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Threshold scattering in two dimensions

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ABSTRACT. — A systematic analysis of low-energy scattering for Schrödinger operators of the type $H = -\Delta \dot{+} V$ in $L^2(\mathbb{R}^2)$ is given. The possibility of zero-energy resonances and zero-energy bound states of H is taken into account explicitly. No spherical symmetry of V is assumed. In particular, a two-variable Laurent expansion around the zero-energy threshold is provided for the transition operator. The first coefficients are written down explicitly. Furthermore the leading behavior of the scattering amplitude and scattering operator is determined. A generalized scattering length is defined and related directly to the threshold behavior of the scattering amplitude. Finally, a two-variable Laurent expansion is derived for the trace of the difference between the full and free resolvents around threshold. This result is used to prove Levinson's theorem in all cases.

RÉSUMÉ. — Nous donnons une analyse systématique de la diffusion à basse énergie pour l'opérateur de Schrödinger $H = -\Delta \dot{+}$ dans $L^2(\mathbb{R}^2)$.

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Nous tenons compte explicitement de la possibilité d'existence d'une résonance ou d'un état lié d'énergie nulle. Nous ne faisons pas d'hypothèse de symétrie sphérique sur V. Nous donnons un développement de Laurent à deux variables autour du seuil d'énergie zéro pour l'opérateur de transition. Les premiers coefficients sont donnés explicitement. De plus les comportements dominants de l'opérateur et de l'amplitude de diffusion sont déterminés. Nous définissons une longueur de diffusion généralisée qui est reliée au comportement au seuil de l'amplitude de diffusion. Nous obtenons aussi un développement de Laurent à deux variables pour la trace de la différence entre la résolvante et la résolvante du problème libre. Ceci nous permet de prouver un théorème de Levinson dans tous les cas.

1. INTRODUCTION

Two-dimensional low-energy phenomena have attracted a lot of interest recently. For example, the introduction of low-energy concepts like partial-wave scattering length and effective range has given new insights in the study of spin-polarized atomic hydrogen adsorbed on a surface. In a lot of these phenomena the occurence of zero-energy resonances or zero-energy bound states plays a significant role. It has been suggested e. g. that such a state might be responsible for the fast surface recombination rate for deuterium atoms below 1 K. For precise references and other examples in this connection we refer to [*2*] [*3*] [*10*] [*11*] [*12*].

The purpose of this paper is then to provide a systematic analysis of low-energy scattering in two dimensions, taking into account explicitly the possibility of zero-energy resonances and/or zero-energy bound states of the Schrödinger Hamiltonian. Such an analysis has been carried out for three dimensions in [*5*] [*6*] [*14*] [*27*] [*33*] and for one dimensions in [*13*] [*15*]. (For an extensive list of references concerning one and three dimensions, see [*8*] [*14*].) From these results it can be seen that the analysis in one dimension is more involved than the one in three dimensions, due to the well-known additional difficulty that the free resolvent has a singularity of the square root type in the limit as the energy tends to zero. In two dimensions the situation will be even more complex because of the logarithmic nature of this singularity.

Different aspects of two-dimensional Schrödinger systems have received already some attention in the past. In particular, ground state properties as a function of the coupling constant have been discussed in [*26*] [*30*] [*38*] [*39*] [*45*]. Bounds for the number of bound states have been obtained [34] [42]. Eigenfunction expansions of the Hamiltonian and their use to study the wave operator and scattering matrix have been considered in [4] [43] [44]. The inverse scattering problem has been treated [16] [18].

Low-energy parametrizations for spherically symmetric scattering on the basis of the scattering length have been derived in [3] [10] [11] [22] [23] [35]. Such results have been employed to obtain a spherically symmetric Levinson theorem [22] [23] [24] [35]. Levinson's theorem for non-spherically symmetric scattering in the absence of zero-energy resonances and zero-energy bound states has been proved first by Cheney [17]. A related local spectral sum rule for the time delay operator has been proved in [37].

Let us now give a short description of the results obtained in this paper. Our treatment and notation parallels the one-dimensional discussion in [13] [15]. Without loss of generality we assume $(v, u) \equiv \int_{\mathbb{R}^2} d^2x V(\underline{x}) \neq 0$. The case $(v, u) = 0$ can be discussed like in [13].

Section 2 studies the occurrence and properties of zero-energy resonances and zero-energy bound states of the Schrödinger Hamiltonian $H = -\Delta \dot{+} V$ in $L^2(\mathbb{R})^2)$. No spherical symmetry of V is assumed. This leads to a classification of essentially four cases: case I, without zero-energy resonances and zero-energy bound-states ($\equiv$ generic case), case II*a* in which an *s*-wave type resonance of multiplicity $M = 1$ occurs, case II*b* containing a *p*-wave type resonance with $1 \leq M \leq 2$, case II*c*, a mixture of both, case III in which zero-energy bound states of multiplicity $N \in \mathbb{N}$ appear and finally case IV, representing mixtures of the cases II and III. This study is based upon the results of Klaus and Simon [30] which are improved in the sense that they are valid here for the class of potentials satisfying

$$(v, u) \neq 0, \int_{\mathbb{R}^2} d^2x(1 + |\underline{x}|^{2+\delta}) | V(\underline{x}) | < \infty, \int_{\mathbb{R}^2} d^2x \, | V(\underline{x}) |^{1+\delta} < \infty$$

for some $\delta > 0$. In the rest of this paper case IV is not treated explicitly for two reasons. Firstly, the results contain nothing new in the sense that they can be read off from the study of the other cases. (E. g. the contributions from zero-energy resonances and zero-energy bound states to Levinson's theorem are simply additive). Secondly, the details are too complex to be written down within a readable paper of reasonable length. However, relevant remarks concerning this case are given at the appropriate places.

In Section 3 we describe in detail the low-energy behavior of the transition operator $T(k)$, following [6] and [15], assuming roughly exponential fall-off for V at infinity. We get two-variable Laurent expansions (Taylor expansions in case I) around zero-energy threshold. In case I we derive recursion relations for the coefficients in this Taylor expansion for $T(k)$.

In the other cases II-III we discuss the first coefficients needed in the sequel. Analogous results for the resolvent and the evolution group of general elliptic differential operators have been obtained by Murata [*31*].

Based on these results, we present in Section 4 Taylor expansions for the scattering amplitude and S-operator. We find that $S(k) \to 1$ as $k \to 0$ in all cases I-IV, in contrast with one and three dimensions [*6*] [*13*] [*15*] [*27*] [*33*]. If V is spherically symmetric we recover the results of [*22*] [*35*].

Section 5 gives, in analogy with [*5*], an appropriate generalization of the definition of scattering length to nonspherically symmetric interactions (see [*10*] [*11*] [*22*] in the case of spherical symmetry). This parameter is directly related to the threshold behavior of the scattering amplitude and explicit formulas are provided.

Finally, in section 6 we derive two-variable Laurent expansions for the trace of the difference between the full and free resolvents around threshold. It is interesting to note that in case II*a* we get a singularity structure comparable to that of case I. We then apply contour integration techniques to prove Levinson's theorem, further assuming that $\mathcal{E}_+ = \mathcal{E} \cap (o, +\infty) = \phi$, $\mathcal{E}$ being the exceptional set including the negative eigenvalues of H. We find that in case II*a* the zero-energy resonances contribute exactly like (zero-energy) bound states. This is in sharp contrast with one (see [*15*]) and three (see e. g. [*8*]) dimensions. We remark that the exponential fall-off condition on V can be relaxed to $V \in L^{4/3}(\mathbb{R}^2)$ and respectively $(1 + |\cdot|)V \in L^1(\mathbb{R}^2)$ in case I, $(1 + |\cdot|^{2+\delta})V \in L^1(\mathbb{R}^2)$ in case II*a* and, roughly, $(1 + |\cdot|^{8+\delta})V \in L^1(\mathbb{R}^2)$, $\delta > 0$, in the other cases. Under somewhat stronger conditions, case I has been discussed before by Cheney [*17*]. These results have recently been used to prove different representations for Krein's spectral shift function [*9*].

A brief outline of the results of this paper has appeared in [*12*].

2. ZERO-ENERGY PROPERTIES OF H

In this section we study zero-energy properties of the two-dimensional Schrödinger operator H, taking into account explicitly the possibility of zero-energy resonances and/or zero-energy bound states of H.

Assume V to be a real-valued measurable function, satisfying hypothesis

(H.1)
$$\int_{\mathbb{R}^2} d^2x(1 + |\underline{x}|^{2+\delta})|V(\underline{x})| < \infty,$$
$$\int_{\mathbb{R}^2} d^2x\,|V(\underline{x})|^{1+\delta} < \infty \text{ for some } \delta > 0,$$
$$\int_{\mathbb{R}^2} d^2x V(\underline{x}) \neq 0. \tag{2.1}$$

The Hamiltonian H in $L^2(\mathbb{R})$ is then defined as the form sum

$$H = H_0 \dotplus \lambda_0 V,\ \lambda_0 \in \mathbb{R}/\{0\},$$
$$H_0 = -\Delta \text{ on } \mathcal{D}(H_0) = H^{2,2}(\mathbb{R}^2).$$

Introducing

$$v(\underline{x}) = |V(\underline{x})|^{1/2},\ u(\underline{x}) = |V(\underline{x})|^{1/2} \operatorname{sign} V(\underline{x}),\ u \cdot v = V \tag{2.3}$$

the transition operator T(*k*) in $L^2(\mathbb{R}^2)$ is defined as

$$T(k) = [1 + \lambda_0 u R_0(k) v]^{-1},\ \operatorname{Im} k \geq 0,\ k \neq 0,\ k^2 \notin \sigma_p(H). \tag{2.4}$$

Here $\sigma_p(\cdot)$ denotes the point spectrum and $R_0(k)$ is the resolvent of H_0

$$R_0(k) = (H_0 - k^2)^{-1},\ \operatorname{Im} k > 0 \tag{2.5}$$

with integral kernel

$$R_0(k, \underline{x}, \underline{y}) = (i/4) H_0^{(1)}(k\,|\underline{x} - \underline{y}|),\ \underline{x} \neq \underline{y} \tag{2.6}$$

($H_0^{(1)}(z)$ being the Hankel function of first kind and order zero [*1*]).

In order to exhibit the singularity of $R_0(k)$ as $k \to 0$, we decompose [*45*]

$$uR_0(k)v = (2\pi)^{-1}[-\ln(k) + (i\pi/2) + \ln(2) + \Psi(1)](v, \cdot)u + M(k),$$
$$\operatorname{Im} k \geq 0,\ k \neq 0 \tag{2.7}$$

where $M(k) \in \mathcal{B}_2(L^2(\mathbb{R}^2))$ (the set of Hilbert-Schmidt operators in $L^2(\mathbb{R}^2)$) for all $\operatorname{Im} k \geq 0$ and $\Psi(z)$ represents the digamma function [*1*]. In particular, the integral kernel of $M(0) \equiv M_{00}$ reads

$$M_{00}(\underline{x}, \underline{y}) = -(2\pi)^{-1} u(\underline{x}) \ln|\underline{x} - \underline{y}|\, v(\underline{y}),\ \underline{x} \neq \underline{y}. \tag{2.8}$$

Next we introduce

$$P = (v, u)^{-1}(v, \cdot)u, \quad Q = 1 - P \tag{2.9}$$

and hence we obtain for T(*k*)

$$T(k) = \{1 + (2\pi)^{-1}\lambda_0[-\ln(k) + (i\pi/2) + \ln(2) + \Psi(1)](v, \cdot)u + \lambda_0 M(k)\}^{-1},$$
$$\operatorname{Im} k \geq 0,\ k \neq 0,\ k^2 \notin \sigma_p(H). \tag{2.10}$$

Since the low-energy behavior of T(*k*) crucially depends on the zero-energy behavior of H we first recall [*15*] [*30*].

LEMMA 2.1. — Let V satisfy (H.1). Assume that -1 is an eigenvalue of $\lambda_0 Q M_{00} Q$ and let

$$\mathcal{V} = \{\phi \in L^2(\mathbb{R}^2) \mid \lambda_0 Q M_{00} Q \phi = -\phi\},$$
$$\mathcal{W} = \{\chi \in \mathcal{V} \mid (v, (z - \lambda_0 M_{00})^{-1}\chi) = 0 \quad \text{for some} \quad |z| > \|\lambda_0 M_{00}\|\}.$$

Then

i) $\mathcal{W}$ is independent of *z*.

ii) $\dim \mathcal{W} = \dim \mathcal{V}$ or $\dim \mathcal{W} = \dim \mathcal{V} - 1$.

iii) $(z - \mu P - \lambda_0 M_{00})^{-1}\chi = (z - \lambda_0 M_{00})^{-1}\chi$ for all $\chi \in \mathscr{W}, \mu \in \mathbb{C}$.
iv) $\lambda_0 M_{00}\chi = -\chi$ for all $\chi \in \mathscr{W}$.
v) If $\phi_0 \in \mathscr{V} \backslash \mathscr{W}$ then

$$\lambda_0 M_{00}\phi_0 = -\phi_0 + (v, u)^{-1}\lambda_0(v, M_{00}\phi_0)u.$$

vi) If $\chi \in \mathscr{V}$ then $(v, (z - \lambda_0 M_{00})^{-1}\chi) = 0$ is equivalent to $(v, M_{00}\chi) = 0$.

Consequently

$$\mathscr{W} = \{ \chi \in \mathscr{V} \mid (v, M_{00}\chi) = 0 \}.$$

This result represents a slightly improved version of Lemma 7.3 of [*30*] and has been proved in [*15*]. Next we state

LEMMA 2.2. — Let V satisfy (H.1). Assume that $\lambda_0 Q M_{00} Q \phi = -\phi$ for some $\phi \in L^2(\mathbb{R}^2)$ and define the function ψ by

$$\psi(\underline{x}) = -(v, u)^{-1}\lambda_0(v, M_{00}\phi) - (2\pi)^{-1}\lambda_0 \int_{\mathbb{R}^2} d^2y \ln|\underline{x} - \underline{y}| v(\underline{y})\phi(\underline{y}). \quad (2.11)$$

Then

i) $\psi \in L^2_{loc}(\mathbb{R}^2)$, $\nabla\psi \in L^2_{loc}(\mathbb{R}^2)$ and $H\psi = 0$ in the sense of distributions.
ii) $u(\underline{x})\psi(\underline{x}) = -\phi(\underline{x})$ a. e.
iii) $\psi + (v, u)^{-1}\lambda_0(v, M_{00}\phi) - (2\pi)^{-1}\lambda_0 \mid \underline{x} \mid^{-2} \underline{x}((\underline{\cdot})v, \phi) \in L^2(\mathbb{R}^2)$, in particular,

$$\psi \in L^2(\mathbb{R}^2) \Leftrightarrow (v, M_{00}\phi) = ((\underline{\cdot})v, \phi) = 0.$$

Proof. — From [*25*], p. 527

$$\int_0^{2\pi} d\varphi \ln [1 - 2a\cos\varphi + a^2] = \begin{cases} 0 & a^2 \leq 1 \\ 2\pi \ln a^2 & a^2 \geq 1 \end{cases} \quad (2.12)$$

and

$$\int_0^{2\pi} d\varphi \ln^2 [1 - 2\cos\varphi + a^2] = 4\pi \sum_{l=1}^{\infty} l^{-2}a^{2l}, |a| \leq 1, a \in \mathbb{R} \quad (2.13)$$

one infers

$$\int_{|\underline{x}| \leq R} d^2x \ln^2 |\underline{x} - \underline{y}| \leq C_R \ln^2(2 + |\underline{y}|), R > 0 \quad (2.14)$$

implying

$$\int_{|\underline{x}| \leq R} d^2x \mid \psi(\underline{x}) \mid^2$$
$$\leq 2\pi R^2 \mid (v, u) \mid^{-2} \mid \lambda_0 \mid^2 \mid (v, M_{00}\phi) \mid^2$$
$$+ 2^{-1}\pi^{-2} \mid \lambda_0 \mid^2 \| \ln^2(2 + |\underline{\cdot}|)V \|_1 \| \phi \|_2^2 < \infty, R > 0. \quad (2.15)$$

Thus $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. From

$$\int_{|\underline{x}|\leq R} d^2x\,|(\nabla\psi)(\underline{x})|^2 \leq (2\pi)^{-2}\lambda_0^2 \int_{\mathbb{R}^4} d^2y d^2z v(\underline{y})v(\underline{z})\,|\phi(\underline{y})|\,|\phi(\underline{z})| \times$$
$$\times \int_{|\underline{x}|\leq R} d^2x\,|\underline{x}-\underline{y}|^{-1}|\underline{x}-\underline{z}|^{-1} \quad (2.16)$$

and from

$$\int_{|\underline{x}|\leq R} d^2x\,|\underline{x}-\underline{y}|^{-1}|\underline{x}-\underline{z}|^{-1} = \int_{|\underline{\xi}|\leq 1} d^2\xi\,|\underline{\xi}-\underline{y}R^{-1}|\,|\underline{\xi}-\underline{z}R^{-1}|$$
$$\leq \int_{|\underline{\xi}|\leq 1+(|\underline{y}|+|\underline{z}|)R^{-1}} d^2\xi\,|\underline{\xi}-\underline{y}R^{-1}|^{-1}|\underline{\xi}-\underline{z}R^{-1}|$$
$$\leq \int_{|\underline{\eta}|\leq 2+2(|\underline{y}|+|\underline{z}|)R^{-1}} d^2\eta\,|\underline{\eta}|^{-1}|\underline{\eta}-(\underline{z}-\underline{y})R^{-1}|$$
$$= \int_0^{2\pi} d\varphi \ln\,[2(\eta^2 - 2\eta\,|\underline{z}-\underline{y}|\,R^{-1}\cos\varphi + |\underline{z}-\underline{y}|^2R^{-2})^{1/2}$$
$$+ 2\eta - 2\,|\underline{z}-\underline{y}|\,R^{-1}\cos\phi\,]\,\Big|_{\eta=0}^{\eta=2+2(|\underline{y}|+|\underline{z}|)R^{-1}}$$
$$\leq C\{\,|\ln(|\underline{z}-\underline{y}|/R)| + \ln[1+(|y|+|z|)R^{-1}] + 1\,\},\ R>0 \quad (2.17)$$

we infer $\nabla\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ using assumptions (2.1) in the same way as one proves $M(k) \in \mathscr{B}_2(L^2(\mathbb{R}^2))$ for $\operatorname{Im} k \geq 0$ [*45*]. This proves *i*). Assertion *ii*) simply follows from Lemma 2.1 *v*) after multiplying eq. (2.11) with $u(\underline{x})$. In order to prove *iii*) we decompose

$$\ln|\underline{x}-\underline{y}| = \ln|\underline{x}_>| + 2^{-1}\ln[1+|\underline{x}_>|^{-2}|\underline{x}_<|^2 - 2|\underline{x}_>|^{-2}\underline{x}_>\cdot\underline{x}_<] \quad (2.18)$$

where

$$\underline{x}_< = \begin{cases} \underline{x}, & |\underline{x}|\leq|\underline{y}| \\ \underline{y}, & |\underline{x}|>|\underline{y}| \end{cases}, \quad \underline{x}_> = \begin{cases} \underline{y}, & |\underline{x}|\leq|\underline{y}| \\ \underline{x}, & |\underline{x}|>|\underline{y}| \end{cases}. \quad (2.19)$$

Then ψ may be rewritten as

$$\psi(\underline{x}) = -(v,u)^{-1}\lambda_0(v, M_{00}\phi) + (2\pi)^{-1}\lambda_0(v, |\underline{x}_>|^{-2}\underline{x}_>\cdot\underline{x}_<\phi)$$
$$-(2\pi)^{-1}\lambda_0(v, \ln|\underline{x}_>|\,\phi) + \tilde{\psi}(\underline{x}) \quad (2.20)$$

where

$$\tilde{\psi}(\underline{x}) = -(2\pi)^{-1}\lambda_0\int_{\mathbb{R}^2} d^2y\,\{\ln[|\underline{x}_>|^{-1}|\underline{x}-\underline{y}|] + |\underline{x}_>|^{-2}\underline{x}_>\cdot\underline{x}_<\}\,v(\underline{y})\phi(\underline{y}). \quad (2.21)$$

From

$$\int_{\mathbb{R}^2} d^2x\,\{\ln[|\underline{x}_>|^{-1}|\underline{x}-\underline{y}|] + |\underline{x}_>|^{-2}\underline{x}_>\cdot\underline{x}_<\}^2 = \pi|\underline{y}|^2/4 \quad (2.22)$$

we conclude that $\tilde{\psi} \in L^2(\mathbb{R}^2)$ since

$$\| \tilde{\psi} \|_2^2 \leq (4\pi^2)^{-1} | \lambda_0 |^2 \| |\underline{\cdot}|^2 V \|_1 \| \phi \|_2^2 < \infty . \tag{2.23}$$

Next, using $(v, \phi) = 0$, we estimate for $\varepsilon > 0$, $|\underline{x}| \geq 2$

$$\begin{aligned} |(v, \ln |\underline{x}_>| \phi)| &\leq |\ln|x|| \int_{|\underline{y}| \geq |\underline{x}|} d^2y v(\underline{y}) |\phi(\underline{y})| + \int_{|\underline{y}| \geq |\underline{x}|} d^2y |\ln|\underline{y}|| v(\underline{y}) |\phi(\underline{y})| \\ &\leq (1 + |\underline{x}|)^{-1-\varepsilon} |\ln|\underline{x}|| \left(\int_{|\underline{y}| \geq |\underline{x}|} d^2y (1 + |\underline{y}|)^{2+2\varepsilon} |V(\underline{y})| \right)^{1/2} \|\phi\|_2 \\ &\quad + (1 + |\underline{x}|)^{-1-\varepsilon} \left(\int_{|\underline{y}| \geq \underline{x}|} d^2y \ln|\underline{y}| (1 + |\underline{y}|)^{2+2\varepsilon} |V(\underline{y})| \right)^{1/2} \|\phi\|_2 \\ &\leq C_\delta |\underline{x}|^{-1-\delta}, \ \delta > 0, \ |\underline{x}| \geq 2 . \end{aligned} \tag{2.24}$$

Similarly, we get

$$\begin{aligned} &|(v, |\underline{x}_>|^{-2} \underline{x}_> \cdot \underline{x}_< \phi) - |\underline{x}|^{-2} \underline{x} \cdot ((\underline{\cdot}) v, \phi)| \\ &\quad \leq |\underline{x}|^{-1} \int_{|\underline{y}| \geq |\underline{x}|} d^2y |\underline{y}| v(\underline{y}) |\phi(\underline{y})| + |\underline{x}| \int_{|\underline{y}| \geq |\underline{x}|} d^2y |\underline{y}|^{-1} v(\underline{y}) |\phi(\underline{y})| \\ &\quad \leq 2 |\underline{x}|^{-1-\delta} \left(\int_{|\underline{y}| \geq |\underline{x}|} d^2y |\underline{y}|^{2+2\delta} |V(\underline{y})| \right)^{1/2} \|\phi|_2, \ \delta > 0, \ |\underline{x}| \geq 1 . \end{aligned} \tag{2.25}$$

Hence *iii*) holds. ■

In view of Lemma 2.1-2.2, we can distinguish the following cases in the zero-energy behavior of H. Assuming that V satisfies (H.1) we have:

CASE I. — (-1) is not an eigenvalue of $\lambda_0 QM_{00}Q$.

CASE II. — (-1) is an eigenvalue of $\lambda_0 QM_{00}Q$ of multiplicity $1 \leq M \leq 3$, $\lambda_0 QM_{00}Q\phi_j = -\phi_j$, $\phi_j \in L^2(\mathbb{R}^2)$ and

a) $M = 1, j = 0, c_1^{(0)} \neq 0$

or

b) $1 \leq M \leq 2$, $j = 1$ if $M = 1$ and $j = 1, 2$ if $M = 2$, $c_1^{(j)} = 0$, $\underline{c}_2^{(j)} \neq 0$, if $M = 2$ then $\underline{c}_2^{(1)}$, $\underline{c}_2^{(2)}$ are linearly independent,

or

c) $2 \leq M \leq 3$, $j = 0, 1$ if $M = 2$ and $j = 0, 1, 2$ if $M = 3$, $c_1^{(0)} \neq 0$, $c_1^{(j)} = 0$, $\underline{c}_2^{(j)} \neq 0$, $j \geq 1$, if $M = 3$, then $\underline{c}_2^{(1)}$, $\underline{c}_2^{(2)}$ are linearly independent,

CASE III. — (-1) is an eigenvalue of $\lambda_0 QM_{00}Q$ of multiplicity $N \in \mathbb{N}$, $\lambda_0 QM_{00}Q\phi_j = -\phi_j$, $\phi_j \in L^2(\mathbb{R}^2)$ and $c_1^{(j)} = \underline{c}_2^{(j)} = 0, 3 \leq j \leq 2 + N$.

CASE IV. — (-1) is an eigenvalue of $\lambda_0 QM_{00}Q$ of multiplicity $M + N$, $1 \leq M \leq 3$, $N \in \mathbb{N}$, $\lambda_0 QM_{00}Q\phi_j = -\phi_j$, $\phi_j \in L^2(\mathbb{R}^2)$ and

a) $M = 1, j = 0, 3, \ldots, 2 + N$, $c_1^{(0)} \neq 0$, $c_1^{(j)} = \underline{c}_2^{(j)} = 0, j \geq 3$

or

b) $1 \leq M \leq 2$, $j = 1, 3, \ldots, 2 + N$ if $M = 1$ and $j = 1, \ldots, 2 + N$ if $M = 2$, $c_1^{(j)} = 0$, $\underline{c}_2^{(j)} \neq 0$, $1 \leq j \leq 2$, $c_1^{(j)} = \underline{c}_2^{(j)} = 0$, $j \geq 3$, if $M = 2$ then $\underline{c}_2^{(1)}$, $\underline{c}_2^{(2)}$ are linearly independent,

or

c) $2 \leq M \leq 3$, $j = 0, 1, 3, \ldots, 2 + N$ if $M = 2$ and $j = 0, \ldots, 2 + N$ if $M = 3$, $c_1^{(0)} \neq 0$, $\underline{c}_1^{(j)} = 0$, $\underline{c}_2^{(j)} \neq 0$, $1 \leq j \leq 2$, $c_1^{(j)} = \underline{c}_2^{(j)} = 0$, $j \geq 3$, if $M = 3$ then $\underline{c}_2^{(1)}$, $\underline{c}_2^{(2)}$ are linearly independent,

Here the coefficients $c_1^{(j)}$, $\underline{c}_2^{(j)}$ are defined by

$$c_1^{(j)} = (v, u)^{-1}(v, M_{00}\phi_j), \quad \underline{c}_2^{(j)} = (2\pi)^{-1}((\underline{\cdot})v, \phi_j) \tag{2.26}$$

where j takes on the values described above.

REMARK 2.3. — *a)* Case I means the absence of zero-energy resonances and zero-energy bound states of H and hence represents the generic case. Cases II*a)*-*c)* describe the various possibilities of zero-energy resonances of H. If V is spherically symmetric then case II*a)* and case II*b)* correspond to zero-energy resonances in the *s*-wave and *p*-wave respectively [*22*] [*35*]. Case III denotes the zero-energy bound state case. In particular, if V is spherically symmetric this case only happens to occur in *d*- and higher waves. Finally, cases IV*a)*-*c)* describe possible admixtures of cases II-III. In all cases II-IV we have $(v, \phi_j) = 0$. By Lemmas 2.1 and 2.2 the above list of cases is complete. As explained in the introduction we restrict our detailed treatment to cases I-III and present some relevant remarks concerning case IV in the following.

b) These case distinctions in two dimensions first appeared in Klaus and Simon [*30*]. Zero-energy resonance wave functions are also discussed in [*31*] [*32*]. Sufficient conditions for $\nabla\psi \in L^2(\mathbb{R}^2)$ in Lemma 2.2 *i)* can be read off from [*7*] [*32*].

3. LOW-ENERGY EXPANSIONS OF T(*k*)

Now we derive the low-energy expansions of T(*k*) for the different cases I-III presented in Section 2. We give recursion relations for the Taylor coefficients of T(*k*) in case I. We closely follow the treatment in [*6*] [*15*].

We start with

LEMMA 3.1. — Assume (H.1) and let $\varepsilon \in \mathbb{C} \setminus \{0\}$ be small enough. Then the norm convergent expansion

$$(1 + \lambda_0 QM_{00}Q + \varepsilon)^{-1} = \varepsilon^{-1}P_0 + \sum_{m=0}^{\infty} (-\varepsilon)^m T_0^{m+1} \tag{3.1}$$

holds. Here P_0 denotes the projection onto the eigenspace of $\lambda_0 QM_{00}Q$ to the eigenvalue -1, i. e.,

$$P_0 = \sum_j (\tilde{\phi}_j, \phi_j)^{-1}(\tilde{\phi}_j, \cdot)\phi_j \tag{3.2}$$

(the range of j depends on the cases I-III and is specified in Section 2) where

$$\lambda_0 QM_{00}Q\phi_j = -\phi_j, \quad \phi_j \in L^2(\mathbb{R}^2), \quad \tilde{\phi}_j = \operatorname{sgn}(V)\phi_j, \quad (\tilde{\phi}_j, \phi_l) = (\tilde{\phi}_j, \phi_j)\delta_{jl}. \tag{3.3}$$

The operator T_0 denotes the corresponding reduced resolvent, viz.

$$T_0 = n - \lim_{\varepsilon \to 0} (1 + \lambda_0 QM_{00}Q + \varepsilon)^{-1}(1 - P_0). \tag{3.4}$$

For the proof of this lemma one can follow the proof of Lemma 3.1 in [*6*] or [*15*] step by step.

Next we strengthen our assumptions on V and replace hypothesis (H.1) by

(H.2)
$$\int_{\mathbb{R}^2} d^2x e^{2a|\underline{x}|} \, | V(\underline{x}) | < \infty ,$$
$$\int_{\mathbb{R}^2} d^2x \, | V(\underline{x}) |^{1+\delta} < \infty \quad \text{for some } a, \delta > 0,$$
$$\int_{\mathbb{R}^2} d^2x V(\underline{x}) \neq 0. \tag{3.5}$$

Taking into account the properties of the Hankel function $H_0^{(1)}(z)$ as $z \to 0$ [*1*] we infer from eq. (2.7).

LEMMA 3.2. — Assume (H.2). Then

$$uR_0(k)v = \lambda_0 \alpha(k)P + \lambda_0 M(k) = \lambda_0 \alpha(k)P + \lambda_0 M_1(k^2) + \lambda_0 M_2(k^2) \ln k \tag{3.6}$$

where

$$\alpha(k) = ((v, u)/2\pi)[- \ln k + (i\pi/2) + \ln 2 + \Psi(1)] \tag{3.7}$$

and $M_j(k^2)$, $j = 1, 2$ are analytic in $\mathscr{B}_2(L^2(\mathbb{R}^2))$-norm with respect to k^2 in $\operatorname{Im} k > -a$. Moreover the expansion

$$uR_0(k)v = \lambda_0 \alpha(k)P + \lambda_0 \sum_{m,n=0}^{\infty} (1/\ln k)^m (k^2 \ln k)^n M_{mn}, \quad | k | \text{ small enough} \tag{3.8}$$

is valid in $\mathscr{B}_2(L^2(\mathbb{R}^2))$-norm. The coefficients M_{mn} have integral kernels

$$M_{m-1,m}(\underline{x}, \underline{y}) = (2\pi)^{-1} 4^{-m} (m!)^{-2} (-1)^{m-1} u(\underline{x}) | \underline{x} - \underline{y} |^{2m} v(\underline{y}), \; m \in \mathbb{N}$$
$$M_{mm}(\underline{x}, \underline{y}) = (4\pi)^{-1} 4^{-m} (m!)^{-2} (-1)^m \{ [i\pi + \ln 4 + 2\Psi(m+1)] \times$$
$$\times u(\underline{x}) | \underline{x} - \underline{y} |^{2m} v(\underline{y}) - 2u(\underline{x}) | \underline{x} - \underline{y} |^{2m} \ln | \underline{x} - \underline{y} | v(\underline{y}) \}, \; \underline{x} \neq \underline{y}, \; m \in \mathbb{N},$$
$$M_{mn}(\underline{x}, \underline{y}) = 0, \; m \leq n-2 \text{ or } m \geq n+1, \; m, n \in \mathbb{N}. \tag{3.9}$$

Next, using these results we determine the low-energy behavior of $T(k)$ in cases I-III.

THEOREM 3.3. — Assume (H.2). Then $T(k)$ has the norm convergent Laurent expansion

$$T(k)=\sum_{m=p}^{\infty}\sum_{n=q}^{\infty}(1/\ln k)^m(k^2\ln k)^n t_{mn},\ 0<|k| \text{ small enough} \tag{3.10}$$

where

$$p=\begin{cases}0 & \text{in cases I, II}b)\\ -1 & \text{in case III, and in case II}c) \text{ if } \underline{c}_2^{(0)}=0\\ -\infty & \text{in case II}a)\text{, and in case II}c) \text{ if } \underline{c}_2^{(0)}\neq 0,\end{cases} \tag{3.11}$$

$$q=\begin{cases}0 & \text{in cases I, II}a)\\ -1 & \text{in cases II}b), c), \text{III}\end{cases} \tag{3.12}$$

and

$$t_{m,0}=0,\ m\le -2 \text{ in case II}a). \tag{3.13}$$

$$t_{m,-1}=0,\ m\ge 1 \text{ in case II}b). \tag{3.14}$$

$$t_{-1,-1}=0,\ t_{m,-1}=0,\ m\ge 1 \text{ in case II}c) \text{ if } \underline{c}_2^{(0)}=0. \tag{3.15}$$

$$t_{m,-1}=0,\ m\ge 1 \text{ or } m\le -1,\ t_{m,0}=0,\ m\le -2 \text{ in case II}c) \text{ if } \underline{c}_2^{(0)}\neq 0. \tag{3.16}$$

Proof of case I. — Recalling eq. (3.8) and employing

$$[1+\lambda_0\alpha(k)P]^{-1}=Q+\beta(k)P,\ \beta(k)=[1+\lambda_0\alpha(k)]^{-1} \tag{3.17}$$

we get, starting from eq. (2.4)

$$\begin{aligned}T(k)&=\{1+\lambda_0\alpha(k)P+\lambda_0M_{00}+\lambda_0k^2\ln kM_{01}+\lambda_0k^2M_{11}+O_{12}(k^4\ln k)\}^{-1}\\ &=\{1+[Q+\beta(k)P][\lambda_0M_{00}+\lambda_0k^2\ln kM_{01}+\lambda_0k^2M_{11}+O_{12}(k^4\ln k)\}^{-1}\\ &\quad\times[Q+\beta(k)P]\\ &=\{1+\lambda_0QM_{00}+\lambda_0\beta(k)PM_{00}+\lambda_0k^2\ln kQM_{01}+\lambda_0\beta(k)k^2\ln kPM_{01}\\ &\quad+\lambda_0k^2QM_{11}+\lambda_0\beta(k)k^2PM_{11}+O_{12}(k^4\ln k)\}^{-1}[Q+\beta(k)P].\end{aligned} \tag{3.18}$$

Here the index of the order symbol $O_{12}(\cdot)$ indicates that $O_{12}(\cdot)$ is analytic in $(1/\ln k, k^2\ln k)$, similarly $O_1(\cdot)$ will indicate analyticity in $1/\ln k$. In addition the symbols $O_{12\varepsilon}(\cdot)$, $O_{1\varepsilon}(\cdot)$ indicate analyticity with respect to an additional parameter ε (to be specified later on in cases II-III). Using

$$(1+AP)^{-1}=1-(v,(1+A)u)^{-1}(v,\cdot)Au,\quad \text{if } (v,(1+A)u)\neq 0,\ A\in\mathscr{B}(L^2(\mathbb{R}^2)) \tag{3.19}$$

($\mathscr{B}(L^2(\mathbb{R}^2))$-the set of bounded operators in $L^2(\mathbb{R}^2)$) and hence

$$(1+\lambda_0QM_{00}P)^{-1}=1-\lambda_0QM_{00}P \tag{3.20}$$

we finally infer

$$\begin{aligned} T(k) &= \{1+\lambda_0 QM_{00}P+\lambda_0 QM_{00}Q+O_{12}(1/\ln k)\}^{-1}[Q+O_1(1/\ln k)] \\ &= \{1+[1-\lambda_0 QM_{00}P][\lambda_0 QM_{00}Q+O_{12}(1/\ln k)]\}^{-1}(1-\lambda_0 QM_{00}P) \\ &\quad \times [Q+O_1(1/\ln k)] \\ &= \{1+\lambda_0 QM_{00}Q+O_{12}(1/\ln k)\}^{-1}[Q+O_1(1/\ln k)]. \end{aligned} \tag{3.21}$$

■

Proof of case IIa). — Repeating the calculation up to eq. (3.21) taking into account higher-order terms gives

$$\begin{aligned} T(k) = \{ &1+\lambda_0 QM_{00}Q+\varepsilon-\varepsilon+\lambda_0\beta(k)PM_{00}-\lambda_0^2\beta(k)QM_{00}PM_{00} \\ &+\lambda_0 k^2 \ln k QM_{01}+\lambda_0\beta(k)k^2\ln k PM_{01}-\lambda_0\beta(k)k^2\ln k QM_{00}PM_{01} \\ &+\lambda_0 k^2 QM_{11}+\lambda_0\beta(k)k^2 PM_{11}-\lambda_0^2\beta(k)k^2 QM_{00}PM_{11} \\ &+O_{12}(k^4\ln k)\}^{-1}(1-\lambda_0 QM_{00}P)[Q+\beta(k)P] \end{aligned} \tag{3.22}$$

where $\varepsilon \in \mathbb{C}\backslash\{0\}$ will be specified later on. Eq. (3.1) and

$$P_0P = 0,\ P_0Q = P_0,\ T_0P = P \tag{3.23}$$

then yield after a lengthy calculation

$$\begin{aligned} T(k) = \{ &1-P_0+\varepsilon^{-1}[(2\pi\lambda_0/(v,u)\ln k)+O_1((1/\ln k)^2)]P_0M_{00}PM_{00} \\ &+\lambda_0\varepsilon^{-1}k^2\ln k P_0M_{01}+\varepsilon^{-1}[(2\pi\lambda_0/(v,u)\ln k) \\ &+O_1((1/\ln k)^2)]k^2\ln k P_0M_{00}PM_{01} \\ &+\lambda_0\varepsilon^{-1}k^2P_0M_{11}+\varepsilon^{-1}[(2\pi\lambda_0/(v,u)\ln k)+O_1((1/\ln k)^2)]k^2P_0M_{00}PM_{11} \\ &+(2\pi\lambda_0/(v,u)\ln k)T_0QM_{00}PM_{00}-(2\pi/(v,u)\ln k)PM_{00} \\ &+\varepsilon^{-1}O_{12}(k^4\ln k)+O_{12\varepsilon}((1/\ln k)^2)\}^{-1}\times \\ &\times\{\varepsilon^{-1}P_0+[(2\pi\lambda_0/(v,u)\ln k)+O_1((1/\ln k)^{-2})]\varepsilon^{-1}P_0M_{00}P+T_0Q \\ &+O_\varepsilon(\varepsilon)Q+O_{1\varepsilon}((1/\ln k))\} \end{aligned} \tag{3.24}$$

(in obvious notation $O_\varepsilon(\cdot)$ indicates analyticity in ε). Next we note that

$$(1+P_0B)^{-1} = 1-\sum_{j,l}(\tilde{\phi},(1+B)\phi)_{jl}^{-1}(B^*\tilde{\phi}_l,\cdot)\phi_j,$$

$$(1+P_0B)^{-1}P_0 = \sum_{j,l}(\tilde{\phi},(1+B)\phi)_{jl}^{-1}(\tilde{\phi}_l,\cdot)\phi_j \tag{3.25}$$

$$B\in\mathscr{B}(L^2(\mathbb{R}^2))$$

if the inverse of the matrix $(\tilde{\phi}_j,(1+B)\phi_l)$ exists.

Now we specialize to case II*a*). Choosing $\varepsilon = 1/\ln k$, then eqs. (3.24),

$$(\tilde{\phi}_0, M_{00}PM_{00}\phi_0) = (v,u)\,|\,c_1^{(0)}\,|^2 \neq 0\,, \tag{3.26}$$

and eq. (3.25), which reads here

$$[1-P_0+(2\pi\lambda_0/(v,u))P_0M_{00}PM_{00}]^{-1}P_0=(1/2\pi\lambda_0|c_1^{(0)}|^2)(\tilde{\phi}_0,\cdot)\phi_0 \quad (3.27)$$

finally prove the theorem in this case. ■

Proof of case IIb). — Choosing $\varepsilon = k^2 \ln k$ and observing $P_0M_{00}P = 0$, eq. (3.24) yields

$$T(k) = [1 - P_0 + \lambda_0 P_0 M_{01} + O_{12}(1/\ln k)]^{-1} \times \times [(k^2 \ln k)^{-1}P_0 + T_0Q + O_{12}(1/\ln k)]. \quad (3.28)$$

Since $(\tilde{\phi}_1, M_{01}\phi_1) = -\pi|\underline{c}_2^{(-1)}|^2 \neq 0$ if $M = 1$ and

$$\det[(\tilde{\phi}_j, M_{01}\phi_l)] = -\pi \det[\overline{\underline{c}_2^{(j)}} \cdot \underline{c}_2^{(l)}] \neq 0$$

if $M = 2$ the use of eq. (3.25) to calculate the inverse in (3.28) completes the proof. ■

Proof of case IIc) if $\underline{c}_2^{(0)} = 0$. — Choosing $\varepsilon = k^2 \ln k$ in eq. (3.24) leads to

$$\begin{aligned}T(k)= \{&1+[1-P_0+(2\pi\lambda_0/(v,u)k^2(\ln k)^2)P_0M_{00}PM_{00}+\lambda_0P_0M_{01}]^{-1}\times\\ &\times O_{12}(1/\ln k)+(1/k^2\ln k)O_1((1/\ln k)^2)P_0M_{00}PM_{00}\}^{-1}\times\\ &\times[1-P_0+(2\pi\lambda_0/(v,u)k^2(\ln k)^2)P_0M_{00}PM_{00}+\lambda_0P_0M_{01}]^{-1}\times\\ &\times[(1/k^2\ln k)P_0+(2\pi/(v,u)k^2(\ln k)^2)P_0M_{00}P + O_{12}(1)]. \quad (3.29)\end{aligned}$$

We know that

$$\begin{aligned}&(\tilde{\phi}, [(2\pi\lambda_0/(v,u)k^2(\ln k)^2)P_0M_{00}PM_{00} + \lambda_0P_0M_{01}]\phi)_{jl}^{-1}\\ &\quad= (1/2\pi\lambda_0|c_1^{(0)}|^2)\delta_{j0}\delta_{l0}k^2(\ln k)^2 - (1/\pi\lambda_0)(\overline{\underline{c}_2^{(j)}}\cdot\underline{c}_2^{(l)})^{-1}(1-\delta_{j0})(1-\delta_{l0}),\\ &\qquad 0 \le j, l \le M-1. \quad (3.30)\end{aligned}$$

Using eq. (3.25) we then get

$$\begin{aligned}&[1 - P_0 + (2\pi\lambda_0/(v,u)k^2(\ln k)^2)P_0M_{00}PM_{00} + \lambda_0P_0M_{01}]^{-1}\\ &\quad= 1 - ([(1/2\pi\lambda_0|c_1^{(0)}|^2)k^2(\ln k)^2(\lambda_0P_0M_{01} - 1)\\ &\quad+ (1/(v,u)|c_1^{(0)}|^2)P_0M_{00}PM_{00}]^*\tilde{\phi}_0,\cdot)\phi_0\\ &\quad- (1/\pi\lambda_0)\sum_{j,l=1}^{M-1}(\overline{\underline{c}_2^{(j)}}\cdot\underline{c}_2^{(l)})^{-1}([\lambda_0P_0M_{01} - 1]\tilde{\phi}_l,\cdot)\phi_j, \quad (3.31)\end{aligned}$$

$$\begin{aligned}&[1 - P_0 + (2\pi\lambda_0/(v,u)k^2(\ln k)^2)P_0M_{00}PM_{00} + \lambda_0P_0M_{01}]^{-1}P_0\\ &\quad= \sum_{j,l=0}^{M-1}[(k^2(\ln k)^2/2\pi\lambda_0|c_1^{(0)}|^2)\delta_{j0}\delta_{l0} -\\ &\quad- (1/\pi\lambda_0)(\overline{\underline{c}_2^{(j)}}\cdot\underline{c}_2^{(l)})^{-1}(1-\delta_{j0})(1-\delta_{l0})](\tilde{\phi}_l,\cdot)\phi_j. \quad (3.32)\end{aligned}$$

Consequently

$$(1/k^2 \ln k)\mathrm{O}_1((1/\ln k)^2)[1 - \mathrm{P}_0 + (2\pi\lambda_0/(v,u)k^2(\ln k)^2)\mathrm{P}_0\mathrm{M}_{00}\mathrm{P}\mathrm{M}_{00} + \lambda_0\mathrm{P}_0\mathrm{M}_{00}]^{-1}\mathrm{P}_0\mathrm{M}_{00}\mathrm{P}\mathrm{M}_{00} = \mathrm{O}_1(1/\ln k) \quad (3.33)$$

and we get

$$\mathrm{T}(k) = \{1 + \mathrm{O}_{12}(1/\ln k)\}^{-1}\left\{\sum_{j,l=0}^{\mathrm{M}-1}[(k^2(\ln k)^2/2\pi\lambda_0\,|\,c_1^{(0)}|^2)\delta_{j0}\delta_{l0} - (1/\pi\lambda_0)(\overline{\underline{c}_2^{(j)}}\cdot\underline{c}_2^{(l)})^{-1}(1-\delta_{j0})(1-\delta_{l0})](\tilde{\phi}_l,\cdot)\phi_j \times \times [(1/k^2\ln k)\mathrm{P}_0 + (2\pi/(v,u)k^2(\ln k)^2)\mathrm{P}_0\mathrm{M}_{00}\mathrm{P}] + \mathrm{O}_{12}(1)\right\}. \quad (3.34)$$

■

Proof of case IIc) if $\underline{c}_2^{(0)} \neq 0$. — Again we choose $\varepsilon = k^2 \ln k$ in eq. (3.24) and without loss of generality we assume $\mathrm{M} = 2$. Then

$$\mathrm{A}_{jl}(k) = (\tilde{\phi}_j, [(2\pi\lambda_0/(v,u)k^2(\ln k)^2)\mathrm{P}_0\mathrm{M}_{00}\mathrm{P}\mathrm{M}_{00} + \lambda_0\mathrm{P}_0\mathrm{M}_{01}]\phi_l), \quad 0 \le j,l \le 1 \quad (3.35)$$

has the inverse

$$\mathrm{A}(k)_{jl}^{-1} = -(1/\pi\lambda_0)|\underline{c}_2^{(1)}|^{-2}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + (1/2\pi\lambda_0|c_1^{(0)}|^2|\underline{c}_2^{(1)}|^2)k^2(\ln k)^2\begin{pmatrix}|\underline{c}_2^{(1)}|^2 & -\overline{\underline{c}_2^{(0)}}\cdot\underline{c}_2^{(1)}\\ -\overline{\underline{c}_2^{(1)}}\cdot\underline{c}_2^{(0)} & |\underline{c}_2^{(0)}|^2\end{pmatrix} - (1/2\pi\lambda_0|c_1^{(0)}|^2|\underline{c}_2^{(1)}|^4)[|\underline{c}_2^{(0)}|^2|\underline{c}_2^{(1)}|^2 - |\overline{\underline{c}_2^{(0)}}\cdot\underline{c}_2^{(1)}|^2]\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + \mathrm{O}_3(k^4(\ln k)^4) \quad (3.36)$$

where $\mathrm{O}_3(\cdot)$ indicates analyticity in $k^2(\ln k)^2$. Insertion of eq. (3.36) into eq. (3.29) employing eq. (3.25) then completes the proof. ■

Proof of case III. — Choosing $\varepsilon = k^2$ and observing

$$\mathrm{P}\mathrm{M}_{00}\mathrm{P}_0 = \mathrm{P}_0\mathrm{M}_{00}\mathrm{P} = \mathrm{P}_0\mathrm{M}_{01}\mathrm{P}_0 = 0$$

in eq. (3.24), yields

$$\mathrm{T}(k) = \{1 - \mathrm{P}_0 + \lambda_0 \ln k\mathrm{P}_0\mathrm{M}_{01} + \lambda_0\mathrm{P}_0\mathrm{M}_{11} + (2\pi\lambda_0/(v,u)\ln k)\mathrm{T}_0\mathrm{Q}\mathrm{M}_{00}\mathrm{P}\mathrm{M}_{00} - (2\pi/(v,u)\ln k)\mathrm{P}\mathrm{M}_{00} + \mathrm{O}_{12}((1/\ln k)^2\}^{-1}[k^{-2}\mathrm{P}_0 + \mathrm{T}_0\mathrm{Q} + \mathrm{O}_{12}(1/\ln k)]. \quad (3.37)$$

Next eq. (3.25) implies

$$[1 - \mathrm{P}_0 + \lambda_0 \ln k\mathrm{P}_0\mathrm{M}_{01} + \lambda_0\mathrm{P}_0\mathrm{M}_{11}]^{-1} = = 1 - \sum_{j,l=3}^{2+\mathrm{N}}(\tilde{\phi}, \lambda_0\mathrm{M}_{11}\phi)_{jl}^{-1}([\lambda_0 \ln k\mathrm{P}_0\mathrm{M}_{01} + \lambda_0\mathrm{P}_0\mathrm{M}_{11} - 1]^*\tilde{\phi}_l,\cdot)\phi_j \quad (3.38)$$

where the inverse matrix $(\tilde{\phi}, \lambda_0 M_{11}\phi)_{jl}^{-1}$ exists since $(\tilde{\phi}_j, M_{11}\phi_l)$ is a positive definite matrix. This can be seen as follows. From [*20*], p. 364 we learn that

$$\int_{\mathbb{R}^4} d^2x d^2y f_1(\underline{x}) \, | \underline{x} - \underline{y} |^2 \ln | \underline{x} - \underline{y} | \, f_2(\underline{y}) = \\ = (2/\pi) \int_{\mathbb{R}^2} d^2p \, | \underline{p} |^{-4} \overline{\hat{f}_1(\underline{p})} \hat{f}_2(\underline{p}) \quad (3.39)$$

if the f_m satisfy the following condition

$$\int_{\mathbb{R}^2} d^2x f_m(\underline{x}) = \hat{f}_m(0) = 0, \int_{\mathbb{R}^2} d^2x \underline{x} \cdot f_m(\underline{x}) = \nabla_{\underline{p}} \hat{f}_m(\underline{p}) = 0, m = 1, 2$$

$\left(\text{where } \hat{f}(\underline{p}) = \int_{\mathbb{R}^2} d^2x f(\underline{x}) e^{i\underline{p}\cdot\underline{x}}, f \in \mathcal{S}(\mathbb{R}^2), \mathcal{S} \text{ denoting the Schwartz space}\right)$.

Thus the l. h. s. of (3.39) is strictly positive under these conditions on f.

Inserting eq. (3.38) into eq. (3.37) (taking the inverse $[\ ]^{-1}$ outside of $\{ \dots \}^{-1}$ on the left, i. e. $[\ ]^{-1} \{ \dots \}^{-1}$ etc.) completes case III. ■

REMARK 3.4. — *a*) One can prove that k^{-2} is the leading singularity of $T(k)$ as $k \to 0$ in cases IV*a*)-*c*), i. e., $t_{-1-1} \neq 0$ and $T(k) \overset{k \to 0}{=} 0(k^{-2})$.

b) It is interesting to note that in case II*a*, the zero-energy resonance causes only a (weak) logarithmic singularity in $T(k)$ as $k \to 0$.

Finally we indicate how to derive recursion relations for the coefficients t_{mn} in analogy to the treatment in [*15*]. For simplicity we confine ourselves to the generic case I.

Defining

$$M^{00}(k) = M(k) - M_{00} \quad (3.40)$$

we have from the definition of $T(k)$

$$[1 + \lambda_0 Q M_{00} Q + \lambda_0 \alpha(k) P + \varepsilon] T(k) = 1 - \lambda_0 M_{00} P T(k) \\ - \lambda_0 P M_{00} Q T(k) - \lambda_0 M^{00}(k) T(k) + \varepsilon T(k). \quad (3.41)$$

Using

$$[1 + \lambda_0 Q M_{00} Q + \lambda_0 \alpha(k) P + \varepsilon]^{-1} = T_0 [1 - \lambda_0 \alpha(k) \beta(k) P] \quad (3.42)$$

we arrive at

$$T(k) = T_0 [Q + \beta(k) P] [1 - \lambda_0 P M_{00} Q T(k) - \lambda_0 M^{00}(k) T(k)] \\ - T_0 [Q + \beta(k) P] \lambda_0 M_{00} P T(k). \quad (3.43)$$

Taking into account $QP = PT_0Q = 0, T_0P = P$ and

$$(1 + \lambda_0 T Q M_{00} P)^{-1} = 1 - \lambda_0 T_0 Q M_{00} P \quad (3.44)$$

one obtains

$$\begin{aligned} T(k) = T_0Q + \beta(k)(1 - \lambda_0T_0QM_{00}P)P - \lambda_0T_0QM^{00}(k)T(k) \\ - \lambda_0\beta(k)(1 - \lambda_0T_0QM_{00}P)PM(k)T(k). \end{aligned} \tag{3.45}$$

Next we expand

$$\beta(k) = -(2\pi/\lambda_0(v, u)) \sum_{n=0}^{\infty} [(i\pi/2) + \ln 2 + \Psi(1) + (2\pi/\lambda_0(v, u)]^n (1/\ln k)^{n+1}, \tag{3.46}$$

$$M(k)T(k) = \sum_{m,n=0}^{\infty} (1/\ln k)^m (k^2 \ln k)^n \sum_{p=0}^{m} \sum_{q=0}^{n} M_{pq} t_{m-p,n-q}, \tag{3.47}$$

(we recall that $M_{pq} = \delta_{p,q-1}M_{q-1,q} + \delta_{p,q}M_{qq}$)

$$\begin{aligned} \beta(k)M(k)T(k) = -(2\pi/\lambda_0(v, u)) \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (1/\ln k)^m (k^2 \ln k)^n \times \\ \times \sum_{r=0}^{m-1} \sum_{p=0}^{r} \sum_{q=0}^{n} [(i\pi/2) + \ln 2 + \Psi(1) + (2\pi/\lambda_0(v, u))]^{m-1-r} M_{pq} t_{r-p,n-q}, \end{aligned} \tag{3.48}$$

$$\begin{aligned} M^{00}(k)T(k) = \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} (1/\ln k)^r (k^2 \ln k)^s M_{01} t_{r,s-1}, \\ + \sum_{m,n=1}^{\infty} (1/\ln k)^m (k^2 \ln k)^n \sum_{p=1}^{m} \sum_{q=1}^{n} M_{pq} t_{m-p,n-q}. \end{aligned} \tag{3.49}$$

Inserting eqs. (3.46)-(3.49) into eq. (3.45) finally yields the recursion relations in case I

$$\begin{aligned} t_{mn} = t_{00}\delta_{m0}\delta_{n0} - (2\pi/\lambda_0(v, u)) [(i\pi/2) + \ln 2 + \Psi(1) + (2\pi/\lambda_0(v, u))]^{m-1} \\ \times (P - \lambda_0T_0QM_{00}P)(1 - \delta_{m0})\delta_{n0} \\ - \lambda_0T_0QM_{01}t_{m,n-1} - \lambda_0T_0Q \sum_{p=1}^{m} \sum_{q=1}^{n} M_{pq}t_{m-p,n-q}(1 - \delta_{m0})(1 - \delta_{n0}) \\ + (2\pi/(v, u))(P - \lambda_0T_0QM_{00}P) \times \\ \times \sum_{r=0}^{m-1} [(i\pi/2) + \ln 2 + \Psi(1) + (2\pi/\lambda_0(v, u))]^{m-1-r} \times \\ \times \sum_{p=0}^{r} \sum_{q=0}^{n} M_{pq}t_{r-p,n-q}(1 - \delta_{m0}), \qquad m, n \in \mathbb{N}_0. \end{aligned} \tag{3.50}$$

Explicitly we have in

CASE I.

$$t_{00} = (1 + \lambda_0 \mathrm{QM}_{00}\mathrm{Q})^{-1}\mathrm{Q}, \tag{3.51}$$

$$t_{10} = - (2\pi/\lambda_0(v, u))(\mathrm{P} - \lambda_0 \mathrm{T}_0 \mathrm{QM}_{00}\mathrm{P}) + (2\pi/(v, u))(\mathrm{P} - \lambda_0 \mathrm{T}_0 \mathrm{QM}_{00}\mathrm{P})\mathrm{M}_{00}\mathrm{T}_{00}, \tag{3.52}$$

$$t_{01} = - \lambda_0 \mathrm{T}_0 \mathrm{QM}_{01} t_{00}, \tag{3.53}$$

$$t_{11} = - \lambda_0 \mathrm{T}_0 \mathrm{QM}_{01} t_{10} - \lambda_0 \mathrm{T}_0 \mathrm{QM}_{11} t_{00} + (2\pi/(v, u))(\mathrm{P} - \lambda_0 \mathrm{T}_0 \mathrm{QM}_{00}\mathrm{P}) \sum_{q=0}^{1} \mathrm{M}_{0q} t_{0,1-q}. \tag{3.54}$$

We also list the first coefficients in the other cases, which are useful in the sequel

CASE II*a*).

$$t_{-10} = (1/2\pi\lambda_0 \,|\, c_1^{(0)} |^2)(\tilde{\phi}_0, \cdot)\phi_0, \tag{3.55}$$

$$t_{m0} = 0, \; m \leq -2. \tag{3.56}$$

CASE II*b*).

$$t_{0-1} = \sum_{j,l=1}^{\mathrm{M}} (\tilde{\phi}, \lambda_0 \mathrm{M}_{01}\phi)_{jl}^{-1}(\tilde{\phi}_l, \cdot)\phi_j, \tag{3.57}$$

$$t_{m-1} = 0, \; m \geq 1 \tag{3.58}$$

CASE II*c*). $\underline{c}_2^{(0)} = 0$.

$$t_{-1-1} = 0, \tag{3.59}$$

$$t_{m-1} = 0, \; m \geq -1, \tag{3.60}$$

$$t_{0-1} = - (1/\pi\lambda_0) \sum_{j,l=1}^{\mathrm{M}-1} (\overline{\underline{c}_2^{(j)}} \cdot \underline{c}_2^{(l)})^{-1}(\tilde{\phi}_l, \cdot)\phi_j, \tag{3.61}$$

$$t_{-10} = (1/2\pi\lambda_0 \,|\, c_1^{(0)} |^2)(\tilde{\phi}_0, \cdot)\phi_0. \tag{3.62}$$

CASE II*d*). $\underline{c}_2^{(0)} \neq 0$.

$$t_{-1-1} = 0, \tag{3.63}$$

$$t_{m-1} = 0, \; m \leq -1, \tag{3.64}$$

$$t_{m0} = 0, \; m \leq -2, \tag{3.65}$$

$$t_{0-1} = - (1/\pi\lambda_0) \sum_{j,l=1}^{\mathrm{M}-1} (\overline{\underline{c}_2^{(j)}} \cdot \underline{c}_2^{(l)})^{-1}(\tilde{\phi}_l, \cdot)\phi_j. \tag{3.66}$$

CASE III.

$$t_{-1-1} = \sum_{j,l=3}^{2+N} (\tilde{\phi}, \lambda_0 M_{11}\phi)_{jl}^{-1}(\tilde{\phi}_l, \cdot)\phi_j, \tag{3.67}$$

$$t_{-10} = - \sum_{j,l=3}^{2+N} (\tilde{\phi}, M_{11}\phi)_{jl}^{-1}(Q^*T_0^*M_{01}^*\tilde{\phi}_l, \cdot)\phi_j. \tag{3.68}$$

4. EXPANSIONS OF ON-SHELL SCATTERING QUANTITIES AROUND THRESHOLD

Given the explicit threshold behavior of $T(k)$ near $k = 0$, we discuss in this Section the low-energy expansions of the on-shell scattering amplitude $f(k, \underline{\omega}, \underline{\omega}')$ and of the scattering operator $S(k)$ in cases I-III.

We first recall the corresponding definitions of $f(k, \underline{\omega}, \underline{\omega}')$ and $S(k)$ in terms of $T(k)$ [*21*]. For simplicity we assume hypothesis (H.2) throughout this section.

The on-shell scattering amplitude is given by

$$f(k, \underline{\omega}, \underline{\omega}') = -\lambda_0 e^{i\pi/4}(8\pi k)^{-1/2}(ve^{ik\underline{\omega}\cdot}, T(k)ue^{ik\underline{\omega}'\cdot})$$
$$k > 0,\ k^2 \notin \mathcal{E}_+,\ \underline{\omega}, \underline{\omega}' \in S^1 \tag{4.1}$$

where $(\cdot, \cdot)$ is the scalar product in $L^2(\mathbb{R}^2)$ and $\mathcal{E}_+$ denotes the exceptional set

$$\mathcal{E}_+ = \{ k^2 > 0 \mid \lambda_0 u R_0(k) v\phi = -\phi, 0 \neq \phi \in L^2(\mathbb{R}^2) \}. \tag{4.2}$$

Hypothesis (H.2) implies that $\mathcal{E}_+$ is a finite, discrete set (implying the absence of the singular continuous spectrum $\sigma_{sc}(H)$ of H). The on-shell scattering operator in $L^2(S^1)$ is then defined by

$$(S(k)\phi)(\underline{\omega}) = \phi(\underline{\omega}) + (ik/2\pi)^{1/2} \int_{S^1} d\omega' f(k, \underline{\omega}, \underline{\omega}')\phi(\underline{\omega}'),$$
$$k > 0,\ k^2 \notin \mathcal{E}_+,\ \phi \in L^2(S^1). \tag{4.3}$$

Next we study $f(k, \underline{\omega}, \underline{\omega}')$. Expanding $T(k)$ and $e^{ik\underline{\omega}\cdot\underline{x}}$ in eq. (4.1) yields the Laurent expansion

$$k^{1/2}f(k, \underline{\omega}, \underline{\omega}') = -\lambda_0 e^{i\pi/4}(8\pi)^{-1/2} \sum_{m=r}^{\infty} \sum_{n=s}^{\infty} (1/\ln k)^m (k \ln k)^n f_{mn}(\underline{\omega}, \underline{\omega}'),$$
$$k > 0 \text{ small enough},\ \underline{\omega}, \underline{\omega}' \in S^1 \tag{4.4}$$

(r, s depending on cases I-III). We investigate in more detail the different cases separately.

THEOREM 4.1. — Assume (H.2). Then $k^{1/2}f(k, \underline{\omega}, \underline{\omega}')$ is analytic in $(1/\ln k, k \ln k)$ near $(0, 0)$ with the following Taylor expansions:

CASE I.

$$\begin{aligned} k^{1/2}f(k, \underline{\omega}, \underline{\omega}') = (\pi/2)^{1/2}e^{i\pi/4} \{ & (1/\ln k) \\ & + [(i\pi/2) + \ln 2 + \Psi(1) + (2\pi/\lambda_0(v, u))](1/\ln k)^2 \\ & + (2\pi/(v, u)^2)[(v, \mathrm{M}_{00}u) - \lambda_0(v, \mathrm{M}_{00}\mathrm{T}_0\mathrm{Q}\mathrm{M}_{00})](1/\ln k)^2 \\ & + \mathrm{O}((1/\ln k)^3) \}, \quad k > 0 \text{ small enough}. \end{aligned} \tag{4.5}$$

CASE IIa).

$$\begin{aligned} k^{1/2}f(k, \underline{\omega}, \underline{\omega}') = (\pi/2)^{1/2}e^{i\pi/4} \{ & i[((\underline{c}_2^{(0)} \cdot \underline{\omega})/c_1^{(0)}) - (\overline{\underline{c}_2^{(0)}} \cdot \underline{\omega}')/\overline{c_1^{(0)}})]k \\ & - (\lambda_0^2/8\pi^2(v, u)^2 \mid v(\cdot), (1 + \lambda_0\mathrm{M}_{00})^{-1}u) \mid^2 k^2 \ln k \\ & - \mid c_1^{(0)} \mid^{-2}(\underline{c}_2^{(0)} \cdot \underline{\omega})/(\overline{\underline{c}_2^{(0)}} \cdot \underline{\omega}')k^2 \ln k + \mathrm{O}(k^2) \}, \\ & k > 0 \text{ small enough}. \end{aligned} \tag{4.6}$$

CASE IIb).

$$\begin{aligned} k^{1/2}f(k, \underline{\omega}, \underline{\omega}') = (\pi/2)^{1/2}e^{i\pi/4} \{ & (1/\ln k) \\ & - 2\pi \sum_{j,l=1}^{\mathrm{M}} (\tilde{\phi}, \mathrm{M}_{01}\phi)_{jl}^{-1}(\underline{c}_2^{(j)} \cdot \omega)(\overline{\underline{c}_2^{(l)}} \cdot \underline{\omega}')(1/\ln k) \\ & + (2\pi/(v, u)^2)[(v, \mathrm{M}_{00}u) - \lambda_0(v, \mathrm{M}_{00}\mathrm{T}_0\mathrm{Q}\mathrm{M}_{00}u)](1/\ln k)^2 \\ & + [(i\pi/2) + \ln 2 + \Psi(1) + (2\pi/\lambda_0(v, u))](1/\ln k)^2 \\ & + \mathrm{O}((1/\ln k)^3) \}, \ k > 0 \text{ small enough}. \end{aligned} \tag{4.7}$$

CASE IIc).

$$\begin{aligned} k^{1/2}f(k, \underline{\omega}, \underline{\omega}') = (2\pi)^{1/2}e^{i\pi/4} & \sum_{j,l=1}^{\mathrm{M}-1} (\underline{c}_2^{(j)} \cdot \underline{c}_2^{(l)})^{-1}(\underline{c}_2^{(j)} \cdot \underline{\omega})(\overline{\underline{c}_2^{(l)}} \cdot \underline{\omega}') \\ & + \mathrm{O}((1/\ln k)^2), \ k > 0 \text{ small enough}. \end{aligned} \tag{4.8}$$

CASE III.

$$\begin{aligned} k^{1/2}f(k, \underline{\omega}, \underline{\omega}') = (\pi/2)^{1/2}e^{i\pi/4} \{ & (1/\ln k) \\ & + (2\pi/(v, u)^2)[(v, \mathrm{M}_{00}u) - \lambda_0(v, \mathrm{M}_{00}\mathrm{T}_0\mathrm{Q}\mathrm{M}_{00}u)](1/\ln k)^2 \\ & + [(i\pi/2) + \ln 2 + \Psi(1) + (2\pi/\lambda_0(v, u))](1/\ln k)^2 \\ & + \mathrm{O}((1/\ln k)^3) \}, \ k > 0 \text{ small enough}. \end{aligned} \tag{4.9}$$

Thus

$$(r, s) = \begin{cases} (1, 0) & \text{in cases I, II}b\text{), }c\text{), III} \\ (1, 1) & \text{in case II}a\text{)} \end{cases} \tag{4.10}$$

Proof of case I. — From eq. (3.51) we see that

$$\mathrm{P}t_{00} = t_{00}\mathrm{P} = 0. \tag{4.11}$$

Using the recursion relations (3.50) for t_{m0} we get

$$\begin{aligned} k^{1/2}f(k, \underline{\omega}, \underline{\omega}') &= c \sum_{m=0}^{\infty} (1/\ln k)^m (v, t_{m0}u) + 0(k) \\ &= -(2\pi c/\lambda_0)(1/\ln k) \\ &- (2\pi c/\lambda_0(v,u))\{ [a(v,u) + (2\pi/\lambda_0)] - \lambda_0(v, \mathrm{M}_{00}t_{10}u)\}(1/\ln k)^2 \\ &+ \mathrm{O}((1/\ln k)^3) \end{aligned} \tag{4.12}$$

Here we used the abbreviations

$$a = (i\pi/2 + \ln 2 + \Psi(1)),\ c = -\lambda_0 e^{i\pi/4}(8\pi)^{-1/2}. \tag{4.13}$$

Since

$$t_{10}\mathrm{P} = -(2\pi/\lambda_0(v,u))(\mathrm{P} - \lambda_0\mathrm{T}_0\mathrm{Q}\mathrm{M}_{00}\mathrm{P}) \tag{4.14}$$

we obtain eq. (4.5). ■

Proof of case IIa). — Tedious but straightforward calculations show

$$\begin{aligned} &\mathrm{P}t_{-10} = t_{-10}\mathrm{P} = 0,\ \mathrm{P}t_{00}\mathrm{P} = \mathrm{P}t_{10}\mathrm{P} = 0, \\ &t_{00}\mathrm{P} = (1/\lambda_0(v,u)c_1^{(0)})(v,\cdot)\phi_0. \end{aligned} \tag{4.15}$$

Moreover, $\lambda_0\mathrm{Q}\mathrm{M}_{00}\mathrm{Q}\phi_0 = -\phi_0$ implies

$$(1 + \lambda_0\mathrm{M}_{00})\phi_0 = c_1^{(0)}u \tag{4.16}$$

and hence

$$(v, (1 + \lambda_0\mathrm{M}_{00})^{-1}u) = 0 \tag{4.17}$$

since $c_1^{(0)} \neq 0$. Thus one infers

$$\mathrm{PT}(k)\mathrm{P} = (\lambda_0/4\pi(v,u)^2)((1+\lambda_0\mathrm{M}_{00}^*)^{-1}v, (\underline{\cdot})u)(v(\underline{\cdot}), (1+\lambda_0\mathrm{M}_{00})^{-1}u)k^2\ln k + \mathrm{O}(k^2). \tag{4.18}$$

Thus

$$\mathrm{P}t_{m0} = t_{m0}\mathrm{P} = \mathrm{P}t_{m0}\mathrm{P} = 0,\ m \in \mathbb{N},\ \mathrm{P}t_{00}\mathrm{P} = 0. \tag{4.19}$$

Collecting all relevant terms in eq. (4.1) then yields eq. (4.6). ■

Proof of case IIb). — In addition to eqs. (3.57) and (3.58) one can show that

$$\begin{aligned} \mathrm{P}t_{00} &= t_{00}\mathrm{P} = 0,\ \mathrm{P}t_{10}\mathrm{P} = -(2\pi/\lambda_0(v,u))\mathrm{P}, \\ \mathrm{P}t_{20}\mathrm{P} &= (2\pi/\lambda_0(v,u))^2[\lambda_0\mathrm{PM}_{00}\mathrm{T}_0\mathrm{QM}_{00}\mathrm{P} - \mathrm{PM}_{00}\mathrm{P}] \\ &- (2\pi/\lambda_0(v,u)[a + (2\pi/\lambda_0(v,u))]\mathrm{P}. \end{aligned} \tag{4.20}$$

Insertion of these results into eq. (4.1) yields eq. (4.7). ■

Proof of case IIc. — In addition to eqs. (3.59)-(3.66), straightforward but rather lengthy calculations yield

$$Pt_{-10} = t_{-10}P = Pt_{00}P = 0\,,$$
$$PM_{00}t_{0-1}M_{01}P = 0,\ PM_{00}t_{-10}M_{00}P = ((v,u)/2\pi\lambda_0)P\,. \qquad (4.21)$$

Collecting all terms in eq. (4.1) then proves eq. (4.8). ■

Proof of case III. — In addition to eqs. (3.67) and (3.68) one can show that

$$Pt_{00}P = 0,\ Pt_{10}P = -(2\pi/\lambda_0(v,u))P\,,$$
$$Pt_{20}P = -\lambda_0^{-1}(2\pi/(v,u))^2[PM_{00}P - \lambda_0 PM_{00}T_0QM_{00}P]$$
$$-(2\pi/\lambda_0(v,u)[a + (2\pi/\lambda_0(v,u))]P \qquad (4.22)$$

which leads to eq. (4.9). ■

REMARK 4.2 *a*). — One can prove in cases IV*a*)-*c*) that

$$k^{1/2}f(k,\underline{\omega},\underline{\omega}') \overset{k\downarrow 0}{=} (1/\ln k)\,.$$

b) Taking into account eq. (4.16), one infers that all orders up to $O(k^2 \ln k)$ in eq. (4.6) are zero if $\underline{c}_2^{(0)} = 0$ and hence $f(k,\underline{\omega},\underline{\omega}') \overset{k\downarrow 0}{=} 0(k^2)$ in case II*a*) if $\underline{c}_2^{(0)} = 0$.

Given Theorem 4.1 it is now easy to read off the low-energy behavior of S(*k*) from eq. (4.3).

THEOREM 4.3. — Assume (H.2). Then S(*k*) is analytic in $(1/\ln k, k\ln k)$ near (0, 0) with the following (norm convergent) Taylor expansions:

CASE I.

$$S(k) = 1 + i\pi(Y_0,\cdot)Y_0(1/\ln k)$$
$$+ (2\pi^2 i/(v,u)^2)[(v, M_{00}u) - \lambda_0(v, M_{00}T_0QM_{00}u)](Y_0,\cdot)Y_0(1/\ln k)^2$$
$$+ i\pi[(i\pi/2) + \ln 2 + \Psi(1) + (2\pi/\lambda_0(v,u))](Y_0,\cdot)Y_0(1/\ln k)^2 + O((1/\ln k)^3)$$
$$k > 0 \text{ small enough} \qquad (4.23)$$

where

$$Y_0(\underline{\omega}) = (2\pi)^{-1/2}\,. \qquad (4.24)$$

CASE II*a*).

$$S(k) = 1 - 2^{-1/2}(c_1^{(0)})^{-1}(Y_0,\cdot)Y_{10}k + 2^{-1/2}(\overline{c_1^{(0)}})^{-1}(Y_{10},\cdot)Y_0k$$
$$-(i/2\pi\,|\,c_1^{(0)}|^2)(Y_{10},\cdot)Y_{10}k^2\ln k$$
$$-(i\lambda_0^2/8\pi(v,u)^2)\,|\,(v(\underline{\cdot}),(1+\lambda_0M_{00})^{-1}u)\,|^2(Y_0,\cdot)Y_0 + O(k^2)\,,$$
$$k > 0 \text{ small enough} \qquad (4.25)$$

where

$$Y_{10}(\underline{\omega}) = \pi^{1/2}\underline{c}_2^{(0)} \cdot \underline{\omega} = (1/2\pi^{1/2})(v\underline{\omega}\cdot, \phi_0). \tag{4.26}$$

CASE II*b*).

$$\begin{aligned} S(k) = {} & 1 + i\pi(Y_0, \cdot)Y_0(1/\ln k) - i \sum_{j,l=1}^{M} (\tilde{\phi}, M_{01}\phi)_{jl}^{-1}(Y_{1l}, \cdot)Y_{1j}(1/\ln k) \\ & + (2\pi^2 i/(v, u)^2)\,[(v, M_{00}u) - \lambda_0(v, M_{00}T_0QM_{00}u)](Y_0, \cdot)Y_0(1/\ln k)^2 \\ & + i\pi\,[(i\pi/2) + \ln 2 + \Psi(1) + (2\pi/\lambda_0(v, u))](Y_0, \cdot)Y_0(1/\ln k)^2 \\ & + O((1/\ln k)^2),\ k > 0 \text{ small enough}, \end{aligned} \tag{4.27}$$

where

$$Y_{1j}(\underline{\omega}) = \pi^{1/2}\underline{c}_2^{(j)} \cdot \underline{\omega} = (1/2\pi^{1/2})(v\underline{\omega}\cdot, \phi_j). \tag{4.28}$$

CASE III.

$$\begin{aligned} S(k) = {} & 1 + i\pi(Y_0, \cdot)Y_0(1/\ln k) \\ & + (2\pi^2 i/(v, u)^2)\,[(v, M_{00}u) - \lambda_0(v, M_{00}T_0QM_{00}u)](Y_0, \cdot)Y_0(1/\ln k)^2 \\ & + i\pi\,[(i\pi/2) + \ln 2 + \Psi(1) + (2\pi/\lambda_0(v, u))](Y_0, \cdot)Y_0(1/\ln k)^2 \\ & + O((1/\ln k)^2),\ k > 0 \text{ small enough}. \end{aligned} \tag{4.29}$$

REMARK 4.4. — *a*) Since $Y_{10}(\underline{\omega}) = 0$ if $\underline{c}_2^{(0)} = 0$, $S(k) \overset{k\downarrow 0}{=} 1 + O(k^2)$ in case II*a*) if $\underline{c}_2^{(0)} = 0$.

b) Expressions (4.5)-(4.9) for $f(k, \underline{\omega}, \underline{\omega}')$ and the corresponding ones for $S(k)$ above, considerably simplify if V is spherically symmetric (cf. [*22*] [*35*] for details). E. g. $Y_{10}(\underline{\omega}) = 0$, $Y_{1j}(\underline{\omega}) \neq 0$ only for the *p*-wave (i. e. angular momentum $l = 1$).

c) We emphasize that $S(k) \overset{k\downarrow 0}{\longrightarrow} 1$ in *all* cases involved (actually this can also be proven in cases IV*a*)-*c*)). This is in sharp contrast to one and three dimensions where $S(k)$ converges to -1 as $k \downarrow 0$ in certain cases [*6*] [*13*] [*15*] [*27*] [*33*].

5. LOW-ENERGY PARAMETERS

Following the three-dimensional treatment in [*5*] we generalize the concept of scattering length for nonspherically symmetric potentials to two dimensions. Furthermore, we relate it to the threshold behavior of the scattering amplitude.

We start with the following definition whose motivation (in the three-dimensional context) can be found in Sect. III of [*5*].

DEFINITION 5.1. — Assume (H.2) and (without loss of generality)

$(0, k_0^2) \cap \mathcal{E}_+ = \emptyset$ for some $k_0 > 0$. In cases I, II*b*) and III we define the scattering length a by

$$a^{-1} = -\Psi(1) - (i\pi/2) + \lim_{0<k<k_0} k \downarrow 0 \{ e^{-i\pi/4}(2k/\pi)^{1/2} (\ln (k/2))^2 \times \\ \times (2\pi)^{-2} \int_{S^1 \times S^1} d\omega d\omega' f(k, \underline{\omega}, \underline{\omega}') - \ln (k/2) \}. \quad (5.1)$$

In case II*a*), *c*), the scattering length does not exist (i. e., $a^{-1} = 0$).

REMARK 5.2. — *a*) If V is spherically symmetric then a reduces precisely to the ordinary two-dimensional scattering length discussed extensively in [*10*] [*11*] [*22*], (cf. also [*3*]).

b) In exactly the same way one can define the effective range parameter for nonspherically symmetric potentials (cf. [*5*] for the analog in three dimensions).

c) Hypothesis (H.2) in Definition 5.1 can be considerably relaxed. In fact, the exponential falloff of V at infinity, $e^{a|\underline{\cdot}|}V \in L^1(\mathbb{R}^2)$ in (H.2), can be replaced by $(1 + |\underline{\cdot}|)^n V \in L^1(\mathbb{R}^2)$ for suitable $n \in \mathbb{N}$ depending on the cases I-III involved.

d) Definition 5.1 also works in case IV*b*) whereas a does not exist in cases IV*a*, *c*) (i. e. $a^{-1} = 0$).

Given the threshold expansion of $f(k, \underline{\omega}, \underline{\omega}')$ in Section 4 it is now easy to calculate a:

THEOREM 5.3. — Assume (H.2). Then

$$a^{-1} = (2\pi/\lambda_0(v, u)) + (2\pi/(v, u)^2) [(v, M_{00}u) - \lambda_0(v, M_{00}T_0QM_{00}u)] \quad (5.2)$$

in cases I, II*b*) and III.

Proof. — We only need to insert eqs. (4.5), (4.7) and (4.9) into eq. (5.1). ■

REMARK 5.4. — *a*) Eq. (5.2) implies that the scattering length a in cases I, II*b*) and III never vanishes.

b) Eq. (5.2) (and hence Remark 5.4*a*)) stays valid also in case IV*b*).

6. TRACE RELATIONS, LEVINSON'S THEOREM

Finally, we investigate the threshold behavior of the trace of the difference between the full and free resolvent. We then use this behavior to derive Levinson's theorem in cases I-III.

In addition to hypothesis (H.2) we need hypothesis

$$\text{(H.3)} \qquad V \in L^{4/3}(\mathbb{R}^2). \qquad (6.1)$$

Then one knows that [9] [17]

$$\| uR_0(k) \|_2 \leq c\,|\,k\,|^{-1},\ \mathrm{Im}\,k > 0, \tag{6.2}$$

$$\| uR_0(k)v \|_2 \leq c\,|\,k\,|^{-1/2}\, \| V \|_{4/3},\ \mathrm{Im}\,k \geq 0,\ k \neq 0, \tag{6.3}$$

$$\| uR_0'(k)v \|_2 \leq c\,|\,k\,|^{-1}\,[\| V \|_1 + |\,k\,|^{1/2}\, \| (1 + |\,\underline{\cdot}\,|)V \|_1], \quad \mathrm{Im}\,k \geq 0,\ k \neq 0. \tag{6.4}$$

For $\mathrm{Im}\,k > 0$ we also recall that

$$\begin{aligned} R(k) - R_0(k) &= -\,\lambda_0 R_0(k) v T(k) u R_0(k) \\ &= -\,\lambda_0 R_0(k) v u R_0(k) + \lambda_0^2 R_0(k) v T(k) u R_0(k) v u R_0(k) \end{aligned} \tag{6.5}$$

where $R(k)$ denotes the full resolvent

$$R(k) = (H - k^2)^{-1},\ \mathrm{Im}\,k > 0. \tag{6.6}$$

This implies

$$\mathrm{Tr}\,[R(k) - R_0(k)] = (\lambda_0 (v, u)/4\pi k^2) + (\lambda_0^2/2k)\,\mathrm{Tr}\,[uR_0'(k) v T(k) u R_0(k) v],\ \mathrm{Im}\,k > 0. \tag{6.7}$$

where we have used

$$\mathrm{Tr}\,[R_0(k) V R_0(k)] = -\,(1/4\pi k^2) \int_{\mathbb{R}^2} d^2x V(\underline{x}),$$

$$R_0(k)^2 = R_0'(k)/2k,\ \mathrm{Im}\,k > 0. \tag{6.8}$$

Inserting the Laurent (resp. Taylor) expansions for $uR_0(k)v$, $(1/2k)uR_0'(k)v$ (in Hilbert-Schmidt norm) and $T(k)$ into eq. (6.7) one obtains

THEOREM 6.1. — Assume (H.2) and (H.3). Then $\mathrm{Tr}\,[R(k) - R_0(k)]$ has the following Laurent expansion in $(1/\ln k, k^2 \ln k)$ around $(0, 0)$

$$\mathrm{Tr}\,[R(k) - R_0(k)] = \sum_{m=\mu}^{\infty} \sum_{n=\nu}^{\infty} (1/\ln k)^m (k^2 \ln k)^n \Delta_{mn}, \quad 0 < |\,k\,| \text{ small enough} \tag{6.9}$$

where

$$(\mu, \nu) = \begin{cases} (-1, -1) & \text{in cases I, II}b) \\ (-2, -1) & \text{in case II, and in case II}c) \text{ if } \underline{c}_2^{(0)} = 0 \\ (-\infty, -1) & \text{in case II}a) \text{ and in case II}c) \text{ if } \underline{c}_2^{(0)} \neq 0 \end{cases} \tag{6.10}$$

and

$$\Delta_{m,-1} = 0,\ m \leq -2 \text{ in case I-III}. \tag{6.11}$$

Explicitly we get

$$\Delta_{-1-1} = \begin{cases} 0 & \text{in cases I, II}a) \\ -\,M & \text{in case II}b) \\ -\,(M - 1) & \text{in case II}c) \\ -\,N & \text{in case III} \end{cases} \tag{6.12}$$

$$\Delta_{0-1} = (\lambda_0/4\pi)(v, u)[(i\pi/2) + \ln 2 + \Psi(1)] - (\lambda_0^2/4\pi)(v, t_{10}M_{00}u). \quad (6.13)$$

(cf. eq. (3.52)) in case I.

Proof of case I. — After multiplying the Laurent (resp. Taylor) expansions of $uR'_0(k)v$, $T(k)$, $uR_0(k)v$, a tedious calculation isolating the most dominating terms in eq. (6.7) leads to

$$\begin{aligned}\mathrm{Tr}\,[R(k) - R_0(k)] &= (\lambda_0/4\pi)(v, u)k^{-2} \\ &\quad - (\lambda_0^2/4\pi)(v, u)\,\mathrm{Tr}\,(Pt_{10}P)\alpha(k)(k^2 \ln k)^{-1} \\ &\quad - (\lambda_0^2/4\pi)(v, u)\,\mathrm{Tr}\,(Pt_{10}M_{00})(k^2 \ln k)^{-1} + O(k^{-2}(\ln k)^{-2}). \quad (6.14)\end{aligned}$$

By eq. (3.52) one gets

$$\mathrm{Tr}\,(Pt_{10}P) = -(2\pi/\lambda_0(v, u)) \quad (6.15)$$

completing the proof. ■

Proof of case IIa). — Isolating again the most singular terms in eq. (6.7) yields after lengthy calculations

$$\begin{aligned}\mathrm{Tr}\,[R(k)-R_0(k)] &= (\lambda_0(v, u)/4\pi)k^{-2} \\ &\quad -(\lambda_0^2/4\pi)\,\mathrm{Tr}\,(Pt_{00}P)\alpha(k)k^{-2} - (\lambda_0^2/4\pi)(v, u)\,\mathrm{Tr}\,(Pt_{10}P)\alpha(k)(k^2 \ln k)^{-1} \\ &\quad -(\lambda_0^2/4\pi)(v, u)\,\mathrm{Tr}\,(Pt_{00}M_{00})k^{-2} + O((k^2 \ln k)^{-1}). \quad (6.16)\end{aligned}$$

The fact that

$$Pt_{00}P = Pt_{10}P = 0, \quad (6.17)$$

$$Pt_{00}M_{00} = (1/\lambda_0(v, u)\overline{c_1^{(0)}})(M_{00}^*\tilde{\phi}_0, \cdot)u \quad (6.18)$$

then completes the proof. ■

Proof of case IIb). — The leading singularities in eq. (6.7) now read

$$\begin{aligned}\mathrm{Tr}\,[R(k)-R_0(k)] &= (\lambda_0(v, u)/4\pi)k^{-2} - (\lambda_0^2/4\pi)(v, u)\,\mathrm{Tr}\,(Pt_{00}M_{00})k^{-2} \\ &\quad -(\lambda_0^2/4\pi)(v, u)\,\mathrm{Tr}\,(Pt_{00}P)\alpha(k)k^{-2} - (\lambda_0^2/4\pi)(v, u)\,\mathrm{Tr}\,(Pt_{10}P)\alpha(k)(k^2 \ln k)^{-1} \\ &\quad +\lambda_0^2\,\mathrm{Tr}\,(M_{01}t_{0-1}M_{00})\,[\ln k + (1/2)](k^2 \ln k)^{-1} + O((k^2 \ln k)^{-1}). \quad (6.19)\end{aligned}$$

Rather involved calculations finally lead to

$$Pt_{00} = 0, \quad (6.20)$$

$$Pt_{10}P = -(2\pi/\lambda_0(v, u))P, \quad (6.21)$$

$$\lambda_0^2\,\mathrm{Tr}\,(M_{01}t_{0-1}M_{00}) = -M \text{ (cf. eq. (3.57))}. \quad (6.22)$$

■

Proof of case IIc). — Now the dominant terms in eq. (6.7) are given by

$$\begin{aligned}\mathrm{Tr}\,[R(k)-R_0(k)] &= (\lambda_0(v, u)/4\pi)k^{-2} - (\lambda_0^2/4\pi)(v, u)\,\mathrm{Tr}\,(Pt_{00}P)\alpha(k)k^{-2} \\ &\quad -(\lambda_0^2/4\pi)(v, u)\,\mathrm{Tr}\,(Pt_{10}P)\alpha(k)(k^2 \ln k)^{-1} - (\lambda_0^2/4\pi)(v, u)\,\mathrm{Tr}\,(Pt_{00}M_{00})k^{-2} \\ &\quad +\lambda_0^2\,\mathrm{Tr}\,(M_{01}t_{0-1}M_{00})\,[\ln k + (1/2)](k^2 \ln k)^{-1} + O((k^2 \ln k)^{-1}). \quad (6.23)\end{aligned}$$

Again tedious calculations yield

$$\lambda_0^2 \operatorname{Tr}(M_{01}t_{0-1}M_{00}) = -(M-1) \text{ (cf. eqs. (3.61), (3.66))}, \quad (6.24)$$

$$Pt_{00}P = 0, \quad (6.25)$$

$$\operatorname{Tr}(Pt_{00}M_{00}) = 1/\lambda_0, \quad (6.26)$$

$$\operatorname{Tr}(Pt_{10}P) = 0. \quad (6.27)$$

■

Proof of case III. — Here the most singular terms in eq. (6.7) turn out to be

$$\begin{aligned}\operatorname{Tr}[R(k)-R_0(k)] = {} & (\lambda_0(v,u)/4\pi)k^{-2} - (\lambda_0^2/4\pi)(v,u)\operatorname{Tr}(Pt_{00}P)\alpha(k)k^{-2} \\ & -(\lambda_0^2/4\pi)(v,u)\operatorname{Tr}(Pt_{10}P)\alpha(k)(k^2\ln k)^{-1} - (\lambda_0^2/4\pi)(v,u)\operatorname{Tr}(Pt_{00}M_{00})k^{-2} \\ & +\lambda_0^2\operatorname{Tr}(M_{01}t_{-1-1}M_{00})[\ln k + (1/2)]k^{-2} \\ & +\lambda_0^2\operatorname{Tr}(M_{11}t_{-1-1}M_{00})[\ln k + (1/2)](k^2\ln k)^{-1} + O((k^2\ln k)^{-1}).\end{aligned} \quad (6.28)$$

Once more quite involved calculations finally give

$$Pt_{00}P = 0, \quad (6.29)$$

$$Pt_{00}M_{00} = (2\pi/(v,u))PM_{01}t_{-1-1}M_{00} \text{ (cf. eq. (3.67))}, \quad (6.30)$$

$$Pt_{10}P = -(2\pi/\lambda_0(v,u))P, \quad (6.31)$$

$$\operatorname{Tr}(M_{01}t_{-1-1}M_{00}) = 0 \text{ (cf. eq. (3.67))}. \quad (6.32)$$

$$\operatorname{Tr}(M_{11}t_{-1-1}M_{00}) = -M/\lambda_0^2 \text{ (cf. eq. (3.67))}. \quad (6.33)$$

REMARK 6.2. — One can also show that

$$\Delta_{-1-1} = \begin{cases} -N & \text{in case IV}a) \\ -(N+M) & \text{in case IV}b) \\ -(N+M-1) & \text{in case IV}c), \end{cases} \quad (6.34)$$

From (6.12) and (6.34) we see that zero-energy resonances of type *a*) never contribute to Δ_{-1-1} in cases II*a*), *c*) IV*a*), *c*) whereas zero-energy resonances of type *b*) contribute to Δ_{-1-1} in cases II*b*), IV*b*), in the same way as zero-energy bound states do. I. e., in contrast to one and three dimensions, there are no factors of 1/2 in Δ_{-1-1} in two dimensions. This fact remains valid in $d \geq 4$ dimensions, since for $d = 4$ the zero-energy resonance contributes like a zero-energy bound state and for $d \geq 5$ there exist no zero-energy resonances of H at all [*28*] [*29*] [*30*] [*31*].

Finally we apply the contour integration method to derive Levinson's theorem in cases I-III.

We define

$$F(k^2) = \operatorname{Tr}[R(k) - R_0(k)] - \Delta_{-1-1}k^{-2} - (\lambda_0/4\pi)(v,u)k^{-2}, \quad k^2 \in \mathbb{C}\setminus[\mathcal{E}\cup\{0\}] \quad (6.35)$$

where

$$\mathcal{E} = \{ k^2 \in \mathbb{R}\backslash\{0\} \mid \lambda_0 u R(k) v\phi = -\phi, 0 \neq \phi \in L^2(\mathbb{R}^2) \} \qquad (6.36)$$

now also includes negative eigenvalues of H. Next we introduce the contour $\Gamma_{\varepsilon,R}$ where

$$\begin{aligned} \Gamma_{\varepsilon,R} &= C^-_{\varepsilon,R} \cup \gamma_\varepsilon \cup C^+_{\varepsilon,R} \cup \bigcup_{j \in I_-} C_{j,\varepsilon} \cup C_{R,\varepsilon}, \\ \gamma_\varepsilon &= \{ \varepsilon e^{i\theta} \mid -\pi/2 \leq \theta \leq \pi/2 \}, \\ C^\pm_{\varepsilon,R} &= \{ k^2 \pm i\varepsilon \mid k^2 \in [0, R] \text{ respectively } k^2 \in [R, 0] \}, \\ C_{j,\varepsilon} &= \{ k^2 + \varepsilon e^{i\theta} \mid -2\pi \leq \theta \leq 0 \}, j \in I_-, \\ C_{R,\varepsilon} &= \{ Re^{i\theta} \mid \arcsin(\varepsilon/R) \leq \theta \leq 2\pi - \arcsin(\varepsilon/R) \}, \\ & \qquad 0 < \varepsilon < R \end{aligned} \qquad (6.37)$$

where I_- indexes the negative eigenvalues $\kappa_j^2 < 0$ of H (i. e. the elements of $\mathcal{E}_- = \mathcal{E} \cap (-\infty, 0)$) and $R > 0$ is chosen large enough such that the open sphere $\{ z \in \mathbb{C} \mid |z| < R \}$ encloses all circles of radius ε around all points in $\mathcal{E}_-$. Then

$$\oint_{\Gamma_{\varepsilon,R}} dk^2 F(k^2) = 0 \qquad (6.38)$$

by Cauchy's theorem. Analyzing the different contributions to the integral over $\Gamma_{\varepsilon,R}$ yields

THEOREM 6.3 (Levinson's theorem). — Assume (H.2), (H.3) and $\mathcal{E}_- = \emptyset$. Then

$$\int_0^\infty dk^2 \operatorname{Im}\{ \operatorname{Tr}[R(k + io) - R_0(k + io)] \} = - \pi N_- + \pi \Delta_{-1-1} - (\lambda_0/4) \int_{\mathbb{R}^2} d^2x V(\underline{x}) \qquad (6.39)$$

where N_- denotes the number of (strictly) negative bound states of H counting multiplicity. Here the integral on the l. h. s. of eq. (6.39) is interpreted as an inproper Riemann integral $\left(\text{i. e.} \lim_{R \to \infty} \int_0^R \dots \right)$.

Proof. — Clearly the part of the integral (6.38) along $\cup_{j \in I_-} C_{j,\varepsilon}$ contributes a factor $2\pi i N_-$. Taking into account expansion (6.9), the integral (6.38) over part of the contour, namely $C^-_{\varepsilon,R} \cup \gamma_\varepsilon \cup C^+_{\varepsilon,R}$, leads to

$$\int_{C^-_{\varepsilon,R} \cup \gamma_\varepsilon \cup C^+_{\varepsilon,R}} dk^2 F(k^2) \stackrel{\varepsilon \downarrow 0}{=} 2i \int_0^R dk^2 \operatorname{Im}\{ \operatorname{Tr}[R(k + io) - R_0(k + io)] \} + i(\lambda_0/2)(v, u) + o(1). \qquad (6.40)$$

Here we used complex conjugation as

$$\mathrm{Tr}\,[\mathrm{R}(k+io)-\mathrm{R}_0(k+io)]=\mathrm{Tr}\,[\mathrm{R}(-k+io)-\mathrm{R}_0(-k+io)],\ \mathrm{Re}\,k>0. \quad (6.41)$$

Furthermore, we note that by expansion (6.9)

$$|\mathrm{Tr}\,[(\mathrm{H}-k^2-i\varepsilon)^{-1}-(\mathrm{H}_0-k^2-i\varepsilon)^{-1}-(\mathrm{H}-k^2+i\varepsilon)^{-1}+(\mathrm{H}_0-k^2+i\varepsilon)^{-1}]|$$
$$\leq \mathrm{C}\varepsilon(k^4+\varepsilon^2)^{-1} \text{ for } \varepsilon, k>0 \text{ sufficiently small}. \quad (6.42)$$

Thus the Riemann integral on the r. h. s. of eq. (6.40) is well defined near zero. Finally we discuss the high-energy contour $\mathrm{C}_{\mathrm{R},\varepsilon}$. Clearly

$$-\int_{\mathrm{C}_{\mathrm{R},\varepsilon}} dk^2 \Delta_{-1-1} k^{-2} \overset{\varepsilon\downarrow 0, \mathrm{R}\to\infty}{=} -2\pi i \Delta_{-1-1} + o(1). \quad (6.43)$$

The remaining integral, viz.

$$\int_{\mathrm{C}_{\mathrm{R},\varepsilon}} dk^2\, \mathrm{Tr}\,[\mathrm{R}(k)-\mathrm{R}_0(k)-(\lambda_0/4\pi)(v,u)k^{-2}] \overset{\varepsilon\downarrow 0, \mathrm{R}\to\infty}{=} o(1) \quad (6.44)$$

has been studied explicitly in Lemma 3.3 of [*9*] (cf. also [*36*]). We remark that for eq. (6.44) to be true, V must only satisfy $(1+|\cdot|)\mathrm{V}\in \mathrm{L}^1(\mathbb{R}^2)$, $\mathrm{V}\in \mathrm{L}^{4/3}(\mathbb{R}^2)$. Since by the estimates (6.3) and (6.4), $\mathrm{Tr}\,[\mathrm{R}(k+io)-\mathrm{R}_0(k+io)]$ is continuous in $k>0$ [*9*], adding up all contributions yields eq. (6.39).

REMARK 6.4. — *a*) From (6.29) and the value of Δ_{-1-1} (recall (6.12) and (6.34)) we see that a zero-energy resonance of the *s*-wave type (case II*a*) does not contribute to Levinson's theorem, while zero-energy resonances of the *p*-wave type (case II*b*) contribute exactly like (zero-energy) bound states. This result has been announced already in [*12*]. It is in sharp contrast with what one finds in one ([*15*]) and three (e. g. [*8*]) dimensions.

b) In the special case of a spherically symmetric interaction V, Levinson's theorem in two-dimensions has been discussed in [*22*] [*23*] [*24*] [*35*].

c) In the general, nonspherically symmetric case, Levinson's theorem in two-dimensions has been derived by Cheney [*17*] in case I (cf. also [*37*]).

d) the l. h. s. of Levinson's theorem (6.39) can be related directly to the on-shell scattering matrix S(*k*) in $\mathrm{L}^2(\mathrm{S}^1)$ via [*17*]

$$2\,\mathrm{Im}\,\{\mathrm{Tr}\,[\mathrm{R}(k+io)-\mathrm{R}_0(k+io)]\} = -i\frac{d}{dk^2}\mathrm{Tr}\,[\ln \mathrm{S}(k)] \quad (6.45)$$

(actually ref. [*17*] requires $\mathrm{V}\in\mathrm{L}^2(\mathbb{R}^2)$ in addition, but presumably this extra condition can be dropped).

e) Higher-order Levinson's theorems could be derived along exactly the same lines after replacing $\mathrm{F}(k^2)$ in (6.35) by

$$k^{-2\mathrm{N}}\left\{\mathrm{Tr}\,[\mathrm{R}(k)-\mathrm{R}_0(k)] - \sum_{m=\mu}^{\mathrm{N}-1}\sum_{n=\nu}^{\mathrm{N}-1}\Delta_{mn}(1/\ln k)^m(k^2\ln k)^n\right\}, \mathrm{N}\in\mathbb{N}_0 \quad (6.46)$$

(see e. g. [*8*] in the three-dimensional context).

f) Clearly our assumptions on the exponential falloff of V at infinity in Theorem 6.3 are not necessary. As long as one guarantees the absence of positive embedded eigenvalues of H (i. e. $\mathcal{E}_- = \emptyset$) [*19*] and the absence of the singular continuous spectrum of H, $\sigma_{sc}(H) = \emptyset$, [*4*] [*40*] [*41*], asymptotic expansions in Theorem 6.1, instead of Laurent expansions, lead to the conditions $V \in L^{4/3}(\mathbb{R}^2)$ and to $(1 + |\cdot|)V \in L^1(\mathbb{R}^2)$ in case I respectively $(1 + |\cdot|^{2+\delta})V \in L^1(\mathbb{R}^2)$ in case II*a*). (This follows directly from eqs. (3.21), (6.3) and (6.4) respectively from Section 2 and eqs (3.24)-(3.27), (6.3) and (6.4)). A rough check in all remaining cases II*b*)-III shows that $V \in L^{4/3}(\mathbb{R}^2)$ and $(1 + |\cdot|^{8+\delta})V \in L^1(\mathbb{R}^2)$ suffices (however, most likely one can improve on this).

g) One can show that Eq. (6.39) remains valid in cases IV*a*)-*c*).

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