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Maximal violation of Bell's inequalities for algebras of observables in tangent spacetime regions

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ABSTRACT. — We continue our study of Bell's inequalities and quantum field theory. It is shown in considerably broader generality than in our previous work that algebras of local observables corresponding to complementary wedge regions maximally violate Bell's inequality in all normal states. Pairs of commuting von Neumann algebras that maximally violate Bell's inequalities in all normal states are characterized. Algebras of local observables corresponding to tangent double cones are shown to maximally violate Bell's inequalities in all normal states in dilatation-invariant theories, in free quantum field models, and in a class of interacting models. Further, it is proven that such algebras are not split in any theory with an ultraviolet scaling limit.

RÉSUMÉ. — Cet article poursuit notre étude des inégalités de Bell en théorie quantique des champs. Nous montrons d'une manière beaucoup plus générale que les algèbres d'observables locales correspondant à des domaines en coins violent au maximum les inégalités de Bell dans tous les états normaux. Nous caractérisons les paires d'algèbres de Von Neumann qui commutent et qui violent maximalement les inégalités de Bell. Nous montrons aussi que les algèbres d'observables correspondant à des cones doubles tangents violent aussi maximalement les inégalités de Bell dans tous les états normaux des théories invariantes par dilatations, dans les modèles des champs libres, et dans des modèles en interaction. Nous montrons de plus que ces algèbres ne sont « split » dans aucune théorie avec une limite de « scaling » ultraviolette.

I. INTRODUCTION

Observables that are modelled by commuting operators are generally agreed to be « independent », i. e. jointly measurable. In local quantum field theory (QFT) the principle of relativistic causality, i. e. the principle that spacelike separated measurements should be « independent », is expressed in the formalism by requiring that observables localized in spacelike separated regions are represented by commuting operators. But just how independent *are* spacelike separated observables in local relativistic quantum theory?

The first result relevant to this question was obtained by Roos [*33*] (see also [*14*]). Based on a property noted by Schlieder [*35*] which is basically a consequence of the positivity of the energy, Roos showed that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are any two spacelike separated regions and $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ are the algebras of observables associated to these regions [*3*], [*25*], then for any pair of states ϕ_i on $\mathcal{A}(\mathcal{O}_i)$, $i = 1, 2$, there exists a state ϕ on $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$, the algebra generated by $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$, such that $\phi(AB) = \phi_1(A)\phi_2(B)$ for all $A \in \mathcal{A}(\mathcal{O}_1)$, $B \in \mathcal{A}(\mathcal{O}_2)$. Thus, the observables in $\mathcal{O}_1$ can be prepared in any arbitrary state, independent of the state of the observables in $\mathcal{O}_2$.

This was, however, a result in the category of C*-algebras, and there is reason to believe that one should in many circumstances restrict the class of admissible states to those that are normal on the local algebras (see, e. g. [*12*], [*7*]). When one makes this restriction, the mathematical question becomes: given any pair of normal states ϕ_i on the von Neumann algebras $\mathcal{A}(\mathcal{O}_i)$, $i = 1, 2$, does there exist a normal product state across $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$, i. e. does there exist a normal state ϕ on $\mathcal{A}(\mathcal{O}_1) \vee A(\mathcal{O}_2)$ such that $\phi(AB) = \phi_1(A)\phi_2(B)$ for any $A \in \mathcal{A}(\mathcal{O}_1)$, $B \in \mathcal{A}(\mathcal{O}_2)$? This is a stronger condition of independence than that considered by Roos.

The answer to the question is known to be *yes* if and only if the algebras are *split*, i. e. there exists a type I factor $\mathcal{M}$ such that $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{M} \subset \mathcal{A}(\mathcal{O}_2)'$ [*6*], [*1*], where the prime indicates that one takes the commutant of the algebra. This split property has been verified for strictly spacelike separated $\mathcal{O}_1$, $\mathcal{O}_2$ in a number of models [*6*], [*17*], [*38*], [*1*]; it has also been derived from assumptions on the thermodynamic behavior of the theory [*8*], [*2*], and it has been shown to be equivalent to the local preparability of arbitrary normal states on $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ [*9*], [*46*].

The answer is also known to be *no* in a number of circumstances [*6*], [*32*], [*14*], [*15*], [*23*], [*42*], [*13*]. In all of these situations but two, the regions $\mathcal{O}_1$, $\mathcal{O}_2$ are spacelike separated, but their closures intersect at one point or in a set of codimension two. We shall call two such regions *tangent*. The breakdown of the strong form of independence is believed to be an ultraviolet effect, and that belief can be seen to be justified in all the known circumstances where the answer is no, excepting one (a pathological case arising in a theory with « too many degrees of freedom » [13], [8]). In this paper we shall offer further evidence for this point of view and provide many more cases with a negative answer.

But first we wish to weave in another thread of development. In a series of papers [39]-[42] we have studied the violation of Bell's inequalities in QFT. Bell's inequalities involve statistical correlations of measurements made on two parts of one system and are derived from two basic assumptions. One is that all measured correlations $p(\alpha, \beta)$ between outcomes α measured on one subsystem and outcomes β measured on the other subsystem can be modelled within a classical probability theory. The second is that this description is « local » in the sense that the choice of a measuring device operating upon one part of the system does not affect the probabilities of outcomes measured on the other part. For a more thorough discussion of the assumptions and the metatheoretical framework within which Bell's inequalities can be derived, see the original preprint version of [40].

Let $\{ \mathscr{A}(\mathcal{O}) \}_{\mathcal{O} \subset \mathbf{R}^4}$ be a net of local observable (von Neumann) algebras satisfying the usual axioms of isotony, locality, Poincaré covariance and the spectrum condition [3], [25] and let ϕ be a state on the C*-algebra $\mathscr{A}$ of quasilocal observables generated by $\{ \mathscr{A}(\mathcal{O}) \}$. Then for $\mathcal{O}_1$, $\mathcal{O}_2$ two spacelike separated regions in Minkowski spacetime, we define the maximal Bell correlation between $\mathscr{A}(\mathcal{O}_1)$ and $\mathscr{A}(\mathcal{O}_2)$ in the state ϕ to be:

$$\beta(\phi, \mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2)) \equiv \frac{1}{2} \sup \phi(\mathrm{A}_1(\mathrm{B}_1 + \mathrm{B}_2) + \mathrm{A}_2(\mathrm{B}_1 - \mathrm{B}_2)), \quad (1.1)$$

where the supremum is taken over all selfadjoint $\mathrm{A}_i \in \mathscr{A}(\mathcal{O}_1)$, $\mathrm{B}_j \in \mathscr{A}(\mathcal{O}_2)$, that satisfy $-1 \leq \mathrm{A}_i \leq 1$, $-1 \leq \mathrm{B}_j \leq 1$, $i, j = 1, 2$. It was shown in [40] that the Clauser-Horne version of Bell's inequalities can be written in algebraic QFT as:

$$\beta(\phi, \mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2)) = 1 . \quad (1.2)$$

It is known [10], [40] that the following inequality must hold *a priori* in algebraic QFT for any state ϕ and regions $\mathcal{O}_1$, $\mathcal{O}_2$ as above:

$$\beta(\phi, \mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2)) \leq \sqrt{2} . \quad (1.3)$$

If equality is attained in (1.3), then we say that Bell's inequalities are *maximally violated* (in the state ϕ by suitable observables in $\mathscr{A}(\mathcal{O}_1)$ and $\mathscr{A}(\mathcal{O}_2)$). If equality holds in (1.3) for *all* normal states ϕ on $\mathscr{A}(\mathcal{O}_1) \vee \mathscr{A}(\mathcal{O}_2)$, then the pair $(\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2))$ will be said to be *maximally correlated.*

Any state ϕ on $\mathscr{A}(\mathcal{O}_1) \vee \mathscr{A}(\mathcal{O}_2)$ that is a weak limit of convex sums of product states across $(\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2))$ satisfies [40]

$$\beta(\phi, \mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2)) = 1 .$$

Thus, if $(\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2))$ is maximally correlated, then the algebras $\mathscr{A}(\mathcal{O}_1)$,

$\mathscr{A}(\mathcal{O}_2)$ are certainly *not* independent in the strong sense indicated above and they are not split. In fact, they are as badly nonsplit as possible. (Note that nonsplitness of $(\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2))$ is a strictly weaker property than maximal correlation of $(\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2))$—see Section 2.) It should be emphasized that even when the algebras are split in QFT, there still exist normal states ϕ such that equality is attained in (1.3) [*40*], [*28*]. Hence, even in the split case, where the strong form of independence obtains, one still has correlations in QFT that are stronger than those admitted by « local » classical theories. Clearly, if the algebras are maximally correlated, then they are as badly nonclassical as possible. The results on maximal correlation proven in this paper and in [*40*]-[*42*] should be understood in both of these contexts.

In [*42*] we characterized von Neumann algebras $\mathscr{B}$ such that $(\mathscr{B}, \mathscr{B}')$ is maximally correlated. There, a surprising equivalence was found with a structure property that had already been studied, for completely different reasons, in operator algebra theory. In Section 2 we present a new structure property for a pair $(\mathscr{A}, \mathscr{B})$ of commuting von Neumann algebras that is equivalent to the pair being maximally correlated. We also show that if $\beta(\phi, \mathscr{A}, \mathscr{B}) = \sqrt{2}$ for *one* faithful, normal state ϕ on $\mathscr{A} \vee \mathscr{B}$, then the pair $(\mathscr{A}, \mathscr{B})$ is maximally correlated. We use this fact in Section 4 to prove that algebras associated to tangent double cones are maximally correlated in a class of quantum field models including all free fields. An independent argument, more geometric in nature, is used in Section 5 to prove the same claim in all dilatation-invariant theories. And in Section 6 we show that algebras associated to a large class of tangent pairs of regions (including tangent double cones) are not split in any theory with an ultraviolet scaling limit. We see the results and methods of Sections 4-6 as further illustrating the ultraviolet nature of the breakdown of independence in the strong sense. In Section 3, using some abstract results explained in Section 2, we revisit the problem of the maximal correlation of algebras associated to complementary wedge regions. We make very significant improvements over [*42*] in the generality of the conditions needed to conclude this maximal correlation of wedge algebras. To close, some conjectures are discussed in Section 7.

We mention that all results in Sections 3, 5 and 6 are valid in two, three or four spacetime dimensions.

II. MAXIMALLY CORRELATED VON NEUMANN ALGEBRAS

In this section we shall prove theorems characterizing maximally correlated pairs of von Neumann algebras that we shall subsequently apply to QFT but that also have independent interest. Throughout this section

$\mathscr{A}$ and $\mathscr{B}$ will be commuting von Neumann algebras acting on a Hilbert space $\mathscr{H}$. $\mathscr{A} \vee \mathscr{B}$ will denote the von Neumann algebra generated by $\mathscr{A}$ and $\mathscr{B}$, and $\mathscr{A}_*$ will denote the predual of $\mathscr{A}$ and will be understood as the set of all normal linear functionals on $\mathscr{A}$. $\mathscr{B}(\mathscr{H})$ will represent the algebra of all bounded operators on $\mathscr{H}$ and $\mathscr{A}'$ will signify the commutant of $\mathscr{A}$ in $\mathscr{B}(\mathscr{H})$.

DÉFINITION. — A pair $(\mathscr{A}, \mathscr{B})$ of commuting von Neumann algebras is called *maximally correlated* if for any state $\phi \in (\mathscr{A} \vee \mathscr{B})_*$ one has $\beta(\phi, \mathscr{A}, \mathscr{B}) = \sqrt{2}$, where

$$\beta(\phi, \mathscr{A}, \mathscr{B}) \equiv \sup \frac{1}{2} \phi(A_1(B_1 + B_2) + A_2(B_1 - B_2)),$$

with the supremum taken over all selfadjoint contractions $A_i \in \mathscr{A}$, $B_j \in \mathscr{B}$, $i, j = 1, 2$.

We recall that if *i*) there exists a type I factor $\mathscr{M}$ such that $\mathscr{A} \subset \mathscr{M} \subset \mathscr{B}'$, or if *ii*) $\mathscr{A}$ or $\mathscr{B}$ is abelian, or if *iii*) $\mathscr{A} \vee \mathscr{B}$ admits a normal state ϕ that is a product state across $(\mathscr{A}, \mathscr{B})$, then $(\mathscr{A}, \mathscr{B})$ is *not* maximally correlated. For in case *i*) there are many normal product states across $(\mathscr{A}, \mathscr{B})$ [*6*], [*1*] (and given such a state ϕ, $\beta(\phi, \mathscr{A}, \mathscr{B}) = 1$ [*40*]), in *ii*) one has $\beta(\phi, \mathscr{A}, \mathscr{B}) = 1$ for *every* state $\phi \in (\mathscr{A} \vee \mathscr{B})^*$ [*40*], and in *iii*) one has $\beta(\phi, \mathscr{A}, \mathscr{B}) = 1$ [*40*].

It is known [*42*] that $(\mathscr{A}, \mathscr{A}')$ is maximally correlated if and only if $\mathscr{A}$ is strongly stable, i. e. $\mathscr{A} \simeq \mathscr{A} \otimes \mathscr{R}_1$ where $\mathscr{R}_1$ is the unique hyperfinite type II_1 factor. Of course, if $(\mathscr{A}, \mathscr{B})$ is maximally correlated, then so is $(\mathscr{A}, \mathscr{A}')$ (and also $(\mathscr{B}, \mathscr{B}')$). Thus, if a pair $(\mathscr{A}, \mathscr{B})$ is maximally correlated, then both $\mathscr{A}$ and $\mathscr{B}$ are strongly stable. The converse is clearly false. It is also worthwhile pointing out that since there are many examples of non-type-I factors that are not strongly stable, this characterization of a maximally correlated pair $(\mathscr{A}, \mathscr{A}')$ entails that the nonsplitness of $(\mathscr{A}, \mathscr{B})$ (see Introduction) is in general strictly weaker than the maximal correlation of $(\mathscr{A}, \mathscr{B})$.

The following result will show itself to be very useful.

THEOREM 2.1. — *Let $(\mathscr{A}, \mathscr{B})$ be a pair of commuting von Neumann algebras acting on a separable Hilbert space. Then the following are equivalent.*

1) *$(\mathscr{A}, \mathscr{B})$ is maximally correlated.*

2) *There exists a faithful state $\omega \in (\mathscr{A} \vee \mathscr{B})_*$ such that $\beta(\omega, \mathscr{A}, \mathscr{B}) = \sqrt{2}$.*

Proof. — The implication 1) $\Rightarrow$ 2) is trivial. To verify the other implication, first note that for $\omega \in (\mathscr{A} \vee \mathscr{B})_*$ a faithful state and $\phi \in (\mathscr{A} \vee \mathscr{B})_*$ an arbitrary normal state, ϕ can be arbitrarily well approximated in norm by elements of the set of all states $\psi \in (\mathscr{A} \vee \mathscr{B})_*$ such that there is some $\lambda > 0$ with $\psi \leq \lambda\omega$. This may be seen by considering the GNS representation $(\pi_\omega, \Omega, \mathscr{H}_\omega)$ of $\mathscr{A} \vee \mathscr{B}$ associated to ω. Since ω is faithful, Ω is cyclic for $\pi_\omega(\mathscr{A} \vee \mathscr{B})'$. $\phi \circ \pi_\omega^{-1}$ determines a normal state on $\pi_\omega(\mathscr{A} \vee \mathscr{B})$, and since Ω is cyclic and separating for $\pi_\omega(\mathscr{A} \vee \mathscr{B})$, $\phi \circ \pi_\omega^{-1}$ is implemented by a vector $\Phi \in \mathscr{H}_\omega$ (Theorem 2.7.9 in [34]). For an arbitrary $\varepsilon > 0$, one can find an $A_\varepsilon \in \pi_\omega(\mathscr{A} \vee \mathscr{B})'$ such that $\| \Phi - \Phi_\varepsilon \| < \varepsilon$, where $\Phi_\varepsilon \equiv A_\varepsilon \Omega$. Then for any positive $x \in \pi_\omega(\mathscr{A} \vee \mathscr{B})$,

$$\langle \Phi_\varepsilon, x\Phi_\varepsilon \rangle = \langle \Omega, A_\varepsilon^* A_\varepsilon x \Omega \rangle \leq \| A_\varepsilon \|^2 \langle \Omega, x\Omega \rangle = \| A_\varepsilon \|^2 \omega(x),$$

and

$$\| \phi \circ \pi_\omega^{-1} - \langle \Phi_\varepsilon, \cdot \Phi_\varepsilon \rangle \| \leq 2\varepsilon .$$

Next, let $\{ A_1^{(n)}, A_2^{(n)}, B_1^{(n)}, B_2^{(n)} \}_{n \in \mathbb{N}}$ be a sequence of selfadjoint contractions such that $A_i^{(n)} \in \mathscr{A}$, $B_j^{(n)} \in \mathscr{B}$, $i, j = 1, 2$, $n \in \mathbb{N}$, satisfying

$$\frac{1}{2}\omega(A_1^{(n)}(B_1^{(n)} + B_2^{(n)}) + A_2^{(n)}(B_1^{(n)} - B_2^{(n)})) \to \sqrt{2}$$

as $n \to \infty$, and let $\phi \in (\mathscr{A} \vee \mathscr{B})_*$ be a state with $\phi \leq \lambda\omega$ for some $\lambda > 0$. Then with

$$T_n \equiv \sqrt{2} - \frac{1}{2}(A_1^{(n)}(B_1^{(n)} + B_2^{(n)}) + A_2^{(n)}(B_1^{(n)} - B_2^{(n)})) \geq 0,$$

one has

$$\phi(T_n) \leq \lambda\omega(T_n) \to 0$$

as $n \to \infty$. Since such states $\phi \in (\mathscr{A} \vee \mathscr{B})_*$ are norm dense in the normal states on $\mathscr{A} \vee \mathscr{B}$, the desired implication follows. ■

Thus, to verify that $(\mathscr{A}, \mathscr{B})$ is maximally correlated, it suffices to check that $\beta(\omega, \mathscr{A}, \mathscr{B}) = \sqrt{2}$ for one faithful state $\omega \in (\mathscr{A} \vee \mathscr{B})_*$. In particular, since in QFT the vacuum state is typically faithful on the local algebras of observables, it suffices to verify that $\beta(\phi_0, \mathscr{A}(\mathscr{O}_1), \mathscr{A}(\mathscr{O}_2)) = \sqrt{2}$ for the vacuum state ϕ_0 (as long as the causal complement of $\mathscr{O}_1 \cup \mathscr{O}_2$ is nonempty) in order to conclude that the pair $(\mathscr{A}(\mathscr{O}_1), \mathscr{A}(\mathscr{O}_2))$ is maximally correlated. This is a satisfying vindication for the physical intuition expressed in [39-41] that if already the vacuum fluctuations are such that Bell's inequalities are maximally violated, then other preparations of the system will lead to violation of Bell's inequalities, as well.

We now shall begin a development that leads to a structural characterization of maximally correlated pairs of commuting von Neumann algebras. Along the way we shall make useful connections to structural properties previously isolated in the theory of operator algebras.

DEFINITION. — Let $\mathscr{A}$ be a C*-algebra with unit 1. Then $N \in \mathscr{A}$ is called a I_2-*generator* if

$$N^2 = 0 \quad \text{and} \quad NN^* + N^*N = 1 .$$

Let $V_\mathscr{A}$ denote the set of I_2-generators in $\mathscr{A}$. Clearly, if N is contained in $V_\mathscr{A}$, then N^*N and NN^* are nonzero complementary projections, i. e. their sum is 1 and their product is 0, and the C*-algebra generated by N is isomorphic to $M_2(\mathbb{C})$ ($= \mathscr{B}(\mathbb{C}^2)$) and contains the unit 1 of $\mathscr{A}$. Conversely, if $\mathscr{A}$ contains a copy of $M_2(\mathbb{C})$ containing 1, then $V_{\mathscr{A}} \neq \emptyset$. Note that if $A_i \in \mathscr{A}$ satisfies $A_i^* = A_i$, $A_i^2 = 1$ and $A_1A_2 + A_2A_1 = 0$ (which is the case if A_1, A_2 are maximal violators of Bell's inequalities in some faithful state on $\mathscr{A}$ (Theorem 2.1 in [40])), then $N \equiv \frac{1}{2}(A_1 + iA_2)$ is an element of $V_{\mathscr{A}}$. Conversely, if $\mathscr{A}$ contains a copy of $M_2(\mathbb{C})$ containing the unit of $\mathscr{A}$, then there exists a state ϕ such that $\beta(\phi, \mathscr{A}, \mathscr{A}') = \sqrt{2}$ [42], [28]. So there is clearly a link between maximal violation of Bell's inequalities by the pair $(\mathscr{A}, \mathscr{A}')$ and the existence of elements in $V_{\mathscr{A}}$.

The next lemma improves Theorem 2.1 (2*b*) in [40] and will be used below. It also extends the applicability of the preceding paragraph to situations where there are no maximal violators, only sequences that saturate the bound $\sqrt{2}$. For example, this occurs typically for wedge algebras because maximal violators in a triple $(\phi, \mathscr{A}, \mathscr{B})$ are in the centralizer of $\mathscr{A}$ (resp. $\mathscr{B}$) in ϕ (Theorem 2.1 (2*b*) in [40]) and centralizers for wedge algebras in the vacuum state are known to be trivial in irreducible vacuum representations [14].

LEMMA 2.2. — *Let $\mathscr{A}$ and $\mathscr{B}$ be commuting von Neumann algebras without type I summands. Then for any four selfadjoint contractions A_1, $A_2 \in \mathscr{A}$ and B_1, $B_2 \in \mathscr{B}$ there exist $\hat{A}_i \in \mathscr{A}$ and $\hat{B}_i \in \mathscr{A}$ $(i = 1, 2)$ such that*

1) $\hat{A}_i^* = \hat{A}_i$, $\hat{A}_i^2 = 1$, $\hat{A}_1\hat{A}_2 + \hat{A}_2\hat{A}_1 = 0$, *and the analogous conditions hold for* $\hat{B}_i$, $i = 1, 2$, *and*

2) *there exists a constant c independent of A_i and B_i such that for any normal state $\omega \in (\mathscr{A} \vee \mathscr{B})_*$ and all $0 < \varepsilon < 1$ the inequality*

$$\frac{1}{2} \cdot \omega(A_1(B_1 + B_2) + A_2(B_1 - B_2)) \geq \sqrt{2}(1 - \varepsilon)$$

implies $\frac{1}{2} \cdot \omega(\hat{A}_1(\hat{B}_1 + \hat{B}_2) + \hat{A}_2(\hat{B}_1 - \hat{B}_2)) \geq \sqrt{2}(1 - c\varepsilon^{1/4})$.

Proof. 1. — The construction of $\hat{A}_i$ (and the analogous construction of $\hat{B}_i$) will be presented in three steps. It will be useful to consider, instead of the operators A_1 and A_2 and their modifications, the respective operators $A \equiv \frac{1}{2}(A_1 + iA_2)$. Then, as mentioned above, conditions 1) are equivalent to saying that $\hat{A} \equiv \frac{1}{2}(\hat{A}_1 + i\hat{A}_2)$ is a I_2-generator.

To begin, consider $A_3 \equiv \frac{i}{2}[A_2, A_1]$ and let P_+ and P_- denote the spectral projections of A_3 for the intervals $[0, \infty)$ and $(-\infty, 0)$, respectively.

Set $C \equiv P_+A_1P_-$, $C_1 \equiv (C^* + C)$, and $C_2 \equiv i(C^* - C)$. Clearly, $C^2 = 0$ and

$$C_1^2 = C_2^2 = CC^* + C^*C \leq P_+A_1^2P_+ + P_-A_1^2P_- \leq P_+ + P_- \leq 1 .$$

Let $C = U|C|$ be the polar decomposition of C. Since $C^2 = 0$, one also has $C^*C \cdot CC^* = 0$, so that the initial and final projections of U are orthogonal, i. e. $U^2 = 0$.

In general, the projection $p \equiv UU^* + U^*U$ can be strictly smaller than 1. Let $\{ p_\alpha \}_{\alpha \in I}$ be a maximal family (containing p) of mutually orthogonal projections in $\mathscr{A}$ of the form $p_\alpha = a_\alpha a_\alpha^* + a_\alpha^* a_\alpha$ for some isometries a_α, and suppose that $q \equiv 1 - \Sigma p_\alpha \neq 0$. Then since $\mathscr{A}$ has no discrete part, q is not Abelian, so that one may use the above construction, starting with suitable $A_i \in q\mathscr{A}q$ to obtain a further projection $r = aa^* + a^*a \neq 0$ with $r \leq q$. This contradicts the maximality of $\{ p_\alpha \}$, and one concludes $\Sigma p_\alpha = 1$. Then $\hat{A} \equiv \Sigma a_\alpha$ is the required isometry with $\hat{A}^*\hat{A} + \hat{A}\hat{A}^* = 1$.

2. Let ω, A_i, B_i be as given in 2). By going to a standard representation of $\mathscr{A} \vee \mathscr{B}$, one may take ω to be given by a vector Ω in a Hilbert space $\mathscr{H}$. Then with $A \equiv \frac{1}{2}(A_1 + iA_2)$ and $B \equiv \frac{1}{2\sqrt{2}}(B_1 + B_2 + i(B_1 - B_2))$ one has

$$\frac{1}{2}\omega(A_1(B_1 + B_2) + A_2(B_1 - B_2)) = 2\sqrt{2}\,\mathrm{Re}\langle A\Omega, B\Omega \rangle \geq \sqrt{2}(1 - \varepsilon) .$$

By a straightforward calculation $\left(\text{note that an irrelevant factor } \frac{1}{2} \text{ is missing}\right.$ from the right hand side of the corresponding equation in step 5 of the proof of Theorem 2.3 in [*42*]),

$$\varepsilon \geq \frac{1}{2}\{ \| A\Omega - B\Omega \|^2 + \| A^*\Omega - B^*\Omega \|^2 \} + \frac{1}{4}\langle \Omega, \{ (1 - A_1^2) + (1 - A_2^2) + (1 - B_1^2) + (1 - B_2^2) \} \Omega \rangle . \quad (2.1)$$

As a convenient shorthand, for vectors $\Phi, \Psi \in \mathscr{H}$ and ε fixed in 2), $\Phi \approx \Psi$ will mean $\| \Phi - \Psi \| \leq c\varepsilon^{1/2}$ for some constant c independent of the given data. Thus, (2.1) implies $A\Omega \approx B\Omega$, $A^*\Omega \approx B^*\Omega$, and $A_i^2\Omega \approx \Omega \approx B_i^2\Omega$ $(i = 1, 2)$. This entails $A^2\Omega \approx AB\Omega = BA\Omega \approx B^2\Omega$ and

$$A^2\Omega \approx \frac{i}{4}(A_1A_2 + A_2A_1)\Omega = \frac{1}{2}(A^2 - A^{*2})\Omega \approx \frac{1}{2}(B^2 - B^{*2})\Omega = \frac{i}{4}(B_1^2 - B_2^2)\Omega \approx 0 .$$

Arguing now precisely as in step 5 of the proof of Theorem 2.3 in [42], one sees that with $A_3 \equiv \frac{-i}{2}[A_1, A_2]$,

$$|\langle \Omega, (1 - A_3^2)\Omega \rangle| \leq c'\varepsilon,$$

for some constant c' independent of the given data. And since $1 - A_3^2 \geq 0$ and $\| 1 - A_3^2 \| \leq 1$, it follows that $(1 - A_3^2)\Omega \approx 0$. In the course of that argument, it was established that $A_1\Omega \approx 2^{-1/2}(B_1 + B_2)\Omega$, which with $A\Omega \approx B\Omega$ implies $A_2\Omega \approx 2^{-1/2}(B_1 - B_2)\Omega$. Thus,

$$\begin{aligned} A_1^2 A_2 \Omega &\approx A_1^2 \cdot 2^{-1/2}(B_1 - B_2)\Omega = 2^{-1/2}(B_1 - B_2)A_1^2\Omega \\ &\approx 2^{-1/2}(B_1 - B_2)\Omega \approx A_2\Omega, \end{aligned}$$

and, similarly, $A_2^2 A_1 \Omega \approx A_1\Omega$.

If $P_\pm$ denotes the spectral projections of A_3 as above, the functional calculus for A_3 yields

$$\left\| P_\pm\Omega - \frac{1}{2}(1 \pm A_3)\Omega \right\|^2 \leq \| (A_3^2 - 1)\Omega \|^2 \leq c \cdot \varepsilon .$$

One concludes that $P_\pm\Omega \approx \frac{1}{2}(1 \pm A_3)\Omega$, and the operator $C \equiv P_+A_1P_-$ satisfies

$$\begin{aligned} C\Omega &\approx P_+A_1 \cdot \frac{1}{2}(1 - A_3)\Omega = P_+ \cdot \frac{1}{2}\left(A_1 + \frac{i}{2}\left[A_1^2A_2 - A_1A_2A_1\right]\right)\Omega \\ &\approx P_+ \cdot \frac{1}{2}(A_1 + iA_1^2A_2)\Omega \approx P_+ \cdot \frac{1}{2}(A_1 + iA_2)\Omega \\ &= P_+A\Omega \approx P_+B\Omega = BP_+\Omega \approx B \cdot \frac{1}{2}(1 + A_3)\Omega \\ &= \frac{1}{2}(1 + A_3)B\Omega \approx \frac{1}{2}(1 + A_3)A\Omega \approx A\Omega, \end{aligned} \tag{2.2}$$

where the final approximate equality is established by expanding out A and A_3 again.

3. Let U be the isometry in the polar decomposition of $C = P_+A_1P_-$ as above. Since $\| C \| \leq 1$ and hence $|C| \geq |C|^2$, it follows that

$$\begin{aligned} \| U\Omega - C\Omega \|^2 &= \langle \Omega, (U^*U + C^*C - 2|C|)\Omega \rangle \\ &\leq \langle \Omega, (U^*U - C^*C)\Omega \rangle . \end{aligned}$$

Adding a similar relation for U^*, one obtains

$$\begin{aligned} \| U\Omega - C\Omega \|^2 + \| U^*\Omega - C^*\Omega \|^2 & \\ &\leq \langle \Omega, (U^*U + UU^*)\Omega \rangle - \langle \Omega, (C^*C + CC^*)\Omega \rangle \\ &\leq \| \Omega \|^2 - (\| C\Omega \|^2 + \| C^*\Omega \|^2) \\ &\leq 1 - \langle \Omega, (A^*A + AA^*)\Omega \rangle + c_1\varepsilon^{1/2} \leq c_2\varepsilon^{1/2}, \end{aligned}$$

using (2.2). Since $U^*U \geq C^*C$ and setting $p \equiv U^*U + UU^*$, one has, by (2.2) again,

$$\| (1-p)\Omega \|^2 \leq 1 - \langle \Omega, (C^*C + CC^*)\Omega \rangle \leq c_2 \varepsilon^{1/2}.$$

By the construction in step 1, $\hat{A} = U + \hat{A}(1-p)$. Hence

$$\| \hat{A}\Omega - A\Omega \| \leq \| C\Omega - A\Omega \| + 2c_2^{1/2}\varepsilon^{1/4} \leq c_3 \varepsilon^{1/4}.$$

Constructing $\hat{B}$ in an analogous manner, one obtains finally

$$\frac{1}{2}\omega(\hat{A}_1(\hat{B}_1 + \hat{B}_2) + \hat{A}_2(\hat{B}_1 - \hat{B}_2)) = 2\sqrt{2}\,\mathrm{Re}\,\langle \hat{A}\Omega, \hat{B}\Omega \rangle$$
$$\geq 2\sqrt{2}\,\mathrm{Re}\,\langle A\Omega, B\Omega \rangle - c_4\varepsilon^{1/4} \geq \sqrt{2}(1 - c\varepsilon^{1/4}). \quad \blacksquare$$

DEFINITION [*31*], [*4*]. — A von Neumann algebra $\mathscr{A}$ is said to have the property L_λ (resp. L'_λ) with $\lambda \in [0, 1/2]$ if for every $\varepsilon > 0$ and any normal state $\phi \in \mathscr{A}_*$ (resp. finite family $\{\phi_i\}_{i=1}^n$ of normal states on $\mathscr{A}$), there exists an $N \in V_{\mathscr{A}}$ such that for any $A \in \mathscr{A}$,

$$| \lambda\phi(AN) - (1-\lambda)\phi(NA) | \leq \varepsilon \| A \| \tag{2.3}$$

(resp. for any $A \in \mathscr{A}$ and $i = 1, \ldots, n$

$$| \lambda\phi_i(AN) - (1-\lambda)\phi_i(NA) | \leq \varepsilon \| A \|).$$

DEFINITION [*5*]. — The asymptotic ratio set $r_\infty(\mathscr{A})$ of a von Neumann algebra $\mathscr{A}$ is the set of all $\alpha \in [0, 1]$ such that $\mathscr{A}$ is W*-isomorphic to $\mathscr{A} \otimes \mathscr{R}_\alpha$, where $\{\mathscr{R}_\alpha\}_{\alpha \in [0,1]}$ is the family of hyperfinite factors constructed by Powers [*30*].

It is known that property L'_λ is strictly stronger than property L_λ [*4*], that property L'_λ implies property $L'_{1/2}$ [*4*], [*5*], and that property L'_λ for $\mathscr{A}$ is equivalent to $\lambda/1 - \lambda \in r_\infty(\mathscr{A})$ [*4*]. Thus, $\mathscr{A}$ has property $L'_{1/2}$ if and only if the pair $(\mathscr{A}, \mathscr{A}')$ is maximally correlated [*42*]. Using Theorem 2.1 in [*40*] one easily sees that if $A_1, A_2 \in \mathscr{A}$ are maximal violators of Bell's inequalities in the normal state ϕ on $\mathscr{A} \vee \mathscr{B}$, where $\mathscr{B} \subset \mathscr{A}'$, then $N \equiv \frac{1}{2}(A_1 + iA_2) \in V_{\mathscr{A}}$ satisfies (2.3) with $\varepsilon = 0$ and $\lambda = 1/2$. And Lemma 2.2 may be used analogously for saturating sequences of self-adjoint contractions. There is clearly an intimate relation between the maximal correlation of pairs $(\mathscr{A}, \mathscr{B})$ and properties L_λ, L'_λ. This shall be explicated further in this section and then capitalized upon in the following section.

First, the following result establishes a characterization of maximally correlated pairs of commuting von Neumann algebras that is reminiscent of the definition of the property L'_λ.

COROLLARY 2.3. — *A pair $(\mathscr{A}, \mathscr{B})$ of commuting von Neumann algebras* *is maximally correlated if and only if for any finite family* $\{ \phi_i \}_{i=1}^n$ *of normal states on* $\mathscr{A} \vee \mathscr{B}$ *and any* $\varepsilon > 0$ *there exist* I_2*-generators* $M \in \mathscr{A}$, $N \in \mathscr{B}$ *such that for* $i = 1, \ldots, n$, $\phi_i(M^*N + MN^*) \geq 1 - \varepsilon$.

Proof. — Given I_2-generators $M \in \mathscr{A}$ and $N \in \mathscr{B}$, set $A_1 \equiv M + M^*$, $A_2 \equiv i(M^* - M)$, $B_1 \equiv e^{-i\pi/4}N + e^{i\pi/4}N^*$, and $B_2 \equiv e^{i\pi/4}N + e^{-i\pi/4}N^*$. Then A_i, B_i are selfadjoint contractions and

$$M^*N + MN^* = \frac{1}{2\sqrt{2}} \{ A_1(B_1 + B_2) + A_2(B_1 - B_2) \} .$$

Hence the implication ($\Leftarrow$) is trivial.

Conversely, if $(\mathscr{A}, \mathscr{B})$ is maximally correlated, neither $\mathscr{A}$ nor $\mathscr{B}$ can have a type I summand (otherwise, one could find a normal product state over $(\mathscr{A}, \mathscr{B})$ supported in that summand). Hence Lemma 2.2 is applicable. Given $\{ \phi_i \}_{i=1}^n$ and $\varepsilon > 0$, set $\omega \equiv \frac{1}{n} \Sigma_{i=1}^n \phi_i$. By assumption, for any $\varepsilon_1 > 0$ there exist admissible $A_i \in \mathscr{A}$, $B_i \in \mathscr{B}$ such that

$$\frac{1}{2} \omega(A_1(B_1 + B_2) + A_2(B_1 - B_2)) \geq \sqrt{2}(1 - \varepsilon_1) .$$

By Lemma 2.2 one can choose the A_i, B_i such that $M \equiv \frac{1}{2}(A_1 + iA_2)$ and $N \equiv \frac{1}{2\sqrt{2}}(B_1 + B_2 + i(B_1 - B_2))$ are I_2-generators and

$$\omega(M^*N + MN^*) \geq 1 - c\varepsilon_1^{1/4} .$$

Since $\| M^*N + MN^* \| \leq 1$, one has

$$\begin{aligned} \phi_i(1 - M^*N - MN^*) &\leq \sum_j \phi_j(1 - M^*N - MN^*) \\ &= n\omega(1 - M^*N - MN^*) \leq nc\varepsilon_1^{1/4} . \end{aligned}$$

Since ε_1 was arbitrary, the proof is complete. ■

By Theorem 2.3 of [*42*] a von Neumann algebra $\mathscr{A}$ has property $L'_{1/2}$ if and only if $\beta(\phi, \mathscr{A}, \mathscr{A}') = \sqrt{2}$ for any state $\phi \in \mathscr{B}(\mathscr{H})_*$. If this condition is weakened to vector states in $\mathscr{H}$, one obtains an alternative characterization of Powers' property $L_{1/2}$.

COROLLARY 2.4. — *For a von Neumann algebra* $\mathscr{A}$ *in standard form, i. e. with a cyclic and separating vector, on a separable Hilbert space* $\mathscr{H}$, *the following conditions are equivalent.*

1) $\mathscr{A}$ *has property* $L_{1/2}$.
2) *For any vector state* $\omega(A) = \langle \Omega, A\Omega \rangle$, $\Omega \in \mathscr{H}$, *one has*

$$\beta(\omega, \mathscr{A}, \mathscr{A}') = \sqrt{2} .$$

3) *For any normal state* $\omega \in \mathscr{A}_*$ *and* $\varepsilon > 0$ *there exist* $A_1, A_2 \in \mathscr{A}$ *and* $\omega_1, \omega_2 \in \mathscr{A}_*$ *such that* $-1 \leq A_i \leq 1$, $-\omega \leq \omega_i \leq \omega$, *and*

$$\frac{1}{2}\{\omega_1(A_1 + A_2) + \omega_2(A_1 - A_2)\} \geq \sqrt{2}(1 - \varepsilon).$$

4) *The set of states whose centralizer in* $\mathscr{A}$ *contains* $\mathscr{R}_1$ *is norm dense in* $\mathscr{A}_*$.

Proof. — 1. 2) $\Leftrightarrow$ 3) : Since $\mathscr{A}$ acts standardly, every normal state ω on $\mathscr{A}$ can be implemented on $\mathscr{A}$ by a vector $\Omega \in \mathscr{H}$ (see, e. g. the appendix of [*13*]). Let $p \in \mathscr{A}'$ denote the projection onto the closure of $\mathscr{A}\Omega$ in $\mathscr{H}$. Since the cyclic representation associated with ω is unique, there exists for every $\omega' \in \mathscr{A}_*$ with $-\omega \leq \omega' \leq \omega$ a unique $B \in p\mathscr{A}'p$ such that $-1 \leq B \leq 1$ and

$$\omega'(A) = \langle \Omega, AB\Omega \rangle, \qquad \text{for all } A \in \mathscr{A}. \tag{2.4}$$

Conversely, for every positive, selfadjoint $B \in \mathscr{A}'$ (2.4) defines a functional $\omega' \in \mathscr{A}_*$. Using these correspondences, statements 2) and 3) are identical.

2. 2) $\Rightarrow$ 1): Let $\omega \in \mathscr{A}_*$ be a state, implemented by the vector $\Omega \in \mathscr{H}$, and let $\varepsilon > 0$. Then by assumption there exist selfadjoint contractions $A_i \in \mathscr{A}$ and $B_i \in \mathscr{A}'$ such that

$$\frac{1}{2}\omega(A_1(B_1 + B_2) + A_2(B_1 - B_2)) \geq \sqrt{2}(1 - \varepsilon).$$

By Lemma 2.2 the A_i can be chosen such that $A \equiv \frac{1}{2}(A_1 + iA_2)$ is a I_2-generator. Then the estimates in the proof of Lemma 2.2 show that for any $C \in \mathscr{A}$, and B as defined in the second part of the proof of Lemma 2.2,

$$|\omega(AC) - \omega(CA)| = \langle \Omega, [A, C]\Omega \rangle = |\langle \Omega, [A - B, C]\Omega \rangle| \leq c\varepsilon^{1/2}\|C\|,$$

for some $c > 0$.

3. 1) $\Rightarrow$ 4): This is shown using methods in the proof of Corollary 8 in [*11*] by a construction very similar to that employed in the proof of Theorem 2.5 below. In order to minimize duplication of arguments, we suppress the details here.

4. 4) $\Rightarrow$ 3): The set of vectors $\Omega \in \mathscr{H}$ for which $\beta(\omega, \mathscr{A}, \mathscr{A}') = \sqrt{2}$ is obviously closed in $\mathscr{H}$. Since the correspondence $\omega \leftrightarrow \Omega$ established in the theory of von Neumann algebras in standard form can be chosen to be continuous (in the norm topology on each [*13*]), it suffices to check condition 3) on a norm dense subset of states in $\mathscr{A}_*$. In particular, one may choose the norm dense subset given in 4).

The centralizer $\mathscr{A}_\omega$ of $\mathscr{A}$ in any such state ω contains $\mathscr{R}_1$ and, thus, a type I_2 factor $\mathscr{M}$. Moreover, there is a conditional expectation $E : \mathscr{A} \to \mathscr{M}$ with $\omega \circ E = \omega$ [*44*]. It is easy to see that the state $\tilde{\omega} \equiv \omega \upharpoonright \mathscr{M}$, which

is the normalized trace on $\mathcal{M}$, has the property in 3) (see, e. g. the argument in step 1 of the proof of Theorem 2.3 in [42]). That is, there exist $A_i \in \mathcal{M}$ and $\tilde{\omega}_i \in \mathcal{M}_*$ with $-\tilde{\omega} \leq \tilde{\omega}_i \leq \tilde{\omega}$ such that $\frac{1}{2}\{\tilde{\omega}(A_1+A_2)+\tilde{\omega}_2(A_1-A_2)\}=\sqrt{2}$.
Then the functionals $\omega_i \equiv \tilde{\omega}_i \circ E$ have the required property. ■

Remarks. — 1. We conjecture that, for a von Neumann algebra $\mathscr{A}$ in standard form on $\mathscr{H}$, $\mathscr{A}$ has property $L_{1/2}$ if and only if $\mathscr{A}$ has no type I summand.

2. An independent proof of 1) ⇒ 3) can be easily put together by combining Lemma 4.1 of [4] with the fact [42] that the normalized trace of $M_2(\mathbb{C})$ satisfies 3).

The main result of this section is the following structural characterization of maximally correlated pairs of commuting von Neumann algebras.

THEOREM 2.5. — *Let $(\mathscr{A}, \mathscr{B})$ be a pair of commuting von Neumann algebras acting on a separable Hilbert space $\mathscr{H}$. Then the pair $(\mathscr{A}, \mathscr{B})$ is maximally correlated if and only if (up to a unitary transformation) $\mathscr{H} = \mathscr{H}_1 \otimes \tilde{\mathscr{H}}$, where $\mathscr{H}_1$ is the standard representation space of the hyperfinite type II_1-factor $\mathscr{R}_1$ [5], and there are von Neumann algebras $\tilde{\mathscr{A}}, \tilde{\mathscr{B}} \subset \mathscr{B}(\tilde{\mathscr{H}})$ such that $\mathscr{A} = \mathscr{R}_1 \otimes \tilde{\mathscr{A}}$ and $\mathscr{B} = \mathscr{R}_1' \otimes \tilde{\mathscr{B}}$.*

Proof. — 1. (⇐) By Theorem 2.3 of [42], $(\mathscr{R}_1, \mathscr{R}_1')$ is maximally correlated. Hence $(\mathscr{A}, \mathscr{B})$ is maximally correlated.

2. (⇒) This converse will be proven by constructing copies of $\mathscr{R}_1$ in $\mathscr{A}$ and $\mathscr{B}$ as an infinite product of matrix algebras. Let $\{\omega_\mu\}_{\mu\in\mathbb{N}}$ be a norm dense sequence in the normal state space of $\mathscr{A} \vee \mathscr{B}$. An inductive construction of sequences $\{N_\nu\}_{\nu\in\mathbb{N}} \subset V_{\mathscr{B}}$, $\{M_\nu\}_{\nu\in\mathbb{N}} \subset V_{\mathscr{A}}$ with the following properties

$$[M_\mu, M_\nu] = [N_\nu, N_\mu] = 0, \qquad \text{for } \mu \neq \nu, \tag{2.5}$$

and

$$\omega_\mu(M_\nu^* N_\nu + M_\nu N_\nu^*) \geq 1 - 2^{-\nu}, \qquad \text{for } \mu \leq \nu, \tag{2.6}$$

will be given first.

For any $\nu \in \mathbb{N}$ let $\mathscr{M}_\nu$ (resp. $\mathscr{N}_\nu$) denote the I_2-factor generated by M_ν (resp. N_ν), and let $\mathscr{C}_\nu \equiv \mathscr{M}_\nu \vee \mathscr{N}_\nu$. Further, let $P_\nu \in \mathscr{C}_\nu$ denote the projection to the positive eigenspace of the selfadjoint isometry $M_\nu^* N_\nu + M_\nu N_\nu^*$, and let $\mathbb{E}_\nu : \mathscr{B}(\mathscr{H}) \to \mathscr{C}_\nu'$ be the partial trace over $\mathscr{C}_\nu$, i. e. the conditional expectation

$$\mathbb{E}_\nu(C) \equiv \int dU dV U^* V^* C V U ,$$

where dU and dV denote the normalized Haar measures of the unitary groups of $\mathcal{M}_\nu$ and $\mathcal{N}_\nu$, respectively. Let $\mathbb{E}_{\leq\nu} \equiv \Pi_{\mu\leq\nu}\mathbb{E}_\mu$. $\mathbb{E}_{\leq\nu}$ projects $\mathcal{A}$ onto the relative commutant

$$\mathcal{A}_\nu \equiv \mathcal{A} \cap \left(\bigvee_{\mu\leq\nu} \mathcal{M}_\mu \right)'$$

and projects $\mathcal{B}$ onto the relative commutant

$$\mathcal{B}_\nu \equiv \mathcal{B} \cap \left(\bigvee_{\mu\leq\nu} \mathcal{N}_\mu \right)'.$$

Moreover, with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ one has $\mathbb{E}_{\leq\nu}(AB) = \mathbb{E}_{\leq\nu}(A)\mathbb{E}_{\leq\nu}(B)$.

From this one concludes that $(\mathcal{A}_\nu, \mathcal{B}_\nu)$ is maximally correlated, since if $\tilde{\omega}$ is a state on $\mathcal{A}_\nu \vee \mathcal{B}_\nu$, $\tilde{\omega} \circ \mathbb{E}_{\leq\nu}$ is a state on $\mathcal{A} \vee \mathcal{B}$, for which one can find admissible operators $A_i \in \mathcal{A}$ and $B_j \in \mathcal{B}$ with

$$\frac{1}{2}\tilde{\omega} \circ \mathbb{E}_{\leq\nu}(A_1(B_1 + B_2) + A_2(B_1 - B_2)) \geq \sqrt{2}(1 - \varepsilon).$$

But then $\tilde{A}_i \equiv \mathbb{E}_{\leq\nu}(A_i) \in \mathcal{A}_\nu$ and $\tilde{B}_j \equiv \mathbb{E}_{\leq\nu}(B_j) \in \mathcal{B}_\nu$ yield

$$\frac{1}{2}\tilde{\omega}(\tilde{A}_1(\tilde{B}_1 + \tilde{B}_2) + \tilde{A}_2(\tilde{B}_1 - \tilde{B}_2)) \geq \sqrt{2}(1 - \varepsilon).$$

Applying now Corollary 2.3 to the finite family $\{ \omega_\mu \circ \mathbb{E}_{\leq\nu} | \mu \leq \nu + 1 \}$ of states on $\mathcal{A}_\nu \vee \mathcal{B}_\nu$ with $\varepsilon \equiv 2^{-\nu-1}$, one obtains the desired I_2-generators $M_{\nu+1} \in \mathcal{A}_\nu$ and $N_{\nu+1} \in \mathcal{B}_\nu$ satisfying (2.5) and (2.6), completing the inductive construction.

3. For any $\nu \in \mathbb{N}$, let $P_{\geq\nu}$ be the weak-* limit of the decreasing family of projections $\{ \Pi_{\mu=\nu}^n P_\nu \}_{n\geq\nu}$. The claim is that $\lim_{\nu\to\infty} P_{\geq\nu} = 1$. To see this, fix any state $\omega_\sigma \in \{ \omega_\mu \}_{\mu\in\mathbb{N}}$. Then for $\mu \geq \sigma$ one has

$$\omega_\sigma(P_\mu) \geq \omega_\sigma(M_\mu^* N_\mu + M_\mu N_\mu^*) \geq 1 - 2^{-\mu}.$$

Consequently, for $\nu \geq \sigma$,

$$\omega_\sigma(1 - P_{\geq\nu}) \leq \Sigma_{\mu\geq\nu}\omega_\sigma(1 - P_\mu) \leq \Sigma_{\mu\geq\nu}2^{-\mu} = 2^{-\nu+1}.$$

Since the sequence $\{ P_{\geq\nu} \}$ is uniformly bounded, one can conclude that $\omega(P_{\geq\nu}) \to 1$ for any normal state $\omega \in (\mathcal{A} \vee \mathcal{B})_*$.

4. Consider now the W*-algebra $\mathcal{C}$ generated by $\{ \mathcal{C}_\nu \}_{\nu\in\mathbb{N}}$. For each ν let $\rho_\nu : \mathcal{C}_\nu \to \mathcal{B}(\mathcal{K}_\nu)$ be a (faithful) representation of $\mathcal{C}_\nu$ on a four-dimensional Hilbert space $\mathcal{K}_\nu$. Let $\mathcal{H}_0 \equiv \otimes_{\nu=1}^\infty (\mathcal{K}_\nu, \Omega_\nu)$ denote the incomplete tensor product of these spaces, where Ω_ν is the vector to which $\rho_\nu(P_\nu)$

projects. Let ρ denote the faithful map from the $*$-algebra $\mathscr{C}_\infty$, generated by $\{\mathscr{C}_\nu\}_{\nu\in\mathbb{N}}$, to $\mathscr{B}(\mathscr{H}_0)$ such that

$$\rho\left(\prod_{\nu=1}^{n} C_\nu\right) = \bigotimes_{\nu=1}^{n} \rho_\nu(C_\nu).$$

The claim is that ρ extends to a W*-isomorphism of $\mathscr{C}$ onto $\mathscr{B}(\mathscr{H}_0)$.

To verify this claim, it suffices to show that the states on $\mathscr{C}_\infty$ obtained by restriction of states in $(\mathscr{A} \vee \mathscr{B})_*$ are precisely those of the form $\psi \circ \rho$ with $\psi \in \mathscr{B}(\mathscr{H}_0)_*$. It is easy to see from the definition of the incomplete tensor product that any state $\omega \in (\mathscr{A} \vee \mathscr{B})_*$, with $\omega(P_{\geq\nu}) = 1$ for some ν, restricts to a state of the form $\psi \circ \rho$ and that, conversely, any state of the form $\psi \circ \rho$ can be approximated in norm by such states. Therefore, it suffices to show that states $\omega \in (\mathscr{A} \vee \mathscr{B})_*$ with $\omega(P_{\geq\nu}) = 1$ for some ν are dense in $(\mathscr{A} \vee \mathscr{B})_*$. But this follows from $P_{\geq\nu} \to 1$, since for each $\phi \in (\mathscr{A} \vee \mathscr{B})_*$, the states $\{\phi_\nu(\cdot) \equiv \phi(P_{\geq\nu})^{-1}\phi(P_{\geq\nu}\cdot P_{\geq\nu})\}$ converge in norm on $\mathscr{A} \vee \mathscr{B}$ to ϕ.

5. Clearly, ρ maps the W*-algebra $\mathscr{M}$ generated by $\{\mathscr{M}_\nu\}$ (and the W*-algebra $\mathscr{N}$ generated by $\{\mathscr{N}_\nu\}$) onto some infinite tensor product factor (and its commutant). In order to identify this factor as $\mathscr{R}_1$ it suffices [5] to note that $\langle \Omega_\nu, \cdot\, \Omega_\nu \rangle$ restricted to $\mathscr{M}_\nu$ is the normalized trace. Since $\mathscr{C}$ is a type I factor, the given Hilbert space $\mathscr{H}$ is isomorphic to $\mathscr{H}_0 \otimes \tilde{\mathscr{H}}$, so that $\mathscr{C} = \mathscr{B}(\mathscr{H}_0) \otimes 1$, $\mathscr{M} = \mathscr{R}_1 \otimes 1$, and $\mathscr{N} = \mathscr{R}_1' \otimes 1$. Then $\mathscr{A} \cap \mathscr{C}' = \mathscr{A} \cap \mathscr{M}'$ is an algebra of the form $1 \otimes \tilde{\mathscr{A}}$ and similarly $\mathscr{B} \cap \mathscr{C}' \simeq 1 \otimes \tilde{\mathscr{B}}$. The only remaining assertion of the theorem is that $\mathscr{A} = \mathscr{R}_1 \otimes \tilde{\mathscr{A}}$, i. e. that $\mathscr{M}$ and $\mathscr{M}' \cap \mathscr{A}$ together generate $\mathscr{A}$, and similarly for $\mathscr{B}$.

6. Let $A \in \mathscr{A}$. Following Araki [4], consider for any $\nu \leq \mu$ the elements $A_{\nu\mu} \equiv \Pi_{k=\nu}^{\mu}\mathbb{E}_k(A)$. Then for fixed ν this determines a bounded sequence with respect to μ, which thus has a weak-$*$ limit point A_ν. By construction, $A_\nu \in \mathscr{A} \cap (\vee_{\mu\geq\nu}\mathscr{M}_\mu)'$. Since $\vee_{\mu<\nu}\mathscr{M}_\mu$ is finite dimensional,

$$\mathscr{A} \cap \left(\bigvee_{\mu\geq\nu} \mathscr{M}_\mu\right)' = \left(\bigvee_{\mu<\nu} \mathscr{M}_\mu\right) \vee \left(\mathscr{A} \cap \left(\bigvee_{\mu\geq 1} \mathscr{M}_\mu\right)'\right).$$

Hence $A_\nu \in \mathscr{M} \vee (\mathscr{A} \cap \mathscr{M}')$. The sequence $\{A_\nu\}_{\nu\in\mathbb{N}}$ is bounded. Thus in order to show that $A_\nu \to A$ in the weak-$*$ topology, it suffices to show that $\omega(A_\nu) \to \omega(A)$ for ω in the dense set of states $\omega \in (\mathscr{A} \vee \mathscr{B})_*$ such that $\omega(P_{\leq\sigma}) = 1$ for some σ. However, since $\mathscr{M}_\mu$ is in the centralizer of any such state for all $\mu \geq \sigma$, one also has $\omega(\mathbb{E}_\mu(A)) = \omega(A)$ for all $A \in \mathscr{A}$. Therefore, for $\sigma \leq \nu \leq \mu$ one has $\omega(A_{\nu\mu}) = \omega(A) = \omega(A_\nu)$. ■

Remarks. — 1. Condition *iii*) in Theorem 2.3 of [42], with $\mathscr{M}$ replaced by $\mathscr{A}$ and $\mathscr{M}'$ replaced by $\mathscr{B}$, is also equivalent to the pair $(\mathscr{A}, \mathscr{B})$ being maximally correlated, by the same argument as in [42].

2. An alternative formulation of the two equivalent properties in Theorem 2.5 is clearly the following. There exists a type I factor $\mathcal{M} \subset \mathcal{A} \vee \mathcal{B}$ such that $\mathcal{A} \cap \mathcal{M}$ and $\mathcal{B} \cap \mathcal{M}$ are (spatially) isomorphic to $\mathcal{R}_1$ and are relative commutants of each other in $\mathcal{M}$.

We close this section by pointing out the following fact that will be used below.

PROPOSITION 2.6. — *Let $(\mathcal{A}, \mathcal{B})$ be a pair of commuting von Neumann algebras and $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be two quasiequivalent representations of $\mathcal{A} \vee \mathcal{B}$. Then $(\pi_1(\mathcal{A}), \pi_1(\mathcal{B}))$ is maximally correlated if and only if $(\pi_2(\mathcal{A}), \pi_2(\mathcal{B}))$ is maximally correlated.*

Proof. — Trivial by Prop. 2.12 in [*43*]. ■

III. COMPLEMENTARY WEDGE ALGEBRAS ARE MAXIMALLY CORRELATED

In this section we revisit the problem addressed in [*42*] and substantially improve the results demonstrated there in Section 3. It was shown in [*42*] that under very general circumstances, $\beta(\phi, \mathcal{A}(W), \mathcal{A}(W')) = \sqrt{2}$ for any wedge region $W \in \mathcal{W}$ and any vector state ϕ. However, we were only able to show that the pair $(\mathcal{A}(W), \mathcal{A}(W'))$ is maximally correlated in markedly less general situations. We present here a proof of the missing element—the proof that under very general circumstances $\mathcal{A}(W)$ is strongly stable—yielding the maximal correlation of the pair $(\mathcal{A}(W), \mathcal{A}(W)')$ very generally. (The maximal correlation of $(\mathcal{A}(W), \mathcal{A}(W'))$ then follows whenever wedge duality holds; for example, whenever a quantum field is locally associated to the net $\{ \mathcal{A}(\mathcal{O}) \}$—see below and [*20*].)

We shall consider in this section a net of observable von Neumann algebras $\{ \mathcal{A}(\mathcal{O}) \}$ satisfying the usual assumptions [*3*], [*25*] of isotony, locality, Poincaré covariance and the spectrum condition in a separable Hilbert space $\mathcal{H}$. $\mathcal{A}$ will denote the C*-algebra generated by $\cup_{\mathcal{O} \subset \mathbb{R}^d} \mathcal{A}(\mathcal{O})$ $(d=2, 3$ or $4)$ and will be called the quasilocal algebra. The « right wedge » is defined by

$$W_R \equiv \{ x \in \mathbb{R}^d \mid |x^0| < x^1 \},$$

where x^0 is the time coordinate. Then the set of all wedges $\mathcal{W}$ is the set of all Poincaré transforms of W_R. $\mathcal{K}$, the set of all double cones, will be the set of the (nonempty) interiors of all intersections of a forward light cone with a backward light cone.

The following theorem is essentially due to Testard [*45*], who sketched the proof of a special case. We modify his proof below.

THEOREM 3.1. — *Let $\{ \mathscr{A}(\mathcal{O}) \}$ be a net of observable algebras in an irreducible vacuum representation such that $[\cup_{\mathcal{O}\in\mathscr{K}} \mathscr{A}(\mathcal{O})]\Omega$ is dense in the representation space $\mathscr{H}$, where Ω is the (up to a factor) unique vacuum vector. Then for each $W \in \mathscr{W}$, $\mathscr{A}(W)$ is a type III_1 factor that has property L'_λ for all $\lambda \in [0, 1/2]$.*

Proof. — Under the above assumptions each wedge algebra $\mathscr{A}(W)$ is nontrivial [*18*] and must be a type III_1 factor [*14*]. Let $\{ V(t) \}_{t\in\mathbb{R}}$ denote the strongly continuous unitary group on $\mathscr{H}$ implementing the velocity transformation subgroup of the Poincaré group that leaves W invariant. Then Ω is the (up to a factor) unique $V(\mathbb{R})$-invariant vector in $\mathscr{H}$ and

$$w - \lim_{|a|\to\infty} V(a)AV(a)^{-1} = \langle \Omega, A\Omega \rangle \cdot 1, \tag{3.1}$$

for every $A \in \mathscr{A}$ (Prop. I.1.6 in [*16*]).

By [*11*], for any $\varepsilon > 0$ and $\lambda \in [0, 1/2]$ there exists a I_2-generator $N \in \mathscr{A}(W)$ such that for every $A \in \mathscr{A}(W)$,

$$| \lambda \langle \Omega, AN\Omega \rangle - (1 - \lambda) \langle \Omega, NA\Omega \rangle | \leq \varepsilon \| A \| .$$

Since Ω is invariant under $V(\mathbb{R})$ and since

$$V(a)\mathscr{A}(W)V(a)^{-1} \equiv \alpha_a(\mathscr{A}(W)) = \mathscr{A}(W)$$

for all $a \in \mathbb{R}$, one also has

$$| \lambda \langle \Omega, A\alpha_a(N)\Omega \rangle - (1 - \lambda) \langle \Omega, \alpha_a(N)A\Omega \rangle | \leq \varepsilon \| A \| , \tag{3.2}$$

for all $A \in \mathscr{A}(W)$ and $a \in \mathbb{R}$. Let $\{ \omega_i \}_{i=1}^n$ be a finite family of normal states on $\mathscr{A}(W)$. Again by [*11*] there exist unitaries U_i, $i = 1, \ldots, n$, in $\mathscr{A}(W)$ such that

$$| \langle \Omega, U_i A U_i^* \Omega \rangle - \omega_i(A) | \leq \varepsilon \| A \| , \tag{3.3}$$

for all $A \in \mathscr{A}(W)$, $i = 1, \ldots, n$. Choosing $b \in \mathbb{R}$ such that

$$\| [\alpha_b(N), U_i]\Omega \| \leq \varepsilon \tag{3.4}$$

and

$$\| [\alpha_b(N), U_i^*]\Omega \| \leq \varepsilon , \tag{3.5}$$

which is possible by (3.1), locality and the cyclicity of Ω for $\mathscr{A}(W)$ (see, e. g. the proof of (A) in [*15*]), one has

$$\begin{aligned} |\lambda\omega_i(A\alpha_b(N)) - (1-\lambda)\omega_i(\alpha_b(N)A)| &\leq \lambda |\omega_i(A\alpha_b(N)) - \langle \Omega, U_i A\alpha_b(N)U_i^*\Omega\rangle| \\ &+ \lambda |\langle \Omega, U_i A\alpha_b(N)U_i^*\Omega\rangle - \langle \Omega, U_i A U_i^*\alpha_b(N)\Omega\rangle| \\ &+ |\lambda\langle \Omega, U_i A U_i^*\alpha_b(N)\Omega\rangle - (1-\lambda)\langle \Omega, \alpha_b(N)U_i A U_i^*\Omega\rangle| \\ &+ (1-\lambda)|\langle \Omega, \alpha_b(N)U_i A U_i^*\Omega\rangle - \langle \Omega, U_i\alpha_b(N)AU_i^*\Omega\rangle| \\ &+ (1-\lambda)|\langle \Omega, U_i\alpha_b(N)AU_i^*\Omega\rangle - \omega_i(\alpha_b(N)A)| \\ &\leq 5\varepsilon \| A \| , \end{aligned}$$

for all $A \in \mathscr{A}(W)$ and $i = 1, \ldots, n$, using (3.2)-(3.5). Since $\alpha_b(N) \in V_{\mathscr{A}(W)}$, the theorem is proved. ■

This has as a corollary the following significant improvement over Theorem 3.3 in [42].

COROLLARY 3.2. — *Let* $\{ \mathscr{A}(\mathscr{O}) \}$ *be a net of local algebras in an irreducible vacuum representation such that* $[\cup_{\mathscr{O} \in \mathscr{K}} \mathscr{A}(\mathscr{O})]\Omega$ *is dense in the representation space* $\mathscr{H}$. *Then for each* $W \in \mathscr{W}$, $\mathscr{A}(W)$ *is strongly stable and* $(\mathscr{A}(W), \mathscr{A}(W)')$ *is maximally correlated.*

Proof. — Since a von Neumann algebra is strongly stable if and only if it has property $L'_{1/2}$ (Theorem 1.3 in [4]), the assertion is an immediate consequence of Theorem 3.1 above and Theorem 2.3 in [42]. ■

Since strong stability and the property L'_λ, $\lambda \in [0, 1/2]$, are isomorphic invariants, Theorem 3.1 and Corollary 3.2 are also true for nets of local algebras in representations such as those occurring in the Doplicher, Haag, Roberts theory of superselection structure [12] and also the « massive single particle » representations of Buchholz and Fredenhagen [7]. Moreover, at the cost of making the weak technical assumption that the local algebras in the net $\{ \mathscr{A}(\mathscr{O}) \}$ are locally generated, i. e. that each $\mathscr{A}(\mathscr{O})$ is generated by $\{ \mathscr{A}(\mathscr{O}_0) \mid \mathscr{O}_0 \in \mathscr{K} \text{ and } \mathscr{O}_0 \subset \mathscr{O} \}$, we can remove the assumption that the representation be irreducible.

THEOREM 3.3. — *Let* $\{ \mathscr{A}(\mathscr{O}) \}$ *be a net of local algebras in a vacuum representation such that at least one Poincaré-invariant vector is cyclic for* $\cup_{\mathscr{O} \in \mathscr{K}} \mathscr{A}(\mathscr{O})$ *and such that the elements of* $\{ \mathscr{A}(\mathscr{O}) \}$ *are locally generated. If every translation-invariant vector in the representation is Poincaré-invariant, then for each* $W \in \mathscr{W}$ $\mathscr{A}(W)$ *is a type III algebra that has property* L'_λ *for all* $\lambda \in [0, 1/2]$. *Hence,* $\mathscr{A}(W)$ *is strongly stable and* $(\mathscr{A}(W), \mathscr{A}(W)')$ *is maximally correlated.*

Proof. — Under the stated assumptions one can decompose the representation into a direct integral of nets of local algebras, each of which satisfies the assumptions of Theorem 3.1 [18]. Then by Prop. 1 of [29], Theorem 1.3 of [4], and the fact [18] that the center of each wedge algebra is equal to the center of the quasilocal algebra $\mathscr{A}$, it follows that $\mathscr{A}(W)$ has property L'_λ for all $\lambda \in [0, 1/2]$. It is type III since every component in its central decomposition is type III (type III_1). ■

Remark. — Because we need the velocity transformations in the proof of Theorem 3.1, we *must* [18] make the assumption in Theorem 3.3 about the Poincaré invariance of all translation-invariant vectors. However, all the hypotheses of Theorem 3.3 are known to hold [19], [20] if the net is locally associated to a quantum field $\varphi(x)$ in the following weak sense:

(C) There exists a test function f_s of compact support (supp $(f_s) \subset \mathcal{O}_s \in \mathcal{K}$), whose Fourier transform vanishes nowhere, for which

$$\langle \varphi(f_s)\Phi, A\Psi \rangle = \langle A^*\Phi, \varphi(f_s)^*\Psi \rangle$$

obtains for all $A \in \mathcal{A}(\mathcal{O}_s)'$ and all $\Phi, \Psi \in \mathcal{P}_{0,s}(\mathbb{R}^d)\Omega$, where $\mathcal{P}_{0,s}(\mathbb{R}^d)$ is the smallest unital *-algebra containing $\varphi(f_s)$ and all of its Poincaré transforms.

Moreover, it is also known [20] that with assumption (C), $\mathcal{A}(W') = \mathcal{A}(W)'$ for all $W \in \mathcal{W}$. We thus have significantly improved on Theorem 3.2 in [42].

With these results we consider the case of wedge algebras to be well in hand. But before we turn our attention to tangent spacetime regions of finite extent, we wish to point out one other consequence of the arguments of this section.

In the theory of the free massless field in four spacetime dimensions, it is known [26], [27] that if V_+ is the forward lightcone with apex at the origin and V_- is the backward lightcone with apex at the origin, then $\mathcal{A}(V_+)$, $\mathcal{A}(V_-)$, $\mathcal{A}(W)$ and $\mathcal{A}(\mathcal{O})$ are all unitarily equivalent for all $W \in \mathcal{W}$ and $\mathcal{O} \in \mathcal{K}$. We thus have the following theorem. (Recall that one has $\mathcal{A}(V_+) = \mathcal{A}(V_-)'$ [26], [27].)

THEOREM 3.4. — *Let* $\{ \mathcal{A}(\mathcal{O}) \}$ *be the net of local observable algebras associated to the free, massless field of arbitrary helicity (for nonzero spin fields, take the net constructed by Hislop* [27]*). Then for any* $W \in \mathcal{W}$ *and* $\mathcal{O} \in \mathcal{K}$, $\mathcal{A}(W)$, $\mathcal{A}(\mathcal{O})$, $\mathcal{A}(V_+)$ *and* $\mathcal{A}(V_-)$ *are type* III_1 *factors with the property* L'_λ *for all* $\lambda \in [0, 1/2]$. *Hence the pairs* $(\mathcal{A}(W), \mathcal{A}(W'))$, $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}'))$ *and* $(\mathcal{A}(V_+), \mathcal{A}(V_-))$ *are each maximally correlated.*

Finally, we mention that in [2] physically motivated conditions are given that entail that local algebras $\mathcal{A}(\mathcal{O})$ are isomorphic to $\mathcal{R} \otimes \mathcal{Z}$, where $\mathcal{R}$ is the unique hyperfinite type III_1 factor and $\mathcal{Z}$ is the center of the algebra $\mathcal{A}(\mathcal{O})$. Since $\mathcal{R}$ is strongly stable, it follows from [42], [29] that under such conditions the pair $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O})')$ is maximally correlated.

IV. TANGENT DOUBLE CONE ALGEBRAS ARE MAXIMALLY CORRELATED IN QUANTUM FIELD MODELS

In a series of steps we shall prove that tangent double cone algebras are maximally correlated in all free quantum field models and in models of interacting fields in the local quasiequivalence class of free fields such as $P(\phi)_2$, Yukawa$_2$ and ϕ^4_3. In order not to take undue space for these results, we shall present details only for hermitian, scalar Bose fields and we shall assume familiarity with our work in [41]. The case of higher spin fields should then be clear from [41]. For mass zero, higher spin free fields we take as the net $\{ \mathcal{A}(\mathcal{O}) \}$ of local algebras the one constructed by Hislop [27].

We have already established [41] that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are tangent double cones and ϕ_0 is the vacuum state for the massless, free field, then

$$\beta(\phi_0, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) = \sqrt{2}. \tag{4.1}$$

Since the vacuum vector is separating for $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$ in any free field model, Theorem 2.1 entails that the pair $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ is maximally correlated. As it is known [21] that free quantum field models of different masses $m \geq 0$ are locally unitarily equivalent in three or four spacetime dimensions, it then follows from Prop. 2.6 that such pairs are maximally correlated in said models, as well. However, this local unitary equivalence between massless and massive free field models breaks down in two spacetime dimensions. In order to include two spacetime dimensional models in our results, and also in order to emphasize the ultraviolet nature of maximal correlation in such models, we present a different argument below. We follow the notation of [41].

THEOREM 4.1. — *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are arbitrary tangent double cones, the pair $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ is maximally correlated in any free quantum field model in two, three or four spacetime dimensions and in the interacting models* $P(\phi)_2$, *Yukawa*$_2$ *and* ϕ_3^4 *(see* [24], [36], [22]*).*

Proof. — 1. Since in free, massless quantum field models (4.1) holds [41], Theorem 2.1 entails that $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ is maximally correlated in such models.

2. Next, a scaling argument is used to verify that (4.1) holds in free, massive quantum field models. Without loss of generality, it may be assumed that the point of tangency of $\mathcal{O}_1$ and $\mathcal{O}_2$ is the origin and that $\mathcal{O}_1 \subset W_R$ and $\mathcal{O}_2 \subset W_R'$.

Let

$$f(x) \rightarrow f_\lambda(x) \equiv f(\lambda^{-1}x)$$

be the induced action of the dilatation group on the test function space $\mathcal{S}(\mathbb{R}^d)$. It is well known that there exists a scaling function $N(\lambda)$ (monotone, nonnegative for $\lambda \in (0, \infty)$) such that for all $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$,

$$\lim_{\lambda \to 0} N(\lambda)^2 W_m^{(2)}(f_{1,\lambda}, f_{2,\lambda}) = W_0^{(2)}(f_1, f_2), \tag{4.2}$$

where $W_m^{(2)}(\cdot,\cdot)$ is the two-point Wightman function of the free field with mass m.

In [41] it was shown that for any mass $m \geq 0$, any $\varepsilon > 0$ and

$\mu \in [1 - \varepsilon, 1 + \varepsilon]$ there exist test functions f_i, g_j with $\operatorname{supp}(f_i) \subset W_R$, $\operatorname{supp}(g_j) \subset W'_R$ satisfying

i) $\| f_j \|_m^2 \approx (1 + \mu^2)/(1 - \mu^2) \approx \| g_j \|_m^2, j = 1, 2,$

ii) $\langle f_1, f_2 \rangle_m \approx i \approx \langle g_1, g_2 \rangle_m,$

iii) $\langle f_2, g_1 \rangle_m \approx 2\mu/(1 - \mu^2) \approx - \langle f_2, g_2 \rangle_m,$

iv) $\langle f_1, g_2 \rangle_m \approx 0 \approx \langle f_2, g_1 \rangle_m,$

where $x \approx y$ means $|x - y| \leq \varepsilon$ and $\langle \cdot, \cdot \rangle_m$ is the scalar product on the test function space induced by the two-point function of the theory of mass m.

Since $\varepsilon > 0$ is arbitrary and since C^∞-functions of compact support are dense in $\mathscr{S}(\mathbb{R}^d)$, it follows that there exist double cones $\hat{\mathcal{O}}_1$, $\hat{\mathcal{O}}_2$ with $\mathcal{O}_1 \subseteq \hat{\mathcal{O}}_1 \subset W_R, \mathcal{O}_2 \subseteq \hat{\mathcal{O}}_2 \subset W'_R$ and test functions $\hat{f}_i$, $\hat{g}_j$ with $\operatorname{supp}(\hat{f}_i) \subset \hat{\mathcal{O}}_1$, $\operatorname{supp}(\hat{g}_j) \subset \hat{\mathcal{O}}_2$, satisfying *i)*-*iv)* for $m = 0$. Of course, for all small enough λ, $\operatorname{supp}(N(\lambda)\hat{f}_{j,\lambda}) \subset \mathcal{O}_1$ and $\operatorname{supp}(N(\lambda)\hat{g}_{j,\lambda}) \subset \mathcal{O}_2$.

By (4.2) one has for all small enough λ

$$\| N(\lambda)\hat{f}_{j,\lambda} \|_m^2 \approx (1 + \mu^2)/(1 - \mu^2) \approx \| N(\lambda)\hat{g}_{j,\lambda} \|_m, \quad j = 1, 2,$$

as well as the analogous relations *ii)*-*iv)* (with equality possibly only within 2ε). Since this can be done for any $\mu \in [1 - \varepsilon, 1 + \varepsilon]$. Prop. 3.1 of [*41*] entails that $\beta(\phi_0^{(m)}, \mathscr{A}_m(\mathcal{O}_1), \mathscr{A}_m(\mathcal{O}_2)) = \sqrt{2}$, where $\phi_0^{(m)}$ is the vacuum state of the free field of mass m. But the vacuum state $\phi_0^{(m)}$ is faithful on $\mathscr{A}_m(\mathcal{O}_1) \vee \mathscr{A}_m(\mathcal{O}_2)$, so that one must conclude that $(\mathscr{A}_m(\mathcal{O}_1), \mathscr{A}_m(\mathcal{O}_2))$ is maximally correlated for $m > 0$.

3. By [*24*], [*36*], [*22*] the representation of the observable algebras in the models $P(\phi)_2$, Yukawa_2 and ϕ_3^4 (the latter with a momentum cutoff) are in the local quasiequivalence class of the corresponding free field theory. Thus Prop. 2.6 completes the proof. ■

V. DILATATION-INVARIANT FIELD THEORIES

It has been known for some time [*14*], [*15*] that in irreducible vacuum representations of dilatation-invariant quantum field theories the algebras associated with wedge regions are type III_1 factors. From recent work (Theorem 2.1 in [*42*]) it then follows that in such theories for any wedge $W \in \mathscr{W}$,

$$\beta(\phi, \mathscr{A}(W), \mathscr{A}(W)') = \sqrt{2}$$

for every vector state ϕ on $\beta(\mathscr{H})$. If the dilatation-invariant theory is, in particular, a free field, then Theorem 3.3 in [*42*] entails that for every $W \in \mathscr{W}$ the pair $(A(W), \mathscr{A}(W'))$ [resp. the pair $(\mathscr{A}(\mathcal{O}), \mathscr{A}(\mathcal{O}'))$ for every $\mathcal{O} \in \mathscr{K}$] is maximally correlated. The same would follow for every dilatation-invariant theory in which the wedge and double cone algebras satisfy duality and are strongly stable. Moreover, we now know from Sections 3 and 4 that in free, massless (hence dilatation-invariant) field theories all pairs of tangent double cone algebras (among others) are maximally correlated.

The aim of this section is to demonstrate that tangent double cone algebras are maximally correlated in *any* dilatation-invariant theory whose wedge algebras satisfy duality. The same results will also hold for a net of field algebras $\{ \mathscr{F}(\mathcal{O}) \}$ [*12*], [*32*] in a dilatation-invariant theory whose wedge algebras satisfy twisted duality, since $\mathscr{A}(\mathcal{O}) \subset \mathscr{F}(\mathcal{O})$ for any open $\mathcal{O} \subset \mathbb{R}^d$. The proof we present in this section also has the advantage of being more geometrical and illustrating the ultraviolet nature of maximal correlation of algebras in such quantum field models.

We thus consider a net of observable von Neumann algebras $\{ \mathscr{A}(\mathcal{O}) \}$ satisfying the usual assumptions of isotony, locality, Poincaré covariance and the spectrum condition in a separable Hilbert space $\mathscr{H}$ with a (up to a factor) unique Poincaré-invariant vector $\Omega \in \mathscr{H}$ that is cyclic for the algebra $\cup_{\mathcal{O} \in \mathscr{K}} \mathscr{A}(\mathcal{O})$. The C*-algebra $\mathscr{A}$ generated by $\cup_{\mathcal{O} \in \mathscr{K}} \mathscr{A}(\mathcal{O})$ is thus irreducible and Ω is cyclic for $\mathscr{A}(\mathrm{W})$ for each $\mathrm{W} \in \mathscr{W}$ [*18*]. We shall also assume the weak technical condition that the wedge algebras are locally generated. A state $\phi \in \mathscr{B}(\mathscr{H})^*$ will be said to be *locally normal* if its restriction to $\mathscr{A}(\mathcal{O})$ is contained in $\mathscr{A}(\mathcal{O})_*$ for every $\mathcal{O} \in \mathscr{K}$.

The dilatation invariance of the theory is expressed by the existence of a strongly continuous, unitary representation $\mathrm{D}(\mathbb{R}_+)$ of the dilatation group on $\mathbb{R}^d$ acting such that

$$\delta_\lambda(\mathscr{A}(\mathcal{O})) \equiv \mathrm{D}(\lambda)\mathscr{A}(\mathcal{O})\mathrm{D}(\lambda)^{-1} = \mathscr{A}(\lambda\mathcal{O}), \quad \lambda > 0$$

where $\lambda\mathcal{O} \equiv \{ \lambda x \mid x \in \mathcal{O} \}$ and $\mathrm{D}(\lambda)\Omega = \Omega$ for any $\lambda \in \mathbb{R}_+$.

THEOREM 5.1. — *Let $\{ \mathscr{A}(\mathcal{O}) \}$ be a net of local von Neumann algebras in an irreducible vacuum representation of a dilatation-invariant theory such that the wedge algebras are locally generated and $\mathscr{A}(\mathrm{W})' = \mathscr{A}(\mathrm{W}')$ for each $\mathrm{W} \in \mathscr{W}$. Then for any tangent double cones $\mathcal{O}_1$, $\mathcal{O}_2$ the pair $(\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2))$ is maximally correlated and thus all double cone algebras are strongly stable.*

Proof. — It is known [*32*] that for any $\mathrm{A} \in \mathscr{A}$, $\delta_\lambda(\mathrm{A})$ converges weakly to $\phi_0(\mathrm{A}) \cdot 1$ as $\lambda \downarrow 0$, where $\phi_0 = \phi_0 \circ \delta_\lambda$ is the vacuum state on $\mathscr{A}$. Thus, for any locally normal state $\phi \in \mathscr{B}(\mathscr{H})^*$, $\phi \circ \delta_\lambda \to \phi_0$ pointwise on $\mathscr{A}$ as $\lambda \downarrow 0$. Without loss of generality, it may be assumed that the point of tangency for $\mathcal{O}_1$ and $\mathcal{O}_2$ is the origin and that $\mathcal{O}_1 \subset \mathrm{W_R}$, $\mathcal{O}_2 \subset \mathrm{W_R'}$.

$\mathscr{A}(\mathrm{W_R})$ is a type III_1 factor [*14*], [*15*]. Hence from Theorem 2.1 of [*42*], $\beta(\phi, \mathscr{A}(\mathrm{W_R}), \mathscr{A}(\mathrm{W_R'})) = \sqrt{2}$ for every vector state ϕ on $\mathscr{B}(\mathscr{H})$. In particular, $\beta(\phi_0, \mathscr{A}(\mathrm{W_R}), \mathscr{A}(\mathrm{W_R'})) = \sqrt{2}$. Let $\varepsilon > 0$ be arbitrary and pick selfadjoint contractions $A_i \in \mathscr{A}(W_R)$, $B_j \in \mathscr{A}(W'_R)$, $i, j = 1, 2$, such that with

$$T_\varepsilon \equiv \frac{1}{2}(A_1(B_1 + B_2) + A_2(B_1 - B_2)),$$

$\phi_0(T_\varepsilon) \geq \sqrt{2} - \varepsilon$. Let also $\delta > 0$ be arbitrary and pick two sufficiently large tangent double cones $\hat{\mathcal{O}}_1$, $\hat{\mathcal{O}}_2$ (with $\mathcal{O}_1 \subset \hat{\mathcal{O}}_1 \subset W_R$ and $\mathcal{O}_2 \subset \hat{\mathcal{O}}_2 \subset W'_R$) such that there exist selfadjoint contractions $\hat{A}_i \in \mathscr{A}(\hat{\mathcal{O}}_1)$, $\hat{B}_j \in \mathscr{A}(\hat{\mathcal{O}}_2)$, i, $j = 1, 2$, satisfying $| \phi_0(T_\varepsilon - \hat{T}_{\varepsilon,\delta}) | < \delta$, where

$$\hat{T}_{\varepsilon,\delta} \equiv \frac{1}{2}(\hat{A}_1(\hat{B}_1 + \hat{B}_2) + \hat{A}_2(\hat{B}_1 - \hat{B}_2))$$

(this is possible by Kaplansky's density theorem [34], [43] and the assumption that the wedge algebras are locally generated).

Then for any locally normal state $\phi \in \mathscr{B}(\mathscr{H})^*$,

$$\phi \circ \delta_\lambda(\hat{T}_{\varepsilon,\delta}) \underset{\lambda \to 0}{\longrightarrow} \phi_0(\hat{T}_{\varepsilon,\delta}) \geq \sqrt{2} - \varepsilon - \delta .$$

But for every $\lambda \in \mathbb{R}_+$, one has $\delta_\lambda(\hat{A}_i) \in \mathscr{A}(\lambda\hat{\mathcal{O}}_1)$ and $\delta_\lambda(\hat{B}_j) \in \mathscr{A}(\lambda\hat{\mathcal{O}}_2)$, and there exists a $\lambda_0 > 0$ such that $\lambda\hat{\mathcal{O}}_1 \subset \mathcal{O}_1$ and $\lambda\hat{\mathcal{O}}_2 \subset \mathcal{O}_2$ for all $\lambda < \lambda_0$. Hence the assertion of the theorem follows at once. ■

Remark. — Note that the proof gives a prescription for choosing a sequence $\{ A_{1,n}, A_{2,n}, B_{1,n}, B_{2,n} \}$ of admissible quadruples that yields maximal violation (i. e. $\beta = \sqrt{2}$) for all (locally) normal states and all tangent double cones at once.

Under general circumstances [33], [14] it is known that there exist product states across $(\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2))$ even if the double cones $\mathcal{O}_1$, $\mathcal{O}_2$ are tangent. The above results, however, entail that such states cannot be normal even when restricted to the « edge of tangency », i. e. they cannot be normal restricted to any algebra $\mathscr{A}(\mathcal{O}_1) \vee \mathscr{A}(\mathcal{O}_2)$ with $\mathcal{O}_1$, $\mathcal{O}_2$ any open tangent spacetime regions, no matter how small $\mathcal{O}_1$, $\mathcal{O}_2$ are, as long as $(\mathcal{O}_1, \mathcal{O}_2)$ contains a pair of tangent double cones, i. e. as long as their interiors are nonempty and they are not too narrow at their point of tangency.

VI. ALGEBRAS FOR TANGENT SPACETIME REGIONS ARE NOT SPLIT IN FIELD THEORIES WITH A SCALING LIMIT

The object of this section is to extend prior results [6], [14], [15], [23], [32] on the nonexistence of normal product states across algebras of local observables associated to tangent spacetime regions. Roberts [32] showed that in a dilatation-invariant theory there exists no (locally) normal product state across $(\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2))$ if $\mathcal{O}_1$ and $\mathcal{O}_2$ are tangent double cones. And

Fredenhagen [23] proved that in a theory with a well-defined ultraviolet scaling limit, there is no normal product state across $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ if $\mathcal{O}_1$ is a double cone and $\mathcal{O}_2$ is a tangent wedge. Both of these extend prior results on complementary wedges [6], [14], [15].

We shall show that in a theory with a well-defined ultraviolet scaling limit there is no (locally) normal product state across $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ whenever $\mathcal{O}_1$ and $\mathcal{O}_2$ are tangent spacetime regions satisfying $\lambda\mathcal{O}_i \subset \mathcal{O}_i$ for all small enough $\lambda > 0$ (or if $(\mathcal{O}_1, \mathcal{O}_2)$ is a Poincaré transform of such a pair), such as tangent double cones or tangent spacelike cones [7] that may be arbitrarily narrow. This will therefore establish a necessary condition for such pairs $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ to be maximally correlated and will, in any case, establish that such algebras are not independent in the strong sense.

Let $\{ \mathcal{A}(\mathcal{O}) \}$ be a net of local von Neumann algebras in a vacuum representation with a quantum field $\varphi(x)$ such that for each open $\mathcal{O} \subset \mathbb{R}^4$ and each test function f with support in $\mathcal{O}$, the closure of the operator $\varphi(f)$ on its standard domain of definition D_1 [37] is affiliated with $\mathcal{A}(\mathcal{O})$ (see, e. g. [20]). Let

$$f(x) \to f_\lambda(x) \equiv f(\lambda^{-1}x)$$

be the induced action of the dilatation group on the test function space. With $\Omega \in \mathcal{H}$ a vacuum vector in D_1, let $\omega(\cdot) \equiv \langle \Omega, \cdot\Omega \rangle / \| \Omega \|^2$ be the associated vacuum state. We make the following assumption about the existence of an ultraviolet scaling limit for this theory.

(A) There exists a scaling function $N(\lambda)$ (monotone, nonnegative for $\lambda \in (0, \infty)$) such that for all test functions f_i

$$\lim_{\lambda \to 0} N(\lambda)\omega(\varphi(f_\lambda)) = W_0^{(1)}(f)\,(= 0),$$
$$\lim_{\lambda \to 0} N(\lambda)^2\omega(\varphi(f_{1,\lambda})\varphi(f_{2,\lambda})) = W_0^{(2)}(f_1, f_2),$$

and

$$\lim_{\lambda \to 0} N(\lambda)^4\omega(\varphi(f_{1,\lambda})\varphi(f_{2,\lambda})\varphi(f_{3,\lambda})\varphi(f_{4,\lambda})) = W_0^{(4)}(f_1, f_2, f_3, f_4),$$

where $\{ W_0^{(j)} \}_{j=1,2,4}$ are the Wightman functions corresponding to the vacuum state of the free, massless field on the Borchers algebra over the test function space.

For any test function f and $a > 0$, let

$$A_a(f) \equiv (1 + a^2 | \overline{\varphi(f)} |^2)^{-1}\overline{\varphi(f)},$$

where $\overline{\varphi(f)}$ denotes the closure of $\varphi(f)$ on D_1. Then for every test function f,

$$\| A_a(f) \| \leq \frac{1}{2a} \tag{6.1}$$

and

$$|\langle \Phi, (A_a(f) - \varphi(f))\Phi \rangle| \leq \frac{a}{2} \| \varphi(f)^*\Phi \| \cdot \| \varphi(f)\Phi \|, \qquad (6.2)$$

for every $\Phi \in D_1$. Furthermore, for every open region $\mathcal{O} \subset \mathbb{R}^4$ and any test function f with support in $\mathcal{O}$, one has $A_a(f) \in \mathcal{A}(\mathcal{O})$.

If f and g are test functions whose supports are spacelike separated, one also observes that

$$\begin{aligned} &|\langle \Phi, (A_a(f)A_a(g) - \varphi(f)\varphi(g))\Phi \rangle| \\ &= |\langle \Phi, [(1 + a^2 |\overline{\varphi(f)}|^2)^{-1}(1 + a^2 |\overline{\varphi(g)}|^2)^{-1} - 1]\varphi(f)\varphi(g)\Phi \rangle| \\ &\leq \frac{a}{2}(\| \varphi(f)^*\Phi \| \cdot \| \varphi(f)\Phi \| + \| \varphi(g)^*\Phi \| \cdot \| \varphi(g)\Phi \|) \\ &+ \frac{a^3}{2} \| \varphi(f)^*\varphi(f)\Phi \| \cdot \| \varphi(g)^*\varphi(g)\Phi \|, \end{aligned} \qquad (6.3)$$

for every $\Phi \in D_1$.

We can now state and prove the main result of this section.

THEOREM 6.1. — *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be (Poincaré transforms of) tangent spacelike separated spacetime regions for which there exists a $\lambda_0 > 0$ such that $\lambda\mathcal{O}_i \subset \mathcal{O}_i$ for all $0 < \lambda < \lambda_0$, $i = 1, 2$. And let $\{ \mathcal{A}(\mathcal{O}) \}$ be a net of local von Neumann algebras in an irreducible vacuum representation, to which is locally associated a quantum field $\varphi(x)$, as above [20]. With assumption* (A), *there is no (locally) normal state ϕ on $\mathcal{A}$ such that $\phi(AB) = \phi(A)\phi(B)$, for all $A \in \mathcal{A}(\mathcal{O}_1)$, $B \in \mathcal{A}(\mathcal{O}_2)$.*

Proof. — Without loss of generality, one may assume that the point of tangency between $\mathcal{O}_1$ and $\mathcal{O}_2$ coincides with the origin. Assume that there is such a (locally) normal product state ϕ. Then for all $0 < \lambda < \lambda_0$ and for all test functions f with support in $\mathcal{O}_1$ and g with support in $\mathcal{O}_2$.

$$\phi(A_a(N(\lambda)f_\lambda)A_a(N(\lambda)g_\lambda)) = \phi(A_a(N(\lambda)f_\lambda)) \cdot \phi(A_a(N(\lambda)g_\lambda)), \qquad (6.4)$$

with $a > 0$ arbitrary. By (6.1), $\{ A_a(N(\lambda)f_\lambda) \}_{\lambda > 0}$ (resp. $\{ A_a(N(\lambda)g_\lambda) \}_{\lambda > 0}$) is uniformly bounded with $a > 0$ fixed. Thus this family has a weak limit point in $\mathcal{B}(\mathcal{H})$. One may therefore replace this family by a convergent subnet. Since one has for any $x \in \mathbb{R}^4$ [47]

$$\bigcap_{\mathcal{O} \ni x} \mathcal{A}(\mathcal{O}) = \mathbb{C}1$$

and also $A_a(N(\lambda)f_\lambda) \in \mathcal{A}(\lambda\mathcal{O}_1)$ (resp. $A_a(N(\lambda)g_\lambda) \in \mathcal{A}(\lambda\mathcal{O}_2)$), one concludes that $A_a(N(\lambda)f_\lambda) - \omega(A_a(N(\lambda)f_\lambda)) \cdot 1$ converges weakly to zero as $\lambda \to 0$. But by (6.2) and assumption (A),

$$\lim_{\lambda \to 0} |\omega(A_a(N(\lambda)f_\lambda)) - W_0^{(1)}(f)| < \frac{a}{2} W_0^{(2)}(\bar{f}, f)^{1/2} \cdot W_0^{(2)}(f, \bar{f})^{1/2}.$$

Letting $a \to 0$, one obtains

$$w - \lim_{a\to 0} w - \lim_{\lambda\to 0} A_a(N(\lambda)f_\lambda) = 0 \tag{6.5}$$

(similarly for $A_a(N(\lambda)g_\lambda)$).

Repeating this argument and using (6.3), one obtains

$$w - \lim_{a\to 0} \omega - \lim_{\lambda\to 0} A_a(N(\lambda)f_\lambda)A_a(N(\lambda)g_\lambda) = W_0^{(2)}(f, g). \tag{6.6}$$

(Note: one can pick subnets so that (6.5), its analogy for $A_a(N(\lambda)g_\lambda)$ and (6.6) all hold.) Hence (6.4)-(6.6) entail $W_0^{(2)}(f, g) = 0$ for all test functions f and g as indicated, which is false. ■

Remarks. — 1) We note that the assumption (A) can clearly be weakened. It is not necessary to assume that the scaling limit is a free, massless field theory. The only conditions on the scaling limit required by the proof above are that for only two test functions f, g as above (and their complex conjugates $\bar{f}$, $\bar{g}$), the limits indicated in (A) exist, the limit for f or g vanishes, and the limit for $N(\lambda)^2\omega(\varphi(f_\lambda)\varphi(g_\lambda))$ does not vanish. One may expect that these conditions will obtain in any theory with a semblance of an ultraviolet scaling limit.

2) If one adds the weak technical assumption that the algebras in the net are locally generated, then one can drop the assumption that the vacuum representation is irreducible, since by employing the results and methods of [*18*] one can decompose the representation into a direct integral of irreducible vacuum representations, each satisfying (A). Then noting that a normal product state would have support in only one component of this direct integral, one can then apply Theorem 6.1 in this component to obtain the contradiction.

Finally, we wish to point out the fact that the results of Theorem 6.1 carry over to (non-vacuum) representations that are in the local quasi equivalence class of a vacuum representation as described in the hypothesis. For example, any massive single particle representation in the sense of [*7*] or any representation in the superselection sectors appearing in the theory of superselection structure of Doplicher, Haag and Roberts [*12*] whose associated vacuum representations [*7*], [*12*] satisfy the hypothesis of Theorem 6.1 would have only nonsplit algebras for tangent spacetime regions as described.

VII. DISCUSSION

The methods and results of Sections 4-6 strongly suggest that the following conjecture is true:

(C) Let $\{\mathcal{A}(\mathcal{O})\}$ be a net of local observable von Neumann algebras in a vacuum representation, to which is locally associated a quantum field and for which assumption (A) in Section 6 holds. Then $(\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2))$ is maximally correlated for any tangent double cones $\mathcal{O}_1, \mathcal{O}_2 \in \mathscr{K}$.

Moreover, it is likely that this will be true for tangent regions $\mathcal{O}_1, \mathcal{O}_2$ in much the same generality as Theorem 5.1 (although the tangent regions should not be toonarrow), and not merely because this theorem established a necessary condition for maximal correlation. But at present we have not found a maximally violating sequence in the massless free quantum field that has a simple expression in terms of fields, and this seems to be necessary to carry over the ideas in this paper to situations described in (A) in Section 6.

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