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Potential scattering in stochastic mechanics (*)

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ABSTRACT. — In this paper we study the sample path trajectories of the diffusions which are to describe potential scattering experiments in stochastic mechanics. Our main result asserts that the behavior of the trajectories is physically correct in that they settle down to straight line motion at constant speed with probability one. This limiting velocity determines a final momentum random variable, and we show that it has the same distribution as does the final momentum for the corresponding quantum state. Finally, we discuss the possible role of such a pathwise description of quantum scattering in Mathematical Physics.

RÉSUMÉ. — On étudie les trajectoires des processus de diffusion qui doivent écrire les expériences de diffusion par un potentiel en mécanique stochastique. Le résultat principal est que le comportement des trajectoires est physiquement correct en ce sens qu'elles tendent vers des mouvements rectilignes uniformes avec probabilité un. La vitesse limite détermine une grandeur aléatoire représentant l'impulsion finale, et on montre qu'elle a la même distribution que l'impulsion finale dans l'état quantique correspondant. Finalement, on discute le rôle possible d'une telle description de la diffusion quantique en termes de chemins en Physique Mathématique.

INTRODUCTION

Nelson's stochastic mechanics provides a description of quantum phenomena in terms of diffusions instead of wave functions. In this paper

(*) Versions of the results contained in this paper were given in part of the author's Princeton thesis, directed by Professor Edward Nelson.

we prove that the sample paths of these diffusions have a physically intuitive behavior in potential scattering problems, and we discuss the role of stochastic mechanical sample path results in mathematical physics. Before precisely stating our results, we will give a brief synopsis of stocastic mechanics in the special case concerning us here. We have endeavored to make this introduction as broadly accessible as possible, so a few technical details are postponed to the main body of the paper.

Consider a point particle of mass m moving in $\mathbb{R}^3$ under the influence of a potential $V(x)$. The kinematical part of stochastic mechanics is that the motion of the particle is to be given by a Markovian diffusion process $t \mapsto \xi(t)$ such that for some time dependent vector field $b : \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3$, called the « drift field », $t \mapsto \xi(t)$ satisfies a stochastic differential equation of the form

$$d\xi(t) = b(\xi(t), t)dt + \sqrt{\frac{\hbar}{m}}\, dw(t) \tag{1}$$

where $t \mapsto w(t)$ is a standard Brownian motion.

The second term on the right hand side of (1) is of a particularly simple form; roughly, this form is motivated by the following physical considerations: The position of the particle at time t is a random variable on account of « quantum fluctuations ». These are to be isotropic and translation invariant, and this implies that the « noise » term in our stochastic differential equation should be the increment of a process which is a constant times a Brownian motion. Since the mean value of $| w(t) - w(s) |^2$ is $| t-s |$, $w(t)$ has units of $(\text{time})^{1/2}$. So, since $\xi(t)$ is to have units of distance, the constant must have units of $(\text{distance})/(\text{time})^{1/2}$. The constant $(\hbar/m)^{1/2}$ has such units; this choice is fully motivated by the dynamical considerations to follow.

The dynamical part of stochastic mechanics is given by the Guerra-Morato variational principle. We will not give details here, but this is a direct and beautiful translation of the Lagrangean variational principle of classical mechanics into the kinematical context just described. A theorem of Guerra and Morato then asserts that a diffusion $t \mapsto \xi(t)$ is critical for their variational principle precisely when there is a solution $\psi(x, t)$ of the Schroedinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left(-\frac{\hbar^2}{2m} \Delta + V(x) \right) \psi(x, t) \tag{2}$$

so that the drift field $b(x, t)$ of $t \mapsto \xi(t)$ is given by

$$b(x, t) = \frac{\hbar}{m} \left(\operatorname{Re} \frac{\nabla \psi}{\psi}(x, t) + \operatorname{Im} \frac{\nabla \psi}{\psi}(x, t) \right) \tag{3}$$

and so that the probability that $\xi(t)$ is in a given measurable set $A \subset \mathbb{R}^3$, $\Pr\{\xi(t) \in A\}$, is given by

$$\Pr\{\xi(t) \in A\} = \int_A |\psi(x, t)|^2 dx \tag{4}$$

Equation (4) can be expressed by saying that $\xi(t)$ has a density $\rho(x, t)$ given by

$$\rho(x, t) = |\psi(x, t)|^2 \tag{5}$$

Note that it is the same potential $V(x)$ which appears in (2) and which governs the motion of the diffusing particle. The choice of $(\hbar/m)^{1/2}$ as the constant appearing in (1) is responsible for the fact that $\hbar$ and m appear in (2) in the usual way. In stochastic terms, $(\hbar/m)^{1/2}$ gives the size of the quantum fluctuations. This said, we will put $\hbar = m = 1$ in the rest of the paper.

Equation (4) is particularly interesting; it says that stochastic mechanics and ordinary quantum mechanics make the same predictions for the same position measurement experiments. Since all measurements are ultimately position measurements, the two theories are experimentally indistinguishable.

The motion of a single spinless particle in $\mathbb{R}^3$ does not at all exhaust the scope of stochastic mechanics; likewise, our discussion of even this simple case has skipped over much. For a detailed discussion, see the book [*1*] by Nelson. The Guerra-Morato variational principle was introduced in [*2*].

The intriguing relation between stochastic mechanics and ordinary quantum mechanics which we have just described raises a host of questions, and there has been much discussion of stochastic mechanics in the recent literature. (See, for instance, the bibliography to [*1*].) Here we are concerned with the following two questions:

1) Do the sample paths of the diffusions in stochastic mechanics have a physically reasonable behavior?

Given a positive answer to this question, we ask:

2) Does the direct probabilistic analysis of stochastic mechanical sample paths provide a new means of studying quantum systems?

Even now, there are only a few results bearing on the first question despite the fact that since the sample paths of the diffusions are supposed to be the possible actual particle motions, the question is basic to stochastic mechanics. Our results in this paper provide a strong positive answer to this question in the context of potential scattering. We will discuss other results bearing on this question later.

We will not answer the second question here; however, at the end of the paper we will be able to rephrase it as a specific mathematical problem.

Turning now to potential scattering, it is eveident that the mathematical structures of stochastic mechanics and ordinary quantum mechanics are sufficiently different that the usual objects of interest—wave operators and the S-matrix—cannot be the focus of attention here. In order to get a grip on potential scattering in this context, we begin with the simplest intuitive picture of a scattering experiment.

Suppose one has two particles, a target and a projectile. The projectile is given a known initial momentum p_i, so that in the beginning of the experiment if moves freely toward the target from far away. After a while, the projectile enters the region where there is an appreciable interaction with the target. To focus on essentials, we suppose the target to be sufficiently heavy that it may be regarded as fixed, and we suppose that the target exerts its influence on the projectile by means of a potential $V(x)$. Eventually, the projectile emerges from the region where the scattering forces are appreciable; and it settles down to free motion with a new momentum, the final momentum p_f.

A theory of potential scattering should provide a precise definition of the final momentum, preferably one that is experimentally accessible, and it should give a method of computing this final momentum given the initial state of the projectile and the interaction between the target and the projectile.

The final momentum is not measured directly in scattering experiments; one only measures positions and times directly. One method of measuring a final momentum is the following: Suppose that the particle is known to be at the scattering center at time 0, and suppose it is detected at $\xi \in \mathbb{R}^3$ at time T. (Perhaps at time T it activates a scintillation counter located at ξ.) If the distance between the target and the detector is much, much greater than the range of the interaction, we assume that during most of its flight, the projectile traveled nearly freely with momentum close to p_f. In this case we should have:

$$\frac{1}{T}\xi \approx p_f$$

Note that we are still using the convention $m = 1$.

Therefore, instead of studying the time evolution of the momentum itself, we will study the time evolution of $\frac{1}{t}\xi(t)$ where $\xi(t)$ is the position of the particle at time t. Our program can be roughly summarized as follows: Given a potential $V(x)$, identify those stochastic mechanical diffusions permitted under $V(x)$ which eventually leave the region where scattering forces are strong, and for these diffusions, show that

$$\lim_{t\to\infty} \frac{1}{t}\xi(t) \equiv p_f$$

exists pathwise with probability one. Shucker [3] has proved such a result when the potential is identically zero. We will give further references to the literature when precise results have been stated.

Here we will prove such a result under fairly general conditions on the potential V(x). The rest of the paper is organized in the following way:

Section I begins with some probabilistic definitions required for a precise discussion of the problem, and it concludes with our main probabilistic estimate. This relates the problem to a question about the asymptotic evolution of familiar quantum observables under the Schroedinger equation.

Section II contains the analysis of the Schroedinger equation which we are led to in the first section. Here we will rely heavily on a result of Kato and Jensen [4]. It is in this section that we specify the class of potentials with which we will work.

Section III begins with a full statement of our main result, and it concludes with a discussion of it in relation to the two questions raised in this introduction.

SECTION I

In the rest of this paper, we will formulate our statements in precise mathematical terms, and we will begin with an explanation of what is meant by a solution to the stochastic differential equation (1). Definitions and discussion of the measure theoretic terms that follow can be found in [5].

Let Ω be the space of all continuous functions $\omega : \mathbb{R} \mapsto \mathbb{R}^n$ given the topology of uniform convergence on compacts. This just means that a neighborhood base at $\omega \in \Omega$ is given by the sets of the form

$$\{ \eta \in \Omega \mid \sup_{t \in K} | \eta(t) - \omega(t) | < \varepsilon \}$$

for $\varepsilon > 0$ and K a compact subset of $\mathbb{R}$. The t configuration function, $\xi(t)$, is defined by

$$\xi(t) : \omega \mapsto \omega(t) \in \mathbb{R}^n$$

Let $\mathscr{B}$ be the Borel field on Ω; then it is a theorem (see [6]) that $\mathscr{B}$ is generated by the configuration functions; that is $\mathscr{B} = \sigma \{ \xi(t) \mid t \in \mathbb{R} \}$. Define the increasing family of σ-fields $\mathscr{P}_t$, $t \in \mathbb{R}$, by $\mathscr{P}_t = \sigma \{ \xi(s) \mid s \leqslant t \}$. $\mathscr{P}_t$ is the past at time t. A function is $\mathscr{P}_t$ measurable in case it depends on configurations up to, and not after, time t. (We speak of configurations because we are thinking of $\mathbb{R}^n$ as the configuration space of some dynamical system.)

Now let Pr be a Borel probability measure on Ω. Then the functions $\xi(t)$ become random variables, and $t \mapsto \xi(t)$ becomes a stochastic process adapted to the filtration $\mathscr{P}_t$. (This just means that each $\xi(t)$ is $\mathscr{P}_t$ measurable.)

Finally, let f be a measurable function on Ω. The integral of f over Ω with respect to Pr,

$$\int_{\Omega} f(\omega)\mathrm{Pr}\,(d\omega)$$

is called the expectation of f, and is denoted by $\mathrm{E}(f)$ or just $\mathrm{E}f$.

In this set up, all the functions involved in defining our diffusion process are always the same, and we get different diffusions by choosing different probability measures Pr. The solution to our stochastic differential equation will then be a probability measure on Ω; nonetheless, we will follow convention and say that under Pr, $t \mapsto \xi(t)$ is a solution of (I) in case

$$t \mapsto w(t) \equiv \xi(t) - \xi(0) - \int_0^t b(\xi(\tau), \tau)d\tau \tag{6}$$

is a Brownian motion started at 0 under Pr. Rearranging terms and formally taking differentials, (6) becomes (1), which is really just a shorthand notation for (6). Further detail can be found in [*1*] and [*7*].

In studying equation (1), the first difficulty to be surmounted is the singular nature of the function $b(x, t)$ in stochastic mechanical problems. A glance at the formula (3) shows that whenever the corresponding solution $\psi(x, t)$ of the Schroedinger equation has a node, the drift field is extremely singular. For a long time, there was no general existence theorem for the solutions of (1) suitable for stochastic mechanics, even in the case where the potential is identically zero.

However, we have recently proved [*7*] an existence theorem for the solutions of (1) which in particular states the following:

Let $\mathrm{V}(x)$ be a Rellich class potential; that is

$$\mathscr{D}(\mathrm{V}) \supset \mathscr{D}(\Delta) \quad \text{and} \quad \| \mathrm{V}\varphi \| \leqslant a \left\| \frac{1}{2}\Delta\varphi \right\| + b \| \varphi \| \qquad \text{for all} \quad \varphi \in \mathscr{D}(\Delta)$$

for some $a < 1$ and $b > 0$, so that $\mathrm{H} \equiv -(1/2)\Delta + \mathrm{V}$ is self adjoint on $\mathscr{D}(\Delta)$. (If A is an operator, then $\mathscr{D}(\mathrm{A})$ denotes the domain of A.) Let $\psi_0 \in \mathrm{L}^2(\mathbb{R}^n)$ be such that $\| x\psi_0 \| < \infty$ and $\| \nabla\psi_0 \| < \infty$. (That is, $x\psi_0$ and the distributional gradient of ψ_0 are square integrable functions, or put differently, ψ_0 is in the form domains of both x^2 and $-\Delta$).

Then if $b(x, t)$ is defined in terms of $\psi(x, t) \equiv \exp(-it\mathrm{H})\psi_0(x)$ by the formula (2), the stochastic differential equation (1) has a solution, and this solution is unique under a natural continuity condition which we will not pause to describe.

Remarks. — First, the condition that $\mathrm{V}(x)$ be Rellich class is really not a restriction; Rellich class includes all the potentials of interest here and most potentials of physical interest, but it is easy to see that the proof given in [*7*] works for a much wider class of potentials.

Next, from (4) it is clear that the condition $\| x\psi_0 \|^2 < \infty$ is equivalent to the condition $E|\xi(0)|^2 < \infty$ on the process $t \mapsto \xi(t)$ at time 0. It is also true, but less clear, that the condition $\| \nabla\psi_0 \| < \infty$ has a probabilistic meaning in terms of the behavior of the diffusion $t \mapsto \xi(t)$ at time 0. We will discuss this later when it is more convenient, but see [7] for details.

Now, given a solution $t \mapsto \xi(t)$ of (1), we are interested in the evolution of $\frac{1}{t}\xi(t)$, so we define

$$\pi(t) \equiv \frac{1}{t}\xi(t)$$

Since $t \mapsto \xi(t)$ satisfies (1), it is easy to see that $t \mapsto \pi(t)$ satisfies

$$\pi(t) - \pi(s) = -\int_s^t \frac{1}{\tau}(\pi(\tau) - b(\xi(\tau), \tau))d\tau + \int_s^t \frac{1}{\tau} dw(\tau) \tag{7}$$

Where the last integral in (7) is an Ito stochastic integral; see [1].

What we want to do is to show that the process $t \mapsto \pi(t)$ settles down to a final value along each path (for a set of paths with probability one), so we want to estimate

$$\Pr\{\sup_{s,t>T} |\pi(t) - \pi(s)| > \varepsilon\}$$

and show that this tends to zero as T tends to infinity. To do this, we will analyze the two terms on the right hand side of (7) separately.

LEMMA 1. — Let $t \mapsto \xi(t)$ be a solution of (1), and let $t \mapsto \psi(t)$ be the corresponding solution of the Schroedinger equation. Let p and x be the usual momentum and positions operators in quantum mechanics. Suppose that

$$\left\| \left(p - \frac{x}{t}\right)\psi(t) \right\| \leqslant g(t)$$

where $\frac{1}{t}g(t)$ is integrable on $[c, \infty)$ for some $0 < c < \infty$. Then

$$\lim_{t\to\infty} \frac{1}{t}\xi(t) \equiv p_f$$

exists pathwise with probability one.

Remarks. — One of the conditions under which we have proved the existence of solutions to (1) was that $\psi(0)$ belong to the form domains of both x^2 and $-\Delta$; that is, $\psi(0) \in \mathscr{Q}(x^2) \cap \mathscr{Q}(-\Delta)$. When the potential is Rellich class, then $\mathscr{Q}(x^2) \cap \mathscr{Q}(-\Delta)$ is invariant under $\exp(-itH)$, so in this case $\left(p - \frac{x}{t}\right)\psi(t)$ is defined and finite for all times t. (See [7].)

Furthermore, assume $V(x)$ is identically zero and that $\psi_0 \in \mathcal{D}(x^2) \cap \mathcal{D}(-\Delta)$. Then the quantum mechanical time evolution is given by

$$\psi(x, t) = (2\pi i t)^{-n/2} \int e^{i|x-y|/2t} \psi_0(y) dy \tag{9}$$

and a simple computation using this formula shows that

$$\left\| \left(p - \frac{x}{t} \right) \psi(., t) \right\| = \frac{1}{t} \| x\psi_0 \| \tag{10}$$

Therefore, in the free case, the condition (8) is satisfied whenever the conditions of our existence theorem are satisfied.

Proof of Lemma 1. — We will estimate the probabilities

$$\Pr \left\{ \int_T^\infty \frac{1}{t} | \pi(t) - b(\xi(t), t) | dt > \varepsilon \right\} \tag{11}$$

$$\Pr \left\{ \sup_{T_1 > T} \left| \int_T^{T_1} \frac{1}{t} dw(t) \right| > \varepsilon \right\} \tag{12}$$

The estimate for (12) is entirely standard; the Ito stochastic integral is a martingale, and Doob's martingale maximal inequality applies and gives us

$$\left\| \sup_{T_1 > T} \left| \int_T^{T_1} \frac{1}{t} dw(t) \right| \right\|_{L^2(\Pr)} \leqslant 2 \sup_{T_1 > T} \left\| \int_T^{T_1} \frac{1}{t} dw(t) \right\|_{L^2(\Pr)}$$

Then since

$$E \left| \int_T^{T_1} \frac{1}{t} dw(t) \right|^2 = \int_T^{T_1} \frac{1}{t^2} dt \leqslant \frac{1}{T}$$

the L^2-Chebychev inequality gives us:

$$\Pr \left\{ \sup_{T_1 > T} \left| \int_T^{T_1} \frac{1}{t} dw(t) \right| > \varepsilon \right\} \leqslant \frac{2}{\varepsilon^2 T} \tag{13}$$

(Ito integrals, martingales, Doob's inequality, and the Chebychev inequalities are discussed in [*1*] and [*8*].)

To estimate (11), we need results special to the problem. Define the time dependent vector fields u and v by

$$u(x, t) = \begin{cases} \operatorname{Re} \dfrac{\nabla\psi}{\psi}(x, t), & \psi(x, t) \neq 0 \\ 0, & \psi(x, t) = 0 \end{cases} \qquad v(x, t) = \begin{cases} \operatorname{Im} \dfrac{\nabla\psi}{\psi}(x, t), & \psi(x, t) \neq 0 \\ 0, & \psi(x, t) = 0 \end{cases} \tag{14}$$

so that $b(x, t) = u(x, t) + v(x, t)$. These vector fields have an important probabilistic interpretation which gives rise to their names: u is called the osmotic velocity, and v is called the current velocity. See [*1*] for the probabilistic meaning of u and v; here we will simply proceed to the following computation:

$$\begin{aligned}\left\|\left(p-\frac{x}{t}\right)\psi(.,t)\right\|^2 &= \int_{\mathbb{R}^n}\left|-i\frac{\nabla\psi(x,t)}{\psi(x,t)}-\frac{x}{t}\right|^2 |\psi(x,t)|^2 dx \\ &= \int_{\mathbb{R}^n}\left|v(x,t)-\frac{x}{t}\right|^2 \rho(x,t)dx + \int_{\mathbb{R}^n} |u(x,t)|^2\rho(x,t)dx \\ &= \mathrm{E}\,|v(\xi(t),t)-\pi(t)|^2 + \mathrm{E}\,|u(\xi(t),t)|^2 \end{aligned} \tag{15}$$

Now to estimate (11) we will use the L^1-Chebychev inequality and the above computation.

$$\begin{aligned}\varepsilon \Pr\left\{\int_T^\infty \frac{1}{t}|\pi(t)-b(\xi(t),t)|\,dt>\varepsilon\right\} &\leqslant \\ \mathrm{E}\int_T^\infty \frac{1}{t}|\pi(t)-b(\xi(t),t)|\,dt &= \int_T^\infty \frac{1}{t}\mathrm{E}\,|\pi(t)-b(\xi(t),t)|\,dt \\ &\leqslant \int_T^\infty \frac{1}{t}\,(\mathrm{E}\,|\pi(t)-b(\xi(t),t)|^2)^{1/2}dt \\ &\leqslant \int_T^\infty \frac{\sqrt{2}}{t}(\mathrm{E}\,|v(\xi(t),t)-\pi(t)|^2+\mathrm{E}\,|u(\xi(t),t)|^2)^{1/2} \\ &= \int_T^\infty \frac{\sqrt{2}}{t}\left\|\left(p-\frac{x}{t}\right)\psi(.,t)\right\| \leqslant \sqrt{2}\int_T^\infty \frac{1}{t}g(t)dt \quad \text{by (8)}.\end{aligned}$$

Therefore

$$\Pr\left\{\int_T^\infty \frac{1}{t}|\pi(t)-b(\xi(t),t)|\,dt>\varepsilon\right\} \leqslant \frac{\sqrt{2}}{\varepsilon}\int_T^\infty \frac{1}{t}g(t)dt \tag{16}$$

Now an application of the Borel-Cantelli lemma finishes the proof. Our estimates guarantee that for each positive integer n, we can find a time T_n large enough that

$$\Pr\left\{\bigcup_{s,t>T_n}\left\{|\pi(t)-\pi(s)| \; 1\frac{1}{n}\right\}\right\} < 2^{-n}$$

Let $A_n \equiv \bigcup_{s,t>T_n}\left\{|\pi(t)-\pi(s)|>\frac{1}{n}\right\}$. (It is not hard to show that A_n is measurable.) Then by the Borel-Cantelli lemma

$$\Pr\left\{\bigcap_{m=1}\bigcup_{n>m} A_n\right\} = 0$$

which is exactly the statement that $\lim_{t\to\infty}\pi(t)$ exists on a set of probability one.

SECTION II

The purpose of this section is to establish an inequality of the form

$$\left\| \left(p - \frac{x}{t}\right) \exp(-it\mathrm{H})\psi_0 \right\| \leqslant \mathrm{C}(1 + |t|)^{-\alpha} \quad \text{for some } \mathrm{C}, \alpha > 0$$

for a large class of potentials V and initial wave functions ψ_0, and then to express our conditions on ψ_0 directly in terms of the corresponding diffusion $t \mapsto \xi(t)$.

We will begin with a quick discussion of the analogous problem in classical mechanics. Our analysis of the quantum problem will follow the classical lines as closely as possible.

Let the potential V(x) be a C^1 function on $\mathbb{R}$ such that

$$|\nabla \mathrm{V}(x)| \leqslant \mathrm{C}(1 + |x|)^{-(1+\varepsilon)} \qquad \text{for all} \quad x \in \mathbb{R}^n$$

and such that

$$|\nabla \mathrm{V}(x) - \nabla \mathrm{V}(y)| \leqslant \mathrm{C}|x - y| \qquad \text{for all} \quad x, y \in \mathbb{R}^n \tag{17}$$

where C and ε are positive constants. A trajectory $t \mapsto \xi(t)$ in $\mathbb{R}^n$ is a solution of Newton's equations with initial velocity v_0 and initial position ξ_0 in case

$$\frac{d^2}{dt^2}\xi(t) = -\nabla \mathrm{V}(\xi(t)) \qquad \frac{d}{dt}\xi(0) = \mathrm{V}_0 \qquad \xi(0) = \xi_0 \tag{18}$$

Since in (17) we suppose V(x) to satisfy a global Lipschitz condition, the initial value problem (18) has unique global solutions. For each of these solutions, the energy $\frac{1}{2}\left(\frac{d}{dt}\xi(t)\right)^2 + \mathrm{V}(\xi(t)) = \mathrm{E}(v_0, \xi_0)$ is a constant of the motion.

We will say that a solution of (18) is a scattering motion in case $\mathrm{E}(v_0, \xi_0) > 0$ and

$$\liminf_{t \to \pm\infty} |\xi(t)| = \infty$$

In this case we will write $(v_0, \xi_0) \in \mathscr{S}$. We will say that a solution of (18) is a bound motion in case

$$\limsup_{t \to \pm\infty} |\xi(t)| < \infty \qquad \square$$

In this case we will write $(v_0, \xi_0) \in \mathscr{B}$. In general there will be solutions of (18) with

$$\lim_{t \to \pm\infty} \inf \xi(t)| = \infty \qquad \text{and} \qquad \limsup_{t \to -\infty} |\xi(t)| < \infty$$

and *vice-versa.* Such motions are negligible in the following sense:

THEOREM (Schwarzschilds' Theorem). — $|\mathbb{R}^{2n} - (\mathcal{S} \cup \mathcal{B})| = 0$.

For the proof, see [*9*]. (If $A \subset \mathbb{R}^n$ is measurable, $|A|$ denotes the Lebesgue measure of A.)

Clearly, bound motions should not be expected to have any sort of final momentum. Moreover, from the definition, it is not immediately clear that we have any control over the rate at which scattering motions move off to infinity. The following lemma, found in Simon's article [*10*], provides such control.

LEMMA 2. — (Simon) If $t \mapsto \xi(t)$ is a scattering motion, then

$$\liminf_{t\to\pm\infty} \frac{1}{t} |\xi(t)| > 0$$

For the proof, see [*10*]; we emphasize that the lower bound this lemma provides us with depends on the particular trajectory considered.

Now, given a solution $t \mapsto \xi(t)$ of (18), let $p(t) = \frac{d}{dt}\xi(t)$ be the momentum at time t, and let $\pi(t)$ denote $\frac{1}{t}\xi(t)$. Define $z(t) = \pi(t) - p(t)$. This last quantity is introduced because $\frac{d}{dt}\xi(t) = -\frac{1}{t}z(t)$, so if $z(t)$ goes to zero fast enough, then $\pi(t)$ will tend to a limit. Finally, we will write $F(t) = -\nabla V(\xi(t))$.

LEMMA 3. — Let $t \mapsto \xi(t)$ be a scattering motion. Then there exists a constant $C > 0$ so that

$$|z(t)| \leqslant C(|t|^{-1} + |t|^{-\varepsilon}) \qquad \text{for all } t \quad \text{with} \quad |t| \geqslant 1$$

Proof. — Differentiating, we find

$$\frac{d}{dt}z(t) + \frac{1}{t}z(t) = F(t) \tag{19}$$

The general solution of (19) on $[1, \infty)$ is given by

$$z(t) = \frac{z(1)}{t} - \frac{1}{t}\int_1^t sF(s)ds$$

By lemma 2, for large s, $sF(s)$ goes as $Cs.s^{-(1+\varepsilon)} = Cs^{-\varepsilon}$. From here the proof is easily completed.

We now give a classical analog of our main result.

THEOREM. — Let V satisfy (17), and let $t \mapsto \xi(t)$ be a scattering motion satisfying (18). Then both of the following limits exist:

$$\lim_{t\to-\infty} \pi(t) \equiv p_i \qquad \lim_{t\to+\infty} \pi(t) \equiv p_f$$

Proof. — If $t_1, t_2 > T > 1$, then $|\pi(t_1)-\pi(t_2)| \leqslant \int_T^\infty \frac{1}{t}|Z(t)|\,dt$ and similarly for $t_1, t_2 < T < -1$.

We now turn to the analogous problem in ordinary quantum mechanics. To avoid dealing with dimension dependent constants, we will now restrict our attention to the physically interesting case of three dimensions. We will also need more stringent conditions on the potential. Let $V(x)$ be a potential on $\mathbb{R}^3$ satisfying

$$\sup_{x\in\mathbb{R}^3} |V(x)|(1+|x|^2)^{\beta/2} < C < \infty$$

$$\sup_{x\in\mathbb{R}^3} |\nabla V(x)|(1+|x|^2)^{\beta/2} < C < \infty$$

for some $\beta > 3$.

For such a potential, $H \equiv -\frac{1}{2}\Delta + V$ is a self adjoint operator on $\mathscr{D}(\Delta)$. We will now prepare to characterize scattering solutions of the Schroedinger equation (2) for such a potential. If $\psi(x,t)$ is a solution of (2), we will often write $\psi(t)$ for $\psi(.,t)$.

Under our conditions, H will have a finite number of eigenvalues, all of them non positive and of finite multiplicity. Let $\mathscr{H}_b$ be the span of the corresponding eigenvectors in $L^2(\mathbb{R}^3)$, and let $\mathscr{H}_c = \mathscr{H}_b^\perp$ so that $L^2(\mathbb{R}^3) = \mathscr{H}_b \oplus \mathscr{H}_c$.

As in the classical case, the scattering motions are to be those that leave every bounded region in both directions of time. We will say that a solution of (2) is a scattering motion in case

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{|x|<R} |\psi(x,t)|^2 dx\,dt = 0$$

for all $R > 0$. We will say that it is a bound motion in case

$$\lim_{R\to\infty} \sup_{t\in\mathbb{R}} \int_{|x|>R} |\psi(x,t)|^2 dx = 0$$

The quantum analog of Schwarzchild's theorem is the RAGE theorem:

THEOREM. — (Ruelle, Amrien, Georgescu, Enss). Let $t \mapsto \psi(t)$ solve (2). Then:

$t \mapsto \psi(t)$ is a scattering motion precisely when $\psi(0) \in \mathscr{H}_c$.
$t \mapsto \psi(t)$ is a bound motion precisely when $\psi(0) \in \mathscr{H}_b$.

For the proof, see [*11*]. The version proved there is much more general than the version above, but we don't need the added generality here.

There is no simple analog of Lemma 2 in quantum mechanics. The problem is that the estimate on the rate at which the particle eventually left the scattering center depended on the particular trajectory considered. In ordinary quantum mechanics, one cannot discuss the time evolution trajectory by trajectory—the option just does not exist. Working much harder, one can get a uniform estimate on the rate at which certain motions leave the scattering center. This is given by the Jensen-Kato local decay theorems in [*4*].

We will now briefly sketch their approach and introduce some terminology and notation.

While $\exp(-itH)$ is of course unitary from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$, it can also be considered as an operator between various weighted Sobolev spaces, and in this case it will no longer be unitary. The work of Jensen and Kato shows that for the proper choice of these spaces, the operator norm of $\exp(-itH)P_c$, where P_c is the orthogonal projection onto $\mathscr{H}_c$ in $L^2(\mathbb{R}^3)$, decays in time as $0(t^{-3/2})$ or as $0(t^{-1/2})$, depending on spectral properties of H. Let $H_{m,s}$ denote the weighted Sobolev space with norm

$$\| \phi \|_{m,s} = \| (1 + |x|^2)^{s/2}(1 - \Delta)^{m/2}\phi \|_{L^2(\mathbb{R}^3)}$$

Under the condition $(1 + |x|^2)^{\beta/2}V(x) < C$ with $\beta > 3$, multiplication by V is a compact operator from $H_{1,0}$ to $H_{-1,\beta'}$ for any β' in $(2, \beta)$. Let $R_0(0)$ be convolution with $(2\pi|x|)^{-1}$; i. e., $\left(-\frac{1}{2}\Delta\right)^{-1}$. Then $R_0(0)$ is a compact operator on, say, $H_{1,-1}$. ($H_{1,-1}$ can be replaced by $H_{1,-s}$ for any $\left(\frac{1}{2}, \beta - \frac{1}{2}\right)$ without otherwise changing anything below.) Let K_λ be the null space of $1 + \lambda R_0(0)V$ on $H_{1,-1}$. By compactness, $K_\lambda = 0$ for all but a discrete set of λ in $\mathbb{R}$. Since $-\frac{1}{2}\Delta$ is injective on $H_{1,-1}$, K_1 coincides with the null space of $-\frac{1}{2}\Delta + V$ on $H_{1,-1}$. If $\phi \in K_1$ happens to be in $L^2(\mathbb{R}^3)$, then it is a zero eigenvector. Otherwise, it is called a zero resonance. Following Jensen and Kato, we will say that 0 is a regular point of H in case the null sace of $1 + R_0(0)V$ is 0 in $H_{1,-1}$. The discussion above shows that this is the generic case. One of their results is the following:

THEOREM. — (Jensen-Kato) Suppose the potential V satisfies the conditions (20), and suppose that 0 is a regular point of H. Let $\psi_0 \in H_{0,3} \cap \mathscr{H}_c$. Then

$$\| \exp(-itH)\psi_0 \|_{0,-3} = 0(t^{-3/2}) \quad \text{as} \quad t \to \infty$$

For the proof, see theorem 10.3 of [*4*]. Note that the finite collection of eigenvectors of H have uniform exponential decay since 0 is a regular point of H. (This follows from standard exponential decay theorems; see for instance [*12*].) Therefore, $H_{0,3} \cap \mathscr{H}_c$ is dense in $\mathscr{H}_c$.

In the rest of the paper, we will work with potentials which satisfy the following condition:

The potential V is such that 0 is a regular point of H. (21)

We now return to our scattering problem. Let x and p denote the usual position and momentum operators on $L^2(\mathbb{R}^3)$. In analogy with the classical case, we define

$$\pi(t) \equiv \exp(itH)\left(\frac{1}{t}x\right)\exp(-itH) \qquad Z(t) \equiv \pi(t) - p(t)$$

where $p(t)$ is of course given by $\exp(itH)p\exp(-itH)$. Differentiating with harmless formality, we obtain

$$\frac{d}{dt}Z(t) + \frac{1}{t}Z(t) = \nabla V(t)$$

Where ∇V is the obvious multiplication operator. Putting

$$F(t) - \exp(itH)\nabla V\exp(-itH)$$

we have

$$\frac{d}{dt}Z(t) + \frac{1}{t}Z(t) = -F(t) \tag{22}$$

which has the same form as (19). (The formality is removed by showing that (22) holds in the operator sense on a dense subset of $\mathscr{H}_c$; this is easy.)

The next lemma provides us with the estimate we seek.

LEMMA 4. — Let the potential V satisfy the conditions (20) and (21), and suppose $\psi_0 \in \mathscr{H}_c \cap H_{0,3} \cap \mathscr{D}(-\Delta)$. Then

$$\left\|\left(p - \frac{x}{t}\right)\exp(-itH)\psi_0\right\| \leqslant C(1 + |t|)^{-1/2} \tag{23}$$

Remark. — Since $H_{0,3} \cap \mathscr{D}(-\Delta) \subset \mathscr{D}(x^2) \cap \mathscr{D}(-\Delta)$, the remarks preceding the proof Lemma 1 imply that $\left(p - \frac{x}{t}\right)\exp(-itH)\psi_0 \in L^2(\mathbb{R}^3)$ for all times t.

Proof of Lemma 4. — Since $z(t)\psi_0$ satisfies

$$Z(t)\psi_0 = \frac{1}{t}Z(1)\psi_0 - \frac{1}{t}\int_1^t sF(s)\psi_0 ds$$

we estimate

$$\begin{aligned} \| F(s)\psi_0 \| &= \| (\nabla V)\exp(-isH)\psi_0 \| \\ &= \| (\nabla V)(1+|x|^2)^{3/2}(1+|x|^2)^{-3/2}\exp(-isH)\psi_0 \| \\ &\geqslant C \| (1+|x|^2)^{-3/2}\exp(-isH)\psi_0 \| = C \| \exp(-isH)\psi_0 \|_{0,-3} \end{aligned}$$

since by our conditions on V, $\sup_{x\in\mathbb{R}^3} |V(x)|(1+|x|^2)^{3/2} < C < \infty$. Since $\psi_0 \in H_{0,3} \cap \mathscr{H}_c$, the Jensen-Kato theorem tells us that $\| F(s)\psi_0 \| = 0(s^{-3/2})$ as s tends to infinity. Therefore

$$\| Z(t)\psi_0 \| \leqslant \frac{1}{t} \| Z(1)\psi_0 \| + \frac{1}{t}\int_1^t s \| F(s)\psi_0 \| ds \leqslant C(1 + |t|)^{-1/2}$$

From (22) and the definition of $z(t)$, we have (23) for positive times; a similar argument takes care of negative times.

We will close this section with a lemma which we will use in next section to identify the distribution of the stochastic mechanical final momentum and the distribution of the ordinary quantum mechanical final momentum.

The distribution of the ordinary quantum mechanical final momentum can be computed using the wave operators $\Omega_\pm$. Under our conditions on the potential, these exist and are complete. This means in particular that

$$\lim_{t\to\infty} \exp(itH_0)\exp(-itH)P_c \equiv \Omega_+^*$$

exists strongly and is a unitary map of $\mathscr{H}_c$ onto $L^2(\mathbb{R}^3)$. Furthermore, it is not difficult to show that for $\psi_0 \in \mathscr{D}(-\Delta)$; i. e., $\| \nabla\psi_0 \| < \infty$,

$$\lim_{t\to\infty} \exp(itH)p\exp(-itH)P_c\psi_0 = \Omega_\pm p \Omega_+^* \psi_0 \tag{24}$$

strongly; $\Omega_+ p \Omega_+^*$ is the quantum mechanical final momentum.

For $\psi_0 \in L^2(\mathbb{R}^3)$, let $\psi(x,t) \equiv \exp(-itH)\psi_0(x)$, and let

$$\phi(x,t) \equiv \exp(-itH_0)\Omega_+^*\psi_0(x) \qquad \text{where} \qquad H_0 \equiv -\frac{1}{2}\Delta .$$

Then

$$\lim_{t\to\infty} \| \psi(.,t) - \phi(.,t) \| = 0 \tag{25}$$

Next let $\tilde{\phi}(x,t) \equiv (it)^{-3/2}\exp(-|x|^2/2)\hat{\phi}_0(x/t)$ where $\hat{\phi}_0$ is the Fourier transform of $\phi(.,0)$. Then Dollard's lemma [*13*] asserts that

$$\lim_{t\to\infty} \| \phi(.,t) - \tilde{\phi}(.,t) \| = 0 \tag{26}$$

and so using this and (25) we have

$$\lim_{t\to\infty} \| \psi(.,t) - \tilde{\phi}(.,t) \| = 0 \tag{27}$$

Now let $\tilde{\rho}(x,t) \equiv |\tilde{\phi}(x,t)|^2$; note that $\int_{\mathbb{R}^3} \tilde{\rho}(x,t)dx = 1$ for all times t. Then

$$\begin{aligned} |\rho - \tilde{\rho}|(x,t) &= |\psi^*(\psi - \tilde{\phi}) - \tilde{\phi}(\tilde{\phi}^* - \psi^*)|(x,t) \\ &\leqslant |\psi^*(\psi - \tilde{\phi})|(x,t) + |\tilde{\phi}(\tilde{\phi}^* - \psi^*)|(x,t) \end{aligned}$$

so integrating and applying the Schwarz inequality, we get:

$$\| \rho(.,t) - \tilde{\rho}(.,t) \|_{L^1(\mathbb{R}^3)} \leqslant 2 \| \psi(.,t) - \tilde{\phi}(.,t) \|_{L^2(\mathbb{R}^3)} \tag{28}$$

Introducing the change of variables $x=kt$, computing the left hand side of (28), and using (27) we get

$$\lim_{t\to\infty} \int | t^3\rho(kt, t) - | \hat{\phi}_0(k) |^2 | dk = 0$$

We have now done all the groundwork for the following lemma.

LEMMA 5. — Let $\psi(t) \equiv \exp(-itH)\psi_0$ with $\psi_0 \in \mathcal{H}_c \cap \mathcal{D}(-\Delta)$. Let $A \subset \mathbb{R}^3$ be a measurable set. Then the probability that the quantum mechanical final momentum is in A is given by

$$\lim_{t\to\infty} t^3 \int_A \rho(kt, t) dk$$

Proof. — Given A, let P_A be the associated spectral projection of the self adjoint operator p. Similarly, let P_A^f be the associated spectral projection of the final momentum operator $\Omega_+ p \Omega_+^*$. Then $P_A^f = \Omega_+ P_A \Omega_+^*$. The probability that the quantum mechanical final momentum is in A is given by

$$\langle \psi, P_A^f \psi_0 \rangle = \langle \psi_0, \Omega_+ P_A \Omega_+^* \psi_0 \rangle = \langle \Omega_+^* \psi_0, P_A \Omega_+^* \psi_0 \rangle = \int_A | \hat{\phi}_0(k) |^2 dk$$

On the other hand, we have from (29) that

$$\lim_{t\to\infty} \int_A t^3\rho(kt, t) dk = \int_A | \hat{\phi}_0(k) |^2 dk$$

which gives us the lemma.

SECTION III

Now we return to stochastic mechanics. In this section, we will assume that we are given a potential V on $\mathbb{R}^3$ satisfying the conditions (20) and (21); and we will assume that we are given a probability measure Pr on Ω under which $t \mapsto \xi(t)$ is a critical diffusion for the Guerra-Morato variational principle, and hence so that there is a solution $t \mapsto \psi(t)$ of the Schroedinger equation which specifies the drift $b(x, t)$ and density $\rho(x, t)$ of $t \mapsto \xi(t)$.

Our quantum mechanical criterion for $t \mapsto \psi(t)$ to be a scattering motion translates directly into probabilistic terms. We will call $t \mapsto \xi(t)$ a scattering motion in case

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} Pr\{ | \xi(t) | < R \} dt = 0 \qquad \text{for all} \quad R > 0 \qquad (30)$$

We emphasize that (30) says nothing about the behavior of the diffusion $t \mapsto \xi(t)$ pathwise.

There is one further point to be clarified. We have mentioned in Section 1 that the condition $\| \nabla\psi(0) \| < \infty$ can be expressed in terms of the diffusion $t \mapsto \xi(t)$. Let u and v be the osmotic and current velocities introduced in equation (14). Then a computation using (14) shows that

$$\| \nabla\psi(0) \|^2 = \mathrm{E}(u^2(\xi(0), 0) + v^2(\xi(0), 0) \tag{31}$$

Again, we refer to [1] for details of the probabilistic meaning of the osmotic and current velocities; and here we just remark that since the finiteness of the left hand side is a time independent property, so is the finiteness of the right and side. (When the potential is Rellich class, $\| \nabla\psi(0) \|$ is uniformly bounded on any compact interval provided only that $\| \nabla\psi(0) \| < \infty$; see [7].)

Collecting our results, we have our main theorem.

THEOREM. — Let V be a potential satisfying the conditions (20) and (21). Let $t \mapsto \xi(t)$ be a stochastic mechanical scattering motion under V, and suppose that

$$\mathrm{E} \mid \xi(0) \mid^6 + \mathrm{E}(u^2(\xi(0), 0) + \mathrm{V}^2(\xi(0), 0)) < \infty$$

Then the following limits exist pathwise with probability one:

$$\lim_{t\to\infty} \frac{1}{t} \xi(t) \equiv p_f \qquad \lim_{t\to-\infty} \frac{1}{t} \xi(t) \equiv p_i \tag{32}$$

Furthermore, the random variable p_f is square integrable and has the same distribution as does the quantum mechanical final momentum for the solution $t \mapsto \psi(t)$ of the Schroedinger equation corresponding to $t \mapsto \xi(t)$, and a similar statement applies to p_i.

Proof. — Since $t \mapsto \xi(t)$ is a scattering motion, the RAGE theorem implies that $\psi(0) \in \mathcal{H}_c$. Then since $\mathrm{E} \mid \xi(0) \mid^6 + \mathrm{E}(u^2(\xi(0), 0) + v^2(\xi(0), 0)) < \infty$ implies that $\psi(0) \in \mathrm{H}_{0,3} \cap \mathcal{D}(-\Delta)$, and since V satisfies the conditions (20) and (21), it follows from Lemma 4 that

$$\left\| \left(p - \frac{x}{t}\right)\psi(t) \right\| < \mathrm{C}(1 + |\, t \,|)^{-1/2}$$

for some finite constant C. Then by Lemma 1, we have that $\lim_{t\to\infty} \frac{1}{t} \xi(t)$ exists pathwise with probability one. A similar argument tales care of negative times.

Next, since the density of $\xi(t)$ is $\rho(x, t)$, the density of $\frac{1}{t}\xi(t)$ is $t^3\rho(kt, t)$: i. e., $\Pr\left\{\frac{1}{t}\xi(t)\in A\right\} = t^3\int_A \rho(tk, t)dk$. Then by (32) and the dominated convergence theorem

$$\Pr\{ p_f \in A \} = \lim_{t\to\infty} \int_A t^3\rho(kt, t)dk$$

and so by Lemma 5, $\Pr\{ p_f \in A \}$ is equal to the probability that the quantum mechanical final momentum for $t \mapsto \psi(t)$ is in the set A. Again, a similar argument works for p_i.

There are some previous results in the direction of this theorem. As mentioned, Shucker has proved a version of this theorem in the case where V is identically zero. His method is quite different from ours. In the case $V = 0$, the quantum mechanical time evolution is given by the convolution formula (9). Shucker uses this formula to obtain pointwise (as oposed to our L^2) control over $b(x, t)$. His approach uses the fact that the Fourier transform of an L^1 function is continuous, and for this reason he requires that $x \mapsto x\psi(x)$ be an L^1 function. This is a rather unnatural condition in the context of stochastic mechanics; it cannot be expressed directly in terms of the diffusion $t \mapsto \xi(t)$.

We emphasize that all the conditions in our theorem are expressed in probabilistic terms. Had we not been insistent on this point and had, for instance, been willing to impose such conditions as « ... $\psi(0)$ is an analytic vector for the dilation operator... », we could have used the results of Perry in [*14*] to prove a version of our theorem even for a wide class of long range potentials.

We also pause to remark that our Lemma 1 together with the remarks that precede its proof suffice to prove a version of Shucker's theorem under the more natural condition $E|\xi(0)|^2 + E(u^2(\xi(0), 0) + v^2(\xi(0), 0)) < \infty$; that is, under the conditions for which we have proven the existence of solutions to (1)

Because Shucker's approach relies heavily on the explicit representation (9), it is very difficult to generalize it to situations with interaction. Recently, Biler has done this in the one dimensional case [*15*]. Here the Schroedinger equation becomes an ordinary differential equation, and this simplifies matters greatly.

There is also an interesting discussion of this subject, from yet another point of view, in [*18*]. I would like to thank the referee for providing me with this reference.

Finally, Nelson [*1*] has treated the case $V = 0$ with the initial condition

$$\psi_0(y) = (2\pi a^2)^{-3/2}e^{-|x|^2/2a^2} \tag{34}$$

In this case the integral in (9) can be computed and one then finds

$$b(x, t) = \frac{t - a}{a^2 + t^2} x$$

The virtue of this special case is that with the above coefficients, the stochastic differential equation (1) is linear, and its solution is reduced to quadratures. Nelson uses this to then explicitly construct p_i and p_f in this case. This gives us a great deal of additional information; in particular it allows us to compute the correlation of p_i and p_f.

The answer [1] is that the correlation matrix of p_i and p_f is given by

$$- \delta^{ij} e^{-\pi}$$

Note that this is independent of the width of the Gaussian packet in (34).

This striking result may unsttle one at first; although the random variables p_i and p_f have the same distribution, they are almost, but not quite, uncorrelated. This means, as Nelson has pointed out, that there is no pathwise analog of the S-matrix in stochastic mechanics, or at least that the obvious analog is not the identity when $V = 0$.

This contradicts neither laboratory experience nor ordinary quantum mechanics—where it is impossible to even make a statement about such correlations. The S-matrix arises in an idealization of scattering experiments which is useful and natural in the context of ordinary quantum mechanics. Taken literaly, however, the S-matrix picture involves an experimenter preparing a very diffuse wave function far, far from the target which bunches up under nearly free evolution to be localized near the target and scattered. In actual experiments, the experimenter produces a reasonably well localized state with reasonably well defined initial momentum—perhaps a minimal uncertainty state. His actual experiment is then described by giving the asymptotic time evolution of this state in the far future. What the asymptotic time evolution would have been in the remote past had the state not been prepared at some finite time does not enter any experiment.

Therefore, we do not regard the lack of an S-matrix as a real problem. Instead of asking « What is the S-matrix? » we will ask « Given the initial state, what is the asymptotic behavior? » The results of this section show that answers to this question in stochastic mechanics and ordinary quantum mechanics provide consistent descriptions of experiment. They also show that any method of computing the final momentum distribution in ordinary quantum mechanics is a method of computing the final momentum distribution in stochastic mechanics.

On the other hand, the Schroedinger equation only enters stochastic mechanics as a trick linearization of the coupled non linear system of partial differential equations for $b(x, t)$ and $\rho(x, t)$ which the Guerra-Morato variational principle leads to. Again, we refer to [1] for details, but just as the Lagrangean variational principle of classical mechanics leads to Newton's second law, the Guerra-Morato variational principle leads to a stochastic version of Newton's second law: Stochastic time derivative operators D and D_* (one for each direction of time) are defined in [1], and in terms of these, the stochastic version of Newton's second law is

$$\frac{1}{2}(DD_* + D_*D)\xi(t) = -\nabla V(\xi(t)) \tag{35}$$

The left hand side of this equation can be expressed in terms of u and v and their derivatives; and Nelson has shown in [16] that a solution of the stochastic differential equation (1) for some drift field $b(x, t)$ satisfies (35), then there is a solution $\psi(x, t)$ of (2) so that $b(x, t)$ is given by (3) and $\rho(x, t)$ is given by (5).

The outstanding problem in stochastic mechanics is to understand the stochastic acceleration and hence equation (35) in direct probabilistic terms. This is the sixth problem in Nelson's list at the end of [1]. The problem is not that the stochastic acceleration isn't well defined; it is. The problem is that the definition is too analytic.

To clarify the issue consider a diffusion which satisfies the stochastic differential equation

$$d\xi(t) = b(\xi(t), t)dt + dw(t), \qquad \xi(0) = \xi_0 \in \mathbb{R}^n \tag{36}$$

where the drift field b has bounded derivitives of all orders. Then

$$D\xi(t) = b(\xi(t), t),$$

and so b may be thought of as a stochastic velocity.

Our probabilistic intuition into the meaning of (36) leads to the following method of constructing approximate sample paths of the solution on, say, the time interval [0, 1].

First fix an integer N, and divide [0, 1] into N equal segments with left endpoints $t_0, \ldots, t_{N-1}$. Next let S_i be a copy of the unit sphere in $\mathbb{R}^n$, and let $\Omega_N = \prod_{i=0}^{N-1} S_i$ with the natural topology, and let μ^N be the normalized uniform measure on Ω_N. Simply denote the points $(\omega_0, \ldots, \omega_{N-1})$ of Ω_N by ω. We then have the N independent identically distributed random variables (isotropic random directions) X_i given by

$$X_i(\omega) \equiv \omega_i$$

Now, to each $\omega \in \Omega_N$ we associate a trajectory $t \mapsto \Lambda_\omega(t)$. Define the $N + 1$ points x_i in $\mathbb{R}^n$ by $x_0 = \xi_0$ and

$$X_{i+1}(\omega) = X_i(\omega) + b(X_i(\omega), t_i)\frac{1}{N} + \sqrt{\frac{1}{N}}\, X_i(\omega)$$

The trajectory $t \mapsto \Lambda_\omega(t)$ is given by connecting these points in sequence with straight line segments to be traversed at constant speed.

The map $\Lambda : \Omega_N \to \Omega$ given by $\omega \mapsto \Lambda_\omega$ is continuous and therefore measurable. Let $\Pr^N$ be the image of μ^N under Λ: that is

$$\Pr^N(A) = \mu^N(\Lambda^{-1}(A))$$

and let Pr be the measure on Ω which solves (36). Prohorkov has shown that in this case

$$\lim_{N \to \infty} \Pr^N = \Pr$$

weakly; see [*16*].

This construction can be, and is, used to generate the sample paths of a diffusion on a computer. Notice in particular that no partial differential equation for a transition function, say, had to to be solved. The method is direct and probabilistic.

Now suppose we are given (35) and a potential V instead of (36) and a drift field $b(x, t)$. At present, the only method of constructing a diffusion satisfying (35) for the given potential V and satisfying (36) for some drift field $b(x, t)$ is to solve the corresponding Schroedinger equation in order to arrive at a familiar stochastic differential equation of type (36) for a given drift field $b(x, t)$.

The problem is to bypass this step with a direct method using the stochastic acceleration to construct sample paths—something like the method sketched above in the stochastic velocity case. The solution of this problem should be greatly facilitated by the results of [*7*] which provide the measures one would try to approximate.

The results of the last section show that such a method of solving (35) would provide a new numerical method of working with quantum scattering problems. There are many open problems in quantum mechanics which analysis with the Schroedinger equation has failed to solve. The hope that direct probabilistic analysis in stochastic mechanics would succeed may still seem optimistic. However, the possibility is certainly worth much further investigation.

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