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Time decay of finite energy solutions of the non linear Klein-Gordon and Schrödinger equations

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ABSTRACT. — We study the time decay of finite energy solutions of the non linear Klein-Gordon (NLKG) equation

$$\square\varphi + \varphi + f(\varphi) = 0$$

in space dimension $n \geq 3$, by using a reformulation of the original method of Morawetz and Strauss previously applied in [*11*] to the non linear Schrödinger (NLS) equation

$$i\frac{d\varphi}{dt} = -\frac{1}{2}\Delta\varphi + f(\varphi).$$

We prove that such solutions satisfy some of the time decay properties of the solutions of the free equation. The assumptions on f cover the case of a single power $f(\varphi) = \lambda\,|\,\varphi\,|^{p-1}\varphi$ with $\lambda > 0$ and $4/n < p-1 < 4/(n-2)$. Our results extend those of Brenner [*4*] [*5*]. We also improve the results of [*11*] for the NLS equation. As an intermediate result, we obtain bounded-

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ness properties in time (in a suitable sense) of finite energy solutions of both equations under fairly weak assumptions on f.

RÉSUMÉ. — On étudie la décroissance en temps des solutions d'énergie finie de l'équation de Klein Gordon non linéaire

$$\Box\varphi + \varphi + f(\varphi) = 0$$

en dimension d'espace $n \geq 3$, en utilisant une reformulation de la méthode originale de Morawetz et Strauss, appliquée précédemment dans [*11*] à l'équation de Schrödinger non linéaire (SNL)

$$i\frac{d\varphi}{dt} = -\frac{1}{2}\Delta\varphi + f(\varphi).$$

On montre que de telles solutions satisfont certaines des décroissances en temps des solutions de l'équation libre. Les hypothèses sur f couvrent le cas d'une seule puissance $f(\varphi) = \lambda \mid \varphi \mid^{p-1}\varphi$ avec $\lambda > 0$ et $4/n < p-1 < 4/(n-2)$. Les résultats étendent ceux de Brenner [*4*] [*5*]. On améliore également les résultats de [*11*] sur l'équation SNL. Comme résultat intermédiaire, on obtient des propriétés de borne uniforme en temps (en un sens convenable) des solutions d'énergie finie des deux équations sous des hypothèses assez faibles sur f.

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1. INTRODUCTION

A large amount of work has been devoted recently to the study of the asymptotic behaviour in time of the solutions of the non linear Klein-Gordon (NLKG) equation [*3*] [*4*] [*5*] [*15*] [*16*] [*17*] [*18*] [*19*] [*20*] [*23*] [*24*]

$$\ddot{\varphi} - \Delta\varphi + \varphi + f(\varphi) = 0 \tag{1.1}$$

and of the non linear Schrödinger (NLS) equation [*8*] [*11*] [*13*] [*19*] [*20*] [*21*] [*22*]

$$i\dot{\varphi} = -\frac{1}{2}\Delta\varphi + f(\varphi) \tag{1.2}$$

where φ is a complex valued function defined in space-time $\mathbb{R}^{n+1}$, the dot denotes the time derivative, Δ is the Laplace operator in $\mathbb{R}^n$ and f is a non linear complex valued function. For simplicity we have taken the mass equal to 1 in the NLKG equation. From now on the equations (1.1) and (1.2) will be referred to as the NLKG and NLS equation respectively. A typical form of the interaction f is the sum of two powers, namely

$$f(\varphi) = \sum_{j=1,2} \lambda_j |\varphi|^{p_j - 1}\varphi \tag{1.3}$$

with $1 \leq p_1 \leq p_2 < \infty$ and $\lambda_j \geq 0$. One of the main results of those investigations is the fact that, for repulsive interactions, arbitrarily large solutions of the NLKG and NLS equations exhibit some of the time decay of solutions of the corresponding free equation (namely the equation with $f = 0$).

For the NLKG equation all the proofs of such results are extensions of that first given in [*15*], which covers the case of dimension $n = 3$ and sufficiently regular solutions. The result was generalised in [*16*] as regards the assumptions on f, and extended in [*3*] to higher dimensions. The next progress was to relax the regularity assumption on the solutions so as to cover the case of arbitrary finite energy solutions (see definition at the end of Section 3). For space dimensions $n \geq 3$ and interaction essentially controled by a single power p typically $p_1 = p_2 = p$ in (1.3)) with $4/n < p - 1 < 4/(n - 1)$, it was proved [*4*] [*5*] that arbitrary large finite energy solutions exhibit some of the time decay of solutions of the free equation. That result was also obtained for $n=3$ and $4/3<p-1<4$ [*6*](*).

For the NLS equation the same method has been applied in [*13*] to the case of space dimension $n = 3$, of an interaction controlled by a single power and of sufficiently regular solutions. That treatment has been extended in [*11*] to any dimension $n \geq 3$, to a class of interactions that contains the special case (1.3) with

$$4/n < p_1 - 1 \leqslant p_2 - 1 < 4/(n - 2) \tag{1.4}$$

and to arbitrary finite energy solutions, again shown to possess some of the decay of solutions of the free equation. The proof, again a descendant of that in [*15*], required a detailed and systematic analysis of the latter, which was split into a number of relatively independent steps, each of which was pushed to its natural limit. The NLS equation can also be treated by the pseudo-conformal invariance method, which yields some related results [*8*] [*21*] [*22*].

For both the NLKG and NLS equations, small data results can be obtained by methods of a more general and more standard character [*17*] [*18*] [*19*] [*20*] [*23*] [*24*].

The purpose of the present paper is twofold. First we reanalyze the NLKG equation and the time decay of its solutions in the light of [*11*]. We introduce a class of spaces which are convenient to characterize the behaviour

(*) See note added in proof.

of solutions. The behaviour in space is described by Besov spaces and the behaviour in time by Birman-Solomjak spaces $l^{\cdot}(L^{\cdot})$ (see (1.6) below). The properties of finite energy solutions of the free equation in terms of those spaces are studied in Section 2, the main result of which is stated as Proposition 2.2. We then study in Section 3, in terms of such spaces, the uniform boundedness in time of arbitrary finite energy solutions of the NLKG equation. The main result, namely Proposition 3.2, holds for arbitrary space dimensions $n \geq 3$ and under rather weak assumptions on f, which reduce to

$$2/n < p_1 - 1 \leqslant p_2 - 1 < 4/(n - 2) \tag{1.5}$$

in the special case (1.3). The method of proof is a direct estimation and does not require the elaborate machinery of [*15*]. In Section 4, we apply that machinery, in the formulation of [*11*], to the NLKG equation. For any dimension $n \geq 3$, we prove that arbitrary finite energy solutions satisfy some of the time decay of solutions of the free equation, as expressed in terms of the spaces mentioned above. The results are stated in Propositions 4.1 and 4.2. The assumptions on f cover the special case (1.3) for p_1 and p_2 in an interval which depends on a parameter $\delta(r)$ (see for instance (4.60)-(4.62)), but which is unfortunately always strictly smaller than (1.4). For a single power however, the entire interval $(4/n, 4/(n - 2))$ is accessible. The decay properties stated in Propositions 4.1 and 4.2 are not the strongest that can be obtained from the method. See on that point the comments at the end of Section 4.

The second purpose of this paper is to improve the results of [*11*] on the NLS equation. That equation is treated in Section 5. We first study the decay properties of solutions of the free equation in terms of the spaces introduced above. The main result is Proposition 5.1 which generalizes Lemma 3.1 of [*11*]. We then turn to the uniform boundedness properties of arbitrary finite energy solutions. The main result, namely Proposition 5.3, holds for arbitrary space dimension $n \geq 3$ and almost under the same assumptions on f as for the NLKG equation, in particular under the condition (1.5) in the special case (1.3). We finally complement the study of the time decay of arbitrary finite energy solutions given in [*11*] (see Section 5) and show that, under the same assumptions on f, finite energy solutions actually exhibit the same decay as obtained in Proposition 5.1 for solutions of the free equation. The main result is stated in Proposition 5.5.

In the framework of scattering theory, the decay properties obtained for both equations imply asymptotic completeness, as soon as there exists a suitable theory of the Cauchy problem at infinity. This is the case for the NLS equation, where such a theory is developped in [*11*]. For the NLKG equation, only partial results are available [*17*] [*18*] [*19*] [*20*] [*23*] [*24*].

We conclude this introduction by giving the main notation used in this paper. We denote by $\|.\|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^n)$. Pairs of conjugate indices are written as r and $\bar{r}$, where $2 \le r \le \infty$ and $r^{-1}+\bar{r}^{-1} = 1$. For any integer k, we denote by $H^k \equiv H^k(\mathbb{R}^n)$ the usual Sobolev spaces. For any interval I of $\mathbb{R}$, for any Banach space B, we denote by $\mathscr{C}(I, B)$ the space of continuous functions from I to B. We denote by $\mathscr{C}^l(I, B)$ $(l \ge 1)$ the space of l times continuously differentiable functions from I to B. For any q, $1 \le q \le \infty$, we denote by $L^q(I, B)$ (resp. $L^q_{loc}(I, B)$) the space of measurable functions φ from I to B such that $\|\varphi(.)\|_B \in L^q(I)$ (resp. $\|\varphi(.)\|_B \in L^q_{loc}(I)$). If I is open we denote by $\mathscr{D}'(I, B)$ the space of vector valued distributions from I to B [*14*]. We shall use the Besov spaces of arbitrary order and the associated Sobolev inequalities [*1*]. We use a notation such as $B^\sigma_{p,q} \equiv B^\sigma_{p,q}(\mathbb{R}^n)$ for those spaces and, for the legibility of the formulas, we denote by $\| . ; B \|$ and $\| . ; L^q(I, B) \|$ the norms in B and in $L^q(I, B)$ when B is such a space. In order to formulate the time decay of the solutions of the NLKG and NLS equations we also introduce the following spaces. For any $t \in \mathbb{R}$, let γ_t be the unit interval with center t. For any m $(1 \le m \le \infty)$ and $q (1 \le q \le \infty)$, for any Banach space B and for any interval $I \subset \mathbb{R}$, we define $l^m(L^q, I, B)$ as the space of measurable functions from I to B for which

$$\| \varphi ; l^m(L^q, I, B) \| \equiv \operatorname*{Sup}_{s \in \gamma_0} \left\{ \sum_{z \in \mathbb{Z}} \left(\int_{\gamma_{s+z} \cap I} dt \, \| \varphi(t) \|_B^q \right)^{m/q} \right\}^{1/m} \tag{1.6}$$

is finite, with obvious modifications if q and/or m is infinite. The spaces $l^m(L^q, I, B)$ are Banach spaces with the norm defined by (1.6). If $B = \mathbb{C}$, we write $l^m(L^q, I)$ for $l^m(L^q, I, \mathbb{C})$. If $I = \mathbb{R}$, I will be omitted in the notation. Those spaces have been introduced by Birman and Solomjak [*2*]. For relevant additional properties we refer to [*11*]. Finally we use the notation $y_\pm = \text{Max}(\pm y, 0)$ for any $y \in \mathbb{R}$. The interaction term f will be assumed to satisfy a number of assumptions which are stated where first needed, namely (H1) before Lemma 3.3 (see (3.17) and (3.18)), (H2) before Proposition 3.2, (H2)′ before Proposition 5.3 and (H3) before Lemma 4.5 (see (4.22)).

2. DECAY ESTIMATES FOR THE FREE MASSIVE KLEIN-GORDON EQUATION

In this section we study the space-time integrability properties of the solutions of the free massive Klein-Gordon equation. We define $\omega_1 = (-\Delta + 1)^{1/2}$. With any r, $1 \le r \le \infty$ we associate the variables $\alpha(r)$, $\delta(r)$ and $\gamma(r)$ defined by

$$\alpha(r) = \gamma(r)/(n-1) = \delta(r)/n = 1/2 - 1/r\,. \tag{2.1}$$

The starting point is an estimate due to Brenner [5] for the operator $\exp(i\omega_1 t)$ for fixed t. For the convenience of the reader, the proof is recalled in the Appendix.

LEMMA 2.1. — Let $2 \leq r \leq \infty$ and $0 \leq \theta \leq 1$. Then

$$\| \exp(i\omega_1 t)\varphi ; B_{r,2}^{-(n+1+\theta)\alpha(r)/2} \| \leqslant \mu(t) \| \varphi ; B_{\bar{r},2}^{(n+1+\theta)\alpha(r)/2} \| \tag{2.2}$$

where

$$\mu(t) = C \operatorname{Min} \{ | t |^{-(\gamma(r)+\theta\alpha(r))}, | t |^{-(\gamma(r)-\theta\alpha(r))_+} \} . \tag{2.3}$$

We shall also need the fact that, for some values of r and θ, the convolution with μ, denoted by $\mu *$, is a bounded operator in suitable combinations of spaces $l^m(L^q)$. We state that property more generally as follows.

LEMMA 2.2. — Let $\mu(t) \equiv \operatorname{Min}(|t|^{-a}, |t|^{-b})$ with $0 \leq a \equiv 2/q < 1$ and $a \leq b$. Then

(1) If $b \equiv 2/m < 1$, $\mu *$ is bounded from $l^2(L^{\bar{q}}) \cap l^{\bar{m}}(L^1)$ to $l^2(L^q) + l^m(L^\infty)$ and *a fortiori* from $l^{\bar{m}}(L^{\bar{q}})$ to $l^m(L^q)$.

(2) If $b = 1$, $\mu *$ is bounded in the same spaces for any $m > 2$.

(3) If $b > 1$, $\mu *$ is bounded from $l^2(L^{\bar{q}})$ to $l^2(L^q)$.

Proof. — We denote the characteristic function of the interval γ_t by the same symbol γ_t. Using the support properties of the convolution, for any $s \in \gamma_0$, for any $z \in \mathbb{Z}$ and for any complex valued measurable function φ, we write

$$\mu * \varphi = \phi_1 + \phi_2$$

where

$$\phi_1 = \sum_{z \in \mathbb{Z}} \gamma_{s+z} \sum_{|z'| \leqslant 1} (\gamma_{s+z-z'}\varphi) * \{ (\gamma_{z'+1/2} + \gamma_{z'-1/2})\mu \}$$

and

$$\phi_2 = \sum_{z \in \mathbb{Z}} \gamma_{s+z} \sum_{|z'| \geqslant 2} (\gamma_{s+z-z'}\varphi) * \{ (\gamma_{z'+1/2} + \gamma_{z'-1/2})\mu \} .$$

Using the Hardy-Littlewood-Sobolev inequality [12] we estimate (for $a > 0$) the contribution of ϕ_1 by

$$\| \gamma_{s+z}\phi_1 \|_q \leqslant c \sum_{|z'| \leqslant 1} \| \gamma_{s+z-z'}\varphi \|_{\bar{q}} .$$

Taking the norm in l^2 we obtain

$$\| \phi_1 ; l^2(L^q) \| \leqslant 3c \| \varphi ; l^2(L^{\bar{q}}) \| . \tag{2.4}$$

Using the pointwise estimate of μ for $|t| > 1$, we estimate ϕ_2 by

$$\| \gamma_{s+z}\phi_2 \|_\infty \leqslant \sum_{|z'| \geqslant 2} (|z'| - 1)^{-b} \| \gamma_{s+z-z'}\varphi \|_1 .$$

From the discrete version of the Hardy-Littlewood-Sobolev inequality if $0 < b < 1$ and of the Young inequality if $b \geq 1$, we obtain

$$\| \phi_2 ; l^m(L^\infty) \| \leqslant C \| \varphi ; l^{\bar{m}}(L^1) \| \tag{2.5}$$

for the values of m announced in cases (1) and (2) and for $m = 2$ in case (3). The lemma now follows immediatly from the estimates (2.4) and (2.5) and from the embeddings of the $l^m(L^q)$ spaces for various values of m and q.

Q. E. D.

Under the same assumptions on μ, one could derive boundedness properties of $\mu *$ between more general combinations of $l^m(L^q)$ spaces. For the applications we have in mind, in particular for the next lemma, we have restricted our attention to the case where the final space is the dual of the initial one.

We next consider the following situation. Let B_0 be a Banach space of functions defined in $\mathbb{R}^n$ and let B be its dual. We suppose that $\mathcal{S}_n \equiv \mathcal{S}(\mathbb{R}^n)$ is continuously embedded and dense in B_0 so that B is naturally embedded into $\mathcal{S}'_n \equiv \mathcal{S}'(\mathbb{R}^n)$. We take the duality between $\mathcal{S}_n$ and $\mathcal{S}'_n$ to be conjugate linear in the first element and linear in the second, so that it reduces to the usual scalar product in L^2 when restricted to $\mathcal{S}_n \times L^2$. Let T_0 be the space $l^{\bar{m}}(L^{\bar{q}})$ or an intersection or a sum of such spaces with $1 \leq \bar{m} < \infty$, $1 \leq \bar{q} < \infty$. Let T be the dual of T_0. Let $T_0(B_0)$ be the space of measurable functions φ from $\mathbb{R}$ to B_0 such that $\| \varphi(.) \|_{B_0} \in T_0$ and let T(B) be defined similarly. In such circumstances $\mathcal{S}_1$ is dense in T_0, $\mathcal{S}_{n+1}$ is dense in $T_0(B_0)$ and T(B) is the dual of $T_0(B_0)$. We can then state the following result.

LEMMA 2.3. — Let B_0, B, T_0, T be as above. Let U(.) be a strongly continuous unitary group in $L^2(\mathbb{R}^n)$. Assume in addition that, for any $t \neq 0$, U(t) maps B_0 into B and that, for any $\varphi \in B_0$, the function $t \mapsto U(t)\varphi$ from $\mathbb{R}$ to B is strongly measurable and satisfies the estimate

$$\| U(t)\varphi \|_B \leqslant \mu(t) \| \varphi \|_{B_0} \tag{2.6}$$

with $\mu \in L^1_{loc}(\mathbb{R}^n)$. Assume in addition that convolution with μ is a bounded operator from T_0 to T. Then, for any $\varphi \in L^2(\mathbb{R}^n)$, $U(.)\varphi \in T(B)$ and $U(.)\varphi$ satisfies the estimate

$$\| U(.)\varphi \|_{T(B)} \leqslant C \| \varphi \|_2 . \tag{2.7}$$

Proof. — By density, it is sufficient to prove (2.7) for $\varphi \in \mathcal{S}_n$ and, by duality and density again, it is sufficient to prove that

$$| \langle \theta, U(.)\varphi \rangle_{n+1} | \leqslant C \| \varphi \|_2 \ \| \theta \|_{T_0(B_0)} \tag{2.8}$$

for any $\theta \in \mathcal{S}_{n+1}$, where $\langle \,.\,, \,.\, \rangle_{n+1}$ denotes the duality in $n+1$ variables. Now

$$| \langle \theta, U(.)\varphi \rangle_{n+1} | = \left| \left\langle \int d\tau U(-\tau)\theta(\tau, .), \varphi \right\rangle_n \right|$$
$$\leqslant \| \varphi \|_2 \left\| \int d\tau U(-\tau)\theta(\tau, .) \right\|_2 . \quad (2.9)$$

The last norm in (2.9) is estimated by

$$\left\| \int d\tau U(-\tau)\theta(\tau, .) \right\|_2^2 = \int dt \left\langle \theta(t, .), \int d\tau U(t-\tau)\theta(\tau, .) \right\rangle_n$$
$$\leqslant \| \theta \|_{T_0(B_0)} \left\| \int d\tau U(.-\tau)\theta(\tau) \right\|_{T(B)} . \quad (2.10)$$

Using (2.6) and the fact that $\mu*$ is a bounded operator from T_0 to T, we can continue (2.10) as

$$\ldots \leqslant C \| \theta \|^2_{T_0(B_0)}$$

which implies (2.8) and thereby completes the proof. Q. E. D.

There is some flexibility in the choice of the assumptions on B_0 and T_0 in Lemma 2.3. The choice made here is both typical and sufficient for our present purpose. Special cases and/or variations can be found in [*9*] [*10*] [*11*].

We are now in a position to determine the space-time integrability properties of the solutions of the free massive Klein-Gordon equation. We first state the basic technical result.

PROPOSITION 2.1. — Let r, ρ and σ satisfy

$$\begin{cases} 2 \leqslant r \leqslant \infty \\ \sigma \equiv \rho + \delta(r) - 1 < 1/2 \\ 2\sigma \leqslant \gamma(r), \end{cases} \quad (2.11)$$

let

$$1/q = \sigma_+ \ (\equiv \mathrm{Max}\,(\sigma, 0)) \quad (2.12)$$

and

$$\begin{cases} 1/m = \mathrm{Min}\,\{ 1/2, \delta(r)/2, \gamma(r) - \sigma \} & \text{if } \mathrm{Min}\,\{ \delta(r)/2, \gamma(r) - \sigma \} \neq 1/2, \\ 1/m < 1/2 & \text{if } \mathrm{Min}\,\{ \delta(r)/2, \gamma(r) - \sigma \} = 1/2. \end{cases} \quad (2.13)$$

Then the map $\varphi \mapsto \omega_1^{-1} \exp(i\omega_1 t)\varphi$ is bounded from $L^2(\mathbb{R}^n)$ to $l^2(L^q, B^\rho_{r,2}) + l^m(L^\infty, B^\rho_{r,2})$ and *a fortiori* from $L^2(\mathbb{R}^n)$ to $l^m(L^q, B^\rho_{r,2})$.

Proof. — From the definition of the Besov spaces and from the Sobolev inequalities it follows that $B^{\sigma'-\delta(r')}_{r',2}$ is continuously embedded in $B^{\sigma-\delta(r)}_{r,2}$ provided $r' \leq r$ and $\sigma' \geq \sigma$. Let $\alpha \equiv \alpha(r)$, $\delta \equiv \delta(r)$, $\gamma \equiv \gamma(r)$ and similarly

$\alpha' \equiv \alpha(r')$, $\delta' \equiv \delta(r')$ and $\gamma' \equiv \gamma(r')$ (see (2.1) for the definition). From the previous embedding and from Lemma 2.1 it follows that

$$\begin{aligned} \| \omega_1^{-1} \exp(i\omega_1 t)\varphi ; B^{\rho}_{\bar{r},2} \| &= \| \exp(i\omega_1 t)\varphi ; B^{\sigma-\delta}_{\bar{r},2} \| \\ &\leqslant C \| \exp(i\omega_1 t)\varphi ; B^{\sigma'-\delta'}_{\bar{r}',2} \| \leqslant C \operatorname*{Min}_{\pm} (| t |^{-(\gamma' \pm \theta\alpha')_+}) \| \varphi ; B^{\delta'-\sigma'}_{\bar{r}',2} \| \\ &\leqslant C \operatorname*{Min}_{\pm} (| t |^{-(\gamma' \pm \theta\alpha')_+}) \| \varphi ; B^{\delta-\sigma}_{\bar{r},2} \| \end{aligned} \tag{2.14}$$

provided we can find r', σ' and θ such that

$$2 \leqslant r' \leqslant r \leqslant \infty, \qquad \sigma' \geqslant \sigma, \tag{2.15}$$

$0 \leq \theta \leq 1$ and

$$\delta' - \sigma' = (n + 1 + \theta)\alpha'/2,$$

or equivalently

$$\gamma' - 2\sigma' = \theta\alpha'.$$

Eliminating θ, we can continue (2.14) as

$$\ldots \leqslant C \operatorname{Min} (| t |^{-2\sigma'_+}, | t |^{-2(\gamma'-\sigma')}) \| \varphi ; B^{\delta-\sigma}_{\bar{r},2} \| \tag{2.16}$$

provided r' and σ' satisfy (2.15) and

$$(n - 2)\alpha' \leqslant 2\sigma' \leqslant \gamma'. \tag{2.17}$$

The result will then follow from Lemmas 2.2 and 2.3 with $B = B^{\sigma-\delta}_{\bar{r},2}$, $T = l^2(L^q) + l^m(L^\infty)$, with q and m defined as in Lemma 2.2 and with $a = 2\sigma'_+$, $b = 2(\gamma' - \sigma')$, provided r' and σ' satisfy (2.15), (2.17) and $\sigma' < 1/2$. It remains to choose r' and σ' for given r and σ. That choice can be made independently in the region $| t | \leq 1$, which yields a and q, and in the region $| t | \geq 1$, which yields b and m, respectively. For $| t | \leq 1$, we choose $\sigma' = \sigma_+$, $r' = 2$ if $n = 1$, and $\gamma' = 2\sigma'$ if $n \geq 2$. That choice satisfies all the required conditions under the assumption (2.11), and yields (2.12). For $| t | \geq 1$ we make the following choice. If $(n - 2)\alpha \leq 2\sigma \leq \gamma$ we take $r' = r$, $\sigma' = \sigma$, so that $b = 2(\gamma - \sigma)$. If $(n - 2)\alpha \geq 2\sigma$, we choose $2\sigma' = (n - 2)\alpha'$ so that $b = \delta'$. Furthermore, for $\gamma \leq 1$, we take $r' = r$, so that $b = \delta$, while for $\gamma > 1$, which occurs only for $n \geq 4$, we take $\sigma' = \operatorname{Max} \{ \sigma, (n-2)/[2(n-1)] \}$ so that $b = \operatorname{Max} \{ 2n\sigma/(n-2), n/(n-1) \} > 1$ in that last case. One checks easily that the previous choices satisfy (2.15), (2.17) and the condition $\sigma' < 1/2$. Q. E. D.

REMARK 2.1. — The choice of (r', σ') is best understood from Fig. 1 (corresponding to the case $n > 3$) where the various conditions are represented in the (γ, σ) plane. The allowed region for (γ', σ') is the triangle $\mathscr{T}$ limited by the lines $\gamma = 2\sigma$, $(n - 2)\alpha = 2\sigma$ and $\sigma = 1/2$. The relation (2.15) expressing that (γ, σ) is controlled by (γ', σ') is equivalent to the fact that (γ, σ) lies in the lower right quadrant with apex at (γ', σ'). In the more complicated region $| t | \geq 1$, for given (γ, σ), one looks for (γ', σ') in the allowed triangle $\mathscr{T}$ such that (γ', σ') controls (γ, σ) and that $\gamma' - \sigma'$ is maxi-

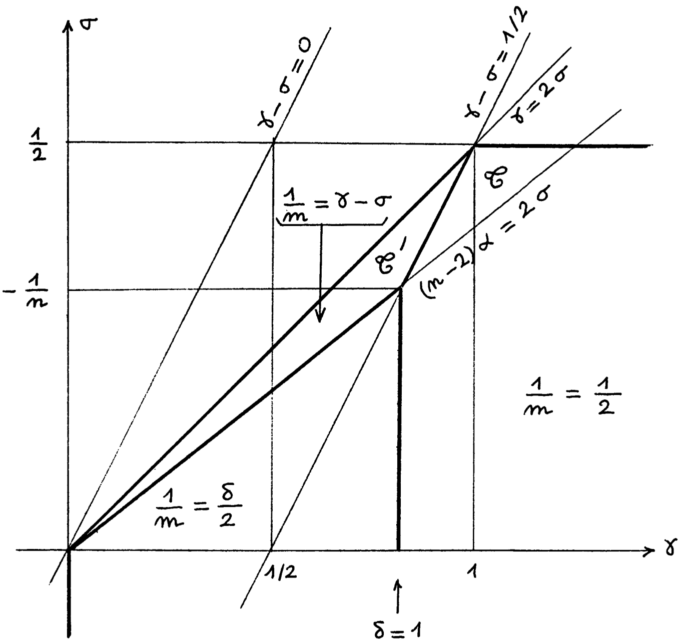

FIG. 1. — *Time decay of the solutions of the free massive KG equation in the* (γ, σ) *plane. The case shown is* $n = 6$.

mal, namely such that (γ', σ') lies as far away as possible from the line $\gamma - \sigma = 0$. In addition there is no point in increasing $\gamma - \sigma$ as soon as it becomes larger than 1/2. The line $\gamma - \sigma = 1/2$ intersects the lines $\gamma = 2\sigma$ and $(n - 2)\alpha = 2\sigma$ at $\gamma = 1$ and $\delta = 1$ respectively and separates out from the triangle $\mathscr{C}$ a smaller left triangle $\mathscr{C}'$. For (γ, σ) in $\mathscr{C}'$, the best choice is obviously $(\gamma', \sigma') = (\gamma, \sigma)$. For (γ, σ) below $\mathscr{C}'$ and $\delta \leq 1$, the best choice consists in taking $\gamma' = \gamma$, with $(n - 2)\alpha' = 2\sigma'$. For (γ, σ) in the remaining region, one can choose any (γ', σ') in $\mathscr{C}\backslash\mathscr{C}'$ that controls (γ, σ). The choice made in the proof is $(\gamma', \sigma') = (\gamma, \sigma)$ if $(\gamma, \sigma) \in \mathscr{C}\backslash\mathscr{C}'$. If $(\gamma, \sigma) \notin \mathscr{C}$, one chooses (γ', σ') on the line $(n - 2)\alpha' = 2\sigma'$ with $\gamma' = \gamma$ if $\gamma \leq 1$, $\sigma' = \sigma$ if that choice yields $\gamma' \geq 1$, and $\gamma' = 1$ in the remaining cases. The resulting values of $1/m$ are $(\gamma - \sigma)/2$ in $\mathscr{C}'$, $\delta/2$ below $\mathscr{C}'$ and as long as $\delta < 1$, and 1/2 in the remaining region.

For the understanding of the case of the full non linear equation, it is convenient to redraw the various regions in the (σ, ρ) plane see Fig. 2).

The space-time integrability properties of the finite energy solutions of the free massive Klein-Gordon equation are now obvious. Let $K_1(t) = \omega_1^{-1} \sin(\omega_1 t)$ and $K_1(t) = \cos(\omega_1 t)$. Let $\varphi_0 \in H^1$ and $\psi_0 \in L^2$.

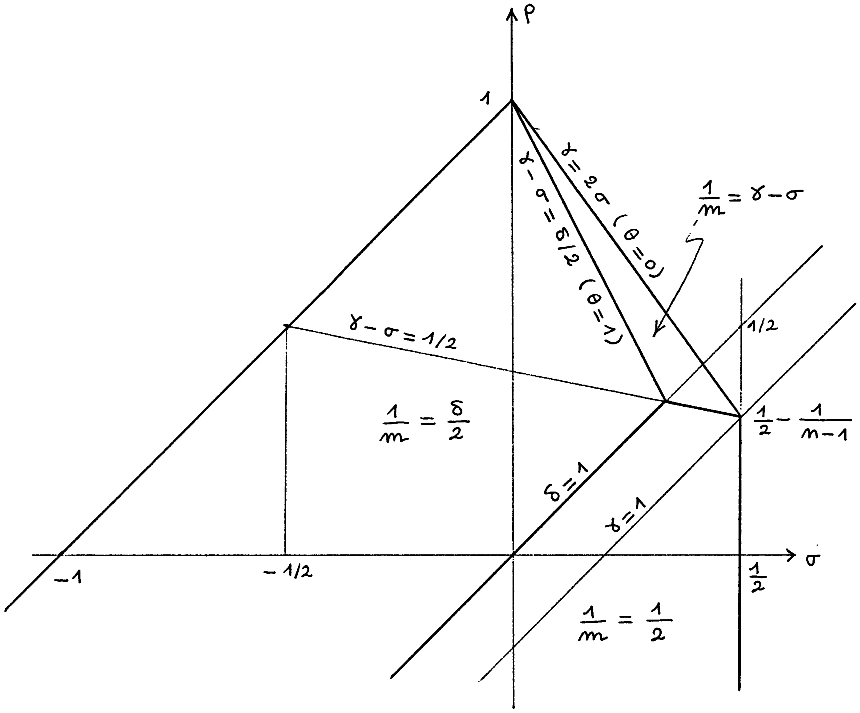

FIG. 2. — *Time decay of the solutions of the free massive KG equation in the (σ, ρ) plane. The case shown is $n = 6$.*

Then the solution $\varphi(t)$ of the free equation with initial condition $\varphi(0) = \varphi_0$ and $\dot{\varphi}(0) = \psi_0$ is formally given by

$$\varphi(t) = \dot{K}_1(t)\varphi_0 + K_1(t)\psi_0 . \tag{2.18}$$

PROPOSITION 2.2. — Let r, ρ, σ, q and m be as in Proposition 2.1. Let $\varphi_0 \in H^1$ and $\psi_0 \in L^2$. Then the map $(\varphi_0, \psi_0) \mapsto \varphi$ given by (2.18) is bounded from $H^1 \oplus L^2$ to $l^2(L^q, B^{\rho}_{r,2}) + l^m(L^{\infty}, B^{\rho}_{r,2})$ and a fortiori to $l^m(L^q, B^{\rho}_{r,2})$.

3. UNIFORM BOUNDS ON THE SOLUTIONS OF THE NLKG EQUATION

In this section we prove that the finite energy solutions of the NLKG equation belong to $l^{\infty}(L^q, B)$ for suitable spaces B, suitable values of q

and suitable assumptions on the interaction f. Preliminary to that result are some estimates on the interaction term which will be used again in the following sections both for the NLKG and NLS equations.

The first result is a minor variation of Lemma 3.2 of [*10*] and will be given without proof.

LEMMA 3.1. — Let $f \in \mathscr{C}^1(\mathbb{C}, \mathbb{C})$ with $|f'(z)| \leq C_0 |z|^{p-1}$ for some p, $1 \leq p < \infty$. Let $0 < \lambda < 1$, let $1 \leq \bar{l} \leq k \leq \infty$, $1 \leq m \leq \infty$ and $1/s = 1/\bar{l} - 1/k$. Then the following inequality holds

$$\| f(\varphi) ; B^{\lambda}_{\bar{l},m} \| \leqslant C \| \varphi ; B^{\lambda}_{k,m} \| \, \| |\varphi|^{p-1} \|_s \tag{3.1}$$

for all φ such that the norms in the right hand side are finite. The same result holds for $\lambda = 1$, $1 < \bar{l} \leq 2 = k = m$.

The basic estimate on the interaction term can now be stated as Lemma 3.2 below. As a first approximation, the reader can take $\eta = 0$ in that Lemma. The case $\eta \neq 0$ will be used only in the proof of Lemma 4.5 below, actually with an additional complication which however cannot be included here without obscuring the main point.

LEMMA 3.2. — Let $f \in \mathscr{C}^1(\mathbb{C}, \mathbb{C})$ with $|f'(z)| \leq C |z|^{p-1}$ for some p, $1 \leq p < \infty$. Let $0 < \lambda \leq 1$, $\rho \leq \lambda$, $\rho < 1$, $1 < \bar{l} \leq 2 \leq r, v < \infty$, $0 \leq \eta \leq \mathrm{Min}\{\delta(l)/n, (p-1)/v\}$ and $0 \leq \sigma \equiv \rho + \delta(r) - 1 < 1$. Then, for all $\varphi \in H^1 \cap B^{\rho}_{r,2}$, $\varphi \in L^v$ if $\eta > 0$, the following inequality holds:

$$\| f(\varphi) ; B^{\lambda}_{\bar{l},2} \| \leqslant M(\| \varphi ; H^1 \|) \| \varphi ; B^{\rho}_{r,2} \|^{\nu} \| \varphi \|_v^{\eta v}, \tag{3.2}$$

where M(.) is a constant depending only on $\| \varphi ; H^1 \|$, provided

$$(p-1)\left(\frac{n}{2} - 1 - \frac{\sigma}{1+\rho_-}\right) \leqslant \delta(l) + \delta(r)\frac{1-\lambda}{1-\rho} + \eta v\left(\delta(v) - 1 - \frac{\sigma}{1+\rho_-}\right)$$

and provided $\nu \geq 0$ satisfies

$$(p-1)\left(\frac{n}{2} - 1\right) \leqslant 1 + \delta(l) - \lambda + \nu\sigma + \eta v(\delta(v) - 1) \tag{3.4}$$

and

$$(p-1)\frac{n}{2} \geqslant \delta(l) + \nu\delta(r) + \rho_+\left(\nu - \frac{1-\lambda}{1-\rho}\right)_+ + \eta v \delta(v). \tag{3.5}$$

Proof. — We estimate the right hand side of (3.2) by Lemma 3.1 as

$$\| f(\varphi) ; B^{\lambda}_{\bar{l},2} \| \leqslant C \| \varphi ; B^{\lambda}_{k,2} \| \, \| |\varphi|^{p-1} \|_s \tag{3.6}$$

with

$$n/s = \delta(l) + \delta(k) \tag{3.7}$$

and we estimate the last norm in (3.6) by the Hölder inequality as

$$\| |\varphi|^{p-1} \|_s \leqslant \| |\varphi|^{p-1-\eta v} \|_u \, \| \varphi \|_v^{\eta v} \tag{3.8}$$

with

$$1/s - 1/u = \eta . \tag{3.9}$$

Since $0 \leq \eta \leq \delta(l)/n$, the conditions (3.7) and (3.9) determine u with $1 \leq u \leq \infty$ for any $k \geq 2$ and $v \geq 2$. We next estimate the norms of φ in $B^{\lambda}_{k,2}$ and $L^{(p-1-\eta v)u}$ by interpolation between the norms in L^2, in H^1 and in $B^{\rho}_{r,2}$, and by using the Sobolev inequalities if necessary. The problem is best visualized in the (σ, ρ) plane (see Figure 3). The interpolation is possible provided

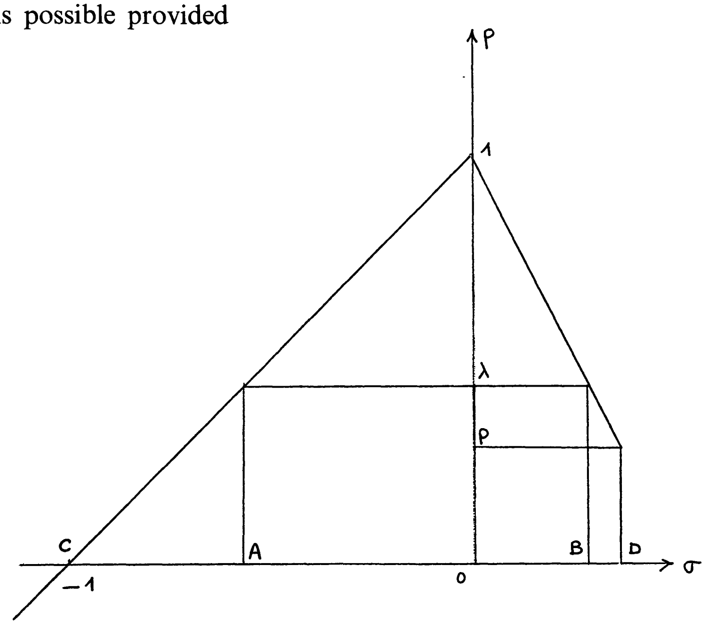

FIG. 3. — *The interpolation in Lemma 3.3. The allowed intervals for $\lambda + \delta(k) - 1$ and $\delta((p - 1 - \eta v)u) - 1$ are AB and CD respectively. The case shown corresponds to $\rho > 0$.*

$$0 \leqslant \delta(k) \leqslant \delta(r)(1 - \lambda)/(1 - \rho) \tag{3.10}$$

and

$$0 \leqslant \delta((p-1-\eta v)u) \leqslant 1 + \mathrm{Min}\left\{ \sigma, \frac{\sigma}{1-\rho} \right\} \equiv 1 + \frac{\sigma}{1+\rho_-} \tag{3.11}$$

or equivalently, after elimination of u through (3.7) and (3.9)

$$(p-1)\left(\frac{n}{2} - 1 - \frac{\sigma}{1+\rho_-}\right) - \delta(l) - \eta v\left(\delta(v) - 1 - \frac{\sigma}{1+\rho_-}\right) \leqslant \delta(k) \leqslant (p-1)n/2 - \delta(l) - \eta v \delta(v) . \tag{3.12}$$

The conditions (3.10) and (3.12) constrain $\delta(k)$ to lie in the intersection of two intervals, both of which are non empty under the assumptions made (in particular $p-1-\eta v \geq 0$). The interpolation is possible, namely the conditions (3.10) and (3.12) are compatible for k, under the conditions (3.3) and

$$(p-1)n/2 \geqslant \delta(l) + \eta v \delta(v). \tag{3.13}$$

When possible, the interpolation yields

$$\| f(\varphi); B^{\lambda}_{\bar{l},2} \| \leqslant C \| \varphi \|_2^{p-\mu-\nu-\eta v} \| \varphi ; H^1 \|^{\mu} \| \varphi ; B^{\rho}_{r,2} \|^{\nu} \| \varphi \|_v^{\eta v} \tag{3.14}$$

where $0 \leq \nu \leq \nu + \mu \leq p - \eta v$ and μ, v are related by the homogeneity condition

$$\frac{n}{\bar{l}} - \lambda = (p - \nu - \eta v)\frac{n}{2} - \mu + \nu\left(\frac{n}{r} - \rho\right) + \eta \tag{3.15}$$

or equivalently

$$(p-1)\frac{n}{2} = \delta(l) - \lambda + \mu + \nu(\delta(r) + \rho) + \eta v \delta(v). \tag{3.16}$$

The minimum value of ν is obtained as follows. Since u is a decreasing function of k (for fixed l, v and η) by (3.7), (3.9), one can always arrange for $\lambda + \delta(k) - 1$ and $\delta((p-1-\eta v)u) - 1$ to have the same sign. If both are negative, one can estimate the norms of φ in $B^{\lambda}_{k,2}$ and in $L^{(p-1-\eta v)u}$, modulo Sobolev inequalities, in terms of the norms of φ in L^2 and H^1, thereby obtaining $\nu = 0$. If both are positive, one can estimate the norms of φ in $B^{\lambda}_{k,2}$ and in $L^{(p-1-\eta v)u}$ in terms of the norms of φ in H^1 and in $B^{\rho}_{r,2}$, thereby obtaining $\mu + \nu = p - \eta v$. Substitution of that relation into (3.16) yields the lower bound (3.4) for ν.

The maximum value of ν is obtained by using the Sobolev inequalities as little as possible, or equivalently by minimizing μ. If $\rho \leq 0$, this is achieved by performing a barycentric decomposition on the degree of derivation, namely by taking $\lambda = \mu + \rho\nu$, thereby obtaining the upper bound (3.5) on ν in that case (the result is then independent of the specific choice of k). If $\rho > 0$, this is achieved by taking k as large as possible and s (or u) as small as possible. If that procedure leads to $\delta((p-1-\eta v)u) = 0$, one performs the same barycentric decomposition as previously. If not, one interpolates the norm in $L^{(p-1-\eta v)u}$ between the norms in L^2 and in $B^{\rho}_{r,2}$ (modulo Sobolev inequalities), and the norm in $B^{\lambda}_{k,2}$ between the norms in H^1 and in $B^{\rho}_{r,2}$ (without using Sobolev inequalities), thereby obtaining $\lambda = \mu + \rho(1-\mu)$. Substitution of that relation into (3.16) yields the upper bound (3.5) for ν when $\rho > 0$. Finally, the lower interpolation condition (3.13) can be omitted, since it is weaker than (3.5) with $\nu \geq 0$.

Q. E. D.

In the study of the boundedness properties of the solutions of the NLKG equation, we shall need the following assumption on the interaction f.

(H1) $f \in \mathscr{C}^1(\mathbb{C}, \mathbb{C})$, $f(0) = 0$ and for all $z \in \mathbb{C}$,

$$\text{Max}\left\{ \left| \frac{\partial f}{\partial z} \right|, \left| \frac{\partial f}{\partial \bar{z}} \right| \right\} \equiv | f'(z) | \leqslant C(| z |^{p_1 - 1} + | z |^{p_2 - 1}) \qquad (3.17)$$

with

$$0 < p_1 - 1 \leqslant p_2 - 1 < 4/(n - 2). \qquad (3.18)$$

We need one more estimate, obtained by combining Lemma 3.2 with estimates on the operator $K_1(t) = \omega_1^{-1} \sin \omega_1 t$ (see Section 2).

LEMMA 3.3. — Let $n \geq 3$. Let f satisfies (H1) with $p_1 - 1 > 2/n$. Let r, ρ and σ satisfy (2.11) with $\sigma \geq 0$, let $\gamma(r) < 1$, $\gamma(r)$ sufficiently close to 1. If $n = 3$, let in addition

$$2\sigma < \text{Min}\,(p_1 - 1, \gamma(r)). \qquad (3.19)$$

Then, there exist $\delta_<$ and $\delta_>$ with $0 \leq \delta_< < 1 < \delta_> (\leq \delta(r))$ and there exists $\nu (0 \leq \nu < 1)$ such that for any $\varphi \in H^1 \cap B^\rho_{r,2}$ and any $t \neq 0$, $K_1(t) f(\varphi) \in B^\rho_{r,2}$ and the following estimate holds

$$\| K_1(t) f(\varphi) ; B^\rho_{r,2} \| \leqslant M(\| \varphi ; H^1 \|)(\underset{\lessgtr}{\text{Min}}\, | t |^{-\delta_\lessgtr})(1 + \| \varphi ; B^\rho_{r,2} \|^\nu) \quad (3.20)$$

where M depends only on the norm of φ in H^1.

Proof. — For definiteness, we assume that $\gamma(r) = 1 - \varepsilon$ with $\varepsilon > 0$, ε sufficiently small, and we derive the estimate (3.20) with $\delta_\gtrless \gtrless 1 \pm \varepsilon$ and $\nu \leq 1 - \varepsilon$. By (3.17), we can decompose f as $f = f_1 + f_2$ with $| f_j'(z) | \leq C | z |^{p_j - 1}$, $j = 1, 2$ and we estimate separately the contributions of f_1 and f_2 to (3.20). For simplicity, we continue the proof with one single power p in f. From Lemma 2.1, elementary properties of the Besov spaces and the Sobolev inequalities, we obtain

$$\| K_1(t) f(\varphi) ; B^\rho_{r,2} \| \leqslant C | t |^{-\delta} \| f(\varphi) ; B^\lambda_{\bar{l},2} \| \qquad (3.21)$$

with $\delta \equiv \gamma(l) + \theta\alpha(l)$ and $\lambda \equiv \sigma + (1 + \theta)\alpha(l)$, under the conditions

$$0 \leqslant \theta \leqslant 1, \qquad 2 < l \leqslant r. \qquad (3.22)$$

We estimate the last norm in (3.21) by Lemma 3.2 with $\eta = 0$ (so that v drops out of the picture) and continue (3.21) as

$$\| K_1(t) f(\varphi) ; B^\rho_{r,2} \| \leqslant C | t |^{-\delta} M(\| \varphi ; H^1 \|) \| \varphi ; B^\rho_{r,2} \|^\nu \qquad (3.23)$$

under the conditions (3.22), (3.3), (3.4), (3.5) and

$$\lambda \equiv \sigma + (1 + \theta)\alpha(l) \leqslant 1. \qquad (3.24)$$

The other preliminary assumptions of Lemma 3.2 are trivially satisfied.

The values of l, θ and ν may depend both on p and on the choice of $>$ or $<$. We are then faced with the following problem: for each p satisfying

$$2/n < p - 1 < 4/(n - 2), \tag{3.25}$$

we have to make two choices of l, θ and ν, corresponding to $>$ and $<$, that ensure the conditions (3.22), (3.3), (3.4), (3.5), (3.24) and in addition $\delta_{\gtrless} \gtrless 1 \pm \varepsilon$ and $0 \leq \nu_{\gtrless} \leq 1 - \varepsilon$.

We now eliminate ν. With everything else held fixed, the conditions $0 \leq \nu \leq 1 - \varepsilon$, (3.4) and (3.5) constrain ν to lie in the intersection of two non empty intervals. Their compatibility reduces to the conditions obtained by crosscomparison of the endpoints of those intervals, namely

$$(p - 1)n/2 \geqslant \delta(l) \tag{3.26}$$

and

$$(p - 1)\left(\frac{n}{2} - 1\right) \leqslant 1 + \delta(l) - \lambda + (1 - \varepsilon)\sigma . \tag{3.27}$$

We next remark that for fixed p, if (l, θ) satisfy the conditions relevant for the case $>$, namely $\delta \geq 1 + \varepsilon$, (3.3), (3.22), (3.24), (3.26) and (3.27), then $(l, 0)$ satisfy the conditions relevant for the case $<$. In fact the condition $\delta_< \leq 1 - \varepsilon$ follows from $\delta_< = \gamma(l) \leq \gamma(r) = 1 - \varepsilon$, the conditions (3.22) and (3.26) are obvious, and the conditions (3.24) and (3.27) become weaker for fixed l when θ decreases. We are therefore left with the task of finding l and θ satisfying the conditions $\delta \geq 1 + \varepsilon$, (3.3), (3.22), (3.24), (3.26), and (3.27) or equivalently, after elimination of θ, of finding l and δ satisfying

$$1 + \varepsilon \leqslant \delta \leqslant \delta(l) \leqslant \delta(r), \tag{3.28}$$

$$\lambda = \sigma + \delta - (n - 2)\alpha(l) \leqslant 1 , \tag{3.29}$$

and the conditions (3.3), (3.26) and (3.27). (The condition $0 \leq \theta \leq 1$ is equivalent to $\gamma(l) \leq \delta \leq \delta(l)$ and follows from (3.28) and the fact that $\gamma(l) \leq \gamma(r) = 1 - \varepsilon$). Since the conditions (3.29), (3.3) and (3.27) involving λ take the form of upper bounds on λ and since λ is increasing in δ, the optimal choice for δ is $\delta = 1 + \varepsilon$, and it remains only to find l satisfying

$$1 + \varepsilon \leqslant \delta(l) \leqslant \delta(r) \tag{3.30}$$

$$\lambda - 1 = \sigma + \varepsilon - (n - 2)\alpha(l) \leqslant 0 \tag{3.31}$$

and the conditions (3.3), (3.26) and (3.27). Since $\sigma \leq \gamma(r)/2 = (1 - \varepsilon)/2$ and $\alpha(l) \geq (1 + \varepsilon)/n$ by (3.30), the inequality in (3.31) is relevant only for $n = 3$, where it reduces to

$$\sigma + \varepsilon \leqslant \alpha(l) \qquad (n = 3) \tag{3.32}$$

which is compatible with (3.30) provided $\sigma + \varepsilon \leq \delta(r)/3 = \gamma(r)/2$, namely provided $2\sigma < \gamma(r)$ and ε is sufficiently small.

In order to prove that for each p satisfying (3.25) there exists l satisfying (3.30), (3.32), (3.3), (3.26) and (3.27), it is sufficient to show that the range of values of p defined by (3.3), (3.26) and (3.27) covers the interval (3.25) when $\delta(l)$ varies in the interval defined by (3.30), (3.32). Now the conditions (3.3), (3.26) and (3.27) are obviously compatible for all $\delta(l)$ satisfying (3.30) and (3.32), and define an interval for p, the endpoints of which are affine increasing functions of $\delta(l)$. Therefore, the accessible values of p fill an interval, the upper (resp. lower) end of which is obtained by substituting in the upper conditions (3.3) and (3.27) for p (resp. in the lower condition (3.26) for p) the upper (resp. lower) value of $\delta(l)$ allowed by (3.30), (3.32). For the upper value $\delta(l) = \delta(r)$, (3.27) reduces to

$$(p - 1)(n/2 - 1) \leqslant 2 - \varepsilon(3 + \sigma) \tag{3.33}$$

which is satisfied under the condition (3.25) for ε sufficiently small, while (3.3) has been shown to hold for such a choice of l, r and p in the proof of Lemma 3.3 of [*10*]. For the lower values of $\delta(l)$, namely

$$\delta(l) = \operatorname{Max}(1 + \varepsilon, (\sigma + \varepsilon)n/(n - 2)),$$

the condition (3.26) becomes

$$(p - 1)n/2 \geqslant \operatorname{Max}(1 + \varepsilon, (\sigma + \varepsilon)n/(n - 2)), \tag{3.34}$$

with the second term in the maximum relevant only for $n = 3$ and $\sigma \geq 1/3$, and the condition (3.34) is satisfied under the condition (3.25) and in addition (3.19) for $n = 3$ provided ε is sufficiently small. Q. E. D.

We are now in a position to prove the basic uniform boundedness result for the solutions of the NLKG equation.

PROPOSITION 3.1. — Let $n \geq 3$. Let f satisfy (H1) with $p_1 - 1 > 2/n$. Let $\varphi \in L^\infty(\mathbb{R}, H^1)$ with $\dot{\varphi} \in L^\infty(\mathbb{R}, L^2)$ be a solution of the NLKG equation in $\mathscr{D}'(\mathbb{R}, H^{-1})$. Let r, ρ, σ and q satisfy (2.11) and (2.12). If $n = 3$, let in addition

$$2\sigma < \operatorname{Min}\{p_1 - 1, (p_1 - 1)\gamma(r), \gamma(r)\}. \tag{3.35}$$

Then $\varphi \in l^\infty(L^q, B^\rho_{r,2})$ and the norm of φ in that space is estimated in terms of the norm of $(\varphi, \dot{\varphi})$ in $L^\infty(\mathbb{R}, H^1 \oplus L^2)$.

Proof. — It is sufficient to prove the result for $\sigma \geq 0$ and $\gamma(r) < 1$, $\gamma(r)$ sufficiently close to 1. The result for $\sigma \geq 0$ and general r with $0 \leq \gamma(r) < 1$ follows by interpolation between that special case and uniform boundedness in H^1, while the result for $\sigma \geq 0$ and $\gamma(r) \geq 1$ follows from the special case and the Sobolev inequalities (Actually the condition (3.35) is obtained from (3.19) by the procedure just described). The result for $\sigma \leq 0$ follows from the result for $\sigma = 0$ and the embedding of the Besov spaces $B^\rho_{r,2}$ for fixed r. From now on we restrict our attention to the special case $\sigma \geq 0$ $\gamma(r) < 1$, $\gamma(r)$ sufficiently close to 1.

Under the assumptions made on f and φ, it has been shown in Lemma 2.1 of [10] that φ satisfies the integral equation

$$\varphi(t) = \dot{K}_1(t)\varphi_0 + K_1(t)\psi_0 - \int_0^t d\tau K_1(t-\tau)f(\varphi(\tau)) \qquad (3.36)$$

with $\varphi_0 = \varphi(0)$ and $\psi_0 = \dot{\varphi}(0)$. The integral in the right hand side of (3.36) can be regarded as a Bochner integral in L^2. Furthermore, we know from a slight variation of Lemma 3.3 of [10] that $\varphi \in L^q_{loc}(\mathbb{R}, B^\rho_{r,2})$ for r, ρ, q satisfying (2.11) and (2.12). Actually the proof of that fact is a simplified version of the present one, in so far as it does not require any integrability property at infinity of the integral in the right hand side of (3.36). We define

$$k_0(t) = \| \dot{K}_1(t)\varphi_0 + K_1(t)\psi_0 \,;\, B^\rho_{r,2} \| , \qquad (3.37)$$

$$k(t) = \| \varphi(t) \,;\, B^\rho_{r,2} \| . \qquad (3.38)$$

We restrict our attention to positive times. Taking the norm in $B^\rho_{r,2}$ of both members of (3.36) and estimating the integrand by Lemma 3.3 and more precisely by (3.20), we see that for $t \geq 0$, $k(t)$ satisfies the integral inequality

$$k \leqslant k_0 + \mu * (1 + k^\nu) \qquad (3.39)$$

with

$$\begin{aligned} \mu(t) &= M(\| \varphi \,;\, L^\infty(\mathbb{R}, H^1) \|) \operatorname*{Min}_{\gtrless} t^{-\delta_\lessgtr} && \text{for} \quad t > 0, \\ \mu(t) &= 0 && \text{for} \quad t \leq 0 . \end{aligned} \qquad (3.40)$$

The main points of that estimate are the integrability of μ in $\mathbb{R}$ and the fact that ν is strictly smaller than one. We now take $a > 0$, multiply both members of (3.29) by the characteristic function χ_a of the interval $[0, a]$, and take the norm in $l^\infty(L^q)$. Applying the Young inequality in the spaces $l^\cdot(L^\cdot)$ (see for instance Lemma 5.6 of [11]) and the inclusion $l^\infty(L^q) \subset l^\infty(L^{\nu q})$ for $\nu \leq 1$, we obtain

$$\| \chi_a k \,;\, l^\infty(L^q) \| \leqslant \| k_0 \,;\, l^\infty(L^q) \| + M(\| \varphi \,;\, L^\infty(\mathbb{R}, H^1) \|) \\ (1 + \| \chi_a k \,;\, l^\infty(L^q) \|^\nu) . \qquad (3.41)$$

The contribution of the free term is controlled by Proposition 2.2. Since $\nu < 1$, the left hand side of (3.41) is bounded uniformly with respect to a. This completes the proof. Q. E. D.

The result of Proposition 3.1 is especially relevant in the situation where energy conservation ensures that all solutions of the Cauchy problem for the NLKG equation with finite energy initial data are uniformly bounded in the energy space, defined below. We introduce the following assumption on f.

(H2). There exists a non negative function $V \in \mathscr{C}^1(\mathbb{C}, \mathbb{R})$ such that $V(0)=0$, $V(z) = V(|z|)$ for all $z \in \mathbb{C}$ and $f(z) = \partial V/\partial \bar{z}$.

The energy space is defined as $H^1 \oplus L^2$ and for any $(\varphi, \psi) \in H^1 \oplus L^2$, the energy is defined by

$$E(\varphi, \psi) = \| \psi \|_2^2 + \| \nabla\varphi \|_2^2 + \| \varphi \|_2^2 + \int dx V(\varphi(x)) . \qquad (3.42)$$

The assumption (H2) formally implies the conservation of the energy (with $\psi = \dot{\varphi}$) for the NLKG equation. Since in the present case the potential V is non negative, the norm in $H^1 \oplus L^2$ is controlled by the energy. Under the assumptions (H1) and (H2). it follows from Proposition 3.2 in [*10*] that for any $t_0 \in \mathbb{R}$, for any $(\varphi_0, \psi_0) \in H^1 \oplus L^2$, there exists a unique $\varphi \in L^\infty(\mathbb{R}, H^1)$ with $\dot{\varphi} \in L^\infty(\mathbb{R}, L^2)$ which solves the NLKG equation in $\mathscr{D}'(\mathbb{R}, H^{-1})$. Clearly for any $t_0 \in \mathbb{R}$, any $\varphi \in L^\infty(\mathbb{R}, H^1)$ with $\dot{\varphi} \in L^\infty(\mathbb{R}, L^2)$ which solves the NLKG equation can be regarded as the unique solution in $L^\infty(\mathbb{R}, H^1)$ of the Cauchy problem for that equation with initial data $\varphi(t_0)$, $\dot{\varphi}(t_0)$ at time t_0. From now on, such solutions will be called finite energy solutions. We can now state the final result of this section.

PROPOSITION 3.2. — Let $n \geq 3$. Let f satisfy (H1) with $p_1 - 1 > 2/n$ and (H2). Let r, ρ, σ and q satisfy (2.11) and (2.12) and in addition (3.35) for $n = 3$. Then any finite energy solution of the NLKG equation belongs to $l^\infty(L^q, B^\rho_{r,2})$ and its norm in that space is estimated in terms of its (conserved) energy.

4. TIME DECAY OF THE SOLUTIONS OF THE NLKG EQUATION

In this section we prove the main result of this paper, namely the fact that for a class of repulsive interactions, all finite energy solutions of the NLKG equation are dispersive in the sense that they satisfy some of the space-time integrability properties previously found for the solutions of the free equation (see Proposition 2.1 and 2.2). The proof relies on the Morawetz-Strauss estimate [*15*] which is directly related to the approximate dilation invariance of the equation (see Lemma 4.3) and on the finiteness of the propagation speed for the NLKG equation (see Lemma 4.2). Combining those two estimates one proves that suitable Besov norms of finite energy solutions of the NLKG equation are arbitrarily small in arbitrarily large time intervals (see Lemma 4.5). That property is exploited through the integral equation (3.36) and for that purpose one needs some additional estimates on the integrand in that equation (see Lemma 4.6). With those estimates available the proof follows step by step the corresponding proof for the NLS equation, given in [*11*]. One first proves that the previous Besov norms of finite energy solutions tend to zero at infinity in time (see Lemma 4.7) and then that they possess the appropriate time decay. The final results are collected in Proposition 4.1 and 4.2.

As a preparation for the proof of the basic estimates we first need to approximate an arbitrary finite energy solution by smooth solutions of a regularized equation. For that purpose we choose an even non negative function $h_1 \in \mathscr{C}^\infty(\mathbb{R}^n)$ with compact support and such that $\| h_1 \|_1 = 1$. For any positive interger j, we define $h_j(x) = j^n h_1(jx)$,

$$f_j(\varphi) = h_j * f(h_j * \varphi)$$

and correspondingly

$$E_j(\varphi, \psi) = \| \psi \|_2^2 + \| \nabla\varphi \|_2^2 + \| \varphi \|_2^2 + \int dx V(h_j * \varphi), \qquad (4.1)$$

where $*$ denotes convolution in $\mathbb{R}^n$. We consider the regularized equation

$$\varphi(t) = \dot{K}_1(t-t_0)(h_j * \varphi_0) + K_1(t-t_0)(h_j * \psi_0) - \int_{t_0}^{t} d\tau K_1(t-\tau) f_j(\varphi(\tau)). \qquad (4.2)$$

The approximation result can be stated as follows.

LEMMA 4.1. — Let f satisfy (H1) and (H2), let $t_0 \in \mathbb{R}$ and let $(\varphi_0, \psi_0) \in H^1 \oplus L^2$. Then

(1) For all $j \in \mathbb{Z}^+$, the equation (4.2) has a unique solution $\varphi_j \in \mathscr{C}(\mathbb{R}, H^1)$. Furthermore $(\varphi_j, \dot{\varphi}_j) \in \mathscr{C}^1(\mathbb{R}, H^{k+1} \oplus H^k)$ for any positive integer k, and φ_j satisfies the differential equation

$$\ddot{\varphi}_j - \Delta\varphi_j + \varphi_j + f_j(\varphi_j) = 0 \qquad (4.3)$$

in H^{k-1}. In addition φ_j satisfies the conservation of energy

$$E_j(\varphi_j(t), \dot{\varphi}_j(t)) = E_j(h_j * \varphi_0, h_j * \psi_0)$$

and $(\varphi_j(t), \dot{\varphi}_j(t))$ is bounded in $H^1 \oplus L^2$ uniformly with respect to j and t.

(2) Let φ be the finite energy solution of the NLKG equation with $\varphi(t_0) = \varphi_0$ and $\dot{\varphi}(t_0) = \psi_0$. Then for any compact interval I and any r, $2 \le r < 2^*$, φ_j converges to φ in $\mathscr{C}(I, L^r)$ when j tends to infinity. For any $t \in \mathbb{R}$, $(\varphi_j(t), \dot{\varphi}_j(t))$ converges to $(\varphi(t), \dot{\varphi}(t))$ strongly in $H^1 \oplus L^2$ when j tends to infinity.

The proof of Lemma 4.1 is the same as that of the analogous result contained in the proof of Proposition 3.2 of [*10*] (where however the mass term, if any, is not included in the free evolution), except for the proof of pointwise strong convergence in $H^1 \oplus L^2$, which follows from weak convergence and the conservation of energy.

We are now in a position to prove the finiteness of the propagation speed in the form of local energy conservation. For any open ball $\Omega = B(x, R)$ of center x and radius R in $\mathbb{R}^n$, for any $t \in \mathbb{R}$, we define $\Omega_\pm(t) = B(x, \mathbb{R} \pm |t|)$, with the convention that $B(x, R)$ is empty if $R \leq 0$. For any measurable set $\Omega \subset \mathbb{R}^n$, for any $(\varphi, \psi) \in H^1 \oplus L^2$, we define

$$E(\varphi, \psi ; \Omega) = \int_\Omega dx(|\psi|^2 + |\nabla\varphi|^2 + |\varphi|^2 + V(\varphi)). \tag{4.4}$$

LEMMA 4.2. — Let f satisfy (H1) and (H2). Let φ be a finite energy solution of the NLKG equation. Then for any open ball $\Omega \subset \mathbb{R}^n$, for any $t \in \mathbb{R}$, the following inequalities hold:

$$E(\varphi(t), \dot\varphi(t) ; \Omega_-(t)) \leqslant E(\varphi(0) ; \dot\varphi(0) ; \Omega) \tag{4.5}$$

and

$$E(\varphi(t), \dot\varphi(t) ; \complement\Omega_+(t)) \leqslant E(\varphi(0), \dot\varphi(0) ; \complement\Omega), \tag{4.6}$$

where $\complement$ denotes the complement in $\mathbb{R}^n$.

Proof. — Without loss of generality, we can assume that $\Omega = B(0, R)$ and that t is positive. The formal proof of (4.5) proceeds as follows. Define

$$\Theta^0(t,x) = |\dot\varphi|^2 + |\nabla\varphi|^2 + |\varphi|^2 + V(\varphi), \tag{4.7}$$

$$\Theta(t,x) = -2\,\mathrm{Re}\,(\bar{\dot\varphi}\nabla\varphi). \tag{4.8}$$

Then

$$\dot\Theta^0 + \nabla.\Theta = 0. \tag{4.9}$$

Integrating (4.9) in the region

$$Q(\Omega, t) = \{ (t', x') \in \mathbb{R}^{n+1} : 0 \leqslant t' < t \quad \text{and} \quad x' \in \Omega_-(t') \}, \tag{4.10}$$

applying Gauss's theorem and taking into account the fact that the vector (θ^0, θ) in $\mathbb{R}^{n+1}$ is time-like and outgoing on the side surface of $Q(\Omega, t)$, one obtains (4.5).

In order to give an actual proof, we first approximate φ by the solution φ_j of the equation (4.2) as described in Lemma 4.1. We then define

$$\Theta_j^0(t,x) = |\dot\varphi_j|^2 + |\nabla\varphi_j|^2 + |\varphi_j|^2 + V(h_j * \varphi_j), \tag{4.11}$$

$$\Theta_j(t,x) = -2\,\mathrm{Re}\,(\bar{\dot\varphi}_j\nabla\varphi_j). \tag{4.12}$$

It follows then from the differential equation (4.3) that

$$\dot\Theta_j^0 + \nabla.\Theta_j = P_j \equiv 2\,\mathrm{Re}\,\{ (h_j * \bar{\dot\varphi}_j) f(h_j * \varphi_j) - \bar{\dot\varphi}_j(h_j * f(h_j * \varphi_j)) \}. \tag{4.13}$$

We now choose a function $m \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ with $m(s) = 0$ for $s \leq 0$, $m(s) = 1$ for $s \geq 1$ and $0 \leq m'(s) \leq 2$, and we define $m_\varepsilon(s) = m(s/\varepsilon)$. From (4.13), we obtain

$$\frac{d}{d\tau}\int dx m_\varepsilon(R - \tau - |x|)\theta_j^0(\tau, x) = -\int dx m'_\varepsilon(R - \tau - |x|) \times (\Theta_j^0(\tau, x) + \hat{x}.\Theta_j(\tau, x)) + \int dx m_\varepsilon(R - \tau - |x|)P_j(\tau, x) \tag{4.14}$$

where $\hat{x} = x/|x|$. Since the vector (θ_j^0, θ_j) in $\mathbb{R}^{n+1}$ is time like with $\theta_j^0 \geq 0$, the first term in the right hand side of (4.14) is negative. Integration of (4.14) between 0 and t then yields

$$\int dx m_\varepsilon(\mathrm{R} - t - |x|)\Theta_j^0(t, x) \leqslant \int dx m_\varepsilon(\mathrm{R} - |x|)\Theta_j^0(0, x) + \int_0^t d\tau \int dx m_\varepsilon(\mathrm{R} - \tau - |x|)\mathrm{P}_j(\tau, x). \quad (4.15)$$

We next let j tend to infinity and ε tend to zero in that order. By Lemma 4.1, part (2), the left hand side and the first term in the right hand side of (4.15) tend to $\mathrm{E}(\varphi(t), \dot{\varphi}(t); \Omega_-(t))$ and $\mathrm{E}(\varphi(0), \dot{\varphi}(0); \Omega)$ respectively. The remainder term in (4.15) can be rewritten as

$$2 \operatorname{Re} \int_0^t d\tau \int dx dy \bar{\dot{\varphi}}_j(\tau, x) h_j(x - y) [f(h_j * \varphi_j)](\tau, y) \times \{ m_\varepsilon(\mathrm{R} - |y| - \tau) - m_\varepsilon(\mathrm{R} - |x| - \tau) \}$$

and is estimated by

$$|\,.\,| \leqslant 4\varepsilon^{-1} \int_0^t d\tau \int dx dy \, |\dot{\varphi}_j(\tau, x)| \, |x - y| \, h_j(x - y) \, |f(h_j * \varphi_j)(\tau, y)| \, .$$

We continue the estimate by using (H1), restricting our attention to the case of a single power $p_1 = p_2 = p$ for brevity. Using the Hölder and Young inequalities, we obtain

$$|\,.\,| \leqslant Ct\varepsilon^{-1} j^{-1+n/2-n/(p+1)} \operatorname*{Sup}_{\tau \in [0,t]} \{ \| \dot{\varphi}_j(\tau) \|_2 \, \| \varphi_j(\tau) \|_{p+1}^p \} \, .$$

Both norms in the right hand side are uniformly bounded in j and τ because of (H2) and energy conservation, so that the right hand side tends to zero when j tends to infinity for fixed ε and t. This completes the proof of (4.5).

The inequality (4.6) follows immediatly from (4.5), from the conservation of energy and the reversibility in time of the NLKG equation.

Q. E. D.

The local conservation of the energy has been stated between the times 0 and t. Since the equation is time translation invariant, the same property holds between any two times.

We now recall the basic decay estimate [*15*]. We define

$$\mathrm{W}_1(z) \equiv \bar{z} f(z) - \mathrm{V}(z) \, . \quad (4.16)$$

Under the assumption (H1), the map $\varphi \mapsto \mathrm{W}_1(\varphi)$ is bounded and Lipschitz continuous from $\mathrm{L}^{p_1+1} \cap \mathrm{L}^{p_2+1}$ to L^1. We also introduce the function $g(x) = (1 + |x|^2)^{-1/2}$, we define $g'(x) = (d/d|x|)(1 + |x|^2)^{-1/2}$ and $g_1(x) = ng(x) + |x| g'(x)$. The function g is positive and decreasing in $|x|$, the function $|x| g(x)$ is increasing and bounded. As a consequence,

$(n-1)g \leq g_1 \leq ng$. Furthermore, an elementary computation shows that $\Delta g_1 \leq 0$ for $n \geq 3$.

LEMMA 4.3. — Let $n \geq 3$. Let f satisfy (H1) and (H2). Let φ be a finite energy solution of the NLKG equation. Then for any s and t in $\mathbb{R}$, $s \leq t$, φ satisfies the inequality

$$\int_s^t d\tau \int dx g_1(x) W_1(\varphi(\tau, x)) \leqslant$$
$$\leqslant \operatorname{Re} \{ \langle \dot{\varphi}(s), (xg.\nabla + \nabla.xg)\varphi(s) \rangle - \langle \dot{\varphi}(t), (xg.\nabla + \nabla.xg)\varphi(t) \rangle \}. \quad (4.17)$$

Proof. — The proof is almost the same for the NLKG and the NLS equation. One first derives the corresponding result for the solution φ_j of the regularized equation (4.2) given by Lemma 4.1 and one then takes the limit $j \to \infty$. Using the fact that $\varphi_j \in \mathscr{C}^1(\mathbb{R}, H^k)$, the differential equation (4.3) and the boundedness of $|x| g(x)$, one computes the time derivative

$$\frac{d}{dt} \operatorname{Re} \langle \dot{\varphi}_j, (xg.\nabla + \nabla.xg), \varphi_j \rangle = \operatorname{Re} \langle \varphi_j, [\Delta, xg].\nabla\varphi_j \rangle$$
$$- \operatorname{Re} \langle f_j(\varphi_j), (xg.\nabla + \nabla.xg)\varphi_j \rangle \quad (4.18)$$

(compare with (5.18) in [*11*]). From there one, the proof is the same as that of the corresponding result for the NLS equation (see Lemma 5.2 in [*11*]). Q. E. D.

In the same way as in [*15*] and [*11*], the estimate of Lemma 4.3 will be used through its following consequence.

LEMMA 4.4. — Let $n \geq 3$. Let f and φ be as in Lemma 4.3 and assume in addition that $W_1 \geq 0$. Then for any $\varepsilon > 0$, $a_0 > 0$ and $l_0 > 0$, there exists $b_0 > 0$, depending only on ε, a_0, l_0 and on the energy E of φ, and there exists c such that $a_0 \leq c \leq b_0 - l_0$ and

$$\int_c^{c+l_0} d\tau \int_{|x| \leqslant 2\tau} dx W_1(\varphi(\tau, x)) \leqslant \varepsilon. \quad (4.19)$$

One can take

$$b_0 = (a_0 + l_0 + 1) \exp \{ 4El_0\varepsilon^{-1}(n-1)^{-1} \}. \quad (4.20)$$

Sketch of proof. — For $W_1 \geq 0$, (4.17) implies

$$\int_s^t d\tau \int dx g(x) W_1(\varphi(\tau, x)) \leqslant 2E(n-1)^{-1} \quad (4.21)$$

by an elementary computation. From there on, the proof is the same as that of Lemma 5.3 of [*11*], with $\psi(\tau)$ in that Lemma replaced by 2τ.

Q. E. D.

The next step in the argument consists in proving that an arbitrary finite energy solution of the NLKG equation is arbitrarily small in arbitrarily large intervals of time provided these intervals are sufficiently far. That result holds both for positive and negative times. For simplicity we state it only for positive times. It requires a repulsivity property of the interaction term f, which we state as follows.

(H3). There exists $C > 0$ and p_4, p_5 with $1 \leq p_4 \leq p_5 < \infty$ such that for all $\rho \in \mathbb{R}^+$

$$W_1(\rho) \geqslant C \operatorname{Min}(\rho^{p_4+1}, \rho^{p_5+1}). \tag{4.22}$$

LEMMA 4.5. — Let $n \geq 3$, let f satisfy (H1) with $p_1 - 1 > 2/n$, (H2) and (H3). Let r, ρ, σ and q satisfy $\sigma > 0$, the conditions (2.11) and (2.12), and in addition (3.35) if $n = 3$. Let φ be a finite energy solution of the NLKG equation. Then for any $\varepsilon > 0$ and any $l > 0$, there exists $a > 0$ such that

$$\| \varphi ; l^\infty(L^q, [a, a + l], B^\rho_{r,2}) \| \leqslant \varepsilon . \tag{4.23}$$

Proof. — As in the case of Proposition 3.1, it is sufficient to prove the result for $\gamma(r) < 1$, $\gamma(r)$ close to 1. The result for general r with $0 < \gamma(r) < 1$ then follows by interpolation between that special case and uniform boundedness in H^1, while the result for $\gamma(r) \geq 1$ follows from the special case by the Sobolev inequalities. From now on, we restrict our attention to the case $\gamma(r) < 1$, $\gamma(r)$ close to 1.

The proof is similar to that of Lemma 5.9 of [*11*]. We estimate φ by using the integral equation (3.36). We define $k_0(t)$ and $k(t)$ by (3.37), (3.38). Since $k_0 \in l^m(L^q)$ by Proposition 2.2, we can choose a_1 such that

$$\| k_0 ; l^\infty(L^q, [a_1, \infty)) \| \leqslant \varepsilon/4 . \tag{4.24}$$

In order to estimate the integral in the right hand side of (3.36), we split the region of integration over τ in the three subregions

$$I_1 = [t - \theta_1, t] \cap \mathbb{R}^+, \qquad I_2 = [t - \theta_2, t - \theta_1] \cap \mathbb{R}^+$$

and

$$I_3 = [0, t - \theta_2] \cap \mathbb{R}^+$$

for some (small) θ_1, $0 < \theta_1 \leq 1$, and some (large) θ_2, to be chosen below. We define, for $i = 1, 2, 3$,

$$k_i(t) = \left\| \int_{I_i} d\tau K_1(t - \tau) f(\varphi(\tau)) ; B^\rho_{r,2} \right\| \tag{4.25}$$

and we estimate successively those three functions of t in $l^\infty(L^q)$. In the intervals I_1 and I_3, namely for k_1 and k_3, we use the estimate (3.20) of Lemma 3.3 and the Young inequality for the spaces $l^\cdot(L^\cdot)$ to obtain

$$\| k_1 ; l^\infty(L^q) \| \leqslant M\theta_1^{1-\delta_<}(1 + \| k ; l^\infty(L^q) \|^\nu) \tag{4.26}$$

and

$$\| k_2 ; l^\infty(L^q) \| \leqslant M\theta_2^{1-\delta_>}(1 + \| k ; l^\infty(L^q) \|^\nu) \tag{4.27}$$

with M depending only on the energy E of φ. Since the last norm in (4.26) and (4.27) is also estimated in terms of E by Proposition 3.2, we can choose θ_1 sufficiently small and θ_2 sufficiently large, depending only on ε and E, to ensure that

$$\| k_1 + k_3 ; l^\infty(L^q) \| \leqslant \varepsilon/4 . \tag{4.28}$$

In the interval I_2, namely for k_2, we estimate the integrand in a slightly different way, using Lemma 3.2 with η positive and small. We decompose f as $f_1 + f_2$ as in the proof of Lemma 3.3 and estimate the contribution of each term separately. We continue the proof in the case of a single power for simplicity. Using Lemma 2.1 with $\theta = 0$ and Lemma 3.1, we obtain with $\varphi = \varphi(\tau)$

$$\begin{aligned} \| K_1(t-\tau)f(\varphi) ; B^\rho_{r,2} \| &\leqslant C \, | t - \tau |^{-\gamma(l)} \| f(\varphi) ; B^\lambda_{\bar{l},2} \| \\ &\leqslant C \, | t - \tau |^{-\gamma(l)} \| \varphi ; B^\lambda_{k,2} \| \| \, | \varphi |^{p-1} \|_s \end{aligned} \tag{4.29}$$

under the conditions $2 \leq l \leq r$ and (3.7), and with $\lambda = \sigma + \alpha(l)$, so that the condition $\lambda \leq 1$ is automatically fulfilled. We next split $\varphi = \varphi_3 + \varphi_4 + \varphi_5$, where

$$\begin{aligned} \varphi_3(\tau, x) &= \varphi(\tau, x) \quad \text{for } | x | > 2\tau \\ &= 0 \qquad \text{otherwise,} \\ \varphi_4(\tau, x) &= \varphi(\tau, x) \quad \text{for } \; | x | \leq 2\tau \;\text{ and }\; | \varphi(\tau, x) | > 1 \\ &= 0 \qquad \text{otherwise} \\ \varphi_5(\tau, x) &= \varphi(\tau, x) \quad \text{for } \; | x | \leq 2\tau \;\text{ and }\; | \varphi(\tau, x) | \leq 1 \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

We estimate separately the contributions of φ_i, $i = 3, 4, 5$, to the last norm in (4.29) by Hölder's inequality in the form (3.8) with the same value of η (for convenience) but with different values v_i, $i = 3, 4, 5$, for v, namely $v_3 = 2$, $v_4 = p_4 + 1$ and $v_5 = p_5 + 1$. The interpolation conditions (3.10) and (3.12) have now to be satisfied (for fixed p and l) by the same k for all three values of v. Those conditions are compatible in k provided the conditions (3.3) and (3.13) hold for all three values of v, and provided in addition the conditions (3.12) on k for various values of v are compatible. The latter requirement is easily seen to be satisfied for η sufficiently small, a sufficient condition being

$$\eta(2 + v_5\delta(v_5)) \leqslant 2/n \tag{4.30}$$

which from now on we assume to hold. Continuing the estimate as in Lemma 3.2, we obtain

$$\begin{aligned} \| K_1(t - \tau)f(\varphi) ; B^\rho_{r,2} \| &\leqslant | t - \tau |^{-\gamma(l)} M(\| \varphi ; H^1 \|) \\ &\quad \times \sum_{3 \leqslant i \leqslant 5} \| \varphi ; B^\rho_{r,2} \|^{v_i} \| \varphi_i \|^{\eta v_i}_{v_i} \end{aligned} \tag{4.31}$$

under the conditions $v_i \geq 0$ and (3.3), (3.4) and (3.5) for v_i, v_i, $i = 3, 4, 5$. Note that contrary to l and k, v_i may depend on i. We next eliminate v as in the proof of Lemma 3.3, but requiring only that $0 \leq v_i \leq 1$, thereby obtaining from (4.31)

$$\| K_1(t-\tau)f(\varphi) ; B^{\rho}_{r,2} \| \leqslant | t - \tau |^{-\gamma(l)} M(\| \varphi ; H^1 \|) \times (1 + \| \varphi ; B^{\rho}_{r,2} \|) \sum_{3 \leqslant i \leqslant 5} \| \varphi_i \|^{\eta v_i}_{v_i} \quad (4.32)$$

provided for $v = v_i$, $i = 3, 4, 5$, there hold the conditions (3.3), (3.13) and

$$(p-1)(n/2-1) \leqslant 1 + \gamma(l) + \eta v(\delta(v) - 1). \quad (4.33)$$

For any p satisfying (3.25) and for $\gamma(r)$ sufficiently close to 1, one sees easily (see the proof of Lemma 3.3) that those conditions can be satisfied for a suitable choice of $l \in [2, r]$ and for $v = v_i$, $i = 3, 4, 5$, provided η is sufficiently small.

By the assumption (H3) and the definition of φ_4, φ_5, the last sum in (4.32) is estimated by

$$\sum_{3 \leqslant i \leqslant 5} \| \varphi_i \|^{\eta v_i}_{v_i} \leqslant \| \varphi_3 \|_2^{2\eta} + C \left\{ \int_{|x| \leqslant 2\tau} dx W_1(\varphi) \right\}^{\eta}. \quad (4.34)$$

We now substitute (4.32) and (4.34) into the definition of k_2 and estimate the time integral by Hölder's inequality in $l^{\cdot}(L^{\cdot})$ spaces. We obtain

$$k_2(t) \leqslant N \left\{ \left\{ \int_{t-\theta_2}^{t} d\tau \| \varphi_3(\tau) \|_2^2 \right\}^{\eta} + \left\{ \int_{t-\theta_2}^{t} d\tau \int_{|x| \leqslant 2\tau} dx W_1(\varphi(\tau)) \right\}^{\eta} \right\} \quad (4.35)$$

where

$$N = \| | . |^{-\gamma(l)} ; l^{(1-\eta)^{-1}}(L^{(1-\eta-\sigma)^{-1}}, [\theta_1, \theta_2]) \| M(\| \varphi ; L^{\infty}(\mathbb{R}, H^1) \|) \times (1 + \| \varphi ; l^{\infty}(L^q, B^{\rho}_{r,2}) \|). \quad (4.36)$$

The first norm in (4.36) is estimated in terms of θ_1 and θ_2, while the next two factors are estimated in terms of the energy of φ by Proposition 3.2. We next estimate the time integrals in (4.35) by Lemma 4.2 and Lemma 4.4 respectively. In fact, it follows from Lemma 4.2 that for each τ

$$\| \varphi_3(\tau) \|_2^2 = \int_{|x| \geqslant 2\tau} dx \, | \varphi(\tau, x) |^2 \leqslant E(\varphi(\tau_{\tau}, \dot{\varphi}(\tau) ; \complement B(0, 2\tau)) \leqslant E(\varphi(0), \dot{\varphi}(0) ; \complement B(0, \tau)) \quad (4.37)$$

so that

$$\int_{t-\theta_2}^{t} d\tau \| \varphi_3(\tau) \|_2^2 \leqslant \theta_2 E(\varphi(0), \dot{\varphi}(0) ; \complement B(0, t - \theta_2)). \quad (4.38)$$

For fixed θ_2, the right hand side of (4.38) tends to zero when t tends to infinity, and the contribution of φ_3 to k_2 can therefore be made $\leq \varepsilon/4$ by taking t sufficiently large, say $t \geq a_2$ ($> \theta_2$). The second integral in (4.35) is estimated by Lemma 4.4, which we apply with $a_0 = \text{Max}(a_1, a_2)$, $l_0 = l + \theta_2$ and ε replaced by $(\varepsilon/4\,\text{N})^{1/n}$ to conclude that there exists $c \geq a_0$ such that the contribution of the second integral in (4.35) to k_2 is smaller than $\varepsilon/4$ for $t \in [a, b]$ with $a = c + \theta_2$ and $b = a + l$, so that

$$\| k_2 ; \mathrm{L}^\infty([a, b]) \| \leqslant \varepsilon/2 . \tag{4.39}$$

Collecting (4.24), (4.28) and (4.39) yields (4.23). Q. E. D.

By interpolation with uniform boundedness of the solutions in H^1, the result of Lemma 4.5 can be extended to other norms corresponding to different values of r, ρ and q with $\sigma \leq 0$. We refrain from a formal statement of this extension, which will not be used in the sequel.

The boundedness and decay results derived so far in Proposition 3.2 and Lemma 4.5 required only (besides repulsivity) the weak assumption (3.25) on p_1 and p_2. In order to proceed further, it is necessary that some norm of an arbitrary finite energy solution satisfy a superlinear integral equation in addition to the sublinear equation used previously (see (3.39)), and for that purpose, stronger assumptions on (p_1, p_2) are needed. The basic estimate, which complements Lemma 3.3, can be stated as follows.

LEMMA 4.6. — Let $n \geq 3$. Let f satisfy (H1). Let r, ρ, and σ satisfy the conditions (cf. (2.11))

$$\begin{cases} \gamma(r) < 1 < \delta(r) \\ 0 < \sigma \equiv \rho + \delta(r) - 1 \leqslant \gamma(r)/2 . \end{cases} \tag{4.40}$$

Assume that

$$(p_2 - 1)\left(\frac{n}{2} - 1 - \frac{\sigma}{1 + \rho_-}\right) < 2\delta(r)\left(1 - \frac{\delta(r)}{n(1 - \rho)}\right), \tag{4.41}$$

$$(p_1 - 1)n/2 > 1 + \delta(r) + (\rho_+/(1 - \rho))(\delta(r) - 1 + 2/n) . \tag{4.42}$$

In addition:

If $n = 3$, assume that $2\sigma < \gamma(r)$ and

$$(p_1 - 1)3/2 > \delta(r) + \rho_+ + 3\sigma . \tag{4.43}$$

If $n \geq 8$ and

$$\delta(r)(n/2 - 1) + 2\rho(n - 1)/n \geqslant n/2 , \tag{4.44}$$

assume that

$$(p_1 - 1)(n^2/(n - 1) - 4\rho) > 4\delta(r) . \tag{4.45}$$

Then there exist $\delta_<$ and $\delta_>$ with $0 \leq \delta_< < 1 < \delta_>$ ($\leq \delta(r)$) and there exist ν_1, $\nu_>$ and $\nu_<$ with

$$\begin{gathered} 1 < \nu_1 \leqslant \nu_> \leqslant \sigma^{-1} \\ \nu_1 \leqslant \nu_< < (1 - \delta_<)\sigma^{-1} + 1 \end{gathered} \tag{4.46}$$

such that for all $\varphi \in H^1 \cap B^\rho_{r,2}$ and any $t \neq 0$, the following estimate holds

$$\| K_1(t) f(\varphi) ; B^\rho_{r,2} \| \leqslant M(\| \varphi ; H^1 \|) \times \begin{cases} |t|^{-\delta_<} (\| \varphi ; B^\rho_{r,2} \|^{\nu_1} + \| \varphi ; B^\rho_{r,2} \|^{\nu_<}) \\ |t|^{-\delta_>} (\| \varphi ; B^\rho_{r,2} \|^{\nu_1} + \| \varphi ; B^\rho_{r,2} \|^{\nu_>}) . \end{cases} \quad (4.47)$$

Proof. — The proof follows closely that of Lemma 3.3. For definiteness we assume that $\gamma(r) \leq 1 - \varepsilon$ and $\delta(r) \geq 1 + \varepsilon$ with $\varepsilon > 0$, ε sufficiently small, and we derive the estimate (4.47) with $\delta_{\gtrless} \gtrless 1 \pm \varepsilon$. As in the proof of Lemma 3.3, we decompose f as $f_1 + f_2$ and estimate the contribution of f_1 and f_2 separately. We continue the proof with a single power p in f and in that case, we derive the estimate (4.47) with one single power $\nu_>$ or $\nu_<$ corresponding to $\delta_>$ or $\delta_<$. For definiteness, we replace (4.46) by

$$\begin{cases} 1 + \varepsilon \leqslant \nu_> \leqslant \sigma^{-1} \\ \varepsilon\sigma \leqslant (\nu_< - 1)\sigma \leqslant 1 - \delta_< - \varepsilon/2 . \end{cases} \quad (4.48)$$

From Lemma 2.1 and Lemma 3.2 with $\eta = 0$, we obtain

$$\| K_1(t) f(\varphi) ; B^\rho_{r,2} \| \leqslant |t|^{-\delta} M(\| \varphi ; H^1 \|) \| \varphi ; B^\rho_{r,2} \|^{\nu}$$

with $\delta \equiv \gamma(l) + \theta\alpha(l)$ under the conditions

$$\begin{cases} 0 \leqslant \theta \leqslant 1, \quad 1 \leqslant \delta(l) \leqslant \delta(r) \\ \lambda \equiv \sigma + (1 + \theta)\alpha(l) \leqslant 1 \end{cases} \quad (4.49)$$

and the conditions (3.3), (3.4), and (3.5). We then have to make two choices of l, θ and ν, corresponding to $>$ and $<$, that ensure in addition the conditions $\delta_{\gtrless} \gtrless 1 \pm \varepsilon$ and (4.48). We next eliminate ν. The conditions (3.4), (3.5) and (4.48) constrain $\nu_>$ and $\nu_<$ to lie in the intersection of two non empty intervals. They are compatible provided, in both cases $>$ and $<$,

$$(p-1)n/2 \geqslant \delta(l) + (1+\varepsilon)\delta(r) + \rho_+(\varepsilon + (\lambda - \rho)/(1-\rho)) \quad (4.50)$$

and in the case $>$,

$$(p - 1)(n/2 - 1) \leqslant 2 + \delta(l) - \lambda \quad (4.51)_>$$

while in the case $<$,

$$(p - 1)(n/2 - 1) \leqslant 2 - 2\theta\alpha(l) - \varepsilon/2 . \quad (4.51)_<$$

The condition $(4.51)_>$ is automatically fulfilled under the conditions (3.18) and (4.49) and need not be considered further. We next remark, as in the proof of Lemma 3.3, that if (l, θ) satisfy the conditions relevant for the case $>$, namely $\delta \geq 1 + \varepsilon$, (4.49), (3.3) and (4.50), then $(l, 0)$ satisfy the conditions relevant for the case $<$. In fact, the condition $\delta_< \leq 1 - \varepsilon$ follows from $\delta_< = \gamma(l) \leq \gamma(r) \leq 1 - \varepsilon$, the conditions $\lambda \leq 1$, (3.3) and (4.50) are satisfied because they become weaker when θ decreases for fixed l, and the condition $(4.51)_<$ for $\theta = 0$ is satisfied under the condition (3.18) for ε sufficiently small.

We are therefore left with the task of finding l and θ satisfying the conditions $\delta \geq 1 + \varepsilon$, (4.49), (3.3) and (4.50), or equivalently, after elimination of θ, of finding l and δ satisfying the conditions (3.28), (3.29), (3.3) and (4.50). As in the proof of Lemma 3.3, the optimal choice for δ is $\delta = 1 + \varepsilon$, and it remains only to find l satisfying (3.3), (4.50), (3.30) and (3.31), the latter condition being relevant only for $n = 3$, in which case it reduces to (3.32). In order to prove that for each p satisfying the assumptions of Lemma 4.6, there exists l satisfying (3.30), (3.32), (3.3) and (4.50), it is sufficient to show that the range of values of p defined by (3.3) and (4.50) covers the interval described in the Lemma when $\delta(l)$ varies in the interval defined by (3.30), (3.32). For fixed l, the conditions (3.3) and (4.50) define an interval for p, both endpoints of which are affine increasing functions of $\delta(l)$. That interval is non empty under the following condition, obtained after an elementary computation by eliminating p between (3.3) and (4.50):

$$\delta(l)(n/2 - 2\rho_+(n-1)/n) \geqslant (n/2 - 1)\delta(r) + \varepsilon[(n/2 - \rho_+)(2-\rho) - (1-\rho_+)\delta(r)]. \tag{4.52}$$

For $\delta(l) = \delta(r)$, that condition reduces to

$$\delta(r)(1 - 2\rho_+(n-1)/n) \geqslant \varepsilon[\,.\,] \tag{4.53}$$

and is always satisfied for ε sufficiently small, since $\rho \leq (n-1)/2n$ in the range (4.40). That fact has two consequences. First, the final allowed interval for p is non empty, namely the upper and lower conditions on p derived below are compatible. Second, the upper end of the allowed interval for p in the previous procedure is obtained by substituting the upper bound $\delta(l) = \delta(r)$ into the upper bound (3.3) for p, which yields

$$(p-1)\left(\frac{n}{2} - 1 - \frac{\sigma}{1+\rho_-}\right) \leqslant 2\delta(r)\left(1 - \frac{\alpha(r) + \varepsilon/2}{1-\rho}\right). \tag{4.54}$$

That condition is satisfied under the assumption (4.41) provided ε is sufficiently small.

As regards the lower end of the allowed interval for p, two cases can occur. If $\delta(l) = 1 + \varepsilon$ satisfies (4.52), namely if

$$(n/2 - 1)\delta(r) + 2\rho_+(n-1)/n \leqslant n/2 - \varepsilon\{\,.\,\}, \tag{4.55}$$

then the interval (3.3)-(4.50) for p is non empty for $\delta(l) = 1 + \varepsilon$, and the lower end of the interval for p is obtained by substituting the lower bound $\delta(l) = 1 + \varepsilon$ into the lower bound (4.50) for p, which yields

$$(p-1)n/2 \geqslant (1+\varepsilon)(1+\delta(r)) + [\rho_+/(1-\rho)][\delta(r) - 1 + 2/n + \varepsilon(1-\rho+2/n)] \tag{4.56}$$

and in addition, if $n = 3$, by substituting $\delta(l) = 3(\sigma + \varepsilon)$ into (4.50), which yields

$$(p - 1)3/2 \geqslant 3(\sigma + \varepsilon) + (1 + \varepsilon)(\delta(r) + \rho_+). \tag{4.57}$$

The conditions (4.56) and (4.57) are satisfied under the assumptions (4.42) and (4.43) provided ε is sufficiently small. If $\delta(l) = 1 + \varepsilon$ does not satisfy (4.52), then the lowest admissible value of $\delta(l)$ is given by equality in (4.52), and the lower end of the interval for p is obtained by equality in (3.3) and (4.50), or equivalently by eliminating $\delta(l)$ between those conditions, which yields

$$(p - 1)[n^2/2 - 2(n - 1)\rho_+] \geqslant 2(n-1)\delta(r) + \varepsilon[n(2-\rho) + (n-2)\delta(r)]. \tag{4.58}$$

That condition is satisfied under the assumption (4.45) for ε sufficiently small, and is relevant under the assumption (4.44). In the (σ, ρ) plane, equality in (4.44) defines a straight line which intersects the strip $\gamma(r) < 1 < \delta(r)$ from the point $\delta(r) = 1$, $\rho = n/(2(n-1))$ to the point $\gamma(r) = 1$, $\rho = [n/(2(n-1))]^2$, and the condition (4.44) is relevant whenever the latter has $\sigma < 1/2$, namely for $n \geq 8$. Q. E. D.

The conditions (4.41)-(4.45) will be discussed after Proposition 4.1. Here we simply remark that (4.42) implies $p_1 - 1 > 4/n$.

At the present stage, we know that under suitable assumptions on ρ, r, q, p_1 and p_2. any finite energy solution of the NLKG equation satisfies the following properties: the norm $k(t)$ defined by (3.38) belongs to $l^\infty(\mathrm{L}^q)$ by Proposition 3.2, is small in large intervals by Lemma 4.5 and satisfies a superlinear integral inequality by Lemma 4.6. By a simple abstract argument, those three properties imply that $k(t)$ tends to zero at infinity in the following sense.

LEMMA 4.7. — Let $n \geq 3$. Let f satisfy (H1), (H2) and (H3). Let r, ρ, and σ satisfy (4.40) and in addition $2\sigma < \gamma(r)$ if $n = 3$. Let p_1 and p_2 satisfy (4.41), (4.42) and (4.43), (4.45) if relevant (see Lemma 4.6). Let $1/q = \sigma$. Let φ be a finite energy solution of the NLKG equation. Let $k(t)$ be defined by (3.38). Then $\| k ; l^\infty(\mathrm{L}^q, [a, \infty)) \|$ tends to zero when a tends to infinity.

Proof. — The assumptions of Lemma 4.7 collect those of Proposition 3.2, Lemma 4.5 and Lemma 4.6. In particular (4.40) implies (2.11), (4.42) implies $p_1 - 1 > 2/n$ and (4.43) implies (3.35). From the integral equation (3.36) and Lemma 4.6, it follows that k satisfies the integral inequality

$$k \leqslant k_0 + \sum_{\gtrless} \mu_{\gtrless} * (k^{\nu_1} + k^{\nu_{\gtrless}}) \tag{4.59}$$

where k_0 is defined by (3.37),

$$\begin{aligned} \mu_{\gtrless}(t) &= \mathrm{M}\,|\,t\,|^{-\delta_{\gtrless}} \quad \text{for} \quad t > 0 \quad \text{and} \quad t \gtrless 1 \\ &= 0 \qquad\qquad\quad \text{otherwise,} \end{aligned}$$

M depends only on the energy of φ, and ν_1, $\nu_{\gtrless}$ are as in Lemma 4.6. By Proposition 2.2, k_0 satisfies the property to be proved for k. From there on, the proof is almost identical with that of Lemma 5.10 and Corollary 5.1 in [*11*] and will be omitted. Q. E. D.

Knowing that $k(t)$ tends to zero at infinity in the sense of the previous Lemma, it is now an easy matter to derive the space time integrability properties of the finite energy solutions of the NLKG equation.

PROPOSITION 4.1. — Let $n \geq 3$. Let f satisfy (H1), (H2) and (H3). Let r, ρ, and σ satisfy (4.40) and in addition $2\sigma < \gamma(r)$ if $n = 3$. Let p_1 and p_2 satisfy (4.41), (4.42) and (4.43), (4.45) if relevant. Let $1/q = \sigma$ and let m be defined by (2.13). Let φ be a finite energy solution of the NLKG equation. Then $\varphi \in l^m(\mathrm{L}^q, \mathrm{B}^{\rho}_{r,2})$.

Indication of proof. — The result follows from Proposition 2.2, from Lemma 4.7 and from the integral inequality (4.59) by a slight variation of Lemma 5.11 in [*11*], which has to be modified as follows: the assumption on k_0 should be reinforced to $k_0 \in l^m(\mathrm{L}^q)$ and the conclusion then states that $k \in l^m(\mathrm{L}^q)$, with suitable minor modifications in the proof. Q. E. D.

Note that the time integrability properties of the norm of φ in $\mathrm{B}^{\rho}_{r,2}$ obtained in Proposition 4.1 are the same as those obtained in Proposition 2.2 for the solutions of the free equation.

We now discuss the assumptions (4.41)-(4.45) on p_1 and p_2. For $\rho \leq 0$ and for fixed r, the lower conditions (4.42) and (4.43) are independent of ρ, the lower condition (4.45) is irrelevant, and the upper condition (4.41) becomes more restrictive when ρ decreases, since it takes the form

$$(p - 1) < 4\alpha(r)(1 - y)/(1 - 2y)$$

with $y = \alpha(r)/(1 - \rho)$. Since in addition the norm in $\mathrm{B}^{\rho}_{r,2}$ is an increasing function of ρ for fixed r, there is no advantage in taking $\rho < 0$. For $\rho \geq 0$ and fixed r, the lower conditions (4.42), (4.43) and (4.45) become more restrictive when ρ increases. On the other hand, the upper condition (4.41) (which is relevant only when stronger than the upper bound on p_2 in (3.18)), becomes weaker when ρ increases, at least for ρ not too large. However, since the allowed interval for p_1, p_2 depends in a rather complicated way on ρ and r, we continue the discussion in the special case $\rho = 0$ only. In that case, Proposition 4.1 reduces essentially to the following result.

PROPOSITION 4.2. — Let $n \geq 3$, let f satisfy (H1), (H2) and (H3). Let r satisfy $\gamma(r) < 1 < \delta(r)$. Let p_1 and p_2 satisfy

$$(p_2 - 1)(n/2 - \delta(r)) < 2\delta(r)(1 - \delta(r)/n), \qquad (4.60)$$

$$(p_1 - 1)n/2 > 1 + \delta(r), \qquad (4.61)$$

and in addition, if $n = 3$,

$$(p_1 - 1)3/2 > 4\delta(r) - 3 . \qquad (4.62)$$

Let φ be a finite energy solution of the NLKG equation. Then $\varphi \in l^2(L^q, L^r)$ where $1/q = \delta(r) - 1$.

We now discuss the conditions on p_1, p_2 in Proposition 4.2. The condition (4.60) is easily seen to become weaker when $\delta(r)$ increases. For $\delta(r)$ close to 1, it reduces to

$$p_2 - 1 < 4(n - 1)[n(n - 2)]^{-1} . \qquad (4.63)$$

On the other hand, it becomes weaker than the upper bound in (3.18) for $\delta(r) > \delta_0$ where δ_0 is the smaller root of the equation

$$(n - 2)\delta^2 - n^2\delta + n^2 = 0 ,$$

namely

$$\delta_0 = n[2(n - 2)]^{-1}[n - (n^2 - 4n + 8)^{1/2}] .$$

In particular $\delta_0 = 3(3 - \sqrt{5})/2 \sim 1{,}146$ for $n = 3$ and $n = 6$, $\delta_0 = 4 - 2\sqrt{2} \sim 1{,}172$ for $n = 4$, $\delta_0 < 1 + 1/n$ for all n and $\delta_0 = 1 + 1/n - 0(1/n^3)$ for large n. The allowed values for $p_1 - 1$ and $p_2 - 1$ in Proposition 4.2 range over an interval, both endpoints of which are non decreasing functions of $\delta(r)$, starting from $(4/n, 4(n-1)/[n(n-2)])$ for $\delta(r)$ close to 1 and reaching $(2(1+\delta_0)/n, 4/(n-2))$ for $\delta(r) = \delta_0$. For $\delta(r) > \delta_0$, the upper end remains fixed at $4/(n-2)$ while the lower end increases according to (4.61) and possibly (4.62). In the case of a single power $p_1 = p_2 = p$, the allowed values of $p - 1$ range over the interval $(4/n, 4/(n-2))$.

Additional decay properties of finite energy solutions of the NLKG equation can be obtained from Proposition 4.1 or from its special case Proposition 4.2 in several ways. Some immediate extensions are obtained as follows. Let φ be a finite energy solution and assume that $\varphi \in l^m(L^q, B^\rho_{r,2})$ as obtained by Proposition 4.1. Then by the Sobolev inequalities and trivial embeddings of Besov spaces, $\varphi \in l^m(L^q, B^{\rho'}_{r',2})$ for any r', ρ' with $r' \geq r$ and $\rho' + \delta(r') \leq \rho + \delta(r)$. Furthermore, by interpolation between that result and uniform boundedness in H^1 implied by energy conservation, it follows that $\varphi \in l^{m'}(L^{q'}, B^{\rho'}_{r',2})$ provided

$$0 \leq \delta(r') \leq \delta(r), \quad (1 - \rho)\delta(r') \leq (1 - \rho')\delta(r), \quad q'\delta(r') = q\delta(r)$$

and

$$m'\delta(r') = m\delta(r) .$$

We refrain from giving more formal statements, especially since the time integrability properties obtained for a given $B^{\rho'}_{r',2}$ norm by that procedure are in general weaker than those of solutions of the free equation.

A better way to extend the results of Proposition 4.1 and 4.2 consists in substituting again the decay there obtained into the integral equation (3.36). We have not checked to what extent under the assumptions of Proposition 4.1 or 4.2 (for some r, ρ), that procedure would yield, for an arbitrary finite energy solution, the time decays obtained in Proposition 2.2 for solutions of the free equation. The proof would require additional estimates of the same kind as in Lemma 3.3 and 4.6, where however the interpolation would make use of four points instead of three. The corresponding program will be carried out in the next section in the simpler case of the NLS equation.

5. TIME DECAY OF THE SOLUTIONS OF THE NLS EQUATION

In this section we study the decay properties of the solutions of the NLS equation. We first derive the decay properties of the solutions of the free equation (see Proposition 5.1), then the uniform boundedness properties of arbitrary finite energy solutions of the NLS equation (see Propositions 5.2 and 5.3) and finally the decay properties of such solutions (see Proposition 5.5).

We recall that the evolution group for the free Schrödinger equation is defined by $U(t) = \exp(i(t/2)\Delta)$. For $t \neq 0$, $U(t)$ is bounded and strongly continuous in t from $L^{\bar{r}}$ to L^r for any $r \geq 2$ with the bound

$$\| U(t)\varphi \|_r \leqslant (2\pi \mid t \mid)^{-\delta(r)} \| \varphi \|_{\bar{r}} \tag{5.1}$$

for all $\varphi \in L^{\bar{r}}$, where $\delta(r) = n/2 - n/r$ (compare with (2.1)). From (5.1) one obtains easily the following estimate.

LEMMA 5.1. — Let $2 \leq r \leq \infty$ and $\eta \geq 0$. Then

$$\| U(t)\varphi ; B_{r,2}^{-\eta} \| \leqslant \mu(t) \| \varphi ; B_{\bar{r},2}^{\eta} \| \tag{5.2}$$

where

$$\mu(t) = C \operatorname{Min} \{ \mid t \mid^{-\delta(r)}, \mid t \mid^{-(\delta(r)-\eta)_+} \} . \tag{5.3}$$

Proof. — From the definition of Besov spaces, from the fact that $U(t)$ commutes with the convolution and from the estimate (5.1) one obtains immediately.

$$\| U(t)\varphi ; B_{r,2}^{\rho} \| \leqslant (2\pi \mid t \mid)^{-\delta(r)} \| \varphi ; B_{\bar{r},2}^{\rho} \| \tag{5.4}$$

for any $r \geq 2$ and $\rho \in \mathbb{R}$. The Sobolev inequalities, the estimate (5.4) and the embeddings of the Besov spaces imply

$$\| U(t)\varphi ; B_{r,2}^{-\eta} \| \leqslant C \| U(t)\varphi ; B_{l,2}^{\lambda} \| \leqslant C \mid t \mid^{-\delta(l)} \| \varphi ; B_{\bar{r},2}^{\eta} \|$$

for any l and λ such that $-\eta + \delta(r) = \lambda + \delta(l)$ and $(\delta(r) - \eta)_+ \leq \delta(l) \leq \delta(r)$.

The estimate (5.2) now follows by taking $\delta(l)=\delta(r)$ and $\delta(l)=(\delta(r)-\eta)_+$.
Q. E. D.

Using Lemmas 5.1, 2.2 and 2.3 we can now determine the space time integrability properties of the solutions of the free Schrödinger equation, thereby generalizing Lemma 3.1 in [9].

PROPOSITION 5.1. — Let r, ρ and σ satisfy

$$\begin{cases} 2 \leqslant r \leqslant \infty \\ \sigma \equiv \rho + \delta(r) - 1 < 1 \\ \rho \leqslant 1, \end{cases} \tag{5.5}$$

let

$$2/q = \sigma_+, \tag{5.6}$$

$$\begin{cases} 2/m = \mathrm{Min}\,(\delta(r), 1) & \text{if} \quad \delta(r) \neq 1, \\ 2/m < 1 & \text{if} \quad \delta(r) = 1. \end{cases} \tag{5.7}$$

Then the map $\varphi \rightarrow U(t)\varphi$ is bounded from $H^1(\mathbb{R}^n)$ to $l^2(L^q, B^\rho_{r,2}) + l^m(L^\infty, B^\rho_{r,2})$ and *a fortiori* to $l^m(L^q, B^\rho_{r,2})$.

Proof. — From Lemmas 5.1, 2.2 and 2.3 it follows that the map $\varphi \mapsto U(t)\varphi$ is bounded from $L^2(\mathbb{R}^n)$ to $l^2(L^q, B^{-\eta}_{r,2}) + l^m(L^\infty, B^{-\eta}_{r,2})$, for $2 \leq r \leq \infty$ and $\eta \geq 0$ with $2/q = (\delta(r) - \eta)_+ < 1$ and $2/m$ given by (5.7). The result now follows by replacing φ by $(1-\Delta)^{1/2}\varphi$ and by taking $\eta = 1 - \rho$.
Q. E. D.

We now turn to the derivation of uniform bounds on the solutions of the NLS equation. For that purpose we need an additional estimate which replaces Lemma 3.3 in the present case. We shall make use of the same assumption (H1) on f (see (3.17) and (3.18)) as for the NLKG equation.

LEMMA 5.2. — Let $n \geq 3$. Let f satisfy (H1) with $p_1 - 1 > 2/n$. Let r, ρ and σ satisfy (5.5) and in addition $0 \leq \rho < 1$, let $\delta(r) > 1$, $\delta(r)$ sufficiently close to 1. Then, there exists $\delta_>$ and $\delta_<$ with $0 \leq \delta_< < 1 < \delta_> (\leq \delta(r))$ and there exists $\nu (0 \leq \nu < 1)$ such that, for any $\varphi \in H^1 \cap B^\rho_{r,2}$ and any $t \neq 0$, $U(t) f(\varphi) \in B^\rho_{r,2}$ and the following estimate holds

$$\| U(t) f(\varphi) ; B^\rho_{r,2} \| \leqslant M(\| \varphi ; H^1 \|)(\mathop{\mathrm{Min}}_{\lessgtr} | t |^{-\delta_\lessgtr})(1 + \| \varphi ; B^\rho_{r,2} \|^\nu), \tag{5.8}$$

where M depends only on the norm of φ in H^1.

Proof. — The proof follows closely that of Lemma 3.3. For definiteness we assume that $\delta(r) = 1 + \varepsilon$, $0 \leq \rho < 1 - \sqrt{\varepsilon}$ with $\varepsilon > 0$ sufficiently small and we derive the estimate (5.8) with $\delta_\gtrless \lessgtr 1 \pm \varepsilon$ and $0 \leq \nu \leq 1 - \varepsilon$. We decompose f as $f_1 + f_2$ as in the proof of Lemma 3.3 and estimate the contributions of each term separately. For simplicity we continue the proof in the case of a single power p in f. From Lemma 5.1 and Lemma 3.2 (with $\eta = 0$) we obtain

$$\begin{aligned} \| U(t)f(\varphi); B^{\rho}_{r,2} \| &\leqslant C|t|^{-\delta(l)} \| f(\varphi); B^{\lambda}_{\bar{l},2} \| \\ &\leqslant C|t|^{-\delta(l)} M(\| \varphi; H^1 \|) \| \varphi; B^{\rho}_{r,2} \|^{\nu} \end{aligned} \tag{5.9}$$

with $\lambda = 1 + \sigma - \delta(l)$, under the conditions $\sigma < \delta(l) \leq \delta(r)$ and (3.3), (3.4), (3.5) (with $\eta = 0$). For each p satisfying (3.25) we have to make two choices of l and ν, corresponding to $>$ and $<$, that ensure in addition to the previous conditions that $\delta(l) = \delta_{\gtrless} \gtrless 1 \pm \varepsilon$ and $0 \leq \nu_{\gtrless} \leq 1 - \varepsilon$. Eliminating ν between $0 \leq \nu \leq 1 - \varepsilon$ and (3.4) and (3.5) yields (3.26) and (3.27), the latter of which becomes

$$(p - 1)(n/2 - 1) \leqslant 2\delta(l) - \varepsilon\sigma . \tag{5.10}$$

Upon substitution of λ, (3.3) becomes

$$(p-1)(n/2-1-\sigma) \leqslant 2\delta(l) - \sigma[\delta(r) - \delta(l)]/(1-\rho) . \tag{5.11}$$

We finally choose $\delta(l) = 1 - \varepsilon$ in the case $<$ and $\delta(l) = \delta(r) = 1 + \varepsilon$ in the case $>$. The remaining conditions (3.26), (5.10) and (5.11) are then satisfied for those two choices of l and for $1 - \rho > \sqrt{\varepsilon}$ provided

$$\begin{aligned} &(p - 1)n/2 \geqslant 1 + \varepsilon \\ &(p - 1)(n/2 - 1) \leqslant 2 - 3\varepsilon \\ &(p - 1)(n/2 - 1 - \sigma) \leqslant 2(1 - \varepsilon - \sqrt{\varepsilon}) . \end{aligned}$$

Those inequalities are satisfied for ε sufficiently small under the condition (3.25). Q. E. D.

We are now in a position to prove the basic uniform boundedness result for the solutions of the NLS equation.

PROPOSITION 5.2. — Let $n \geq 3$. Let f satisfy (H1) with $p_1 - 1 > 2/n$. Let $\varphi \in L^{\infty}(\mathbb{R}, H^1)$ be a solution of the NLS equation in $\mathscr{D}'(\mathbb{R}, H^{-1})$. Let r, ρ and q satisfy (5.5), $\rho < 1$ and (5.6). Then $\varphi \in l^{\infty}(L^q, B^{\rho}_{r,2})$ and the norm of φ in that space is estimated in terms of the norm of φ in $L^{\infty}(\mathbb{R}, H^1)$.

Proof. — It is sufficient to prove the result for $\rho \geq 0$ and $\delta(r) > 1$, $\delta(r)$ sufficiently close to 1. The result in the general case follows from the special case by interpolation with uniform boundedness in H^1, the Sobolev inequalities and the embeddings of the Besov spaces. Under the assumptions made on f and φ, it has been shown in Lemma 2.1 of [9] that φ satisfies the integral equation

$$\varphi(t) = U(t)\varphi_0 - i \int_0^t d\tau U(t - \tau) f(\varphi(\tau)) \tag{5.12}$$

with $\varphi_0 = \varphi(0)$. The integral in the right hand side of (5.12) can be regarded as a Bochner integral in H^{-1}. Furthermore we know from a variation of Lemma 3.3 of [9] that $\varphi \in L^q_{loc}(\mathbb{R}, B^\rho_{r,2})$ for r, ρ and q satisfying (5.5), $\rho < 1$ and (5.6). The difference is that here we use the more convenient Besov spaces instead of Sobolev spaces and the more efficient Lemmas 3.1 and 3.2 instead of Lemma 3.2 of [9]. In any case the proof of that property is a simplified version of that of the present proposition, in so far as it does not require any integrability property at infinity of the integral in the right hand side of (5.12).

We define

$$k_0(t) = \| U(t)\varphi_0 ; B^\rho_{r,2} \| , \tag{5.13}$$

$$k(t) = \| \varphi(t) ; B^\rho_{r,2} \| . \tag{5.14}$$

From the integral equation (5.12) and Lemma 5.2 it follows that k satisfies the integral inequality (3.39) with μ defined by (3.40). From theore on the end of the proof is identical with that of Proposition 3.1. Q. E. D.

As for the NLKG equation, the result of Proposition 5.2 is especially relevant if f satisfies the following assumption:

(H2′). There exists a function $V \in \mathscr{C}^1(\mathbb{C}, \mathbb{R})$ such that $V(0)=0$, $V(z)=V(|z|)$ for all $z \in \mathbb{C}$, $f(z) = \partial V/\partial \bar{z}$ and, for some p_3, $1 \leq p_3 < 1 + 4/n$, and all $\rho \in \mathbb{R}^+$, V satisfies the inequality

$$V(\rho) \geqslant - C(\rho^2 + \rho^{p_3+1}). \tag{5.15}$$

In that case the L^2 norm and the energy, now defined by

$$E(\varphi) = \frac{1}{2} \| \nabla\varphi \|_2^2 + \int dx V(\varphi(x)) \tag{5.16}$$

are conserved and the H^1 norm of φ is estimated in terms of the L^2 norm and of the energy of φ (see for instance Lemma 3.2 in [7]). Under the assumptions (H1) with $p_1 - 1 > 2/n, p_1 - 1 \geq 4(n-4)/[n(n-2)]$ if $n > 6$, and (H2′), it follows from Proposition 3.2 in [9] that for any $t_0 \in \mathbb{R}$, for any $\varphi_0 \in H^1$, there exists a unique $\varphi \in L^\infty(\mathbb{R}, H^1)$ which solves the NLS equation in $\mathscr{D}'(\mathbb{R}, H^{-1})$ with the initial condition $\varphi(t_0) = \varphi_0$. Such solutions will be called finite energy solutions. We can restate the previous boundedness result as follows.

PROPOSITION 5.3. — Let $n \geq 3$. Let f satisfy (H1) with $p_1 - 1 > 2/n$, $p_1 - 1 \geq 4(n-4)/[n(n-2)]$ if $n > 6$, and (H2′). Let r, ρ and q satisfy (5.5), $\rho < 1$, and (5.6). Then any finite energy solution of the NLS equation belongs to $l^\infty(L^q, B^\rho_{r,2})$ and its norm in that space is estimated in terms of its (conserved) L^2-norm and energy.

We now turn to the study of the time decay properties of the finite energy solutions of the NLS equation and prove that such solutions satisfy the same decay properties as those of the free equation. For that purpose we rely heavily on Section 5 of [*11*] from which we derive the following intermediate result.

PROPOSITION 5.4. — Let $n \geq 3$. Let f satisfy (H1) with $p_1 - 1 > 4/n$, (H2) and (H3). Let r satisfy

$$1 < \delta(r) < (p_1 - 1)n/2 - 1 \tag{5.17}$$

and let $2/q = \delta(r) - 1$. Then any finite energy solution of the NLS equation belongs to $l^2(L^q, L^r)$.

Indication of proof. — The result is a slight reinforcement of Corollary 5.2 of [*11*] which, under the same assumptions, states the weaker property that $\varphi \in l^{q'}(\mathbb{R}, L^r)$ for $2 < q' \leq q$. The proof of this proposition is the same as that leading to Corollary 5.2 of [*11*] with the only exception of Lemma 5.11 of [*11*] which has to be modified as follows: the assumption on k_0 has to be reinforced to $k_0 \in l^2(L^q)$ and the conclusion then states that $k \in l^2(L^q)$, with suitable minor modifications in the proof. Q. E. D.

Proposition 5.4 provides us with some time decay of finite energy solutions. We now improve that decay by plugging the result just obtained again into the integral equation (5.12). For technical reasons it is convenient to proceed in two steps. The first one consists in obtaining the free decay for the norms in $L^{r'}$ with $\delta(r') < 1$.

LEMMA 5.3. — Let $n \geq 3$. Let f satisfy (H1) with $p_1 - 1 > 4/n$. Let $\varphi \in L^\infty(\mathbb{R}, H^1)$ be a solution of the NLS equation in $\mathcal{D}'(\mathbb{R}, H^{-1})$ (in particular, let f satisfy (H2') and φ be a finite energy solution). Assume that $\varphi \in l^2(L^q, L^r)$ for $1 < \delta(r) \leq \delta_1$ for some $\delta_1 > 1$ and $2/q = \delta(r) - 1$. Then $\varphi \in l^{m'}(L^\infty, L^{r'})$ for $0 \leq \delta(r') = 2/m' < 1$.

Proof. — Since $\varphi \in L^\infty(\mathbb{R}, H^1)$, it is sufficient to prove the result for $\delta(r') = \delta'$ close to 1, say $\delta' = 1 - \varepsilon$ with $\varepsilon > 0$ and small. As in the proof of Lemma 3.3 we decompose f as $f_1 + f_2$ and estimate the contribution of each term separately. We continue the proof in the case of a single power $p = p_1 = p_2$ for simplicity. From (5.1) we obtain

$$\| U(t)f(\varphi) \|_{r'} \leqslant C \,| t |^{-\delta'} \| f(\varphi) \|_{\bar{r}'} \leqslant C \,| t |^{-\delta'} \| \varphi \|_{p\bar{r}'}^{p}. \tag{5.18}$$

Under the assumption (H1), by taking ε sufficiently small, we can ensure that

$$(p - 1)(n/2 - \delta') \leqslant 2\delta',$$

so that $p\bar{r}' \leq r' \,(< 2n/(n - 2))$.

We interpolate the last norm in (5.18) between the norms of φ in L^2, H^1 and L^r for some r with $1 < \delta(r) \leq \delta_1$, as

$$\| U(t)f(\varphi) \|_{r'} \leqslant C \,| t |^{-\delta'} \| \varphi \|_2^{p-\mu-\eta} \| \nabla\varphi \|_2^{\mu} \| \varphi \|_r^{\eta} \tag{5.19}$$

with

$$(p-1)n/2 = \delta' + \mu + \eta\delta(r) \tag{5.20}$$

and $\mu \geq 0, \eta \geq 0$. The result now follows from the integral equation (5.12), by taking the norm in $l^{m'}(\mathrm{L}^\infty, \mathrm{L}^{r'})$, estimating the free term by Proposition 5.1 and the integral by (5.19) and the Hardy-Littlewood-Sobolev inequality [*12*], provided

$$2\eta/q \equiv \eta(\delta(r)-1) < 2(1-\delta') \tag{5.21}$$

and

$$2\eta/m \equiv \eta \geqslant 2 - \delta' + 2/m' = 2 - \delta' . \tag{5.22}$$

We choose $\eta = 2 - \delta'$. The condition (5.21) then reduces to

$$\delta(r) - 1 < 2(1-\delta')/(2-\delta')$$

and is satisfied by taking for instance $\delta(r) = 1 + \varepsilon$. The condition (5.20) then holds with $\mu \geq 0$ provided

$$(p-1)n/2 \geqslant \delta' + (2-\delta')\delta(r) = 2 + \varepsilon + \varepsilon^2$$

which is satisfied for $p_1 - 1 > 4/n$ and ε sufficiently small. Q. E. D.

The second step in the improvement of the time decay is given in the following Lemma.

LEMMA 5.4. — Let $n \geq 3$. Let f satisfy (H1) with $p_1 - 1 > 4/n$. Let $\varphi \in \mathrm{L}^\infty(\mathbb{R}, \mathrm{H}^1)$ be a solution of the NLS equation in $\mathscr{D}'(\mathbb{R}, \mathrm{H}^{-1})$ (in particular, let f satisfy (H2′) and φ be a finite energy solution). Assume that $\varphi \in l^{m'}(\mathrm{L}^\infty, \mathrm{L}^{r'})$ for all r' such that $0 \leq \delta(r') = 2/m' < 1$. Then, for all r, ρ, q and m satisfying (5.5), $\rho < 1$, (5.6) and (5.7), $\varphi \in l^m(\mathrm{L}^q, \mathrm{B}^\rho_{r,2})$.

Proof. — It is sufficient to prove the result in the special case where $\delta(r)$ is close to 1 and $\sigma \geq 0$. The result in the region $0 \leq \sigma < 1$, $\delta(r) > 1$, follows from that special case (with $\delta(r) > 1$) by the Sobolev inequalities. The result in the region $0 \leq \sigma < 1$, $0 \leq \delta(r) < 1$ follows from the special case (with $\delta(r) < 1$) by interpolation with uniform boundedness in H^1. The result for $\sigma \leq 0$ follows from the result for $\sigma = 0$ by the embedding of the Besov spaces $\mathrm{B}^\rho_{r,2}$ for fixed r. From now on, we concentrate on the case where $\delta(r)$ is close to 1, $\sigma \geq 0$ and $\rho < 1$ (we shall take later $\delta(r) = 1 \pm \varepsilon$ and $\rho \leq 1 - \sqrt{\varepsilon}$ for $\varepsilon > 0$ and sufficiently small).

The proof consists again in substituting the available decay into the integral equation (5.12), and for that purpose we need additional estimates of the integrand similar to but more complicated than those of Lemma 5.2. Let r' satisfy $\delta(r') < 1$, $\delta(r')$ close to 1 (we shall take later $\delta(r') \equiv \delta' = 1 - \varepsilon$). As in the proof of Lemma 3.3, we decompose f as $f_1 + f_2$ and estimate the contribution of each term separately. We continue the proof in the case of a single power $p=p_1=p_2$ for simplicity. From Lemma 5.1 and Lemma 3.1, we obtain

$$\| U(t)f(\varphi);B^{\rho}_{r,2} \| \leqslant C\,|t|^{-\delta(l)} \| \varphi;B^{\lambda}_{k,2} \| \| |\varphi|^{p-1} \|_s \qquad (5.23)$$

with $\lambda = 1 + \sigma - \delta(l)$, under the condition $\sigma < \delta(l) \leq \delta(r)$ and (3.7) which can be rewritten as

$$(p-1)[n/2 - \delta((p-1)s)] = \delta(l) + \delta(k)\,. \qquad (5.24)$$

We impose $\delta(l) < 1$, $\delta(l)$ close to 1 (we shall take later $\delta(l)=1-\varepsilon$), and we estimate the last two norms in (5.23) by interpolation between the norms of φ in L^2, H^1, $B^0_{r',2}$ and $B^{\rho}_{r,2}$ in the following way. Since for p and l fixed, s is a decreasing function of k, we can always arrange for the points $(\lambda, \delta(k))$ and $(0, \delta((p-1)s))$ to lie on the same side of the line $\delta(r)=(1-\rho)\delta'$ in the $(\rho, \delta(r))$ variables. If both points can be made to lie in the region $\delta(r) \leqslant (1-\rho)\delta'$ (case 1), we interpolate the two norms barycentrically, namely without using the Sobolev inequalities, between the norms in L^2, H^1 and $B^0_{r',2}$. If both points can be made to lie in the region $\delta(r) \geq (1-\rho)\delta'$ (case 2), we choose $(p-1)s = r'$ and we interpolate the norm in $B^{\lambda}_{k,2}$ barycentrically between the norms in H^1, $B^0_{r',2}$ and $B^{\rho}_{r,2}$. It follows from (5.24) that case 1 occurs provided

$$(p-1)n/2 \geqslant \delta(l) \qquad (5.25)$$

and

$$(p-1)(n/2 - \delta') \leqslant \delta(l) + (1-\lambda)\delta'\,. \qquad (5.26)$$

In that case, we obtain from (5.23)

$$\| U(t)f(\varphi);B^{\rho}_{r,2} \| \leqslant C\,|t|^{-\delta(l)} \| \varphi;H^1 \|^{\lambda} \| \varphi;B^0_{r',2} \|^{\eta} \| \varphi \|_2^{p-\lambda-\eta} \qquad (5.27)$$

with

$$(p-1)n/2 = \delta(l) + \eta\delta'\,. \qquad (5.28)$$

Similarly it follows from (5.24) that case 2 occurs provided

$$(p-1)(n/2 - \delta') \leqslant \delta(l) + \delta(r)(1-\lambda)/(1-\rho) \qquad (5.29)$$

and

$$(p-1)(n/2 - \delta') \geqslant \delta(l) + (1-\lambda)\delta'\,. \qquad (5.30)$$

In that case, we obtain from (5.23)

$$\| U(t)f(\varphi);B^{\rho}_{r,2} \| \leqslant C\,|t|^{-\delta(l)} \| \varphi;H^1 \|^{\mu} \| \varphi;B^0_{r',2} \|^{p-\mu-\nu} \times \| \varphi;B^{\rho}_{r,2} \|^{\nu} \qquad (5.31)$$

with $\lambda = \mu + \nu\rho$ and

$$(p-1)n/2 = \delta(l) - \lambda + \mu + \nu(\rho + \delta(r)) + (p-\mu-\nu)\delta'$$

or equivalently, after elimination of μ

$$(p - 1)(n/2 - \delta') = \delta(l) + (1 - \lambda)\delta' + \nu(\delta(r) - (1 - \rho)\delta'). \quad (5.32)$$

We continue the proof assuming for the time being that the $B^0_{r',2}$ norm of the given solution φ satisfies the free time decay, namely $\varphi \in l^{m'}(L^\infty, B^0_{r',2})$ with $2/m' = \delta'$ (This assumption is stronger than that of the Lemma, since the embedding $B^0_{r',2} \subset L^{r'}$ goes in the wrong direction). Let k_0 and k be defined by (5.13) and (5.14) and let χ_a be the characteristic function of the (time) interval $[0, a]$. From the integral equation (5.12), where we estimate the integrand by (5.27) or (5.31) depending on the case at hand and the time integral by the Hardy-Littlewood-Sobolev inequality [*12*] we obtain in case 1

$$\begin{aligned} \| \chi_a k ; l^m(L^q) \| \leqslant & \| k_0 ; l^m(L^q) \| \\ & + M(\| \varphi ; L^\infty(\mathbb{R}, H^1) \|) \| \varphi ; l^{m'}(L^\infty, B^0_{r',2}) \|^\eta \end{aligned} \quad (5.33)$$

under the conditions

$$\eta\delta' \geqslant 2(1 - \delta(l)) + 2/m \quad (5.34)$$

and

$$0 < 2(1 - \delta(l)) + 2/q , \quad (5.35)$$

while in case 2

$$\begin{aligned} \| \chi_a k ; l^m(L^q) \| \leqslant \| k_0 ; l^m(L^q) \| + M(\| \varphi ; L^\infty(\mathbb{R}, H^1) \|) \\ \times \| \varphi ; l^{m'}(L^\infty, B^0_{r',2}) \|^{p-\mu-\nu} \| \chi_a k ; l^m(L^q) \|^\nu \end{aligned} \quad (5.36)$$

under the conditions

$$(p - \lambda - \nu(1 - \rho))\delta' \geqslant 2(1 - \delta(l)) + (1 - \nu)2/m \quad (5.37)$$

and

$$0 < 2(1 - \delta(l)) + (1 - \nu)2/q . \quad (5.38)$$

It follows then from (5.33) or (5.36), after taking the limit $a \to \infty$ and provided we can take $\nu < 1$ in case 2, that $\varphi \in l^m(L^q, B^\rho_{r,2})$.

We are then faced with the task of showing that for the relevant values of r and ρ and for any value of p in the interval

$$4/n < p - 1 < 4/(n - 2), \quad (5.39)$$

one can choose $\delta(l)$ and δ' with $\sigma < \delta(l) \leq \delta(r)$ and $\delta' < 1$ such that:

. in case 1, namely under the condition (5.26), there hold (5.25) and (5.35), while η defined by (5.28) satisfies (5.34);

. in case 2, namely under the condition (5.30), there holds (5.29), while ν defined by (5.32) satisfies (5.37), (5.38) and $\nu < 1$.

We recall that we are interested in the values $\delta(r) = 1 \pm \varepsilon$, $\rho \leq 1 - \sqrt{\varepsilon}$, we choose $\delta(l) = \delta' = 1 - \varepsilon$ and we impose $\nu \leq 1 - \varepsilon$ in case 2.

We first consider case 1. The condition (5.25) follows trivially from (5.39), (5.35) is obvious, and elimination of η between (5.28) and (5.34) yields the condition

$$(p-1)n/2 \geqslant 2 - \delta(l) + 2/m \tag{5.40}$$

which reduces to $(p-1)n/2 \geq 2 + \varepsilon$ in the worse case $\delta(r) = 1 + \varepsilon$, and therefore follows from (5.39) for ε sufficiently small.

We next consider case 2. Now (5.38) is obviously satisfied for $v < 1$. The condition (5.29) reduces to

$$(p-1)(n/2 - 1 + \varepsilon) \leqslant 2(1-\varepsilon)$$

in the case $\delta(r) = 1 - \varepsilon$ and to the stronger condition

$$(p-1)(n/2 - 1 + \varepsilon) \leqslant 2 - 2\varepsilon(1+\varepsilon)/(1-\rho)$$

in the case $\delta(r) = 1 + \varepsilon$. The latter is satisfied for $\rho \leq 1 - \sqrt{\varepsilon}$ provided

$$(p-1)(n/2 - 1 + \varepsilon) \leqslant 2\{1 - (1+\varepsilon)\sqrt{\varepsilon}\}$$

which follows from (5.39) for ε sufficiently small. By (5.32), the condition $v \leq 1 - \varepsilon$ reduces to

$$(p-1)(n/2 - 1 + \varepsilon) \leqslant (1-\varepsilon)(2 - \varepsilon\rho)$$

which also follows from (5.39) for ε sufficiently small. The condition (5.37) becomes, after partially eliminating v with (5.32)

$$(p-1)n/2 \geqslant 2 - \delta(l) + 2/m + v(\delta(r) - 2/m) \tag{5.41}$$

and reduces in the worse case $\delta(r) = 1 + \varepsilon$ to $(p-1)n/2 \geq 2 + (1+v)\varepsilon$ which follows from (5.39) and from $v \leq 1$ for ε sufficiently small.

Therefore, in both cases 1 and 2, all the relevant conditions follow from (5.39). This completes the proof that $\varphi \in l^m(L^q, B^\rho_{r,2})$ for $\delta(r) = 1 \pm \varepsilon$, $\sigma \geq 0$ and $\rho \leq 1 - \sqrt{\varepsilon}$, with ε sufficiently small, under the assumption that $\varphi \in l^{m'}(L^\infty, B^0_{r',2})$. It remains to derive that assumption from those of the Lemma. This is achieved by using a minor variation of the method just followed, taking now $\delta(r) = \delta(r') = \delta(l) = 1 - \varepsilon$ and $\lambda = \rho = 0$. In that case the estimate (5.23) is replaced by

$$\| U(t)f(\varphi)\,;B^0_{r,2} \| \leqslant C\,|\,t\,|^{-\delta(r)}\, \| f(\varphi)\,;B^0_{\bar{r},2} \| \leqslant C\,|\,t\,|^{-\delta(r)}\, \| f(\varphi) \|_{\bar{r}} \tag{5.42}$$

where we have used (5.2), the embedding $B^0_{\bar{r},2} \supset L^{\bar{r}}$ and we have avoided the use of Lemma 3.1, which does not apply since $\lambda = 0$. The last norm in (5.42) is then estimated by Hölders' inequality as

$$\| f(\varphi) \|_{\bar{r}} \leqslant \| \varphi \|^{\eta}_{r} \| \varphi \|^{p-\eta}_{2}$$

with $(p-1)n/2 = (1+\eta)\delta(r)$ and the proof is completed as in case 1 of the main argument. Q. E. D.

By putting together Proposition 5.4 and Lemmas 5.3 and 5.4 we obtain the following decay properties of the finite energy solutions of the NLS equation for repulsive interactions.

PROPOSITION 5.5. — Let $n \geq 3$. Let f satisfy (H1) with $p_1 - 1 > 4/n$, (H2) and (H3). Let r, ρ, q and m satisfy (5.5), $\rho < 1$, (5.6) and (5.7). Then any finite energy solution of the NLS equation belongs to $l^m(L^q, B^{\rho}_{r,2})$.

We conclude this section with some remarks on Section 5 of [*11*]. By a systematic use of Besov spaces instead of Sobolev spaces and in particular by the use of the estimates of Lemmas 3.1 and 3.2 one can generalize the main body of the argument of that section leading to Corollary 5.2 to a direct study of the time decay of the norm of φ in $B^{\rho}_{r,2}$ for $\delta(r) > 1$, $\delta(r)$ close to 1 and $0 \leq \sigma \equiv \rho + \delta(r) - 1 < 1$. This extension parallels the treatment given for the NLKG equation in Section 4. The main steps are a generalization of Lemma 5.9 of [*11*] analogous to Lemma 4.5 above and a generalization of Lemma 5.5 of [*11*] analogous to Lemma 4.6 above. The proofs are similar to but simpler than those of Lemmas 4.5 and 4.6. That extension however does not lead to any improvement of the final result contained in Proposition 5.5.

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Note added in proof.

That result was subsequently extended to all $n \geqslant 3$ and to interactions f controlled by a single power $p \in (1 + 4/n, 1 + 4/(n - 2))$ in the following paper:

[*6 a*] P. Brenner, *Space-time means and non linear Klein-Gordon equations*, preprint, 1985.

APPENDIX

Proof of Lemma 2.1. — We first recall the definition of Besov spaces [1]. Let $\hat{\psi} \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq \hat{\psi} \leq 1$, $\hat{\psi}(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\psi}(\xi) = 0$ for $|\xi| \geq 2$ and define for any $j \in \mathbb{Z}$

$$\hat{\varphi}_j(\xi) = \hat{\psi}(2^{-j}\xi) - \hat{\psi}(2^{-j+1}\xi)$$

so that

$$\operatorname{Supp} \hat{\varphi}_j \subset \{ \xi : 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1} \}$$

and, for any ξ, $\hat{\psi}(\xi) + \sum_{j \geqslant 1} \hat{\varphi}_j(\xi) = 1$ with at most two non vanishing terms in the sum.

For any $\rho \in \mathbb{R}$, $1 \leq r \leq \infty$ and $1 \leq q \leq \infty$, we define the Besov space $B^\rho_{r,q} \equiv B^\rho_{r,q}(\mathbb{R}^n)$ as the Banach space

$$B^\rho_{r,q} = \left\{ v \in \mathcal{S}'(\mathbb{R}^n) : \| v \,;\, B^\rho_{r,q} \| \equiv \left\{ \| \psi * v \|_r^q + \sum_{j \geqslant 1} 2^{\rho j q} \| \varphi_j * v \|_r^q \right\}^{1/q} < \infty \right\} \tag{A.1}$$

(with obvious modifications if $q = \infty$), where ψ and φ_j are the inverse Fourier transforms of $\hat{\psi}$ and $\hat{\varphi}_j$, and $*$ denotes convolution in $\mathbb{R}^n$. In order to establish (2.2) (where we replace φ by v to avoid confusion) we shall estimate the norms of $\psi * (\exp(i\omega_1 t)v)$ and of $\varphi_j * (\exp(i\omega_1 t)v)$ in L^r by interpolation between the L^2 norms, which do not depend on t, and the L^∞ norms. Using the fact that

$$\sum_{k=j,\, j\pm 1} \hat{\varphi}_k \hat{\varphi}_j = \hat{\varphi}_j$$

and

$$(\hat{\psi} + \hat{\varphi}_1)\hat{\psi} = \hat{\psi}$$

and the Young inequality, we obtain

$$\| \psi * (\exp(i\omega_1 t)v) \|_\infty \leqslant (\| \exp(i\omega_1 t)\psi \|_\infty + \| \exp(i\omega_1 t)\varphi_1 \|_\infty) \| \psi * v \|_1 \tag{A.2}$$

and

$$\| \varphi_j * (\exp(i\omega_1 t)v) \|_\infty \leqslant \sum_{k=j,\, j\pm 1} \| \exp(i\omega_1 t)\varphi_k \|_\infty \| \varphi_j * v \|_1 . \tag{A.3}$$

Now

$$\| \exp(i\omega_1 t)\varphi_k \|_\infty = 2^{nk} \operatorname{Sup}_x \left| \int d\xi \exp\{ it2^k(\xi^2 + 2^{-2k})^{1/2} - ix.\xi \} \hat{\varphi}_0(\xi) \right| \tag{A.4}$$

can be estimated by the stationary phase method as

$$\| \exp(i\omega_1 t)\varphi_k \|_\infty \leqslant C 2^{nk} \{ \operatorname{Min}(1, (|t| 2^k)^{-(n-1)/2}) \} \{ \operatorname{Min}(1, (|t| 2^{-k})^{-1/2}) \} \tag{A.5}$$

and similarly

$$\| \exp(i\omega_1 t)\psi \|_\infty \leqslant C \operatorname{Min}(1, |t|^{-n/2}), \tag{A.6}$$

so that, for any $\lambda \in \mathbb{R}$,

$$\| \psi * (\exp(i\omega_1 t)v) \|_\infty \leqslant \mu_\infty(t) \| \psi * v \|_1 \tag{A.7}$$

and

$$\| \varphi_j * (\exp(i\omega_1 t)v) \|_\infty \leqslant \mu_\infty(t) 2^{\lambda j} \| \varphi_j * v \|_1 , \tag{A.8}$$

where

$$\mu_\infty(t) = C \operatorname{Sup}_{k \geqslant 0} 2^{(n-\lambda)k} \{ \operatorname{Min}(1, (|t| 2^k)^{-(n-1)/2}) \} \{ \operatorname{Min}(1, (|t| 2^{-k})^{-1/2}) \} . \tag{A.9}$$

Now, for $\lambda \geq (n + 1)/2$, the Supremum in (A.9) is finite and reduces to

$$\begin{aligned} \mu_\infty(t) &= C\,|\,t\,|^{-(n-\lambda)_+} && \text{for} \quad |\,t\,| \leqslant 1 \\ &= C\,\mathrm{Max}\,\{\,|\,t\,|^{-n/2}, |\,t\,|^{-(\lambda-1)}\,\} && \text{for} \quad |\,t\,| \geqslant 1\,. \end{aligned} \tag{A.10}$$

By interpolation between the pair (L^∞, L^1) and the pair (L^2, L^2) we obtain from (A.7) and (A.8)

$$\|\, \psi * (\exp(i\omega_1 t)v)\,\|_r \leqslant \mu(t)\,\|\,\psi * v\,\|_{\bar r} \tag{A.11}$$

and

$$2^{-\lambda j\alpha(r)}\,\|\,\varphi_j * (\exp(i\omega_1 t)v)\,\|_r \leqslant \mu(t) 2^{\lambda j \alpha(r)}\,\|\,\varphi_j * v\,\|_{\bar r} \tag{A.12}$$

with

$$\mu(t) = \mu_\infty(t)^{2\alpha(r)} \tag{A.13}$$

and $\alpha(r)$ given by (2.1). Taking the norms of (A.11) and (A.12) in l^2 and choosing $\lambda = (n + 1 + \theta)/2$ with $0 \leq \theta \leq 1$ yields (2.2), with (A.13) reduces to (2.3). Q. E. D.

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