\gamma m_2 i_3(y)) \in \mathbb{R}^6 \quad \text{for} \quad m = {}^t(m_1, m_2) \in \mathbb{R}^2.$$

To check the property (6.10) choose $f_3 = {}^t(i_3, i_3) \in \text{Im } \pi_{\text{norm}}$. Then $P(i_3)f_3 = {}^t(1,1) \in \mathbb{R}^2$ and the definition (6.11) of L implies that $LP(f_3) = -\gamma f_3$.

Since Im π_{norm} is one dimensional linear space this proves the property (6.10). Next we shall use the relation

$$(6.12)\quad \mathrm{LP}(\eta) = -\gamma\langle \eta, i_3\rangle \mathrm{I} + \gamma \mathrm{A}(\eta)\mathrm{A}(i_3) \qquad \text{for any} \qquad \eta \in \mathbb{R}^3,$$

which can be verified directly from the form of the differential operators P, A and the definition (6.11) of the operator L. The relation (6.12) yields

$$(6.13)\quad \begin{cases} a) \ \mathrm{LP}(i_3) = -\gamma \mathrm{I} + \gamma \mathrm{A}(i_3)\mathrm{A}(i_3), \\ b) \ \text{the operator } \mathrm{LP}(\nabla_{y'}) - \gamma \mathrm{A}(\nabla_{y'})\mathrm{A}(i_3) + \gamma\langle i_3, \nabla_{y'}\rangle \mathrm{I} \text{ is a} \\ \quad \text{differential operator of order 0.} \end{cases}$$

Combining (6.8) and (6.13), we obtain

$$(\partial_{y_0} + \gamma\langle i_3, \nabla_{y'}\rangle)u - \mathrm{A}(n)(\mathrm{I} + \gamma\mathrm{A}(i_3))\partial_{y_3}u - \mathrm{A}(\nabla_{y'})(\mathrm{I}+\gamma\mathrm{A}(i_3))u = -\mathrm{D}(u)$$

with D being a differential operator of order 0.

This completes the proof of the Proposition.

Our next goal will be to construct a parametrix to the mixed problem (6.6) near the fixed point

$$\rho^* \in \mathrm{WF}(f) \qquad \text{with} \qquad \rho^* = (y_0^*, y_1^*, y_2^*, \eta_0^*, \eta_1^*, \eta_2^*)$$

and f defined in the new variables y by

$$(6.14)\quad f(y, s, \omega) = \delta'(y_0 + s - \langle y', \omega'\rangle - \omega_3\mu(y_0, y'))\pi_-(\omega)$$

Now our choice of the coordinate system implies that

$$(6.15)\qquad \eta^{*\prime} \equiv (\eta_1^*, \eta_2^*) = -\omega'\eta_0^*/(1 - \omega_3\partial_t\mu).$$

It is not difficult to calculate locally the coordinates of the unit outern normal $\mathrm{N}(x)$ at $x \in \Gamma(s, \omega)$. Since $\Gamma(s, \omega)$ is defined locally near x^* by $\phi(\langle x, \omega\rangle - s, x) = 0$, where $\phi(t, x) = x_3 - \mu(t, x')$ and the outern domain for $\Gamma(s, \omega)$ is determined locally by $\phi < 0$, we conclude that $\mathrm{N}(x)$ is parallel to $-\omega\partial_t\phi - \nabla_x\phi$. On the other hand, the equality

$$\mathrm{N}(x^*) = -\omega$$

(6.1) and the concrete form of the function ϕ lead to

$$(6.16)\quad \omega' = (\omega_1, \omega_2) = 0, \qquad \omega_3 = 1, \qquad \eta^{*\prime} = 0, \qquad \eta_0^* \neq 0$$

in the coordinate system we have chosen. To construct a microlocal parametrix of the mixed problem (6.6) we use the projections π_{norm} and $\pi_{\tan}$ introduced above. Setting $w_{\text{norm}} = \pi_{\text{norm}}(w)$ and $w_{\tan} = \pi_{\tan}(w)$ and applying the operator π_{norm} to the first equation in (6.6) we obtain microlocally the equality

$$(6.17)\qquad w_{\text{norm}} = \mathrm{K}(w_{\tan}),$$

where K is a pseudodifferential operator of order 0, defined microlocally near ρ^*, whose principal symbol at ρ^* is 0. Applying the projection $\pi_{\tan}$ to the first equation in (6.6) and substituting w_{norm} from (6.17) we get

$$\pi_{\tan}A(n)\pi_{\tan}D_{y_3}w_{\tan} + Q(y, D_{y_0}, D_{y'})w_{\tan} = 0, \tag{6.18}$$

where $D_{y_j} = -i\partial_{y_j}$ and $Q(y, D_{y_0}, D_{y'})$ is a pseudodifferential operator of order 1, whose principal symbol at ρ^* is

$$Q(\rho^*) = -\eta_0^*\pi_{\tan}(y^*)[I + \gamma A(\omega)]^{-i}\pi_{\tan}(y^*).$$

To construct a parametrix for (6.18), we shall prove that the system in (6.18) has characteristic surfaces of constant multiplicity. More precisely, we shall prove that the roots of the equation

$$\det\{\pi_{\tan}A(n)\pi_{\tan}\lambda + Q(y, \eta_0, \eta')\} = 0 \tag{6.19}$$

have constant multiplicity for $\rho = (y, \eta_0, \eta')$ close to ρ^*. Indeed, choosing $\rho = \rho^*$ we see direclty from (6.19), (6.16) that the roots of the equation for this case are

$$\lambda_\sigma(\rho^*) = \eta_0^*/(\gamma + \sigma 1), \qquad \sigma = \pm \tag{6.20}$$

and they have multiplicity 2. Since the roots of (6.19) are continuous functions of ρ, we conclude that any root λ of (6.19) satisfy the property $\lambda\partial_{y_0}\mu - \eta_0 \neq 0$ for ρ close to ρ^* and λ arbitrary root of (6.19). Then the above decomposition of w into $w_{\text{norm}} + w_{\tan}$ implies that any root to (6.19) is also a root to the equation

$$\det\{A(n)\lambda + A(\eta') - (\eta_0 + \gamma\langle e_3, \eta'\rangle)[I - \gamma A(i_3)]^{-1}\} = 0.$$

The transformation constructed in the proof of proposition 6.1 guarantees the above equation is equivalent to

$$\det\{(\eta_0 - \lambda\partial_{y_0}\mu)I - A(\eta' + \lambda n) - LP(\eta' + \lambda n)\} = 0.$$

Setting $\xi_0 = \eta_0 - \lambda\partial_{y_0}\mu$, $\xi = \eta' + \lambda n$ we see that $\xi_0 \neq 0$ and ξ is close to λn. By using the projections $\pi_\pm(\xi)$, $\pi_0(\xi)$ and the property

$$\operatorname{Im} LP(\xi) \subseteq \operatorname{Im} \pi_0(\xi), \qquad LP(\xi) = LP(\xi)\pi_0(\xi)$$

we see that the matrix $\xi_0 I - A(\xi) - LP(\xi)$ in the basis $\pi_+(\xi)$, $\pi_-(\xi)$, $\pi_0(\xi)$ has the form diag $((\xi_0 - |\xi|)I_2, (\xi_0 + |\xi|)I_2, \xi_0 I_2 - LP(\xi)\pi_0(\xi))$. The last matrix $\xi_0 I_2 - LP(\xi)\pi_0(\xi)$ is an invertible one since $\xi_0 \neq 0$, ξ is close to $\lambda\omega$ and $P(\xi)\pi_0(\xi) \equiv 0$. Thus λ is a root to the equation

$$[(\lambda\partial_{y_0}\mu - \eta_0)^2 - (\eta' + \lambda n(y))^2]^2 = 0 \tag{6.21}$$

and the roots of (6.19) have constant multiplicity 2.

Denotes these roots by $\lambda_\pm(\rho)$. The corresponding projections on the eigenspaces are

$$\Pi_\pm(\rho) = (2\pi i)^{-i}\int_{|z-\lambda_\pm(\rho)|=\varepsilon}(\pi_{\tan}A(n)\pi_{\tan}z + Q(\rho))^{-1}dz,$$

with $\varepsilon > 0$ sufficiently small. Decompositing $w_{\tan}$ into $w_{\tan} = w_+ + w_-$, where $w_\pm = \Pi_\pm w$, $\Pi_\pm$ being a pseudodifferential operator of order 0 with principal symbol $\Pi_\pm(\rho)$. Then the first equation in (6.6) reads

$$D_{y_3}w_\pm - \lambda_\pm w_\pm = R_\pm(w),$$

where $R_\pm$ are pseudodifferential operators of order 0. Applying Taylor's decoupling procedure (see [*30*], ch. IX, p. 205, or [*31*]) we get

$$D_{y_3}w_\pm - \lambda_\pm w_\pm = \hat{R}_\pm(w_\pm), \tag{6.22}$$

where $\hat{R}_\pm$ are pseudodifferential operators of order 0. The propagation of singularities of $w_+(y)$ is along the bicharacteristic curves of the principal symbol of the operator $D_{y_3} - \lambda_+(y, D_{y_0}, D_{y'})$. Choosing $w_- = 0$, we can arrange the last equation in (6.6) modulo smooth function in a small conic neighbourhood of ρ^*. We construct w_+ in the form

$$w = w_+ = \Pi_+ J(g),$$

where J is a local Fourier operator

$$J(g) = \int e^{i\phi(y,\eta_0,\eta')}a(y, \eta_0, \eta')\hat{g}(\eta_0, \eta')d\eta_0 d\eta'.$$

The local integral operator satisfies in a small conic neighbourhood $\mathscr{W} \subset \mathbb{R} \times T^*(\mathbb{R}^3) - \{0\}$ of ρ^* the property

$$[D_{y_3} - \lambda_+ - \hat{R}_+]\Pi_+ J(f) \in C^\infty(\mathscr{W}).$$

The phase function ϕ is a solution to the equation

$$\partial_{y_3}\phi = \lambda_+(y, \partial_{y_0}\phi, \partial_{y'}\phi)$$

with initial data $\phi|_{y_3=0} = \sum_{j=0}^{2} \eta_j y_j$. The symbol $a(\rho)$ is classical one of order 0 and $a(\rho) = I$ for $y_3 = 0$. The distribution function g will be determined from the boundary condition in (6.6) having the form

$$\Lambda([I + \gamma A(i_3)]^{-i}\,\Pi_+ g + f) \equiv 0 \qquad \text{on} \qquad y_3 = 0.$$

The principal symbol of $[I + \gamma A(i_3)]^{-i}\,\Pi_+$ at $\rho = \rho^*$ is $(i + \gamma)^{-i}\,\pi_+(\omega)$. One sufficient condition which enables one to solve the pseudodifferential equation with respect to g is

$$\text{(H)} \quad \operatorname{Ker} M(y^*) \cap \operatorname{Im} \pi_+(\omega) = \{0\}.$$

Then any $v \in \mathbb{C}^6$ can be represented in the form

$$v = -\,[I + \gamma A(i_3)]^{-i}\Pi_+(\rho)b(v) + h, \qquad h \in \operatorname{Ker} \Lambda(y) \tag{6.23}$$

for ρ close to ρ^*, $b(\rho)$ is the linear operator in $\mathbb{C}^6$ defined by the above equality. Hence g can be represented in the form $g = b(f)$, where b is a pseudodifferential operator of order 0 with principal symbol $b(\rho)$. Finally, we obtain

$$w(y_0, y', 0) = (\mathrm{I} + \mathrm{K})\Pi_+ b(f),$$

and

$$u(y_0, y', 0) = \mathrm{L}(f),$$

where L is a pseudodifferential operator, whose principal symbol $\sigma_{\mathrm{L}}(\rho)$ at ρ^* is the linear operator defined by

$$\Lambda(y^*)(\mathrm{I} + \sigma_{\mathrm{L}}(\rho^*)) = 0, \qquad \mathrm{Im}\ \sigma_{\mathrm{L}}(\rho^*) \subseteq \mathrm{Im}\ \pi_+(\omega). \tag{6.24}$$

This completes the construction of the parametrix. Especially, in the case of a perfect conductor the principal symbol $\sigma_{\mathrm{L}}(\rho^*)$ restricted to Im $\pi_-(\omega)$ has the simple form

$$\sigma_{\mathrm{L}}(\rho^*) = \frac{1-\gamma}{1+\gamma}\begin{pmatrix} -\mathrm{I}_3 & 0 \\ 0 & \mathrm{I}_3 \end{pmatrix}. \tag{6.25}$$

Indeed, the boundary matrix Λ for this case is (3×6) matrix

$$\Lambda = [\mathrm{D}(v_x)\, v_t \pi(v_x/|\, v_x\,|)],$$

where

$$\mathrm{D}(\xi)v = \xi \times v, \qquad \pi(\xi) = -\, \mathrm{D}(\xi)\mathrm{D}(\xi).$$

Then we have the equality $\Lambda(y^*) = [-\,\mathrm{D}(\omega)\,\gamma\pi(\omega)]$. Let $v = (v', v'') \in \mathbb{R}^6$, $v' \in \mathbb{R}^3$ and $v \in \mathrm{Im}\ \pi_-(\omega)$. Let $\sigma_{\mathrm{L}}(\rho^*)v = w$, $w = (w', w'') \in \mathrm{Im}\ \pi_+(\omega)$. Then the property (2.2) implies that $v' \perp \omega$, $v'' = \omega \times v'$ and $w' \perp \omega$, $w'' = -\,\omega \times w'$. Now the equality $\Lambda(y^*)[v + w] = 0$ can be written down in the form $\omega \times [(-\,1 + \gamma)v' - (1 + \gamma)w'] = 0$ and we derive that $\sigma_{\mathrm{L}}(\rho^*)$ has the form (6.25) on Im $\pi_-(\omega)$. This completes the construction of a microlocal parametrix needed in the next section.

7. LEADING SINGULARITY OF THE SCATTERING KERNEL AND THE INVERSE SCATTERING PROBLEM

The representation of the scattering (echo) kernel $\mathbb{S}^\#(s', \omega', s, \omega)$ was obtained in section 4. Throughout this section we assume $\omega' = -\,\omega$ and $\omega \in \mathbb{S}^2$ fixed. The main goal of this section is to investigate the singularities with respect to $\sigma' \in \mathbb{R}$ of the scattering kernel. First, we need

LEMMA 7.1. — $\mathrm{Supp}_{s'}\ \mathrm{S}^\# \subseteq (-\infty, s + h(s)]$, where $h(s) = -2\min\{\langle x, \omega\rangle;\ x \in \Gamma(s, \omega)\}$.

Proof. — The representation formula of the scattering kernel implies that $(t', x') \in \mathrm{supp}\ \mathrm{G}$, $t' + s' + \langle x', \omega\rangle = 0$ for some $(t', x') \in \Sigma$ provided $s' \in \mathrm{supp}\ \mathrm{S}^\#$. On the other hand, we have shown in section 4 that

$$\mathrm{G}(t, x, s, \omega) = -\,\delta'(t + s - \langle x, \omega\rangle)\pi_-(\omega) + \partial_s^4 w(t, x, s, \omega),$$

where $w(t, x, s, \omega)$ is the solution to the mixed problem (3.2). Denote by $C_{t,x}$ the cone

$$C_{t,x} = \{ (\tau, y) \in Q ; | x - y | < t - \tau \}$$

and choose $(t'', x'') \in \Sigma$ so that $(t', x') \in C_{t'',x''}$. If

$$\text{(7.1)} \qquad \operatorname{supp}_{t,x} [d(t + s - \langle x, \omega \rangle] \cap C_{t'',x''} = \emptyset ,$$

by using an integration by parts in the cone $C_{t'',x''}$ (see Theorem 4.2 in [8]) we see that $G = w = 0$ in $C_{t'',x''}$. This contradicts the assumption (7.1). Therefore, choosing (t'', x'') tending to (t', x') so that (7.1) is not true, one can find $(\hat{t}, \hat{x}) \in \operatorname{supp}_{t,x} [d(t + s - \langle x, \omega \rangle]$, such that $(\hat{t}, \hat{x}) \in \overline{C}_{t',x'}$-the closure of the cone $C_{t',x'}$ and

$$\text{(7.2)} \qquad \hat{t} + s \geq \langle \hat{x}, \omega \rangle .$$

Our choice of (t', x') guarantees that $\langle x', \omega \rangle + t' + s = 0$. Combining this equality, (7.2) and the fact that $(\hat{t}, \hat{x}) \in C_{t',x'}$ we get

$$s' = - \langle x', \omega \rangle - t' + \hat{t} - \hat{t} \leq -2 \langle \hat{x}, \omega \rangle + | \hat{x} - x' | + \hat{t} - t' + s \leq s + h(s) .$$

This completes the proof of the lemma.

In order to study the singularities of the scattering kernel near $s' = s + h(s)$, choose a cut-off function $\varphi(\sigma) \in C_0^\infty(\mathbb{R})$ with $\varphi(\sigma) = 0$ for $|\sigma| \geq 1$, $\varphi(\sigma) = 1$ for $|\sigma| \leq 1/2$ and consider the filterred scattering amplitude (see [20])

$$\text{(7.3)} \qquad a(\lambda) = \int_{-\infty}^{+\infty} S^\#(s', -\omega, s, \omega) \varphi\left(\frac{s' - s - h(s)}{\delta}\right) e^{-i\lambda s'} ds' .$$

where the integral in the right hand side is taken in the sense of distributions. Our goul is to investigate the asymptotic behabiour of $a(\lambda)$ as λ tends to ∞. The substitution of the representation formula of $S^\#$ into (7.3) gives

$$\text{(7.4)} \qquad a(\lambda) = \frac{i\lambda}{4\pi^2} \pi_-(\omega) \int_{\partial Q} [A(\nu_x) - \nu_t I] u \varphi_\delta e^{i\lambda(t + \langle x, \omega \rangle)} d S_{t,x}$$

with $\varphi_\delta(t, x) = \varphi([\langle x, \omega \rangle - t - s - h(s)]/\delta)$. First, we turn to the

Proof of Theorem 1. — The assertion *a)* follows directly from lemma 7.1. To prove *b)* consider the set

$$R(s, \omega) = \{ x \in \Gamma(s, \omega) ; \langle x, \omega \rangle = r(s, \omega) \} ,$$

where $r(s, \omega) = \min \{ \langle y, w \rangle ; y \in \Gamma(s, \omega) \}$. By using a partition of the unity on $R(s, \omega)$ and the construction of the parametrix near the points of $R(s, \omega)$ we get $G|_{\partial Q} = L(f)$, where L is zero order classical pseudodifferential operator defined locally near $R(s, \omega)$ and $f = \delta'(t + s - \langle x, \omega \rangle) \pi_-(\omega)$. Since the principle symbol of the operator $\lambda_+(y, D_{y_0}, D_{y'})$, introduced in the previous section, is real valued from transport equations for the amplitudes of the local Fourier integral operator, which is a parametrix for $D_{y_3} + \lambda_-$, we conclude that the symbol σ_L of L has asymptotic expansion

$\sigma_L \sim \Sigma_{k=0}^{\infty} i^k \sigma_{-k}$, where σ_{-k} are real valued homogeneous of degree $-k$ functions. Choose $v = (v', v'')$ so that

$$v' \perp \omega, \qquad v'' = \omega \times v''.$$

Hence $v \in \operatorname{Im} \pi_-(\omega)$ and set $v^* = (v', -v'')$. Then $v^* \in \operatorname{Im} \pi_+(\omega)$. Consider the term $I(\lambda) = \langle a(\lambda)v, v^* \rangle$. Then modulo $O(\lambda^{-\infty})$ we have

$$I(\lambda) = \frac{i\lambda}{4\pi^2} \int_{\partial Q} [A(v_x) - v_t I] \langle L(f)v, v^* \rangle \varphi_\delta e^{i\lambda(t + \langle x, \omega \rangle)} dS_{t,x}.$$

By using the localization principle (Theorem 5.1), we get

$$I(\lambda) = \frac{i\lambda}{4\pi^2} \int_{\partial Q \cap W_0} \langle f(v), L^* [A(v_x) - v_t I] v^* \varphi_\delta e^{i\lambda(t + \langle x, \omega \rangle)} \rangle dS_{t,x},$$

where L^* is the operator adjoint to L and W_0 is a small neighbourhood of R_0. If the boundary ∂Q is defined by $x_3 = \mu(t, x')$ locally near R_0, we have the equality

$$I(\lambda) = \frac{i\lambda}{4\pi^2} \int_D \langle f(v), L^* [A(v_x) - v_t I] v^* \psi_\delta e^{i\lambda(t + \langle x, \omega \rangle)} \rangle \, dt dx',$$

where $\psi_\delta = \varphi_\delta (1 + |\nabla \mu|^2)^{1/2}$ and D is a compact set in $\mathbb{R}^3$. Applying the formula for the action of a pseudodifferential operator on an exponential type function, we are in situation to apply the following result of Soga [*29*].

THEOREM 7.2. — Let

$$I(\lambda) = \int_D \beta(x, \lambda) \exp(i\lambda\psi(x)) dx,$$

where D is a compact set in $\mathbb{R}^n$, $\psi(x)$ is real valued and $\beta(x, \lambda) \sim \Sigma_{k=0}^{\infty} (i\lambda)^{-k} \beta_j(x)$ as λ tends to ∞ with $\beta_j(x)$ real valued and supp $\beta_j \subset D$. Assume $\beta_0(x) \geq 0$, $\beta_0(x) > 0$ on the set $\{ x \in D;\ \psi(x) = \min_{y \in D} \psi(y) \}$. Then there exists $m \in \mathbb{R}$ depending only on the dimension n so that

$$\lambda^m I(\lambda) \notin L^2(1, \infty).$$

In our case the phase function $\psi(x)$ has minimum on $R(s, \omega)$ and the integral $iI(\lambda)/\lambda$ has the form given in the above theorem with

$$\beta_0(x') = \langle v, \sigma_L^*(\rho^*) [A v_x) - v_t I](1 + \gamma^2)^{1/2} v^* \rangle.$$

In the previous section we have shown that on $R(s, \omega)$ we have

$$|v_x| = (1 + \gamma^2)^{-1/2}, \qquad v_x / |v_x| = -\omega$$

and $\langle v, \sigma_L^* v^* \rangle = d |v|^2$, where d is nonzero. Hence $\beta_0(x') > 0$ on R_0. Applying the result of Soga we complete the proof of theorem 1.

Proof of Theorem 2. — To construct a dense subset T of $\mathbb{R} \times \mathbb{S}^2$ consider the Gauss map

$$x \in \partial\Omega(t) \to v_x(t, x)/ | v_x(t, x) | \in \mathbb{S}^2.$$

Applying Sard's theorem we find a dense subset $K(t) \subset \mathbb{S}^2$, such that for $\omega \in K(t)$ there are only finite number of isolated points $x_1, \ldots, x_k \in \partial\Omega(t)$ such that $\langle x_j, \omega \rangle = \min \{ \langle x, \omega \rangle ; x \in \Gamma(s, \omega) \}$ and the boundary $\partial\Omega(t)$ is strictly convex near x_j, $j = 1, \ldots, k$.

Next we need

LEMMA 7.4 ([7]). — Suppose $r_{\partial\Omega(t)}(\omega)$ and $r(s, \omega)$ are defined by

$$r_{\partial\Omega(t)}(\omega) = \min \{ \langle x, \omega \rangle, \ x \in \partial\Omega(t) \},$$
$$r(s, \omega) = \min \{ \langle x, \omega \rangle ; x \in \Gamma(s, \omega) \}.$$

Then the equality $t + s = r_{\partial\Omega(t)}(\omega)$ implies that $r_{\partial\Omega(t)}(\omega) = r(s, \omega)$.

Proof. — Any point $x^* \in \partial\Omega(t)$ with $\langle x^*, \omega \rangle = r_{\partial\Omega(t)}(\omega)$ belongs to $\Gamma(s, \omega)$ for $s = \langle x, \omega \rangle - t$. Hence we, have

$$r_{\partial\Omega(t)}(\omega) \geq r(s, \omega). \tag{7.5}$$

To prove the opposite inequality choose $\hat{x} \in \Gamma(s, \omega)$, such that $\langle \hat{x}, \omega \rangle = r(s, \omega)$. Let $\hat{x} \in \partial\Omega(\hat{t})$. Then $\hat{t} + s = \langle \hat{x}, \omega \rangle$ and from (7.5) we get $\hat{t} \leq t$. If $\hat{t} \neq t$, choose $x^* \in \partial\Omega(t)$, such that $\langle x^*, \omega \rangle = t + s = r_{\partial\Omega(t)}(\omega)$. Since (t, x^*) and $(\hat{x}, \hat{t})$ are on the arrival plane, we have the relations

$$| \hat{t} - t | = | \langle \hat{x} - x^*, \omega \rangle | \leq | \hat{x} - x^* |. \tag{7.6}$$

On the other hand, the points (t, x^*) and $(\hat{x}, \hat{t})$ are on the boundary ∂Q and $t \neq \hat{t}$. Therefore the inequality in (7.6) contradicts the assumption (H_1) that the boundary is timelike. Thus we get $\hat{t} = t$ and the proof is complete.

The above lemma suggests us to consider the functions

$$t(s, \omega) = r(s, \omega) - s, \qquad s(t, \omega) = r_{\partial\Omega(t)}(\omega) - t. \tag{7.7}$$

Lemma 7.4 guarantees that the unique solution to the equation $s(t, \omega) = s$ with respect to t is $t = t(s, \omega)$. Set

$$T = \{ (s(t, \omega), \omega) ; \omega \in K(t) \}.$$

Since $K(t)$ is dense in $\mathbb{S}^2$, we conclude that T is dense in $\mathbb{R} \times \mathbb{S}^2$. Turning again to the proof of theorem 2, we assume $(s, \omega) \in T$ fixed and $t^* = t(s, \omega)$. We see that Lemma 7.4 yields the set

$$R(s, \omega) = \{ x \in \Gamma(s, \omega) ; \langle x, \omega \rangle = r(s, \omega) \}$$

coincides with

$$R_0(t^*, \omega) = \{ x \in \partial\Omega(t^*) ; \langle x, \omega \rangle = r_{\partial\Omega(t^*)}(\omega) \}$$

for $t^* = t(s, \omega)$ determined by (7.7). The definition of the set T implies that

$R(s, \omega) = R_0(t^*, \omega)$ consists of a finite number of isolated points $x_1, \ldots, x_k$, such that $\partial\Omega(t^*)$ and $\Gamma(s, \omega)$ are strictly convex near x_j, $j = 1, \ldots, k$.

To obtain the leading term of the asymptotic expansion of $a(\lambda)$ we quote Proposition 5.3 with $\omega' = -\omega$ and see that given any $\varepsilon > 0$, one can choose $\delta = \delta(\varepsilon) > 0$ sufficiently small so that $[A(v_x) - v_t I]G\varphi_\delta$ in (7.4) can be replaced by

$$G_\varepsilon = [A(v_x) - v_t I]G\xi_\varepsilon, \tag{7.8}$$

where

$$\xi_\varepsilon = \Sigma_{j=1}^{k} \varphi(|x - x^j|/\varepsilon)\varphi(|t - t^*|/\varepsilon)\varphi_\delta(t, x).$$

Indeed, their difference is a smooth function according to Proposition 5.3 and the fact that x^j, $j = 1, \ldots, k$, form the set $R(s, \omega)$. Integrating by parts with respect to t into the integral

$$\int_{\partial Q} [G_\varepsilon - (A(v_x) - v_t I)G\varphi_\delta] e^{i\lambda(t + \langle x, \omega \rangle)} dS_{t,x},$$

we see that the above integral is $O(|\lambda|^{-\infty})$ and

$$a(\lambda) = i\lambda(2\pi)^{-2}\pi_+(\omega) \int_{\partial Q} G_\varepsilon e^{i\lambda(t + \langle x, \omega \rangle)} dS_{t,x} + O(|\lambda|^{-\infty}). \tag{7.9}$$

The support of G_ε lies in a small neighbourhood $\mathscr{V}$ of R_0. Thus one can choose the coordinate system as in section 6, i. e. ∂Q is defined by $x_3 = \mu(t, x')$ with $x^j = (x^{j'}, 0)\nabla_{x'}\mu(t^*, x^{j'}) = 0$ and Q is defined locally by $x_3 < \mu(t, x')$. The property (6.16) enables one to simplify the integral in (7.9) and obtain

$$a(\lambda) = i\lambda(2\pi)^{-2}\pi_+(\omega) \int G_\varepsilon e^{i\lambda(t + \mu)}(1 + |\nabla\mu|^2)^{1/2} dt dx', \tag{7.10}$$

where $\nabla\mu = (\partial_t\mu, \nabla_{x'}\mu)$. Combining (7.8), (7.9) and (7.10) and choosing $v = (v', v'') \in \mathbb{C}^6$ to be a constant vector we derive

$$\langle a(\lambda)v, v \rangle = \frac{i\lambda}{4\pi^2} \int_{\mathbb{R}^3} \langle \pi_+(\omega)[A(v_x) - v_t]G\xi_\varepsilon e^{i\lambda(t+\mu)}\sqrt{1 + |\nabla\mu|^2} v, v \rangle dt dx' + O(|\lambda|^{-\infty}).$$

Replacing G by the parametrix $L(f)$ constructed in the previous section we obtain $\langle a(\lambda)v, v \rangle = i\lambda(2\pi)^{-2} \langle v, I(\lambda)v \rangle$, with

$$I(\lambda) = \int_{\mathbb{R}^3} f L^*([A(v_x) - v_t]\pi_+(\omega)\xi_\varepsilon e^{-i\lambda(t+\mu)}(1 + |\nabla\mu|^2)^{1/2}) dt dx',$$

where L^* is the pseudodifferential operator adjoint to L. The action of a pseudodifferential operator on exponential type function $e^{-i\lambda F(t,x')}A(t, x')$ has the expansion

$$L^*(e^{i\lambda F}A) = \sigma_{L^*}(t, x', \partial_t F, \partial_{\nabla_{x'}} F) A e^{i\lambda F} + O(|\lambda|^{-i}),$$

where σ_{L^*} is the principal symbol of L^*. Thus we derive

$$I(\lambda) = \int_{\mathbb{R}^3} \sigma_{L^*}[A(v_x) - v_t]\pi_+(\omega)\xi_\varepsilon e^{-i\lambda(t+\mu)}\sqrt{1 + |\nabla\mu|^2}dtdx' + O(|\lambda|^{-1})$$

with $\sigma_{L^*} = \sigma_{L^*}(t, x, 1 + \partial_t\mu, \nabla_x, \mu)$. Since

$$f = \delta'(t + s - \mu)\pi_-(\omega) = (1 - \partial_t\mu)^{-1}\partial_t\delta(t + s - \mu)\pi_-(\omega),$$

we can use the equality

$$\int \delta(t + s - \mu(t, x'))f(t, x')dtdx' = \int f(t(x'), x')(1 - \partial_t\mu)^{-1}dx',$$

where $t(x')$ is the local solution to the equation $t + s = \mu(t, x')$ near $(t^*, x) \in R(s, \omega)$ and $f(t, x')$ is any smooth smooth function with compact support. Thus we are going to the equality

$$\langle a(\lambda)v, v \rangle = (\lambda/2\pi)^2 \langle \pi_+(\omega)J(\lambda)\pi_-(\omega)v, v \rangle$$
$$+ \text{ terms involving lower powers of } \lambda.$$

Here

$$J(\lambda) = \int_{\mathbb{R}^2} e^{i\lambda F(x')} \frac{(1 + |\nabla\mu|^2)^{1/2}(1 + \partial_t\mu)}{(1 - \partial_t\mu)^2}(A(v_x) - v_t)\xi_\varepsilon\sigma_L dx',$$
$$F(x') = t(x') + \mu(t(x'), x').$$

Applying stationary phase method ([12], ch. VII, p. 260) and using the fact that

$$|\det F_{x'x'}(x^{j'})| = 4K(x^j) \quad \text{with} \quad K(x^j)$$

being the Gauss curvature of x^j, we obtain

$$a(\lambda) = \frac{-i\lambda}{4\pi} \sum_{j=1}^{k} K(x^j)^{-1/2}b(\omega)(1 + \gamma_j)(1 - \gamma_j)^{-1}e^{-i\lambda[s+h(s)]} + O(1).$$

This completes the proof of theorem 2.

Proof of Corollary 3. — Theorem 1 guarantees that

$$\max \operatorname{sing\,supp}_{s'} S^\#(s', -\omega, s, \omega) = s + h(s, \omega).$$

Therefore, one can recover

$$(7.11) \qquad h(s, \omega) = -2 \min \{ \langle x, \omega \rangle ; x \in \Gamma(s, \omega) \}.$$

Applying Lemma 7.4 we conclude that the equality

$$(7.12) \quad \min \{ \langle x, \omega \rangle ; (x, t^*) \in \partial Q \} = \min \{ \langle x, \omega \rangle ; x \in \Gamma(\sigma, \omega) \}$$

holds with $t^* = -s - 1/2h(s, \omega)$. From (7.12) one can recover $t^* = -s - h(s, \omega)/2$ and $\min \{ \langle x, \omega \rangle ; (x, t^*) \in \partial Q \}$ from the back-scattering data. On the other hand, the support function of $\partial Q \cap \{ t = t^* \}$

determines the convex hull of the obstacle $\partial Q \cap \{ t = t^* \}$. Therefore, one can recover the convex hull of the obstacle at all times from the back-scattering data.

This proves the Corollary.

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