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J. YNGVASON

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Algebraic quantum field theory and noncommutative moment problems II

by

J. YNGVASON
Science Institute, University of Iceland

ABSTRACT. — Methods are presented for constructing strongly positive, linear functionals on partially symmetric tensor algebras. The results are applied to the quotient algebra $\underline{\mathcal{S}}/\mathcal{I}_c$, where $\underline{\mathcal{S}}$ is Borchers test function algebra and $\mathcal{I}_c$ the locality ideal. In particular it is shown that this algebra has a separating family of finite dimensional representations and that the kernels of the translationally invariant, strongly positive functionals on the algebra have a minimal intersection.

1. INTRODUCTION

The subject of this paper is a construction of strongly positive functionals on Borchers' tensor algebra $\underline{\mathcal{S}}$ modulo the locality ideal $\mathcal{I}_c$. The interest in such functionals is due to the fact that they correspond to states on a quasi-local C*-algebra that is associated with $\underline{\mathcal{S}}/\mathcal{I}_c$. The paper is a sequel to a joint paper [*1*] of the author with J. Alcántara, and I refer to [*1*] for a discussion of the general setting for these investigations.

In [*1*] bounded representations of the algebra $\underline{\mathcal{S}}/\mathcal{I}_c$ were constructed in the following way: First the problem was reduced to a study of partially symmetric tensor algebras over finitely dimensional spaces using a method introduced in [*2*]. In a second step bounded representations of such algebras were obtained by embedding them into group algebras. In section 3 of the present paper an alternative for this second step is proposed. Here the partially symmetric tensor algebras are embedded into a tensor product of full tensor algebras. In this way one can show that $\underline{\mathscr{S}}/\mathscr{I}_c$ admits a separating family of finite dimensional representations, and one obtains also a large set of explicitly given strongly positive functionals. I expect that these functionals will eventually lead to a simple characterization of the strongly positive functionals on $\underline{\mathscr{S}}/\mathscr{I}_c$: My conjecture is that a functional on $\underline{\mathscr{S}}/\mathscr{I}_c$ is strongly positive if and only if it gives rise to a strongly positive representation (see (2.9)) of all abelian subalgebras of $\underline{\mathscr{S}}/\mathscr{I}_c$.

While this conjecture has not yet been proven, several results on the size of the cone of strongly positive functionals have been obtained. In theorem 4.7 is proven that every linear functional on $\underline{\mathscr{S}}/\mathscr{I}_c$ with the property that the singularities of the n-point distribution are bounded independently of n can be written as a linear combination of strongly positive functionals. Also it is possible to generalize a construction of translationally invariant, positive functionals developed in [*3*] so that includes strongly positive functionals as well.

In the course of the proofs of these statements a topology $\hat{\tau}$ on $\underline{\mathscr{S}}/\mathscr{I}_c$ is introduced that seems to be naturally adopted to the strongly positive functionals. An analogous topology on the totally symmetric tensor algebra was studied in [*4*]. In order to prove that continuity properties of the action of a group on $\underline{\mathscr{S}}/\mathscr{I}_c$ carry over to representations of the C*-algebras associated with $\underline{\mathscr{S}}/\mathscr{I}_c$ it is desirable to have a better understanding of this topology. These matters will be dealt with in a separate paper.

2. SOME ELEMENTARY OPERATIONS WITH STRONGLY POSITIVE FUNCTIONALS

(2.1) Let $\mathfrak{A}$ be a *-algebra and denote its positive cone by $\mathfrak{A}^+ = \{ \Sigma a_i^* a_i \mid a_i \in \mathfrak{A} \}$. A linear functional on $\mathfrak{A}$ is called positive if it is positive on $\mathfrak{A}^+$. Let Γ be a family of C*-seminorms on $\mathfrak{A}$ and denote by $\mathfrak{A}_\Gamma^+$ the closure of $\mathfrak{A}^+$ in the topology $\mathscr{T}_\Gamma$ defined by the seminorms in Γ. A linear functional on $\mathfrak{A}$ is called Γ-*strongly positive* if it is positive on $\mathfrak{A}_\Gamma^+$. We denote the set of all Γ-strongly positive functionals by $\mathfrak{A}_\Gamma^{+\prime}$. The following proposition follows directly from the bipolar theorem.

(2.2) PROPOSITION. — *$\mathfrak{A}_\Gamma^{+\prime}$ is a weakly closed, convex cone in the dual space of $\mathfrak{A}$. The $\mathscr{T}_\Gamma$-continuous, positive linear functionals on $\mathfrak{A}$ form a dense subcone of $\mathfrak{A}_\Gamma^{+\prime}$.*

Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$ and denote by $\Gamma \mid \mathfrak{Q}$ the family of C*-seminorms on $\mathfrak{B}$ obtained from Γ by restriction. If ω is a Γ-strongly positive functional on $\mathfrak{A}$, then the restriction $\omega \mid \mathfrak{B}$ is $\Gamma_{\mathfrak{B}}$-strongly positive on $\mathfrak{B}$. Let $\mathfrak{A}_{\Gamma}^{+\prime} \mid \mathfrak{B}$ denote the set of all such restrictions of Γ-strongly positive functionals to $\mathfrak{B}$.

(2.3) PROPOSITION. — *$\mathfrak{A}_{\Gamma}^{+\prime} \mid \mathfrak{B}$ is a dense subcone of $\mathfrak{B}_{\Gamma|\mathfrak{B}}^{+\prime}$.*

Proof. — It is clear that $\mathfrak{A}_{\Gamma}^{+\prime} \mid \mathfrak{B} \subset \mathfrak{B}_{\Gamma}^{+\prime}$. On the other hand, if $\omega \in \mathfrak{B}_{\Gamma|\mathfrak{B}}^{+\prime}$, then ω is by prop. 2.2 the weak limit of positive functionals that are $\Gamma \mid \mathfrak{B}$-continuous. The assertion now follows from the fact that a positive functional on a subalgebra of a C*-algebra has a continuous, positive extension to the whole algebra [*5*].

(2.4) REMARK. — It is generally not true that every $\Gamma \mid \mathfrak{B}$-strongly positive functional on a subalgebra $\mathfrak{B}$ can be extended to a Γ-strongly functional on the whole algebra, cf. theorem 9 in [*6*]. (The algebras in this example are tensor algebras where the usual notion of positivity coincides with that of strong positivity, cf. [*7*], theorem 2).

(2.5) Let $\mathfrak{A}^{(1)}$ and $\mathfrak{A}^{(2)}$ be *-algebras and $\Gamma^{(i)}$ a family of C*-seminorms on $\mathfrak{A}^{(i)}$, $i = 1, 2$. Let $\varphi : \mathfrak{A}^{(1)} \to \mathfrak{A}^{(2)}$ be a linear map that is positive, i. e. $\varphi(\mathfrak{A}^{(1)+}) \subset \mathfrak{A}^{(2)+}$, and continuous w. r. t. the topologies $\mathcal{T}_{\Gamma^{(1)}}$ and $\mathcal{T}_{\Gamma^{(2)}}$. Then φ maps $\mathfrak{A}_{\Gamma}^{(1)+}$ into $\mathfrak{A}_{\Gamma}^{(2)+}$ and its adjoint φ^*, defined by $\varphi^*\omega = \omega \circ \varphi$, maps the strongly positive functional on $\mathfrak{A}^{(2)}$ into those on $\mathfrak{A}^{(1)}$.

(2.6) In particular φ could be a *-homomorphism $\mathfrak{A}^{(1)} \to \mathfrak{A}^{(2)}$. If p_2 is a C*-seminorm on $\mathfrak{A}^{(2)}$, then $p_1 = \varphi^* p = p \circ \varphi$ is a C*-seminorm on $\mathfrak{A}^{(1)}$. If $\varphi^*\Gamma_2 \subset \Gamma_1$, then φ is continuous w. r. t. the topologies $\mathcal{T}_{\Gamma^{(1)}}$ and $\mathcal{T}_{\Gamma^{(2)}}$.

(2.7) Another example is provided by the mappings $\varphi_b : \mathfrak{A} \to \mathfrak{A}$, defined by

$$\varphi_b(a) = b^*ab$$

with $b \in \mathfrak{A}$ fixed. These mappings are positive and continuous w. r. t. any family of C*-seminorms on $\mathfrak{A}$.

(2.8) Next we consider tensor products. Let $\mathfrak{A}^{(1)}$, $\mathfrak{A}^{(2)}$, $\Gamma^{(1)}$, $\Gamma^{(2)}$ be as above. The algebraic tensor product $\mathfrak{A}^{(1)} \otimes \mathfrak{A}^{(2)}$ is in a natural way a *-algebra, and $\mathfrak{A}^{(1)}$ and $\mathfrak{A}^{(2)}$ are embedded in $\mathfrak{A}^{(1)} \otimes \mathfrak{A}^{(2)}$ as commuting subalgebras, if they have unit elements. On the other hand, there are in general many families of C*-seminorms on $\mathfrak{A}^{(1)} \otimes \mathfrak{A}^{(2)}$ that induce the topology $\mathcal{T}_{\Gamma^{(i)}}$ on $\mathfrak{A}^{(i)}$, $i = 1, 2$. Suppose $p_i \in \Gamma^{(i)}$, $i = 1, 2$ and consider a C*-seminorm p on $\mathfrak{A}^{(1)} \otimes \mathfrak{A}^{(2)}$ with $p(a_1 \otimes a_2) = p_1(a_1)p(a_2)$ (cf. [*8*], ch. IV). We have then

$$p_1 \otimes_{\min} p_1 \leq p \leq p_1 \otimes_{\max} p_2$$

where the C*-seminorms $p_1 \otimes_{\max} p_2$ and $p_1 \otimes_{\min} p_2$ are defined in the following way:

$$p_1 \otimes_{\max} p_2(a) = \sup_{\pi} \| \pi(a) \|$$

where the sup is taken over all representations π of $\mathfrak{A}^{(1)} \otimes \mathfrak{A}^{(2)}$, such that $\pi(a_i) \leq p_i(a_i)$ for $a_i \in \mathfrak{A}^{(i)}$. On the other hand $p_1 \otimes_{\min} p_2$ is defined by a tensor product representation of $\mathfrak{A}^{(1)} \otimes \mathfrak{A}^{(2)}$. Let π_i be a representation of $\mathfrak{A}^{(i)}$ on a Hilbert space $\mathscr{H}_i$ with $\| \pi_i(a) \| = p_i(a)$, $i = 1, 2$. Let $\pi_1 \otimes \pi_2$ denote the tensor product representation on $\mathscr{H}_1 \otimes \mathscr{H}_2$. Then

$$p_1 \otimes_{\min} p_2(a) := \| \pi_1 \otimes \pi_2(a) \| .$$

The C*-norm $p_1 \otimes_{\max} p_2$ is in general difficult to deal with, since it involves representations that are not explicitly known. We shall therefore have to be content with the minimal tensor product, and we define

$$\Gamma^{(1)} \otimes_{\min} \Gamma^{(2)} = \{ p_1 \otimes_{\min} p_2 \mid p_i \in \Gamma^{(i)}, i = 1, 2 \} .$$

(2.9) For the following discussion it is convenient to introduce the concept of a *strongly positive representation.* Let $\mathfrak{A}$ be a *-algebra and π a *-representation of $\mathfrak{A}$ (in general unbounded) defined on a domain $\mathscr{D}$ in a Hilbert space $\mathscr{H}$. If Γ is a family of C*-seminorms on $\mathfrak{A}$, we call π a Γ-strongly positive representation, if the functionals $a \mapsto \langle \psi, \pi(a)\psi \rangle$ are Γ-strongly positive for all $\psi \in \mathscr{D}$. Because of (2.7) it suffices to require this for some cyclic subspace of $\mathscr{D}$. The following proposition is a simple corollary of the fact that Γ-strongly positive functionals are precisely those that can be approximated by $\mathscr{T}_\Gamma$-continuous positive functionals. These latter functionals are given by vectors in representations that are bounded by C*-seminorms in Γ.

(2.10) PROPOSITION. — *Let π_1 be a $\Gamma^{(i)}$-strongly positive representation of $\mathfrak{A}^{(i)}$ on a domain $\mathscr{D}_i$ in a Hilbert space $\mathscr{H}_i$. Then $\pi_1 \otimes \pi_2$ is a $\Gamma^{(1)} \otimes_{\min} \Gamma^{(2)}$-strongly positive representation of $\mathfrak{A}^{(1)} \otimes \mathfrak{A}^{(2)}$ on $\mathscr{D}_1 \otimes \mathscr{D}_2$.*

(2.11) COROLLARY. — Let $\omega_{jk}^{(i)}$, $i = 1, 2, j, k \in \mathbb{N}$ be linear functionals on $\mathfrak{A}^{(i)}$ such that $\Sigma \mid \omega_{jk}(a) \mid^2 < \infty$ for all $a \in \mathfrak{A}^{(i)}$ and such that

$$\sum_{j,k} \bar{\lambda}_j \omega_{jk}^{(i)} \lambda_k$$

is $\Gamma^{(i)}$-positive for all sequences $(\lambda_j) \in l_2$. Then the functional

$$\omega = \sum_{j,k} \omega_{kj}^{(1)} \otimes \omega_{jk}^{(2)}$$

is a $\Gamma^{(1)} \otimes_{\min} \Gamma^{(2)}$-strongly positive functional on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$.

Proof. — This follows from (2.10) and a generalized GNS-construction with the matrices $\{ \omega^{(i)}_{jk} \}$ (cf. e. g. [*9*], theorem 2.3): There are representations π_i on domains $\mathscr{D}_i$ and vectors $\Omega^{(i)}_j$, $i = 1, 2$; $j \in \mathbb{N}$, such that

$$\omega^{(i)}_{jk} = \langle \Omega^{(i)}_j, \pi_i(.)\Omega^{(i)}_k \rangle .$$

The condition on ω_{jk} implies by (2.7) that π_i is strongly positive, and the statement follows from (2.10).

Conversely we have

(2.12) PROPOSITION. — *Functionals of the form*

$$\omega = \sum_{j,k=1}^{\mathrm{N}} \omega^{(1)}_{kj} \otimes \omega^{(2)}_{kj}$$

with $\mathscr{T}_{\Gamma^{(i)}}$-continuous functionals $\omega^{(i)}_{kj}$, $k, j = 1, \ldots, \mathrm{N} < \infty$ and $\Sigma \bar{\lambda}_k \omega^{(i)}_{kj} \lambda_j$ positive for all (λ_j) are dense in the cone of $\Gamma^{(1)} \otimes_{\min} \Gamma^{(2)}$-strongly positive functionals on $\mathfrak{A}^{(1)} \otimes \mathfrak{A}^{(2)}$. If one of the algebras is abelian, then functionals of the form

$$\omega = \sum_{j=1}^{\mathrm{N}} \omega^{(1)}_j \otimes \omega^{(2)}_j$$

with positive and $\mathscr{T}_{\Gamma^{(i)}}$-continuous functionals $\omega^{(i)}_j$ are already dense.

Proof. — This is again a consequence of the fact that Γ-strongly positive functionals can be approximated by $\mathscr{T}_\Gamma$-continuous positive functionals. Every positive functional on $\mathfrak{A}^{(1)} \otimes \mathfrak{A}^{(2)}$ that is continuous w. r. t. a seminorm in $\Gamma^{(1)} \otimes_{\min} \Gamma^{(2)}$ is given by a vector state in a tensor product representation, and every vector in the tensor product can be approximated by a finite sum of product vectors. The statement for the abelian case follows from [*8*], theorem 4.14.

We shall now consider special algebras, the partially symmetric tensor algebras $\mathrm{S}_\rho(\mathrm{E})$ studied in [*2*] and [*1*].

Let E be a linear space with an antilinear involution *. Let ρ be a *-invariant relation on E. The algebra $\mathrm{S}_\rho(\mathrm{E})$ is defined as the quotient algebra $\mathrm{T}(\mathrm{E})/\mathscr{I}_\rho$, where T(E) is the full tensor algebra and $\mathscr{I}_\rho$ is the two sided ideal generated by $a \otimes b - b \otimes a$, $(a, b) \in \rho$. $\mathrm{S}_\rho(\mathrm{E})$ is a natural way a *-algebra with unit. If E is a topological vector space it is understood that $\mathrm{E}^{\otimes n}$ is completed w. r. t. a suitable tensor product topology and T(E) is the direct sum of these completed spaces. Also $\mathscr{I}_\rho$ is taken to be the closed ideal generated by the commutators.

$\mathrm{S}_\rho(\mathrm{E})$ is a graded algebra:

$$\mathrm{S}_\rho(\mathrm{E}) = \bigoplus_{n \geq 0} \mathrm{S}_\rho(\mathrm{E})_n$$

where $S_\rho(E)_n := E^{\otimes n}/\mathscr{I}_\rho \cap E^{\otimes n}$. If ω is a linear functional on $S_\rho(E)$ we denote by ω_n its restriction to $S_\rho(E)_n$.

Let $\omega^{(1)}$ and $\omega^{(2)}$ be two linear functionals on $S_\rho(E)$. Their s-product ([*10*], cf. also [*11*]) is defined by

$$(\omega^{(1)} s \omega^{(2)})_n(a_1 \ldots a_n) = \sum_{\substack{\text{parti-}\\ \text{tions}}} \omega_k^{(1)}(a_{i_1} \ldots a_{i_k})\omega_l^{(2)}(a_{j_1} \ldots a_{j_l})$$

for $a_i \in E = S_\rho(E)_1$, where the sum is over all partitions of $\{ 1 \ldots n \}$ into subsets $\{ i_1, \ldots, i_k \}, \{ j_1, \ldots, j_l \}$ with $i_1 < \ldots < i_k ; j_1 < \ldots < j_l$ including the empty sets.

(2.13) Proposition. — *$\omega^{(1)} s \omega^{(2)}$ is a well defined linear functional on $S_\rho(E)$. If Γ is a family of C*-seminorms on $S_\rho(E)$ and $\omega^{(1)}, \omega^{(2)}$ are Γ-strongly positive then $\omega^{(1)} s \omega^{(2)}$ is also Γ-strongly positive.*

Proof. — Consider the linear map

$$\varphi : a \mapsto a \otimes \mathbb{1} + \mathbb{1} \otimes a$$

of $E = S_\rho(E)_1$ into $S_\rho(E) \otimes S_\rho(E)$. It is clear that $[a, b] = 0$ in $S_\rho(E)$ implies $[\varphi(a), \varphi(b)] = 0$. Hence φ extends to a homomorphism

$$S_\rho(E) \to S_\rho(E) \otimes S_\rho(E)$$

also denoted by φ. We have $\omega^{(1)} s \omega^{(2)} = (\omega^{(1)} \otimes \omega^{(2)}) \circ \varphi$ so $\omega^{(1)} s \omega^{(2)}$ is a well defined linear functional on $S_\rho(E)$. Also, it is clear that φ is continuous if $S_\rho(E)$ is equipped with the topology defined by Γ and $S_\rho(E) \otimes S_\rho(E)$ with the topology defined by $\Gamma \otimes_{\min} \Gamma$. By (2.6) and (2.11) it follows that $\omega^{(1)} s \omega^{(2)}$ is Γ-strongly positive if this holds for $\omega^{(1)}$ and $\omega^{(2)}$.

Next we consider multiplication of functionals by sequences of positive type, i. e. sequences $\{ \gamma_n \}_{n \geq 0}$ of complex numbers such that $(\gamma_{n+m})_{n,m}$ is a positive semidefinite matrix.

(2.14) Proposition. — *Let ω be a linear functional on $S_\rho(E)$ and $\{ \gamma_n \}$ a sequence of positive type. If ω is Γ-strongly positive, w. r. t. a family of C*-seminorms Γ, then the same holds for the functional $\omega_{\{\gamma_n\}}$ defined by $(\omega_{\{\gamma_n\}})_m = \gamma_m \omega_m$.*

Proof. — The sequence $\{ \gamma_n \}$ defines a linear functional γ on the algebra $\mathbb{C}[X]$ of polynomials in one indeterminate:

$$\gamma : \Sigma a_n X^n \mapsto \Sigma \gamma_n a_n .$$

Moreover, this functional is strongly positive by the solution of the 1-dimensional moment problem. If we define a homomorphism $\xi : S_\rho(E) \to S_\rho(E) \otimes \mathbb{C}[X]$ by

$$\xi(a) = a \otimes X$$

for $a \in \mathrm{E}$ and canonical extension to $\mathrm{S}_\rho(\mathrm{E})$, we may write

$$\omega_{\{\gamma_n\}} = (\omega \otimes \gamma) \circ \xi .$$

The assertion thus follows from (2.6) and (2.11).

(2.15) For the construction of invariant, strongly positive functionals a result on *conditionally strongly positive functionals* is needed. A functional ω on $\mathrm{S}_\rho(\mathrm{E})$ is called conditionally strongly positive (w. r. t. a family of C*-seminorms Γ) if ω is positive on all elements of $\mathrm{S}_\rho(\mathrm{E})_\Gamma^+$ that have a vanishing component in $\mathrm{S}_\rho(\mathrm{E})_0 = \mathbb{C}$.

The following variant of a theorem of Hegerfeldt ([*12*], theorem 2.1) holds:

(2.16) PROPOSITION. — *If ω is hermitean and conditionally strongly positive, then* $\exp|_s\omega = \sum_n \frac{1}{n!}\omega s \ldots s\omega$ *is strongly positive.*

Proof. — First we note that by the bipolar theorem we can approximate ω by $\mathscr{T}_\Gamma$-continuous, conditionally strongly positive functionals. Since the mapping $\omega \mapsto \exp|_s\omega$ is weakly continuous it is therefore sufficient to prove the assertion for $\mathscr{T}_\Gamma$-continuous functionals. We can then follow closely the proof in [*12*]. By the Jordan decomposition we can write $\omega = \omega_1 - \omega_2$ where ω_1, ω_2 are $\mathscr{T}_\Gamma$-continuous and positive functionals. We have then $|\omega(a)| \leq \omega_3(a^*a)^{1/2}$ for all a, with $\omega_3 = \text{const.}(\omega_1 + \omega_2)$. Pick an $\varepsilon > 0$ and consider the functional

$$\omega^{(\varepsilon)} = \omega + \varepsilon\cdot\omega_3 .$$

We have for all $a = (a_0, \underline{a})$, $a_0 \in \mathbb{C}$, $\underline{a} \in \bigoplus_{n\geq 1} \mathrm{S}_\rho(\mathrm{E})_n$

$$\omega^{(\varepsilon)}(a^*a) \geq \varepsilon\omega_3(\underline{a}^*\underline{a}) - 2\cdot(1+\varepsilon)\cdot|a_0|\cdot\omega_3(\underline{a}) + \varepsilon\cdot\omega_3(a_0^2) \geq -\mathrm{C}_\varepsilon$$

for some constant $\mathrm{C}_\varepsilon < \infty$. If 1 denotes the functional $(1, 0, 0, \ldots) \in \mathrm{S}_\rho(\mathrm{E})_\Gamma^{+\,1}$ we have therefore

$$1 + \omega^{(\varepsilon)}/n \in \mathrm{S}_\rho(\mathrm{E})_\Gamma^{+\,\prime}$$

for $n > \mathrm{C}_\varepsilon$. (Note that $1 + \omega^{(\varepsilon)}/n$ is $\mathscr{T}_\Gamma$-continuous.) Hence by (2.13) and (2.2) we have also that

$$\exp|_s\omega^{(\varepsilon)} = \lim_{n\to\infty}(1 + \omega^{(\varepsilon)}/n)s \ldots s(1 + \omega^{(\varepsilon)}/n)$$

is strongly positive, and this holds then also for

$$\exp|_s\omega = \lim_{\varepsilon\to 0}\exp|_s\omega^{(\varepsilon)} .$$

3. EMBEDDING OF $S_\rho(E)$ INTO $T(E) \otimes \ldots \otimes T(E)$

(3.1) In this section we consider a finite dimensional space E and suppose that ρ is a relation on a basis $\{ e_1, \ldots, e_N \}$ of E, cf. [2]. We can think of ρ as a symmetric, reflexive relation on $\{ 1, \ldots, N \}$ and picture it as a graph with N vertices $1, \ldots, N$ and links between the pairs $(i,j) \in \rho$. Let ρ^c denote the complementary relation which we also picture as a graph: A pair (i,j) is linked in ρ^c iff it is not linked in ρ, iff e_i, e_j do *not* commute in $S_\rho(E)$. A subgraph γ of ρ^c will be called *totally connected*, if all pairs of vertices in γ are joined by a link. An isolated vertex is also considered as a totally connected subgraph.

(3.2) Let γ_k, $k = 1, \ldots, M$ be totally connected subgraphs of ρ^c such that every $i \in \{ 1, \ldots, N \}$ is a vertex os some γ_k. (In this case we write $i \in \gamma_k$.) Let $E^{(k)}$ be the subspace of E spanned by the vectors e_i with $i \in \gamma_k$, and let $T(E^{(k)})$ be the corresponding tensor algebra. We now define a homomorphism

$$\varphi_\rho : T(E) \to T(E^{(1)}) \otimes \ldots \otimes T(E^{(M)})$$

in the following way:

$$\varphi_\rho(e_i) = \ldots \otimes e_i \otimes \ldots \otimes \mathbb{1} \otimes \ldots \otimes e_i \otimes \ldots$$

where the k-th factor is either e_i, if $i \in \gamma_k$, or the unit element $\mathbb{1} \in T(E^{(k)})$, if $i \notin \gamma_k$. This definition fixes φ_ρ uniquely on all of T(E).

(3.3) PROPOSITION. — $\ker \varphi_\rho = \mathscr{I}_\rho$.

Proof. — We recall some notations and results from [2]. If $I = (i_1, \ldots, i_n)$ is a multiindex; $i_1 \in \{ 1, \ldots, N \}$ we write $e_I = e_{i_1} \otimes \ldots \otimes e_{i_n}$. It is also convenient to define $e_\phi = \mathbb{1}$. If $J = (j_1, \ldots, j_n)$ we write $I \underset{\rho}{\sim} J$ if J can be transformed into I by « admissible permutations », i. e. if there is a sequence J_l of multiindices, $(j_{1l}, \ldots, j_{nl})$, $l = 0, \ldots, L$ with $J_0 = J$, $J_L = I$ and J_l and J_{l+1} differing only in a single pair

$$(j_{ml}, j_{m+1,l}) = (j_{m+1,l+1}, j_{m,l+1}) \in \rho .$$

In [2], lemma 3.1, it was shown that $\Sigma \alpha_I \varphi_I \in \mathscr{I}_\rho$ if and only if

$$\sum_{J \sim I} \alpha_J = 0$$

for all I.

Now from the definition of φ_ρ it follows that

$$\varphi_\rho(e_I) = e_{I^{(1)}} \otimes \ldots \otimes e_{I^{(M)}}$$

where the multiindex $I^{(k)}$ is obtained from I by deleting all indices except those belonging to the totally connected subgraph γ_k. (Note that $e_\phi := \mathbb{1}$.)

Since the elements $e_{I^{(1)}} \otimes \ldots \otimes e_{I^{(M)}}$ form a linearly independent set, we see that

$$\Sigma \alpha_I e_I \in \ker \varphi_\rho \quad \text{iff} \quad \sum_{\substack{J^{(1)}=I^{(1)} \\ \vdots \\ J^{(M)}=I^{(M)}}} \alpha_J = 0 \quad \text{for all I}$$

The assertion thus follows once the following lemma is proven.

(3.4) LEMMA. — $I \sim J$ *if and only if* $I^{(k)} = J^{(k)}$ *for all* k.

Proof of Lemma. — Since an admissible permutation cannot change the order of indices belonging to a totally connected subgraph, it is clear that $I \sim J$ implies $I^{(k)} = J^{(k)}$ for all k. Conversely, assume that $I^{(k)} = J^{(k)}$ for all k. First we remark that if an index i appears in I ν_i times, then it also appears ν_i-times in $I^{(k)}$, for every k with $i \in \gamma_k$. Since every i belongs to some γ_k, we conclude that J must be a permutation of I_0. Suppose now that $i_1 \neq j_m$. If $(j_l, j_m) \notin \rho$ for some $l < m$, then there exists a maximal totally connected subgraph γ_k of ρ^c containing j_l and $j_m = i_1$. For this k we obviously have $I^{(k)} = (i_1, \ldots) \neq J^{(k)} = (\ldots, i_1, \ldots)$ contrary to hypothesis. Hence we conclude that $(j_l, j_m) \in \rho$ for all $l < m$, so we can permute these indices and obtain $J' = (j'_1, \ldots, j'_n)$ with $j'_1 = j_m = i_1$.

By repeating this procedure we gradually transform J into I, so $I \sim J$. This completes the proof of the lemma and thus of prop. 3.3.

Since φ_ρ annihilates $\mathscr{I}_\rho$, it induces a homomorphism

$S_\rho(E) \to T(E^{(1)}) \otimes \ldots \otimes T(E^{(M)})$. We thus obtain the following

(3.5) COROLLARY. — $S_\rho(E)$ *is isomorphic to a subalgebra of* $T(E^{(1)}) \otimes \ldots \otimes T(E^{(M)})$.

(3.6) REMARK. — The only condition which the totally connected subgraphs γ_k have to fullfill is that each i should occur in some γ_k. In particular we might choose for the γ_k all connected subgraphs with two vertices and all isolated points. In order to keep the number of factors in the tensor product as small as possible one could also choose to consider only *maximal* totally connected subgraphs, i. e. those who are not properly contained in larger totally connected subgraphs.

(3.7) PROPOSITION. — *For every nonzero element* $a \in S_\rho(E)$, *there is a finite dimensional representation* π *of the algebra with* $\pi(a) \neq 0$.

Proof. — The finite dimensional representations of T(E) separate points, cf. [*13*] and [*7*]. The same holds therefore for the tensor product $T(E) \otimes \ldots \otimes T(E)$ and any of its subalgebras, in particular $S_\rho(E)$ by (3.5).

4. STRONGLY POSITIVE FUNCTIONALS ON $\underline{\mathcal{S}}/\mathcal{I}_c$

It was shown in [2] that for every nonzero element $a \in \underline{\mathcal{S}}/\mathcal{I}_c$ there is a homomorphism Φ of $\underline{\mathcal{S}}/\mathcal{I}_c$ into a partially symmetric tensor algebra $S_p(E)$ with E finite dimensional such that $\Phi(a) \neq 0$. Combining this fact with proposition (3.7) we obtain

(4.1) PROPOSITION. — *For every nonzero element $a \in \underline{\mathcal{S}}/\mathcal{I}_c$ there is a finite dimensional representation π of the algebra with $\pi(a) \neq 0$.*

The finite dimensional representation of $\underline{\mathcal{S}}/\mathcal{I}_c$ can be used to give an alternative proof of theorems 3.4 and 3.5 in [1]. Moreover, since every positive, linear functional on T(E) is strongly positive ([7], thm. 2), we may use (2.11) and (3.5) to obtain a large collection of strongly positive functionals on $\underline{\mathcal{S}}/\mathcal{I}_c$. We shall now establish some results on the existence of strongly positive functionals with specific properties.

(4.2) Let $\mathcal{T}$ be a locally convex topology on a *-algebra $\mathfrak{A}$ and let Γ be a family of $\mathcal{T}$-continuous C*-seminorms. Define a locally convex topology $\hat{\mathcal{T}}$ on $\mathfrak{A}$ by the seminorms

$$\hat{q}(a) = \sup_{\chi} \frac{|\chi(a)|}{q^0(\chi)}$$

where the supremum is taken over all $\mathcal{T}_\Gamma$-continuous, positive functionals χ, q is a $\mathcal{T}$-continuous seminorm, and

$$q^0(\chi) = \sup_{b} \frac{|\chi(b)|}{q(b)}.$$

Alternatively, we could say that $\hat{\mathcal{T}}$ is the topology of uniform convergence on $\mathcal{T}$-equicontinuous sets of $\mathcal{T}_\Gamma$-continuous states. Obviously $\hat{q} \leq q$, and if $p \leq q$, then we have $q^0 \leq p^0$ and thus $\hat{p} \leq \hat{q}$. If $p \in \Gamma$, then $\hat{p} = p$.

(4.3) The seminorms $\hat{q}$ are monotonous w. r. t. the cone $\mathfrak{A}_\Gamma^+$: If $0 \leq a \leq b$, then $\hat{q}(a) \leq \hat{q}(b)$. This means that $\mathfrak{A}_\Gamma^+$ is a *normal cone* for the topology $\hat{\mathcal{T}}$ [15]; in particular every $\hat{\mathcal{T}}$-continuous, linear functional is a linear combination of Γ-strongly positive functionals.

(4.4) The natural topology on $\mathfrak{A} = \underline{\mathcal{S}}/\mathcal{I}_c$ is the Mackey topology τ, and eventually we would like to have an explicit description of $\hat{\tau}$ in a similiar way as was done for the totally symmetric tensor algebra in [4]. However, for $\underline{\mathcal{S}}/\mathcal{I}_c$ the topology $\hat{\tau}$ seems to be complicated to describe in terms of Schwartz-norms on $\mathcal{S}$. We shall therefore be content here with a weaker result. Consider the topology τ_∞ on $\underline{\mathcal{S}}$ that is defined by all seminorms of the form

$$\sum_n c_n \|\cdot\|_k^{\otimes n}$$

where c_n is an arbitrary sequence of positive numbers and $\|\,.\,\|_k$ is a Schwartz-norm on $\mathcal{S}$ that is *independent of* n [*14*]. The tensor product can be either the ε- or the π-product for seminorms since $\mathcal{S}$ is a nuclear space. We denote the corresponding quotient topology on $\underline{\mathcal{S}}/\mathcal{I}_c$ also by τ_∞. Every τ-continuous C*-seminorm on this algebra is obviously τ_∞-continuous, and we define $\hat{\tau}_\infty$ by using the family of all such C*-seminorms.

(4.5) PROPOSITION. — *The topology τ_∞ and $\hat{\tau}_\infty$ coincide on $\underline{\mathcal{S}}/\mathcal{I}_c$.*

Proof. — The topology $\hat{\tau}_\infty$ is weaker than τ_∞ by (4.2). One has to show that conversely there is for every τ_∞-continuous seminorm q another τ_∞-continuous seminorm q' with $q \leq \hat{q}'$. The proof is similar to that of theorem 4.6 in [*2*] so I can be brief. As a first step one shows that if $\hat{q}$ is a τ_∞-continuous seminorm, then also the seminorms

$$a \mapsto \sum_{n \geq 0} c_n \hat{q}(a_n) \tag{*}$$

are $\hat{\tau}_\infty$-continuous for arbitrary sequences $\{ c_n \}$ of positive numbers. Here a_n denotes the homogeneous components of $a \in \underline{\mathcal{S}}/\mathcal{I}_c$. As in [*2*] this goes by constructing a sequence of positive type $\{ d_n \}$ such that

$$\Sigma c_n \hat{q}(a_n) \leq \sup_\chi \frac{|\chi_{\{d_n\}}(a)|}{q^0(a)}$$

where $\chi_{\{d_n\}}(a) = \sum_m d_m \chi(a_m)$. By (2.14) the functionals $\chi_{\{d_n\}}$ can be approximated by $\mathcal{T}_\Gamma$-continuous positive functionals. An inspection of the proof of (2.14) also shows that if χ is majorized by a seminorm $a \mapsto \Sigma q(a_n)$ then one can choose the $\mathcal{T}_\Gamma$-continuous, positive functionals that approximate $\chi_{\{d_n\}}$ in such a way that they are majorized by the seminorm $q'(a) = \sum_n d_n q(a_n)$.

Hence we have

$$\sum_n c_n \hat{q}(a_n) \leq \hat{q}'(a).$$

To complete the proof of the proposition we have to show that every

τ_∞-continuous seminorm can be dominated by a seminorm of the form (*). This, however, follows essentially from the fact established in theorem 3.5 in [1] that the continuous C*-norms induce the original Fréchet-topology on $(\underline{\mathcal{S}/\mathcal{I}_c})_n$ for all n. One has only to note one additional feature that is implicit in the proof in [1]: A Schwartz-norm $\|\cdot\|_k^{\otimes n}$ on $(\underline{\mathcal{S}/\mathcal{I}_c})_n$ is continuous w. r. t. a C*-seminorm $p^{(n)}$ that can be chosen to be continuous w. r. t. the τ_∞-norm $\sum_n \|\cdot\|_k^{\otimes n}$ on $\underline{\mathcal{S}/\mathcal{I}_c}$. Hence all the Schwartz-norms $\|\cdot\|_k^{\otimes n}$, $n = 1, 2, \ldots$ are continuous w. r. t. the τ_∞-continuous C*-norm $\sup_n p^{(n)} := p = \hat{p}$ and the proof is complete.

By (4.3) we have the following corollaries

(4.6) THEOREM. — $(\underline{\mathcal{S}/\mathcal{I}_c})_\Gamma^+$ *is a normal cone for the topology* τ_∞.

(4.7) THEOREM. — *Every* τ_∞*-continuous linear functional on* $\underline{\mathcal{S}/\mathcal{I}_c}$ *can be written as a linear combination of strongly positive and* τ_∞*-continuous linear functionals.*

Using the fact that $\underline{\mathcal{S}/\mathcal{I}_c}$ with the topology τ_∞ is a nuclear space, one proves also in the same way as in theorem 4.8 in [2].

(4.8) THEOREM. — *For every* τ_∞*-continuous seminorm* q *on* $\underline{\mathcal{S}/\mathcal{I}_c}$ *there is a strongly positive,* τ_∞*-continuous linear functional* ω *with* $q(a) \leq \omega(a^*a)^{1/2}$ *for all* $a \in \underline{\mathcal{S}/\mathcal{I}_c}$.

As a last topic we consider strongly positive functionals that are invariant with respect to the natural action α_x of the translation group $\mathbb{R}^d$ on $\underline{\mathcal{S}/\mathcal{I}_c}$. If ω is a positive functional on $\underline{\mathcal{S}/\mathcal{I}_c}$ we denote its kernel by $\mathrm{K}(\omega)$ and its left kernel by $\mathrm{L}(\omega) = \{ a \mid \omega(a^*a) = 0 \}$.

In [3], theorem 3.3 it was proven that the intersection of the left kernels of all translationally invariant, positive functionals on $\underline{\mathcal{S}/\mathcal{I}_c}$ is the zero element, and the intersection of the kernels is the linear space $\mathcal{K}_{\mathbb{R}^d}$ = closure of $\{ a - \alpha_x a \mid a \in \underline{\mathcal{S}/\mathcal{I}_c}, x \in \mathbb{R}^d \}$. Because of theorem 4.7 and prop. (2.16), we can use exactly the same method as in [3] to prove the following strengthening of this result:

(4.9) THEOREM. — $\cap\, \mathrm{K}(\omega) = \mathcal{K}_{\mathbb{R}^d}$ and $\cap\, \mathrm{L}(\omega) = \{ 0 \}$, *where the intersection is over all strongly positive and translationally invariant,* τ_∞*-continuous linear functionals on* $\underline{\mathcal{S}/\mathcal{I}_c}$.

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