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G. C. MCVITTIE

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Elliptic functions in spherically symmetric solutions of Einstein's equations

by

G. C. McVITTIE
Mathematical Institute, The University, Canterbury,
Kent, CT2 7NF, England

ABSTRACT. — The metrics considered are of the form

$$ds^2 = y^2dt^2 - S^2(t)e^{\eta}\{dr^2 + f^2(r)d\Omega^2\},$$

where y, η are functions of $z = \ln Q(r) - \ln S(t)$, $y = 1 - \frac{1}{2}d\eta/dz$ and f, Q are functions to be determined, S(t) being left arbitrary. Previous work (McVittie 1933, 1966, 1967) has shown that solutions of Einstein's equations in this case can be reduced to those of three ordinary second-order differential equations, one of which—the y-equation—determines y, and hence η, as functions of z, while the other two give f and Q. This last pair have been dealt with in McVittie 1967. But hitherto only certain elementary function solutions of the y-equation have been found through the trial and error discovery that dy/dz can be a quadratic function of y. In Part I of the present paper, the y-equation, which contains two constants α and β^2, is solved under certain conditions in terms of elliptic functions. When $\alpha \neq 0$, this is only possible if α and β^2 are related by $8\alpha^2 + 5\alpha + \beta^2 = N\alpha^2$ where $N = 2$ or 14; for other values of N the functions satisfying the y-equation are unknown. When $\alpha = 0$, elliptic function solutions are always possible. Previously obtained metrics, with one exception, are shown to arise when the elliptic functions degenerate to elementary functions and a number of new metrics of this kind are produced. The exceptional case, called the Peculiar Integral, can be shown to be the singular solution of the y-equation when $\alpha = 0$, but a proof that this is so when $\alpha \neq 0$ has not been found.

In Part II, papers published between 1967 and 1980 by authors whose

names occur in the subheadings, (*i*) to (*viii*), of Sec. 8 are shown to include metrics found in Part I, now called « McV-metrics ». Direct coordinate-transformation is used to convert a Part II to a McV-metric, in all but one case. This occurs in (*iv*) of Sec. 8 where an indirect procedure had to be employed because the finite transformation equations were not found.

PART I

THE SIMILARITY METHOD FOR SPHERICALLY SYMMETRIC METRICS

1. Introduction

The origin of this investigation lies in my having noticed early in 1980 that P. A. M. Dirac (1979) had rediscovered, by an ingenious method of his own, one member of a class of solutions of Einstein's field equations I had found in 1933 (McVittie 1933; hereinafter McV33). This early paper of mine had been used by Noerdlinger and Petrosian (1971) in their work on the effect of cosmological expansion on self-gravitating ensembles of particles and it received a favourable notice in A. K. Raychaudhuri's (1979) book on cosmology. Dirac makes an important comment about the Schwarzschild solution, which I would rephrase thus: If we believe that the metric of the universe is of Robertson-Walker type, then at large distances from the central body of the Schwarzchild space-time, the metric should tend to that of a non-empty cosmological model universe and not to the empty and flat Minkowski metric. Of course, to achieve this, the Schwarzschild metric must be modified to some extent, as indeed happens in all the McV33 metrics. These may be expressed in physically dimensionless variables (McVittie 1979) as

$$ds^2 = \mathrm{A}dt^2 - \mathrm{B}(d\bar{r}^2 + \bar{r}^2 d\Omega^2), \tag{1.1}$$

where $\bar{r}$ is the radial coordinate, and

$$\mathrm{A}(\bar{r}, t) = \left[1 - \frac{m}{2\bar{r}} e^{-g/2}\left(1 + \frac{\bar{r}^2}{4\mathrm{R}^2}\right)^{1/2}\right]^2 \times \left[1 + \frac{m}{2\bar{r}} e^{-g/2}\left(1 + \frac{\bar{r}^2}{4\mathrm{R}^2}\right)^{1/2}\right]^{-2},$$

$$\mathrm{B}(\bar{r}, t) = e^g\left[1 + \frac{m}{2\bar{r}} e^{-g/2}\left(1 + \frac{\bar{r}^2}{4\mathrm{R}^2}\right)^{1/2}\right]^4 \left(1 + \frac{\bar{r}^2}{4\mathrm{R}^2}\right)^{-2},$$

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2,$$

m and $1/R^2$ are constants and g is an arbitrary function of t. The Dirac (1979) result corresponds to setting $1/R^2 = 0$. However, as was shown later (McVittie 1966; hereinafter McV66) a much simpler form of the metric is obtained in terms of a constant k and radial coordinate r where

$$\left.\begin{array}{ll}\dfrac{1}{R^2} = k, & \bar{r} = \{ 2\tan(r/2), \quad r, \quad 2\tanh(r/2) \}, \\ & \text{according as } k = (+1, 0, -1).\end{array}\right\} \tag{1.2}$$

This turns (1.1) into

$$ds^2 = \left(\frac{1 - Q/S}{1 + Q/S}\right)^2 dt^2 - S^2(1 + Q/S)^4(dr^2 + \Sigma^2 d\Omega^2), \tag{1.3}$$

where

$$\left.\begin{array}{l} S(t) = e^{g/2}, \\ \sum = (\sin r, \quad r, \quad \sinh r), \quad Q = \left(\dfrac{m}{4\sin\frac{1}{2}r}, \ \dfrac{m}{2r}, \ \dfrac{m}{4\sinh\frac{1}{2}r}\right)\end{array}\right\} \tag{1.4}$$

for the three values of k. It will be observed that in regions where Q/S is small compared with unity, the metric (1.3) becomes a Robertson-Walker cosmological metric whose scale-factor is S and curvature constant is k. A year later (McVittie 1967; hereinafter McV67) it was shown that (1.3) was itself a special case of a class of similarity solutions whose definition will be given in Sec. 2.

In correspondence with Professor Dirac (1980) he remarked à propos of McV33, that had he known of this paper earlier, he would have been saved a great deal of work. It occurred to me that this might be true of other investigators also, especially as McV66 and McV67 are rarely referred to in the literature. It turns out that a number of re-discoveries have taken place since 1967 and they will form the subject of Part II. But I also wondered if the lack of interest might be due to the trial and error methods employed in both McV33 and McV67. Solutions appear out of the blue and reasons for their existence are not apparent. Hence in Part II have completed the investigation I began some fifty years ago, in so far as this can be done in terms of known functions. These prove to be the elliptic functions and their « degenerate » elementary function forms. A number of new solutions have been found and, perhaps more importantly, the reasons for the existence of the solutions in McV67 have been elucidated.

2. **Basic equations.**

The general spherically symmetric metric may be written

$$ds^2 = e^{\nu} d\bar{t}^2 - e^{\omega} dr^2 - e^{\psi} d\Omega^2, \tag{2.1}$$

where all variables and functions are dimensionless and ν, ω, ψ are functions of $(\bar{t}, r)$. If a « dot » denotes a partial derivative with respect to $\bar{t}$ and a prime one with respect to r, the $T^{\bar{t}}_{r}$, $T^{r}_{\bar{t}}$ components of the energy-tensor vanish provided that

$$2\dot{\psi}' + \psi'\dot{\psi} - \psi'\dot{\omega} - \nu'\dot{\psi} = 0\,. \tag{2.2}$$

This is, of course, the condition that a co-moving coordinate-system be possible. One way of integrating the equation is to assume that

$$\dot{\omega} = \dot{\psi}\,, \tag{2.3}$$

which leads to

$$e^{\psi/2} = e^{\omega/2} f(r)\,, \tag{2.4}$$

$$e^{\nu/2} = \dot{\omega}/\mathrm{T}(\bar{t}) \tag{2.5}$$

where f and T are arbitrary functions of their arguments. The adoption of (2.3) also means that the 3-space $\bar{t}$=constant in (2.1) is conformal to

$$d\sigma^2 = dr^2 + f^2 d\Omega^2 \tag{2.6}$$

which is of constant curvature when

$$f(r) \equiv \Sigma(r) = (\sin r, r, \sinh r)\,. \tag{2.7}$$

The discussion of (2.3) will be resumed in Part II. In Part I the specialization employed in McV67 will be adopted. It is equivalent to assuming, in addition to (2.3), (2.4) and (2.5), that

$$e^{\omega/2} = \mathrm{S}(\bar{t})e^{\eta/2}\,, \tag{2.8}$$

$$e^{\nu/2} = y\,, \tag{2.9}$$

where η and y are functions of the similarity variable z defined by

$$e^{z} = \mathrm{Q}(r)/\mathrm{S}(\bar{t})\,. \tag{2.10}$$

In these formulae η, y and Q are functions of their respective variables and are to be determined from Einstein's equations while S is to remain an arbitrary function of the time, a situation analogous to that in a Robertson-Walker cosmological metric.

The equations (2.8), (2.9) and (2.5) yield

$$y = \frac{2\dot{\mathrm{S}}}{\mathrm{S}}\frac{1}{\mathrm{T}}\left(1 - \frac{1}{2}\eta_z\right), \tag{2.11}$$

where a suffix z, here and elsewhere, means the derivative with respect to z. Thus in the metric (2.1) there occurs the combination

$$e^{\nu}d\bar{t}^2 = \left(1 - \frac{1}{2}\eta_z\right)^2\left(\frac{2\dot{\mathrm{S}}}{\mathrm{S}}\frac{d\bar{t}}{\mathrm{T}(\bar{t})}\right)^2$$

and therefore no generality is lost if the time-coordinate $\bar{t}$ is replaced by t where

$$dt = \frac{2\dot{S}}{S}\frac{1}{T(\bar{t})}d\bar{t}. \tag{2.12}$$

Hence also, S may be regarded as a function of t. In summary the metric of McV67 has the essential form

$$\left.\begin{aligned} ds^2 &= y^2dt^2 - S^2(t)e^{\eta}\{dr^2 + f^2(r)d\Omega^2\}, \\ y &= 1 - \frac{1}{2}\eta_z, \qquad e^z = Q(r)/S(t), \end{aligned}\right\} \tag{2.13}$$

y and η being functions of z. In the sequel, use will often be made of the integral

$$\eta = 2z - 2\int y(z)dz + \ln K, \tag{2.14}$$

where K is an arbitrary constant of integration.

The remaining components of the energy tensor that are not identically zero are T^t_t and T^r_r, T^θ_θ, T^ϕ_ϕ, the last three representing the stress in the material. The only further use of Einstein's field equations, apart from the vanishing of T^t_r, T^r_t, that will be made is the condition of the isotropy of stress, namely,

$$T^r_r = T^\theta_\theta = T^\phi_\phi. \tag{2.15}$$

As is shown in the appendix to McV67 this leads to

$$(f_{rr}/f - f_r^2/f^2 + 1/f^2)y + \{Q_{rr}/Q - Q_rf_r/(fQ)\}(y - y^2 + y_z) + \{\{y(1-y)(y-2) - (3-y)y_z + y_{zz}\}(Q_r/Q)^2 = 0 \tag{2.16}$$

where the suffix r denotes a derivative with respect to r. Since r and z are independent variables their separation in (2.16) is possible by the introduction of two constants a and b. The equation breaks up into the three equations

$$Q_{rr}/Q - Q_rf_r/(fQ) = a(Q_r/Q)^2, \tag{2.17}$$

$$f_{rr}/f - f_r^2/f^2 + 1/f^2 = b(Q_r/Q)^2, \tag{2.18}$$

$$y_{zz} + (a - 3 + y)y_z + y\{a + b - 2 - (a-3)y - y^2\} = 0. \tag{2.19}$$

The success of this operation depends on preserving the form (2.6) for the spatial part of the metric, in other words, it is not permissible to transform r to $\bar{r}$ so that

$$d\sigma^2 = dr^2 + f^2(r)d\Omega^2 = P(\bar{r})(d\bar{r}^2 + \bar{r}^2d\Omega^2), \tag{2.20}$$

in spite of the fact that such a transformation is mathematically possible. This point is overlooked in the one-sentence reference to McV67 found in Kramer *et al.* (1980).

The solutions of (2.17), (2.18) for the interlocking functions f and Q are described in Sec. 2 of the Appendix to McV67 and are not repeated here, though some further reference to them will be made in Part II. Attention will be concentrated on equation (2.19) for y which, of course, carries with it the determination of η through (2.14). It has already been mentioned that only particular solutions of (2.19) have hitherto been available. It is now possible to find the sets of one-parameter solutions to which these particular solutions belong and also to find the primitive of (2.19) in certain cases. Since elliptic functions will be involved, a summary of some of the properties of these functions is given here (Abramowitz and Stegun 1972).

The elliptic functions in question are the Weierstrass P-functions which arise whenever (2.19) basically involves the differential equation

$$\begin{aligned} P_x^2(x) &= 4P^3(x) - g_2 P(x) - g_3 , \\ &= 4(P - e_1)(P - e_2)(P - e_3) . \end{aligned} \tag{2.21}$$

Here $P_x = dP/dx$ and we shall be concerned with real values of x, g_2 and g_3. The roots e_1, e_2, e_3 of the cubic satisfy

$$e_1 + e_2 + e_3 = 0 , \tag{2.22}$$

$$- 4(e_1e_2 + e_1e_3 + e_2e_3) = g_2 , \tag{2.23}$$

$$4e_1e_2e_3 = g_3 . \tag{2.24}$$

The differential equation (2.21) possesses a « homogeneity » property, the strict proof of which will be found in Whittaker and Watson (1920). Suffice it to say here that, if $P(x)$ satisfies (2.21) and if

$$\overline{P}(\overline{x}) = \frac{1}{\lambda^2} P(x), \qquad \overline{x} = \lambda x , \tag{2.25}$$

where λ is a constant, then $\overline{P}(\overline{x})$ satisfies

$$\left(\frac{d\overline{P}}{d\overline{x}}\right)^2 = 4\overline{P}^3(\overline{x}) - \overline{g}_2\overline{P}(\overline{x}) - \overline{g}_3 , \tag{2.26}$$

where

$$\overline{g}_2 = \lambda^{-4} g_2 , \qquad \overline{g}_3 = \lambda^{-6} g_3 . \tag{2.27}$$

When $g_2 = 0$ and $g_3 > 0$ it follows that $\overline{g}_3 = 1$ if $\lambda = g_3^{1/6}$, and the relation between P and $\overline{P}$ may also be written as

$$P(x ; 0, g_3) = g_3^{1/3}\overline{P}(g_3^{1/6}x ; 0, 1) . \tag{2.28}$$

This is the equianharmonic case of the P-function.

The discriminant of the cubic on the right hand side of (2.21) is

$$\Delta = g_2^3 - 27g_3^2 = 16(e_1 - e_2)^2(e_3 - e_1)^2(e_2 - e_3)^2 . \tag{2.29}$$

When two roots of the cubic are equal Δ vanishes, and conversely. The P-function then reduces to an elementary function in three different ways:

i) when $g_2 = 0,\ g_3 = 0 : (e_1 = e_2 = e_3 = 0)$

then

$$P(x) = x^{-2}; \tag{2.30}$$

ii) when $g_2 > 0,\ g_3 > 0 : (e_1 = 2\varepsilon,\ e_2 = e_3 = -\varepsilon,\ \varepsilon > 0)$
then

$$\left.\begin{aligned} g_2 &= 12\varepsilon^2, \qquad g_3 = 8\varepsilon^3, \\ P(x) &= -\varepsilon + 3\varepsilon\{\sin[(3\varepsilon)^{1/2}x]\}^{-2}; \end{aligned}\right\} \tag{2.31}$$

iii) when $g_2 > 0,\ g_3 < 0 : (e_1 = e_2 = \varepsilon,\ e_3 = -2\varepsilon,\ \varepsilon > 0)$
then

$$\left.\begin{aligned} g_2 &= 12\varepsilon^2, \qquad g_3 = -8\varepsilon^3, \\ P(x) &= \varepsilon + 3\varepsilon\{\sinh[(3\varepsilon)^{1/2}x]\}^{-2}. \end{aligned}\right\} \tag{2.32}$$

The three P-functions just listed follow the convention of textbooks on elliptic functions, namely, that the constant of integration in the solution of (2.21) is zero. However, in the sequel it will be necessary to extend the definition (2.30) by the use of the addition-theorem for elliptic functions (Whittaker & Watson 1920) which states that

$$P(x + x_0) = \frac{1}{4}\left\{\frac{P_x(x) - P_x(x_0)}{P(x) - P(x_0)}\right\}^2 - P(x) - P(x_0).$$

If $P(x) = x^{-2}$, $P_x(x) = -2x^{-3}$, and correspondingly for x_0, it follows after some calculation that

$$P(x + x_0) = (x + x_0)^{-2} \tag{2.33}$$

which can replace (2.30) in case (*i*).

3. The equation for y. First integrals.

The manipulation of the equation (2.19) for y is simplified if y, a and b are respectively replaced by V, α and β^2 where

$$y = \mathrm{V} + \gamma - \frac{5\alpha}{2}, \qquad \alpha = \frac{a-3}{5}, \qquad \beta^2 = b + 1, \tag{3.1}$$

γ being a constant to be determined later. Nevertheless certain results will be expressed in terms of a, b since the expressions for f and Q found in McV67 involve these constants. The equation (2.19) then reads

$$\begin{aligned} \mathrm{V}_{zz} &+ \left(\mathrm{V} + \gamma + \frac{5\alpha}{2}\right)\mathrm{V}_z - \mathrm{V}^3 + \left(\frac{5\alpha}{2} - 3\gamma\right)\mathrm{V}^2 \\ &+ \left\{\left(\frac{25\alpha^2}{4} + 5\alpha + \beta^2\right) + 5\alpha\gamma - 3\gamma^2\right\}\mathrm{V} \\ &- \left(\gamma - \frac{5\alpha}{2}\right)\left\{\gamma^2 - \left(\frac{25\alpha^2}{4} + 5\alpha + \beta^2\right)\right\} = 0 \end{aligned} \tag{3.2}$$

and it resembles the equations (6.30) to (6.35) in Kambe's compendium (1971). He appears to have found them in the works of P. Painlevé and E. L. Ince — at the references he gives — but warns the reader that he has either not verified the solutions he quotes or that he has had to correct errors and misprints. It has therefore seemed best to derive the solutions of (3.2) ab initio by noticing that the equation is identifiable with one or other of Kamke's equations if γ is chosen to satisfy

$$\gamma + 5\alpha/2 = 3(5\alpha/2 - 3\gamma) \tag{3.3}$$

whence

$$\gamma = \frac{1}{2}\alpha . \tag{3.4}$$

The equation for y becomes

$$\left.\begin{aligned} & y = V - 2\alpha , \\ & V_{zz} + (V + 3\alpha)V_z - V^3 + \alpha V^2 + CV - 2\alpha(C - 2\alpha^2) = 0 , \\ & C = 8\alpha^2 + 5\alpha + \beta^2 = \frac{1}{25}(8a^2 - 23a + 22) + b . \end{aligned}\right\} \tag{3.5}$$

In the determination of y from this equation α and β (or a and b) will be regarded as given constants. The primitive of the equation for V will therefore involve two additional constants—its constants of integration. When α and β, and therefore C, have arbitrary values, the functions satisfying (3.5) are apparently unknown. Nevertheless, a particular first integral does exist, and is called (A.29) in McV67, which is

$$V_z = C - \frac{7}{4}\alpha^2 - \left(V + \frac{1}{2}\alpha\right)^2 , \tag{3.6}$$

from which V is obtainable by quadratures. This differential equation will be called the Peculiar Integral. It will be shown in the sequel that the primitive of (3.5) is obtainable for certain values of C and, in these cases, the Peculiar Integral leads to the singular solution of (3.5). It had also been found by trial and error that, for certain values of C, other particular first integrals existed, called (A.26), (A.27) and (A.28) in McV67. These were of the form

$$V_z = A_0^* + A_1^* V + A_2^* V^2 , \tag{3.7}$$

where the A_i^* were constants. It is convenient to begin by showing that all the particular first integrals of the form (3.7) are those found in McV67. The treatment is, however, different according as $\alpha \neq 0$ or $\alpha = 0$.

i) When $\alpha \neq 0$ it is possible to introduce U, Z and the constants z_0 and N by

$$Z = \alpha(z - z_0), \qquad U = V/\alpha, \qquad C = N\alpha^2 . \tag{3.8}$$

The equation (3.5), the Peculiar Integral (3.6) and equation (3.7) become, respectively,

$$\left.\begin{aligned} &y = \alpha(\mathrm{U} - 2)\,, \\ &\mathrm{U_{ZZ}} + (\mathrm{U} + 3)\mathrm{U_Z} - \mathrm{U}^3 + \mathrm{U}^2 + \mathrm{NU} - 2(\mathrm{N} - 2) = 0\,, \end{aligned}\right\} \tag{3.9}$$

$$\mathrm{U_Z} = \left(\mathrm{N} - \frac{7}{4}\right) - \left(\mathrm{U} + \frac{1}{2}\right)^2, \tag{3.10}$$

$$\mathrm{U_Z} = \mathrm{A}_0 + \mathrm{A}_1\mathrm{U} + \mathrm{A}_2\mathrm{U}^2\,, \tag{3.11}$$

where the A_i are constants. By evaluating $\mathrm{U_{ZZ}}$ in terms of U from the last equation and substituting into (3.9) one finds the algebraic relation

$$\begin{aligned} &\{\mathrm{A}_0(\mathrm{A}_1+3)-2(\mathrm{N}-2)\} + \{\mathrm{A}_0(2\mathrm{A}_2+1)+\mathrm{A}_1(\mathrm{A}_1+3)+\mathrm{N}\}\,\mathrm{U} \\ &\qquad + (3\mathrm{A}_2+1)(\mathrm{A}_1+1)\mathrm{U}^2+(2\mathrm{A}_2^2+\mathrm{A}_2-1)\mathrm{U}^3 \equiv 0\,, \end{aligned} \tag{3.12}$$

which must be satisfied for all values of U. The coefficient of U^3 vanishes if A_2 is either -1 or $1/2$ and then the coefficient of U^2 can vanish only if $\mathrm{A}_1 = -1$, since $3\mathrm{A}_2+1 \neq 0$ for either value of A_2. The constant term is zero if $\mathrm{A}_0 = \mathrm{N}-2$ and finally the coefficient of U vanishes provided that either $\mathrm{A}_2 = -1$ and N has any value, or $\mathrm{A}_2 = 1/2$ and $\mathrm{N} = 2$, $\mathrm{A}_0 = 0$. Hence the possibilities are:

$$\mathrm{N} \text{ arbitrary}, \mathrm{A}_0 = \mathrm{N} - 2, \mathrm{A}_1 = -1, \mathrm{A}_2 = -1\,;$$

or

$$\mathrm{N} = 2, \mathrm{A}_0 = 0, \mathrm{A}_1 = -1, \mathrm{A}_2 = 1/2\,.$$

The corresponding particular first integrals are

$$\mathrm{U_Z} = \left(\mathrm{N} - \frac{7}{4}\right) - \left(\mathrm{U} + \frac{1}{2}\right)^2, \qquad (\text{any N}) \tag{3.13}$$

$$\mathrm{U_Z} = -\mathrm{U} + \frac{1}{2}\mathrm{U}^2\,, \qquad (\mathrm{N} = 2) \tag{3.14}$$

the first of which is the Peculiar Integral. It is the only particular first integral when $\mathrm{N} \neq 2$, but there are two such integrals when $\mathrm{N} = 2$.

In order to compare these results with their McV67 versions a return is made to y, z, a and b. From (3.5) and $\mathrm{C} = \mathrm{N}\alpha^2$ it follows that for any N

$$\left.\begin{aligned} \beta^2 - 1 = b &= (\mathrm{N} - 8)\alpha^2 - 5\alpha - 1 \\ &= \frac{1}{25}\left(\mathrm{N} - \frac{7}{4}\right)(a - 3)^2 - \frac{1}{4}(a - 1)^2\,. \end{aligned}\right\} \tag{3.15}$$

The equation (3.13) becomes

$$\begin{aligned} \frac{dy}{dz} &= \frac{\mathrm{N} - 8}{25}(a - 3)^2 - (a - 3)y - y^2 \\ &= (a + b - 2) - (a - 3)y - y^2\,, \end{aligned} \tag{3.16}$$

which is therefore the $(y; a, b)$ form of the Peculiar Integral. Since $a \neq 3$, this integral is (A.29) of McV67 when $a + b - 2 \neq 0$, and is (A.27) when $a + b - 2 = 0$, or $N = 8$ [1]. On the other hand when $N = 2$, it follows from (3.15) that

$$b = -\frac{1}{25}(6a^2 - 11a + 4), \tag{3.17}$$

the Peculiar Integral is

$$\frac{dy}{dz} = -\left\{\frac{6}{25}(a-3)^2 + (a-3)y + y^2\right\},$$

and the equation (3.14) is

$$\frac{dy}{dz} = \frac{1}{5}(a-3)y + \frac{1}{2}y^2, \tag{3.18}$$

which is (A.26) in McV67.

ii) When $\alpha = 0$, the equations (3.5) do not require the use of V and it is also possible to introduce the constant z_0 and to write

$$Z = z - z_0. \tag{3.19}$$

Then the equation for y is

$$\left.\begin{aligned} y_{ZZ} + yy_Z - y^3 + \beta^2 y &= 0, \\ C = \beta^2 &= b + 1, \end{aligned}\right\} \tag{3.20}$$

where $\beta^2 \gtreqless 0$ since b may have any value. The Peculiar Integral (3.6) and equation (3.7) may then be written respectively as

$$y_Z + y^2 - \beta^2 = 0, \tag{3.21}$$

$$y_Z = B_0 + B_1 y + B_2 y^2, \tag{3.22}$$

where the B_i are constants. Proceeding as was done for (3.11), it turns out that

$$B_0 = \beta^2, \qquad B_1 = 0, \qquad B_2 = -1$$

or

$$B_0 = -\frac{1}{2}\beta^2, \qquad B_1 = 0, \qquad B_2 = \frac{1}{2}.$$

The first set of values reproduces the Peculiar Integral (3.21); the second provides another first integral of (3.20), namely,

$$2y_Z - y^2 + \beta^2 = 0. \tag{3.23}$$

Since $\alpha = 0$ is $a = 3$ and $\beta^2 = b + 1$ it follows that (3.21) is (A.29) of McV67, while (3.23) is (A.28).

The four cases in McV67 in which y is expressible in terms of elementary

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[1] The metric obtained from (3.13) with $N = 8$ was analysed in detail in McVittie and Stabell (1967).

functions are therefore covered by the equations (3.13), (3.14), (3.21) and (3.23). These are all the elementary function solutions of the basic equation (3.5) when y_z is a quadratic function of y.

4. The functions y and η for $\alpha \neq 0$.

This is the general case and is the most intractable. The primitive of (3.9) for U, and therefore for y, has not been found but one-parameter solutions can be determined. In the first place, there is the set obtainable by integrating the equation (3.13) which defines the Peculiar Integral, and also by integrating (3.14) for which N = 2. Many of these can be found in the literature, for example in McVittie and Stabell (1967, 1968). A complete list is given below, in which items (*a*) to (*c*) refer to the Peculiar Integral while (*d*) refers to (3.14). Once U is obtained, y follows from $y = \alpha(U - 2)$ and η from (2.14). The values of β^2 and b, derived from (3.15) are also listed.

$$\left.\begin{aligned} &a)\ \left(N-\frac{7}{4}\right)=j^2>0,\ \beta^2-1=b=\frac{1}{4}\{(2j-5)\alpha-2\}\{(2j+5)\alpha+2\},\\ &y=-\frac{\alpha}{2}\{(5-2j)\Gamma e^{2\alpha jz}+(5+2j)\}(1+\Gamma e^{2\alpha jz})^{-1},\\ &e^{\eta}=Ke^{(2+5\alpha+2\alpha j)z}(1+\Gamma e^{2\alpha jz})^{-2},\\ &\Gamma=e^{-2\alpha j z_0}=\text{constant},\qquad j=\pm\left(N-\frac{7}{4}\right)^{\frac{1}{2}}. \end{aligned}\right\} \quad (4.1a)$$

$$\left.\begin{aligned} &b)\quad \left(N-\frac{7}{4}\right)=0,\qquad \beta^2-1=b=-\frac{1}{4}(5\alpha+2)^2,\\ &y=\left(1-\frac{5}{2}\alpha\Gamma-\frac{5}{2}\alpha z\right)(\Gamma+z)^{-1},\\ &e^{\eta}=Ke^{(2+5\alpha)z}(\Gamma+z)^{-2},\\ &\Gamma=-z_0=\text{constant}. \end{aligned}\right\} \quad (4.1b)$$

$$\left.\begin{aligned} &c)\ \left(N-\frac{7}{4}\right)=-k^2,\ k^2>0,\ \beta^2-1=b=-\frac{1}{4}\{4k^2\alpha^2+(5\alpha+2)^2\}\\ &y=-\frac{\alpha}{2}\{(5-2k\Gamma)\cos x+(5\Gamma+2k)\sin x\}\{\cos x+\Gamma\sin x\}^{-1},\\ &e^{\eta}=Ke^{(2+5\alpha)z}(\cos x+\Gamma\sin x)^{-2},\\ &x=\alpha kz,\ \Gamma=\tan(\alpha k z_0)=\text{constant},\ k=\pm\left(\frac{7}{4}-N\right)^{\frac{1}{2}}. \end{aligned}\right\} \quad (4.1c)$$

$$\left.\begin{aligned} &d)\quad N=2,\ \beta^2-1=b=-(3\alpha+1)(2\alpha+1),\\ &y^2=4\alpha^2\Gamma^2(\Gamma+e^{-\alpha z})^{-2},\\ &e^{\eta}=Ke^{2(1+2\alpha)z}(\Gamma+e^{-\alpha z})^4,\\ &\Gamma=-e^{-\alpha z_0}=\text{constant}. \end{aligned}\right\} \quad (4.1d)$$

In these solutions N is fixed as soon as (α, β^2) are given. Thus they contain only one arbitrary constant, Γ, and are therefore one-parameter solutions of the basic equation (3.5).

It will now be shown that there are other one-parameter solutions obtainable by a method suggested by Kamke's treatment of his equations (6.30) to (6.35). Let

$$\zeta = e^{-Z} = e^{-\alpha(z-z_0)}, \qquad U = -\zeta w(\zeta), \tag{4.2}$$

so that

$$\left.\begin{aligned} U_Z &= -\zeta U_\zeta = \zeta(w + \zeta w_\zeta), \\ U_{ZZ} &= -\zeta(w + 3\zeta w_\zeta + \zeta^2 w_{\zeta\zeta}). \end{aligned}\right\} \tag{4.3}$$

Substitution into (3.9) then gives

$$\left.\begin{aligned} &y = -\alpha(\zeta w + 2), \\ &w_{\zeta\zeta} + ww_\zeta - w^3 + \frac{N-2}{\zeta^2}w + \frac{2(N-2)}{\zeta^3} = 0. \end{aligned}\right\} \tag{4.4}$$

Let it be assumed that

$$w = \frac{v' + h'}{v + h} \qquad ' \equiv \frac{d}{d\zeta} \tag{4.5}$$

where $v(\zeta)$ is the Weierstrass P-function with invariants g_2 and g_3, $h(\zeta)$ is a function to be determined and the prime is introduced for brevity. The assumption means that

$$\begin{aligned} (v')^2 &= 4v^3 - g_2 v - g_3, \\ v'' &= 6v^2 - \frac{1}{2}g_2, \\ v''' &= 12vv'. \end{aligned}$$

Making use of these relations one finds that

$$w_\zeta = \frac{6v^2 - \frac{1}{2}g_2 + h''}{v + h} - \frac{(v' + h')^2}{(v + h)^2},$$

$$w_{\zeta\zeta} = \frac{12vv' + h'''}{v + h} - \frac{3(v' + h')\left(6v^2 - \frac{1}{2}g_2 + h''\right)}{(v + h)^2} + \frac{2(v' + h')^3}{(v + h)^3}.$$

After substitution into the equation for w in (4.4) and re-arrangement of terms there comes

$$\begin{aligned} &\left\{12h + \frac{N-2}{\zeta^2}\right\}vv' - \left\{\frac{d}{d\zeta}\left(12h + \frac{N-2}{\zeta^2}\right)\right\}v^2 \\ &+ \left\{h''' + \frac{N-2}{\zeta^2}h' + \frac{4(N-2)}{\zeta^3}h\right\}v + \left\{g_2 - 2h'' + \frac{N-2}{\zeta^2}h\right\}v' \qquad (4.6) \\ &+ \left\{hh''' + \left(g_2 - 2h'' + \frac{N-2}{\zeta^2}h\right)h' + \frac{2(N-2)}{\zeta^3}h^2\right\} = 0. \qquad (4.6) \end{aligned}$$

If now h is chosen so that the coefficients of vv' and of v^2 vanish, it follows that

$$h = -\frac{N-2}{12}\frac{1}{\zeta^2}. \tag{4.7}$$

The coefficients of v and of v' become, respectively,

$$h''' + \frac{N-2}{\zeta^2}h' + \frac{4(N-2)}{\zeta^3}h = \frac{1}{6}(N-2)(14-N)\frac{1}{\zeta^5},$$

$$g_2 - 2h'' + \frac{N-2}{\zeta^2}h = g_2 + \frac{1}{12}(N-2)(14-N)\frac{1}{\zeta^4},$$

while the term independent of v and v' reduces to

$$+\, g_2\frac{N-2}{6}\frac{1}{\zeta^3}.$$

Therefore the equation (4.6) is satisfied if

$$g_2 = 0 \qquad \text{and} \qquad N = 14 \qquad \text{or} \qquad N = 2.$$

The first condition means that $v(\zeta)$ is the equianharmonic P-function of Sec. 2. The values $N = 14$ and $N = 2$ correspond to the only two cases for which the solution of the equation (3.9) for U is free of movable critical points (Ince 1956*a*). Detailed results for the two cases are given below.

i) CASE $N = 14$

Here, by (4.2), (4.5) and (4.7), it follows that

$$\left.\begin{aligned} U &= -\zeta\frac{v_\zeta + 2\zeta^{-3}}{v - \zeta^{-2}} = -\zeta\frac{d}{d\zeta}[\ln(v - \zeta^{-2})] \\ v_\zeta^2 &= 4v^3 - g_3\,. \end{aligned}\right\} \tag{4.8}$$

Reference to (2.26), (2.28) shows that ζ is replaceable by $\bar{\zeta}$ and v by $\bar{v}$ where

$$\bar{\zeta} = g_3^{1/6}\zeta, \qquad v = g_3^{1/3}\bar{v}(\bar{\zeta}) \tag{4.9}$$

and $\bar{v}$ satisfies

$$(d\bar{v}/d\bar{\zeta})^2 = 4\bar{v}^3 - 1\,.$$

It is easily seen that

$$U = -\bar{\zeta}\frac{d}{d\bar{\zeta}}[\ln\{P(\bar{\zeta};0,1) - \bar{\zeta}^{-2}\}]. \tag{4.10}$$

But by (4.2) and (4.9)

$$\bar{\zeta} = (g_3^{1/6}e^{\alpha z_0})e^{-\alpha z}$$

and therefore $U(z)$ involves the independent constants g_3 and z_0 always in the combination $g_3^{1/6}e^{\alpha z_0}$. If this combination be denoted by Γ, then

$$U(z) = +\frac{1}{\alpha}\frac{d}{dz}\left[\ln\left\{P(\Gamma e^{-\alpha z};0,1) - \frac{e^{2\alpha z}}{\Gamma^2}\right\}\right],$$

the functions y and η are obtained from (3.9) followed by (2.14) and they are

$$\left.\begin{aligned} y^2 &= \left\{ \frac{d}{dz} [\ln \{ P(\Gamma e^{-\alpha z}; 0, 1) - e^{2\alpha z}/\Gamma^2 \} - 2\alpha z] \right\}^2, \\ e^\eta &= K e^{2(1+2\alpha)z} \{ P(\Gamma e^{-\alpha z}; 0, 1) - e^{2\alpha z}/\Gamma^2 \}^{-2}. \end{aligned}\right\} \tag{4.11}$$

This method of solution fails when $g_3 = 0$ if v, by analogy with (2.30), is defined as ζ^{-2}, for then U is indeterminate. However v can be defined by analogy with (2.33) as

$$v = (2\Gamma + \zeta)^{-2}$$

where Γ is a non-zero constant, $z_0 = 0$ and $\zeta = e^{-\alpha z}$ in (4.2)-since only one arbitrary constant can occur in the expression for y. Then U is obtained from (4.8), y from $y = \alpha(U - 2) > 0$ and η from (2.14) to achieve the result

$$\left.\begin{aligned} y^2 &= \alpha^2(1 + 2\Gamma e^{\alpha z})^{-2}(1 + \Gamma e^{\alpha z})^{-2}, \\ e^\eta &= K\Gamma^{-2} e^{2(1-\alpha)z}(1 + 2\Gamma e^{\alpha z})^4(1 + \Gamma e^{\alpha z})^{-2}. \end{aligned}\right\} \tag{4.12}$$

It is not included among the elementary solutions of McV67 because they pre-suppose that y_z is a quadratic function of y whereas here a more complicated relationship occurs. In fact the expression for y may also be written

$$\Gamma e^{\alpha z} = -\frac{3}{4} \pm \frac{1}{4}\left(1 + \frac{8\alpha}{y}\right)^{1/2}$$

whose derivative produces

$$\Gamma e^{\alpha z} = \mp \left(1 + \frac{8\alpha}{y}\right)^{-1/2} y_z/y^2 .$$

Equating the right-hand sides results in

$$(4y_z + y^2 + 8\alpha y)^2 - 9y^2(y^2 + 8\alpha y) = 0 . \tag{4.13}$$

A second elementary function solution is (4.1*a*) with $N = 14$ or $j = \pm 7/2$ and so is derived from the Peculiar Integral. Also $N = 14$ in (3.15) means that

$$\beta^2 - 1 = b = (6\alpha + 1)(\alpha - 1)$$

which, of course, is associated with (4.11) and (4.12) as much as with (4.1*a*).

ii) CASE $N = 2$.

In this case $h = 0$ by (4.7) and (4.2), (4.3) lead, by a method similar to that used for $N = 14$, to

$$U = -\zeta v_\zeta/v, \qquad v_\zeta^2 = 4v^3 - g_3 . \tag{4.14}$$

Hence the expressions for y and η are obtainable from (4.11) by omitting the terms in $e^{2\alpha z}/\Gamma^2$ which represent h. Thus, with $y = \alpha(U - 2)$ also

$$\left.\begin{aligned} y^2 &= \left\{ \frac{d}{dz} [\ln \{ P(\Gamma e^{-\alpha z}; 0, 1) \} - 2\alpha z] \right\}^2, \\ e^\eta &= K e^{2(1+2\alpha)z} \{ P(\Gamma e^{-\alpha z}; 0, 1) \}^{-2}, \end{aligned}\right\} \tag{4.15}$$

with $\Gamma = g_3^{1/6} e^{\alpha z_0}$ as before. These therefore provide a third set of one-parameter solutions which, with one exception, were not found in McV67. The exceptional one occurs when $g_3 = 0$, $(\Delta = 0)$. If by analogy with (2.30) the choice $v = \zeta^{-2}$ is made, the trivial solution $y = 0$ of (2.19) is reached via (4.14). But if v is identified with

$$v = (\Gamma + \zeta)^{-2}, \tag{4.16}$$

where Γ is a non-zero constant, and if $z_0 = 0$, $\zeta = e^{-\alpha z}$ in (4.2), then (4.1d) is recovered. Thus (4.15) and (4.1d) are connected by a particular choice of the value of g_3. The special case of (4.1d) in which $a = \frac{1}{2}$, $b = 0$, $\left(\alpha = -\frac{1}{2}, \beta^2 = 1\right)$ is labelled (A.30) in McV67 and was there used to investigate a collapse problem.

The other elementary function solution arises from the Peculiar Integral and is (4.1a) with $N = 2$ or $j = \pm\frac{1}{2}$. Substitution of $j = +\frac{1}{2}$ in (4.1a) produces

$$\left.\begin{aligned} \beta^2 - 1 = b &= -(2\alpha + 1)(3\alpha + 1) \\ y^2 &= \alpha^2(2\Gamma + 3e^{-\alpha z})^2(\Gamma + e^{-\alpha z})^{-2} \\ e^\eta &= 4K e^{(1+2\alpha)z}(\Gamma + e^{-\alpha z})^{-2}. \end{aligned}\right\} \tag{4.17}$$

The corresponding results for $j = -\frac{1}{2}$ are obtainable from these by writing $\overline{\Gamma}$ for $1/\Gamma$ and $\overline{K} = K/\Gamma^2$ thus producing no essential difference. Comparison with (4.1d) shows that, though the values of b are the same, the solutions are otherwise different.

In this section therefore it has been possible to obtain one-parameter solutions of the basic equation (3.5) for y when $\alpha \neq 0$ which involve the equianharmonic P elliptic function. It is also shown how the elementary function solutions of McV67—apart from those produced by the Peculiar Integral—are deduced from them. But the one-parameter solutions only occur when α and β^2 depend on one another in the two ways represented by the values 2 and 14 of the constant $N = C\alpha^{-2}$. Evaluation of the Peculiar Integral provides further elementary function solutions. But whether this Integral does or does not lead to a singular solution of (3.5) has not been established.

5. The functions y, η when $\alpha = 0$.

In this case (3.19), (3.20) apply, namely,

$$\left.\begin{aligned} Z &= z - z_0, \qquad \beta^2 = b + 1, \\ y_{ZZ} &+ y y_Z - y^3 + \beta^2 y = 0. \end{aligned}\right\} \tag{5.1}$$

This equation for y is also one which is free from moveable critical points (Ince 1956*b*). It will shortly be shown that y is expressible in terms of a P-function $w(z)$ where

$$y = \frac{1}{n} w_Z (w - m)^{-1}, \qquad w_Z^2 = 4w^3 - g_2 w - g_3, \tag{5.2}$$

and n, m, g_2, g_3 are constants. In the course of the proof it will turn out that m and g_2 are related by

$$g_2 = 12m^2. \tag{5.3}$$

Accepting this for the moment, certain results useful in the sequel can be established. By (5.2) and some calculation

$$y_Z = \frac{1}{n}\left\{ \frac{w_{ZZ}}{w-m} - \frac{w_Z^2}{(w-m)^2} \right\} = \frac{1}{n}\left\{ 2(w-m) + \frac{8m^3 + g_3}{(w-m)^2} \right\},$$

$$y^2 = \frac{1}{n^2}\left\{ 4(w-m) + 12m - \frac{8m^3 + g_3}{(w-m)^2} \right\}.$$

Hence

$$n y_Z + n^2 y^2 - 12m = 6(w - m), \tag{5.4}$$

$$2n y_Z - n^2 y^2 + 12m = 3\frac{(8m^3 + g_3)}{(w-m)^2}. \tag{5.5}$$

Thus when it is known *a priori* that $g_3 + 8m^3 = 0$, (5.5) yields an equation of type (3.22). But if this constant does not vanish, (5.4) and (5.5) yield a first integral of the equation (5.1) for y, namely,

$$\left(2y_Z - ny^2 + 12\frac{m}{n}\right)\left(y_Z + ny^2 - 12\frac{m}{n}\right)^2 = \frac{108}{n^3}(8m^3 + g_3), \tag{5.6}$$

provided, of course, that consistency with (5.2), (5.3) has been secured. It also follows from (5.3) that the discriminant of the equation for w in (5.2) is

$$\Delta = 27(2^6 m^6 - g_3^2)$$

and therefore vanishes if

$$g_3 = \pm 8m^3. \tag{5.7}$$

To show that (5.2) provides a solution of (5.1), the first two derivatives of y are calculated; they are

$$y_Z = \frac{1}{n}\left\{ 2(w-m) + \frac{1}{2}(g_2 - 12m^2)(w-m)^{-1} + (g_2 m + g_3 - 4m^3)(w-m)^{-2} \right\}$$

$$y_{ZZ} = y\left\{ 2(w-m) - \frac{1}{2}(g_2 - 12m^2)(w-m)^{-1} - 2(g_2 m + g_3 - 4m^3)(w-m)^{-2} \right\}.$$

Substitution into (5.1) turns this equation into

$$\frac{y}{n^2}\,[2(n^2+n-2)(w-m) + (n^2\beta^2 - 12m) + \frac{1}{2}(2+n-n^2)(g_2 - 12m^2)(w-m)^{-1} + (1+n-2n^2)(g_2 m + g_3 - 4m^3)(w-m)^{-2}] = 0.$$

Non-trivial solutions for y occur when each term in the square bracket vanishes by a suitable determination of the constants n, m, g_2 and g_3. Two sets of values are possible, namely:

CASE A: $n=1$, $m=\frac{\beta^2}{12}$, $g_2 = 12m^2 = \frac{\beta^4}{12}$, g_3 is arbitrary;

CASE B: $n=-2$, $m=\frac{1}{3}\beta^2$, $g_2 = 12m^2 = \frac{4}{3}\beta^4$, $g_3 = -8m^3 = -\frac{8}{27}\beta^6$.

Since the constants of Case B imply that $g_3 + 8m^3 = 0$, equation (5.6) is not derivable; instead (5.5) yields

$$y_Z + y^2 - \beta^2 = 0$$

which is the Peculiar Integral (3.21) when $\alpha = 0$. It is also evident that in both cases g_2 cannot be negative and (5.3) holds.

The primitive of the equation (5.1) must clearly arise from Case A as this contains the arbitrary constant g_3 to accompany the other arbitrary constant $-z_0$. The first integral (5.6) is now derivable from (5.4) and (5.5) and it will also be assumed to hold « in the limit » when $g_3 + 8m^3$ tends to zero. The identification of the Peculiar Integral with the singular solution of (5.6) may be established as follows. Let the constants in (5.6) refer to Case A and let

$$p = y_Z, \qquad X = 2p - ny^2 + 12\frac{m}{n}, \qquad Y = p + ny^2 - \frac{12m}{n}$$

so that (5.6) becomes

$$F(p, y) \equiv XY^2 - \frac{108}{n^3}(8m^3 + g_3) = 0.$$

This is one of the three necessary conditions for the existence of a singular solution (Ince 1956*c*). The other two are

$$\frac{\partial F}{\partial p} = 6pY = 0, \qquad p\frac{\partial F}{\partial y} = 6npy\left(p - ny^2 + \frac{12m}{n}\right)Y = 0.$$

The three equations can be satisfied only if

$$\left.\begin{aligned} &8m^3 + g_3 = 0, \\ &Y \equiv y_Z + ny^2 - 12m/n = 0. \end{aligned}\right\} \tag{5.8}$$

The values of n and m yield

$$g_3 = -8m^3 = -\frac{\beta^6}{216}, \qquad y_Z + y^2 - \beta^2 = 0, \tag{5.9a, b}$$

and (5.7) indicates that the elliptic functions reduce to elementary ones; they will be found in subsection (*iv*) below. The equation (5.9*b*) is clearly the same as the Peculiar Integral (3.21) which is therefore shown to be a singular integral of the equation for y.

With the constants $n=1$, $m=\beta^2/12$, $g_2=\beta^4/12$, the equations (5.1), (5.2) give the primitive as

$$y = w_Z\left(w - \frac{\beta^2}{12}\right)^{-1}, \qquad w_Z^2 = 4w^3 - \frac{\beta^4}{12}w - g_3. \tag{5.10}$$

Hence, with the aid of (2.14) for η

$$\left.\begin{aligned} y^2 &= \left(4w^3 - \frac{\beta^4}{12}w - g_3\right)\left(w - \frac{\beta^2}{12}\right)^{-2}, \\ e^\eta &= Ke^{2z}\left(w - \frac{\beta^2}{12}\right)^{-2}. \end{aligned}\right\} \tag{5.11}$$

The equations (5.4), (5.5), (5.6) are, respectively,

$$y_Z + y^2 - \beta^2 = 6\left(w - \frac{\beta^2}{12}\right), \tag{5.12}$$

$$2y_Z - y^2 + \beta^2 = 3\left(\frac{\beta^6}{216} + g_3\right)\left(w - \frac{\beta^2}{12}\right)^{-2}, \tag{5.13}$$

$$(2y_Z - y^2 + \beta^2)(y_Z + y^2 - \beta^2)^2 = 108\left(\frac{\beta^6}{216} + g_3\right). \tag{5.14}$$

It is clear that (5.12) is not identifiable with the Peculiar Integral (3.21) by assigning some particular value to the arbitrary constant g_3 in the primitive, thus confirming that (3.21) is indeed singular. On the other hand, the first integral (3.23) is obtainable from the primitive by setting $g_3 = -\beta^6/216$ in (5.13).

In the formulae (5.1) to (5.14) the constant β^2 may be positive, negative or zero. These possibilities may be illustrated by considering those cases of the general elliptic function solution (5.11) that satisfy (5.7), which now reads

$$g_3 = \pm \frac{\beta^6}{216}, \tag{5.15}$$

and also those cases obtained from $\beta^2 = 0$.

i) $\beta^2 > 0$.—Consider first the value $g_3 = -\beta^6/216 < 0$ so that (5.10) is

$$\left.\begin{aligned} y &= w_Z\left(w - \frac{\beta^2}{12}\right)^{-1} \\ w_Z^2 &= 4w^3 - \frac{\beta^4}{12}w + \frac{\beta^6}{216} = 4\left(w - \frac{\beta^2}{12}\right)^2\left(w + \frac{\beta^2}{6}\right). \end{aligned}\right\} \tag{5.16}$$

Since $g_2 > 0$, $g_3 < 0$, it follows that (2.32) applies and that

$$\varepsilon = \frac{\beta^2}{12}, \quad w(Z) = \frac{\beta^2}{12} + \frac{\beta^2}{4}\{\sinh(\beta Z/2)\}^{-2}, \quad y = -\beta \coth(\beta Z/2). \tag{5.17}$$

Thus finally, with $\Gamma = -e^{-\beta z_0}$, the equations (5.11) become

$$\left.\begin{aligned} y^2 &= \beta^2\left(\frac{1 - \Gamma e^{\beta z}}{1 + \Gamma e^{\beta z}}\right)^2, \\ e^{\eta} &= \frac{K}{16\Gamma^2} e^{2(1-\beta)z}(1 + \Gamma e^{\beta z})^4 \end{aligned}\right\} \tag{5.18}$$

in all of which β can be taken to be the positive square root of $\beta^2 = b + 1 > 0$. The last result, with $K = 16\Gamma^2$, is the same as (A.31) of McV67 which is now shown to arise by the degeneration of the *P*-function solution (5.11) to elementary functions. The same is true for the further specialization $\beta^2 = 1$, which makes (5.18) identical with the metrics of McV33 in their McV66 form (1.3) since $\alpha = 0$ and $\beta^2 = 1$ can produce the expressions (1.4) for f and Q.

With β^2 still kept positive, the second value of g_3 which makes $\Delta = 0$ is

$$g_3 = \frac{\beta^6}{216} > 0$$

and so by (5.10)

$$\left.\begin{aligned} y &= w_Z(w - \beta^2/12)^{-1}, \\ w_Z^2 &= 4w^3 - \frac{\beta^4}{12}w - \frac{\beta^6}{216} = 4(w + \beta^2/12)^2(w - \beta^2/6). \end{aligned}\right\} \tag{5.19}$$

Hence (2.31) applies with

$$\varepsilon = \frac{\beta^2}{12}, \qquad w = -\frac{\beta^2}{12} + \frac{\beta^2}{4}\{\sin(\beta Z/2)\}^{-2}$$
$$= -\frac{\beta^2}{12} + \frac{\beta^2}{2}(1-\cos\beta Z)^{-1} \tag{5.20}$$

and it follows that

$$\left.\begin{aligned} y &= \frac{w_Z}{(w-\beta^2/12)} = -\frac{3\beta\sin\beta Z}{(1-\cos\beta Z)(2+\cos\beta Z)}, \\ e^\eta &= \mathrm{K}e^{2z}\left(\frac{1-\cos\beta Z}{2+\cos\beta Z}\right)^2. \end{aligned}\right\} \tag{5.21}$$

It is possible to replace Z by

$$\overline{Z} = Z - \frac{\pi}{2\beta} = z - z_0 - \frac{\pi}{2\beta},$$

and to obtain finally

$$\left.\begin{aligned} y^2 &= 9\beta^2\frac{1-\sin\beta\overline{Z}}{(1+\sin\beta\overline{Z})(2-\sin\beta\overline{Z})^2}, \\ e^\eta &= \mathrm{K}e^{2z}\left(\frac{1+\sin\beta\overline{Z}}{2-\sin\beta\overline{Z}}\right)^2, \\ \overline{Z} &= z - \left(z_0 + \frac{\pi}{2\beta}\right). \end{aligned}\right\} \tag{5.22}$$

This solution was not detected by the methods of McV33 and McV67 because $y_z (\equiv y_Z)$ does not depend quadratically on y. Instead (5.21) connects y_Z with y by (5.6) with $n = 1$, $m = \frac{\beta^2}{12}$ and $g_3 = \frac{\beta^6}{216}$ which is

$$(2y_Z - y^2 + \beta^2)(y_Z + y^2 - \beta^2)^2 = \beta^6. \tag{5.23}$$

ii) $\beta^2 < 0$. — If $\overline{\beta}^2 = -\beta^2 > 0$, it follows that $m = -\frac{\overline{\beta}^2}{12}$ and that the primitive (5.10) is given by

$$y = \frac{w_Z}{w + \overline{\beta}^2/12}, \qquad w_Z^2 = 4w^3 - \frac{\overline{\beta}^4}{12}w - g_3.$$

The discriminant of the equation for w vanishes by (5.7) if

$$g_3 = \pm\frac{\overline{\beta}^6}{216}.$$

Consider first the value $g_3 = -\frac{\bar{\beta}^6}{216}$. Comparison with the equations for w in (5.16) and (5.17) shows that

$$\varepsilon = \frac{\bar{\beta}^2}{12}, \qquad w = \frac{\bar{\beta}^2}{12} + \frac{\bar{\beta}^2}{4}\{\sinh(\bar{\beta}Z/2)\}^{-2},$$
$$= \frac{\bar{\beta}^2}{12} + \frac{\bar{\beta}^2}{2}(\cosh\bar{\beta}Z - 1)^{-1},$$

but the expression for y is

$$y = \frac{w_Z}{w + \bar{\beta}^2/12} = -\frac{3\bar{\beta}\sinh\bar{\beta}Z}{(\cosh\bar{\beta}Z - 1)(\cosh\bar{\beta}Z + 2)}$$

and thus, with the aid of (2.14),

$$\left.\begin{aligned} y^2 &= 9\bar{\beta}^2\frac{(\cosh\bar{\beta}Z + 1)}{(\cosh\bar{\beta}Z - 1)(\cosh\bar{\beta}Z + 2)^2}, \\ e^{\eta} &= Ke^{2z}\left(\frac{\cosh\bar{\beta}Z - 1}{\cosh\bar{\beta}Z + 2}\right)^2, \\ Z &= z - z_0. \end{aligned}\right\} \tag{5.24}$$

The equations corresponding to (5.12) to (5.14) contain $-\bar{\beta}^2$ in place of β^2 and the right-hand sides of the first two cannot vanish if $g_3 = -\bar{\beta}^6/216$. The third equation, of course, reads

$$(2y_Z - y^2 - \bar{\beta}^2)(y_Z + y^2 + \bar{\beta}^2)^2 = -\bar{\beta}^6. \tag{5.25}$$

The results (5.24), (5.25) are the analogues of (5.21), (5.23) when β^2 has a negative value. They remained undetected in McV67 because of the y_Z, y relationship (5.25).

Secondly, consider the value $g_3 = +\frac{\bar{\beta}^6}{216}$ so that the P-function $w(Z)$ satisfies

$$w_Z^2 = 4w^3 - \frac{\bar{\beta}^4}{12}w - \frac{\bar{\beta}^6}{216}.$$

Comparison with (5.19), (5.20) shows that

$$w = -\frac{\bar{\beta}^2}{12} + \frac{\bar{\beta}^2}{2}\frac{1}{(1 - \cos\bar{\beta}Z)}$$

and hence

$$y = \frac{w_Z}{w + \bar{\beta}^2/12} = -\frac{\bar{\beta}\sin\bar{\beta}Z}{1 - \cos\bar{\beta}Z}.$$

Since the constant z_0 may be altered to $\bar{z}_0$ by

$$\bar{\beta}Z = \bar{\beta}z - \bar{\beta}z_0 = \bar{\beta}(z - \bar{z}_0) + \pi$$

no generality is lost by employing $(z - \bar{z}_0)$ which will also be denoted by Z in the formula for y, so that

$$y = \frac{\bar{\beta} \sin \bar{\beta} Z}{1 + \cos \bar{\beta} Z} = \bar{\beta} \tan (\bar{\beta} Z/2).$$

Thus, with the aid of (2.14) there comes

$$\left.\begin{aligned} y^2 &= \bar{\beta}^2 \frac{1 - \cos \bar{\beta} Z}{1 + \cos \bar{\beta} Z} = \bar{\beta}^2 \tan^2 (\bar{\beta} Z/2), \\ e^\eta &= K e^{2z} (1 + \cos \bar{\beta} Z)^2 = 4 K e^{2z} \cos^4 (\bar{\beta} Z/2), \end{aligned}\right\} \quad (5.26)$$

which is the analogue of the solution (5.18) for a negative value of β^2. The equation (5.13), with $-\bar{\beta}^2$ in place of β^2 and $g_3 = \bar{\beta}^6/216$, is

$$2y_Z - y^2 - \bar{\beta}^2 = 0.$$

Thus the result (5.26) could have been deduced from (A.28) of McV67 had negative values of $(b + 1)$ been considered. The formula for y in (5.26) is to be found, with $Z \equiv z$, in Knutsen and Stabell (1979).

iii) $\beta^2 = 0$. — The equation (5.10) now shows that $w(Z)$ is the equianharmonic P-function that satisfies

$$w_Z^2 = 4w^3 - g_3.$$

With a notation similar to that used in (2.28) it follows that

$$w(Z; 0, g_3) = C_1^2 w(u; 0, 1)$$

where

$$C_1 = g_3^{1/6}, \qquad C_2 = z_0 g_3^{1/6}, \qquad u = C_1 z - C_2.$$

Again from (5.10)

$$y = \frac{d}{dZ} [\ln w(Z; 0, g_3)] = C_1 \frac{d}{du} [\ln w(u; 0, 1)]$$

and hence by (2.14)

$$\eta = 2z - 2 \ln w(u; 0, 1) + \ln K.$$

Therefore the functions which occur in the metric (2.13) are

$$\left.\begin{aligned} y^2 &= C_1^2 \left\{ \frac{d}{du} [\ln w(u; 0, 1)] \right\}^2 \\ e^\eta &= K \frac{e^{2z}}{w^2(u; 0, 1)}, \qquad u = C_1 z - C_2, \end{aligned}\right\} \quad (5.27)$$

where C_1, C_2 are the two arbitrary constants of integration in the primitive of the equation for y, which is now (5.1) with $\beta^2 = 0$.

The condition for the degeneration of w to elementary functions is $g_3 = 0$ and it gives with the aid of (2.33)

$$w = (z - z_0)^{-2}, \qquad y = -2(z - z_0)^{-1}.$$

Thus, also using (2.14), we have

$$y^2 = \frac{4}{(z - z_0)^2}, \qquad e^\eta = \mathrm{K}e^{2z}(z - z_0)^4 . \tag{5.28}$$

This expression for y also occurs in Knutsen & Stabell (1979).

iv) Lastly the singular solution (Peculiar Integral) (5.9*b*) gives rise by quadratures to the following results:

$$\left.\begin{array}{l} \beta^2 > 0: \quad \Gamma = e^{-2\beta z_0}, \qquad y^2 = \beta^2\left(\dfrac{1 - \Gamma e^{2\beta z}}{1 + \Gamma e^{2\beta z}}\right)^2, \\ e^\eta = \mathrm{K}e^{2(1+\beta)z}(1 + \Gamma e^{2\beta z})^{-2} ; \end{array}\right\} \tag{5.29}$$

$$\left.\begin{array}{l} \beta^2 = -\bar\beta^2,\ \bar\beta^2 > 0: \qquad y^2 = \bar\beta^2 \tan^2 \bar\beta \mathrm{Z}, \quad e^\eta = \mathrm{K}e^{2z} \sec^2 \bar\beta \mathrm{Z} \\ \mathrm{Z} = z - z_0 ; \end{array}\right\} \tag{5.30}$$

$$\beta^2 = 0: \qquad y^2 = (z - z_0)^{-2}, \qquad e^\eta = \mathrm{K}e^{2z}(z - z_0)^{-2} . \tag{5.31}$$

It has been shown in this section that the primitive of the equation (3.5) for y is attainable when $\alpha = 0$. The primitive produces a two-parameter set of solutions involving the P elliptic function. In certain cases they degenerate to elementary function solutions which include those in McV33 and a number of new ones whose non-detection in McV67 is explained. The Peculiar Integral (3.21) is also shown to be the first integral of the singular solution of the equation for y.

6. **Summary.**

In conclusion, it may be pointed out that the new results consist of the metrics expressed in terms of elliptic functions, equations (4.11/12/15) for the $\alpha \neq 0$ case, and equations (5.11/27) for the $\alpha = 0$ case. Some elementary function solutions (5.22/24/26), are also new in the sense that they are absent from McV67. The remaining solutions of this kind are also found in McV67 or are deducible by quadratures from the first integrals there given. All such solutions are shown to originate in the degeneration of elliptic to elementary functions.

It may well be asked why almost fifty years were needed for the completion of this investigation. The only explanation I can offer is that the attainment of a first integral of a non-linear second order-differential equation may block progress. In fact equations (26) and (28) of McV33 are, respectively [2], (5.1) with $\beta^2 = 1$ and (5.14) with an arbitrary, unanalyzed

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([2]) By writing γ for y and x for -Z.

constant A^3 on the right-hand side. The obvious way to future progress appeared to lie, not only for me, but, as I have been told, for others also, through this first integral. The present investigation shows that this path is illusory; instead a return must be made to the basic second-order equation. Nevertheless the determination of the primitive of the equation for y in the case of $\alpha \neq 0$ has still to be found, only one-parameter solutions having been attained. The problem remains that of finding the general solution of the equation (4.4).

It is necessary in Part II to have a brief name for a metric produced by combining a (y^2, e^η) pair with a permissible (f, Q) pair obtained by solving the equations (2.17) and (2.18). I hope that it will not seem presumptious if I call it a « McV-metric ».

PART II

McV-METRICS IN THE LITERATURE

7. Introductory remarks.

One of the ways of arriving at spherically symmetric metrics in the post-1967 literature is by the use of a method originally most clearly developed by Kustaanheimo and Qvist (1948). It employs the sequence of equations (2.2) to (2.4) in a coordinate system $(\bar{t}, \bar{r}, \theta, \phi)$ in which the metric is

$$ds^2 = e^\nu d\bar{t}^2 - e^\omega(d\bar{r}^2 + \bar{r}^2 d\Omega^2) \tag{7.1}$$

so that (2.5) becomes

$$e^\nu = \left(\frac{\partial\omega}{\partial\bar{t}}\right)^2 \frac{1}{T^2(\bar{t})}. \tag{7.2}$$

The specialization represented by (2.8), (2.9) and (2.10) is not made: instead it is shown that the isotropy of stress equation (2.15) can be reduced to

$$L_{xx} = F(x)L^2, \tag{7.3}$$

where

$$x = \bar{r}^2, \qquad L = e^{-\omega/2} \tag{7.4}$$

and F(x) is an arbitrary function of integration. There are, of course, other ways of attacking the problem which can be found in the papers cited in Sec. 8 (*i*)-(*vii*) below. A noticeable feature of these investigations is that, whatever the method used, the resulting metrics often involve only one arbitrary function of the time and so should be transformable into McV-metrics. It must be emphasized that the investigations in question do not *merely* arrive at metrics; they also deal with such important matters as the physical and geometrical meanings of the space-times attained.

To establish the equivalence of a McV-metric and one found in the literature requires the complete specification of the former. This means that in (2.13) not only y^2 and e^η must be known, but also f and Q. Hence the equations (2.17) and (2.18) must be solved as is done in the Appendix to McV67. When a, b are replaced, respectively, by $5\alpha + 3$ and $\beta^2 - 1$, the successive integrals of (2.17) may be taken to be

$$\mathrm{A}\mathrm{Q}_r\mathrm{Q}^{-(5\alpha+3)} = f, \qquad \mathrm{Q} = \left\{ \mathrm{C}_1 - \frac{1}{n\mathrm{A}}\int f\,dr \right\}^{-n}, \qquad n = \frac{1}{2+5\alpha}, \quad (7.5)$$

where A, C_1 are constants of integration and it is assumed that $\alpha \neq -2/5$. The possibility $\alpha = -2/5$, corresponding to $a = 1$, is dealt with in McV67. Substitution of f from (7.5) into (2.18) determines Q. However, there is one possibility in which f is determinable independently of Q, namely, when $\beta^2 = 1$ or $b = 0$. In this case the possible solutions of (2.18) for f are shown in the first line of Table I. The second line contains the corresponding expressions for Q obtained from (7.5), and the third line defines the constant C_2. With these values of f, it is clear that the metric

$$d\sigma^2 = dr^2 + f^2(r)d\Omega^2$$

found in (2.13) is that of a 3-space of constant curvature.

TABLE I. — *Expressions for f, Q when $\beta^2 = 1$ and $n = \dfrac{1}{2+5\alpha}$.*

f	$\sin r$	r	$\sinh r$
Q	$(\mathrm{C}_1 + \mathrm{C}_2 \cos r)^{-n}$	$(\mathrm{C}_1 + \mathrm{C}_2 r^2)^{-n}$	$(\mathrm{C}_1 + \mathrm{C}_2 \cosh r)^{-n}$
$1/\mathrm{C}_2$	$n\mathrm{A}$	$-2n\mathrm{A}$	$-n\mathrm{A}$

8. **Identifications.**

The general method of carrying out the equivalences is to find the coordinate transformation from the $(\bar{t}, \bar{r})$ of (7.1) to the (t, r) of a McV-metric and then to equate the metric coefficients each to each. The procedure is best illustrated by the examples which follow:

i) Bonnor and Faulkes (1967).

Their equations (2.3) and (2.7) produce a metric that may be written as

$$\left.\begin{aligned} ds^2 &= \{ 1+(1+\bar{\alpha})(\mathrm{H/F})^{\bar{\alpha}} \}^2 \{ 1+(\mathrm{H/F})^{\bar{\alpha}} \}^{-2} dt^2 \\ &\qquad - \bar{\alpha}^2 \mathrm{F}^2 \{ 1 + (\mathrm{H/F})^{\bar{\alpha}} \}^{-2} (dr^2 + r^2 d\Omega^2) \\ \mathrm{H} &= \{ \bar{\alpha} - 1 + \bar{\alpha} r^2/(4\mathrm{R}^2) \}^{1/\bar{\alpha}}, \end{aligned}\right\} \quad (8.1)$$

where $\bar{\alpha}$, R^2 are arbitrary constants, F is an arbitrary function of t and the coordinates are those of a McV-metric which, inspection of Part I suggests, is likely to be (4.1*a*) with f, Q given by the second column of Table I. This metric is in full

$$\begin{aligned} ds^2 = \frac{\alpha^2}{4}(5+2j)^2 \left\{ 1+\frac{5-2j}{5+2j}\Gamma\left(\frac{\mathrm{Q}}{\mathrm{S}}\right)^{2\alpha j} \right\}^2 \left\{ 1+\Gamma\left(\frac{\mathrm{Q}}{\mathrm{S}}\right)^{2\alpha j} \right\}^{-2} dt^2 \\ - \mathrm{KS}^2\left(\frac{\mathrm{Q}}{\mathrm{S}}\right)^{2+(2j+5)\alpha} \left\{ 1+\Gamma\left(\frac{\mathrm{Q}}{\mathrm{S}}\right)^{2\alpha j} \right\}^{-2} (dr^2+r^2d\Omega^2). \quad (8.2) \end{aligned}$$

If the two metrics are identical, the factors multiplying $(dr^2+r^2d\Omega^2)$ yield

$$\mathrm{K} = \bar{\alpha}^2, \qquad \mathrm{S} \equiv \mathrm{F},$$

$$2 + (2j + 5)\alpha = 0, \quad (8.3)$$

$$2\alpha j = \bar{\alpha}, \quad (8.4)$$

$$\Gamma^{1/2\alpha j}(\mathrm{C}_1 + \mathrm{C}_2 r^2)^{-n} = \{ \bar{\alpha} - 1 + \bar{\alpha} r^2/4\mathrm{R}^2 \}^{1/\bar{\alpha}} \quad (8.5)$$

and the only additional relation provided by the equalisation of the coefficients of dt^2 is

$$\Gamma(5-2j)(5+2j)^{-1} = 1+\bar{\alpha}. \quad (8.6)$$

The equation (8.3) is the condition that $\beta^2=1$ which makes the use of the second column of Table I possible. With equation (8.4) it yields

$$\bar{\alpha} = 2\alpha j = -(2 + 5\alpha) = -\frac{1}{n}, \quad (8.7)$$

while (8.6) becomes

$$(1 - \Gamma)(1 + 5\alpha) = 0.$$

Since $\bar{\alpha}$, and therefore α, is arbitrary, the last equation is solved by $\Gamma=1$. Finally the $\bar{\alpha}$-power of the equation (8.5) leads to

$$\mathrm{C}_1 = \bar{\alpha} - 1 = -(3 + 5\alpha), \qquad \mathrm{C}_2 = \frac{\bar{\alpha}}{4\mathrm{R}^2} = -\frac{2 + 5\alpha}{4\mathrm{R}^2}.$$

The identification process therefore connects the constants of (8.1) with those of (8.2) by the equations

$$\bar{\alpha} = -(2 + 5\alpha), \qquad 4\mathrm{R}^2 = -(2 + 5\alpha)/\mathrm{C}_2, \quad (8.8)$$

but it also restricts the constants of the McV-metric (8.2) as follows:

$$\Gamma = 1, \qquad \mathrm{K} = (2 + 5\alpha)^2, \qquad \mathrm{C}_1 = -(3 + 5\alpha). \quad (8.9)$$

Identification is also possible with the $\alpha = 0$ McV-metric whose y^2, e^η are given by (5.29) with $\beta = -1$. Hence $\beta^2 = 1$ and so the second column of Table 1 may again be used with $n = \frac{1}{2}$. The McV-metric in full is

$$ds^2 = \left(\frac{1-\Gamma(Q/S)^{-2}}{1+\Gamma(Q/S)^{-2}}\right)^2 dt^2 - K\{1+\Gamma(Q/S)^{-2}\}^{-2}(dr^2+r^2d\Omega^2)$$

and equating these coefficients with those of (8.1) eventually produces

$$\left.\begin{array}{lll} \bar\alpha = -2, & R^2 = -(2\Gamma C_2)^{-1}, & F = S \\ K = 4, & \Gamma C_1 = -3. & \end{array}\right\} \qquad (8.10)$$

Both (4.1*a*) and (5.29) are derived from the Peculiar Integral and therefore, in the $\alpha = 0$ alternative, from the singular solution. Bonnor and Faulkes, in their equation (3.7) impose for physical reasons the condition

$$\frac{1}{2} < \bar\alpha < 1$$

and therefore the (8.10) alternative would be unacceptable to them.

ii) Faulkes (1969).

The space-time which Faulkes analyzes in most detail has a metric (his equation (13)) equivalent to

$$ds^2 = \frac{E^2 d\bar t^2}{(E - r^2/4k^2)^2} - (E - r^2/4k^2)^4(dr^2 + r^2 d\Omega^2),$$

where k^2 is a constant and E is an arbitrary function of $\bar t$. The time may be rescaled by $\bar t = 2\alpha t$ and the metric becomes

$$ds^2 = \{1 - r^2/(4k^2E)\}^{-2}4\alpha^2 dt^2 - E^4\{1 - r^2/(4k^2E)\}^4(dr^2 + r^2 d\Omega^2).$$

Inspection of (4.1*d*) shows that $\alpha = -\frac{1}{2}$ makes $\beta^2 = 1$ and so permits the use of the second column of Table I. The McV-metric is

$$ds^2 = \left\{1 + \frac{1}{\Gamma}\left(\frac{Q}{S}\right)^{\frac{1}{2}}\right\}^{-2} dt^2 - K\Gamma^4 S^2\left\{1 + \frac{1}{\Gamma}\left(\frac{Q}{S}\right)^{\frac{1}{2}}\right\}^4 (dr^2 + r^2 d\Omega^2)$$

with

$$Q = (C_1 + C_2 r^2)^2.$$

Hence the two metrics are identical if

$$K\Gamma^4 = 1, \qquad S^{\frac{1}{2}} = -E, \quad C_1 = 0, \quad \frac{C_2}{\Gamma} = \frac{1}{4k^2}.$$

Faulkes' metric is reducible to a McV-metric that is not obtained from the Peculiar Integral.

iii) Banerjee and Banerji (1976).

Their metric (1*a*), with obvious modifications of notation, is

$$d\bar{s}^2 = \{ H(\bar{t}) + \bar{a}(1 + \bar{k}\bar{r}^2)^{-1} \}^{-2} d\bar{t}^2 - \{ H(\bar{t}) + \bar{a}(1 + \bar{k}\bar{r}^2)^{-1} \}^4 \frac{d\bar{r}^2 + \bar{r}^2 d\Omega^2}{(1 + \bar{k}\bar{r}^2)^2}, \qquad (8.11)$$

where $H(\bar{t})$ is an arbitrary function of $\bar{t}$ and $\bar{a}$, $\bar{k}$ are constants. If $\bar{r}$, $\bar{t}$ and $d\bar{s}$ are re-scaled appropriately, $\bar{k}$ may be replaced by $\pm \frac{1}{4}$ or 0, the third possibility leading to a Robertson-Walker metric. Conversion to a McV-metric will be worked out for $\bar{k} = +\frac{1}{4}$, with the $\bar{k} = -\frac{1}{4}$ case following on similar lines. Coordinates r, t are introduced by

$$\bar{r} = 2 \tan \frac{1}{2} r, \qquad t = \int \frac{d\bar{t}}{H(\bar{t})},$$

and (8.11) becomes

$$ds^2 = \left\{ 1 + \frac{\bar{a}}{2}\left(\frac{1 + \cos r}{H(t)}\right) \right\}^{-2} dt^2 - H^4(t) \left\{ 1 + \frac{\bar{a}}{2}\left(\frac{1 + \cos r}{H(t)}\right) \right\}^4 (dr^2 + \sin^2 r d\Omega^2). \qquad (8.12)$$

Now consider a McV-metric whose y^2, e^η are given by (4.1*d*) with $\alpha = -\frac{1}{2}$ so that $\beta^2 = 1$ and $n = -2$ in Table I. Combine these with the f, Q found in the first column of Table I. The McV-metric is therefore

$$ds^2 = \left\{ 1 + \frac{1}{\Gamma}\left(\frac{C_1 + C_2 \cos r}{S^{\frac{1}{2}}(t)}\right) \right\}^{-2} dt^2 - K\Gamma^4 S^2 \left\{ 1 + \frac{1}{\Gamma}\left(\frac{C_1 + C_2 \cos r}{S^{\frac{1}{2}}}\right) \right\}^4 (dr^2 + \sin^2 r d\Omega^2)$$

which is identifiable with (8.12) if

$$S = H^2, \qquad \frac{C_2}{\Gamma} = \frac{1}{2}\bar{a}, \qquad C_1 = C_2, \qquad K\Gamma^4 = 1.$$

The last two equations, of course, impose a slight restriction on the choice of the (4.1*d*) McV-metric with $\alpha = -\frac{1}{2}$.

iv) Chakravarty, Choudhury and Banerjee (1976).

This paper will be referred to as CCB. The authors divide their solutions into four cases labelled A, B, C and D. Cases A, B and the first solution of C are respectively identical with Faulkes (1969), Banerjee and Banerji (1976) and Nariai (1967) and so are known to be convertible to McV-metrics. The second solution of Case C (CCB equation (6)), in a slightly modified notation, gives (7.1) with

$$e^{-\frac{1}{2}\omega} = \frac{1}{4}\tau^2\left(\frac{1}{3} + \tan^2\bar{\eta}\right)(\pm\,\tau\bar{r}^2 + \varepsilon)^{\frac{1}{2}}, \tag{8.13}$$

where τ, ε are arbitrary constants and

$$e^{\bar{\eta}} = \mathrm{H}(\bar{t})(\pm\tau\bar{r}^2 + \varepsilon)^{\frac{1}{4}}, \tag{8.14}$$

H being an arbitrary function of $\bar{t}$. Hence

$$\begin{aligned} e^{\omega} &= \frac{36}{\tau^4}\left(\frac{1+\cos 2\bar{\eta}}{2-\cos 2\bar{\eta}}\right)^2(\pm\,\tau\bar{r}^2 + \varepsilon)^{-1} \\ &= \frac{36}{\tau^4}\frac{e^{4\bar{\eta}}}{\mathrm{H}^4}\left(\frac{1+\cos 2\bar{\eta}}{2-\cos 2\bar{\eta}}\right)^2(\pm\,\tau\bar{r}^2 + \varepsilon)^{-2}. \end{aligned} \tag{8.15}$$

The first of these expressions shows that ω depends on $\bar{t}$ only through $\bar{\eta}$ and by (8.14)

$$\frac{\partial\bar{\eta}}{\partial\bar{t}} = \frac{1}{\mathrm{H}}\frac{d\mathrm{H}}{dt}.$$

Thus in (7.1) the first term on the right becomes, by (7.2),

$$e^{\nu}d\bar{t}^2 = \left\{\frac{4}{\mathrm{T}(\bar{t})}\frac{1}{\mathrm{H}(\bar{t})}\frac{d\mathrm{H}}{d\bar{t}}\,d\bar{t}\right\}^2 \frac{9\sin^2 2\bar{\eta}}{(1+\cos 2\bar{\eta})^2(2-\cos 2\bar{\eta})^2}.$$

Clearly by a suitable choice of the arbitrary function $\mathrm{T}(\bar{t})$ the expression in curly brackets may be replaced by the differential of a new time-coordinate t, and $\bar{\eta}$ and H be henceforward regarded as functions of t. Thus the first term on the right of (7.1) becomes

$$e^{\nu}d\bar{t}^2 = \frac{9\sin^2 2\bar{\eta}}{(1+\cos 2\bar{\eta})^2(2-\cos 2\bar{\eta})^2}\,dt^2. \tag{8.16}$$

The terms on the right-hand side of (7.1) which involve the differentials of the space-coordinates are proportional, by the second expression for e^{ω} in (8.15), to

$$d\sigma^2 = \frac{d\bar{r}^2 + \bar{r}^2 d\Omega^2}{(\varepsilon \pm \tau\bar{r}^2)^2}.$$

This is well-known to be the metric of a 3-space of constant curvature for all values of ε and τ and for either choice of sign. The method of conversion to a McV-metric is sufficiently illustrated by the case of positive curvature. Specifically let the plus sign be chosen with

$$\varepsilon > 0, \quad \tau > 0, \quad \bar{r} = (\varepsilon/\tau)^{\frac{1}{2}} \tan \frac{1}{2} r ,$$

which give, by (8.14) also,

$$d\sigma^2 = \frac{1}{4\tau\varepsilon}(dr^2 + \sin^2 r d\Omega^2) ,$$
$$e^{4\bar{\eta}} = 2\varepsilon \mathrm{H}^4(t)(1 + \cos r)^{-1} . \tag{8.17}$$

These results introduced into (8.15), together with (8.16) show finally that the CCB metric is

$$\begin{aligned} ds^2 = & \frac{9 \sin^2 2\bar{\eta}}{(1 + \cos 2\bar{\eta})^2 (2 - \cos 2\bar{\eta})^2} dt^2 \\ & - \frac{9}{\varepsilon\tau^5} \frac{e^{4\bar{\eta}}}{\mathrm{H}^4(t)} \left(\frac{1 + \cos 2\bar{\eta}}{2 - \cos 2\bar{\eta}}\right)^2 (dr^2 + \sin^2 r d\Omega^2) , \end{aligned} \tag{8.18}$$

$\bar{\eta}$ being given by (8.17).

Now consider the $\alpha = 0$ McV-metric whose y^2, e^η are given by (5.21) with $\beta = 1$, $z_0 = \pi$ and whose f, Q are obtained from the first column of Table I. The metric is

$$\begin{aligned} ds^2 = & \frac{9 \sin^2 z}{(1 + \cos z)^2(2 - \cos z)^2} dt^2 \\ & - \mathrm{KS}^2 e^{2z} \left(\frac{1 + \cos z}{2 - \cos z}\right)^2 (dr^2 + \sin^2 r d\Omega^2) \end{aligned} \tag{8.19}$$

$$n = \frac{1}{2}, \qquad \mathrm{Q} = (\mathrm{C}_1 + \mathrm{C}_2 \cos r)^{-\frac{1}{2}} .$$

Identification with (8.18) requires that

$$\mathrm{S} = \mathrm{H}^{-2}, \qquad \mathrm{K} = \frac{9}{\varepsilon\tau^5}, \qquad z = 2\bar{\eta} , \tag{8.20a}$$

the last of which necessitates, by (8.17) also, that

$$e^z = \frac{(\mathrm{C}_1 + \mathrm{C}_2 \cos r)^{-\frac{1}{2}}}{\mathrm{S}} \equiv e^{2\bar{\eta}} = \mathrm{H}^2(2\varepsilon)^{\frac{1}{2}}(1 + \cos r)^{-\frac{1}{2}}$$

or that

$$\mathrm{C}_1 = \mathrm{C}_2 = \frac{1}{2\varepsilon} . \tag{8.20b}$$

Case D of CCB contains nine solutions for $e^{-\omega/2}$ each of which includes one arbitrary function of $\bar{t}$, called $\phi(\bar{t})$. It is to be expected that the metrics are convertible to McV-metrics. Consider for example CCB's equation (8*c*) in which

$$e^{\omega} = \frac{\varepsilon^2}{\tau^6} \frac{\sinh^2 \psi}{\left(\frac{1}{3} + \tan^2 \bar{\eta}\right)^2},$$

$$2\bar{\eta} = \phi(\bar{t}) + \psi(\bar{r}^2),$$

$$\coth \psi = (\pm\, \varepsilon^2 \bar{r}^2 + \lambda)/\tau = \zeta, \tag{8.21}$$

where ε^2, λ, τ are constants. The method of conversion to a McV-metric is again sufficiently illustrated by assuming that the three constants are positive and by choosing the positive sign in (8.21).
The introduction of r by

$$\bar{r} = \left(\frac{\lambda + \tau}{\varepsilon^2}\right)^{\frac{1}{2}} \tan (r/2)$$

yields, after some calculation

$$e^{2\psi} = \frac{\zeta + 1}{\zeta - 1} = \left(\frac{\tau}{\lambda + \tau}\right)^{-1} \left(\frac{\lambda}{\tau} - \cos r\right)^{-1}, \tag{8.22}$$

$$1 + \zeta = \frac{\lambda + \tau}{\tau} \sec^2 (r/2),$$

$$e^{\omega} = \frac{9}{4} \frac{\varepsilon^2 e^{-2\phi}}{\tau^4(\lambda + \tau)^2} \cdot e^{4\bar{\eta}} \left(\frac{1 + \cos 2\bar{\eta}}{2 - \cos 2\bar{\eta}}\right)^2 \cos^4 (r/2).$$

Hence

$$e^{\omega}(d\bar{r}^2 + \bar{r}^2 d\Omega^2) = \frac{9}{4} \frac{e^{-2\phi}}{\tau^4(\lambda + \tau)} . e^{4\eta} \left(\frac{1 + \cos 2\bar{\eta}}{2 - \cos 2\bar{\eta}}\right)^2 (dr^2 + \sin^2 r d\Omega^2).$$

Since ω is now known, e^{ν} may be calculated from (7.2), and the time $\bar{t}$ replaced by t where

$$dt^2 = \frac{4}{T^2(\bar{t})} \left(\frac{d\phi}{d\bar{t}}\right)^2 d\bar{t}^2.$$

The final result is that the metric of CCB case (8*c*) is

$$\begin{aligned} ds^2 = {} & \frac{9}{(1 + \cos 2\bar{\eta})^2 (2 - \cos 2\bar{\eta})^2} dt^2 \\ & - \frac{9}{16} \frac{e^{-2\phi}}{\tau^4(\lambda + \tau)} \cdot e^{4\bar{\eta}} \left(\frac{1 + \cos 2\bar{\eta}}{2 - \cos 2\bar{\eta}}\right)^2 (dr^2 + \sin^2 r d\Omega^2) \end{aligned} \tag{8.23}$$

which may be identified with (8.19) by the requirements that

$$S = e^{-\phi}, \qquad K = \frac{9}{16} \frac{1}{\tau^4(\lambda + \tau)}, \qquad z = 2\bar{\eta},$$

the third of which gives, by (8.22) also,

$$e^{z} = \frac{(\mathrm{C}_1 + \mathrm{C}_2 \cos r)^{-\frac{1}{2}}}{\mathrm{S}} \equiv e^{2\bar{\eta}} = e^{\phi}e^{\psi} = e^{\phi}\left(\frac{\lambda}{\lambda+\tau} - \frac{\tau}{\lambda+\tau}\cos r\right)^{-\frac{1}{2}}$$

whence

$$\mathrm{C}_1 = \frac{\lambda}{\lambda+\tau}, \qquad \mathrm{C}_2 = -\frac{\tau}{\lambda+\tau}.$$

Therefore the apparently different metrics defined by CCB equations (6) and (8*c*) are both reducible to the McV-metric (8.19) with the rather trivial difference that in the first C_1 and C_2 are equal while in the second $\mathrm{C}_1 - \mathrm{C}_2 = 1$.

The group of CCB metrics defined by the equations (9*a*, *b*, *c*) of Case D is more interesting for two reasons. Firstly it turns out that the metrics are the only examples so far found in the literature which correspond to McV-metrics involving values of β^2 different from unity. Secondly, the transformation from $\bar{r}$ to r is extremely complicated and has to be avoided. These points are illustrated by the analysis of CCB 9(*a*) in which

$$\left.\begin{aligned} e^{\omega} &= \frac{\varepsilon^2}{\tau^6}\frac{\cos^4\bar{\eta}}{\sec^2\psi} = \frac{\varepsilon^2}{\tau^6}\frac{\cos^4\bar{\eta}}{1+\tan^2\psi}, \\ \tan\psi &= (\pm\,\varepsilon^2\bar{r}^2 + \lambda)/\tau, \qquad 2\bar{\eta} = \phi(\bar{t}) + \psi(\bar{r}^2), \end{aligned}\right\} \tag{8.24}$$

and, by (7.2),

$$e^{\nu} = \frac{4}{\mathrm{T}^2(\bar{t})}\left(\frac{d\phi}{d\bar{t}}\right)^2 \frac{1-\cos 2\bar{\eta}}{1+\cos 2\bar{\eta}},$$

where ε^2, λ, τ are arbitrary constants. To fix ideas, the + sign will be adopted in the definition of $\tan\psi$. The time $\bar{t}$ will be replaced by t where

$$k^2dt^2 = \frac{4}{\mathrm{T}^2(\bar{t})}\left(\frac{d\phi}{d\bar{t}}\right)^2 d\bar{t}^2$$

and k is an arbitrary constant. Finally the coordinate $\bar{r}$ is replaced by $\bar{\rho}$, where

$$\bar{\rho} = \varepsilon^2\bar{r},$$

and the constants δ, γ are introduced by

$$\delta = \varepsilon^2\tau, \qquad \gamma = \lambda\varepsilon^2.$$

The CCB metric may then be written as

$$ds^2 = k^2\left(\frac{1-\cos 2\bar{\eta}}{1+\cos 2\bar{\eta}}\right)dt^2 - \frac{\varepsilon^2}{4\tau^4}(1+\cos 2\bar{\eta})^2\frac{d\bar{\rho}^2 + \bar{\rho}^2d\Omega^2}{\delta^2+(\bar{\rho}^2+\gamma)^2}, \tag{8.25}$$

$\bar{\eta}$ now being regarded as a function of t and $\bar{\rho}$.

Consider next the $\alpha = 0$ McV-metric given by (5.26) with $\bar{z}_0 = 0$. This metric becomes, with the aid of (7.5) and the substitution $r = A^{\frac{1}{2}}\rho$,

$$ds^2 = \bar{\beta}^2\left(\frac{1 - \cos\bar{\beta}z}{1 + \cos\bar{\beta}z}\right)dt^2$$
$$- KA(1 + \cos\bar{\beta}z)^2Q^2(\rho)\{d\rho^2 + (Q_\rho/Q^3)^2d\Omega^2\}. \qquad (8.26)$$

Clearly if (8.25) is convertible into (8.26), it must be the case that

$$Qd\rho = \{\delta^2 + (\bar{\rho}^2 + \gamma)^2\}^{-\frac{1}{2}}d\bar{\rho}$$

and

$$Q_\rho/Q^2 = \{\delta^2 + (\bar{\rho}^2 + \gamma)^2\}^{-\frac{1}{2}}\bar{\rho},$$

two relations which show that

$$\frac{dQ}{Q} = \frac{1}{2}\{\delta^2 + (\bar{\rho}^2 + \gamma)^2\}^{-1}d(\bar{\rho}^2).$$

Hence, by (8.24) also,

$$\ln Q(\bar{\rho}) = \frac{1}{2\delta}\arctan\left(\frac{\bar{\rho}^2 + \gamma}{\delta}\right) = \frac{1}{2\delta}\psi(\bar{\rho}). \qquad (8.27)$$

It must also be the case that

$$\bar{\beta}z = 2\bar{\eta}$$

which means that

$$\left(\frac{Q}{S}\right)^{\bar{\beta}} = e^{\phi(t)}e^{\psi(\bar{\rho})}.$$

Therefore

$$S = e^{-\phi/\bar{\beta}}, \qquad Q(\bar{\rho}) = e^{\psi(\bar{\rho})/\bar{\beta}}$$

the second of which is consistent with (8.27) if

$$\bar{\beta} = 2\delta = 2\tau\varepsilon^2. \qquad (8.28)$$

Lastly the equality of the constant multipliers gives

$$k = \bar{\beta} = 2\tau\varepsilon^2, \qquad KA = \varepsilon^2/(4\tau^4).$$

If the minus sign had been selected in the definition of $\tan\psi$ the only significant change would have been the replacement of (8.28) by $\bar{\beta} = -2\tau\varepsilon^2$. The CCB cases (9*b*) and (9*c*) can also be converted to McV-metrics by a method similar to the foregoing.

v) Glass and Mashhoon (1976).

These authors through their equations (2), (5), (6), (7), (11) and (12) produce metrics equivalent to

$$\left.\begin{aligned} d\bar{s}^2 &= \left(\frac{1-\phi}{1+\phi}\right)^2 d\bar{t}^2 - (1+\phi)^4 \bar{f}^2(\bar{t}) \left\{ \frac{d\rho^2 + \rho^2 d\Omega^2}{(\delta + \gamma\rho^2)^2} \right\}, \\ \phi &= \frac{\lambda_0}{2}\left(\frac{\gamma\rho^2+\delta}{\bar{\alpha}\rho^2+\bar{\beta}}\right)^{\frac{1}{2}} \frac{1}{\bar{f}(\bar{t})}, \end{aligned}\right\} \tag{8.29}$$

where $\bar{\alpha}, \bar{\beta}, \gamma, \delta, \lambda_0$ are constants and $\bar{f}$ is an arbitrary function of $\bar{t}$. Whether γ is positive, zero or negative the terms in curly brackets are those of a 3-space of constant curvature. To fix ideas the case particularly studied by Glass and Masshoon will be considered, by a method useable for any combination of positive and negative values of $\bar{\alpha}, \bar{\beta}, \delta$. The case is defined by $\gamma = -\varepsilon, \varepsilon > 0$. It will also be assumed that $\lambda_0, \bar{\alpha}, \bar{\beta}$ and δ are all positive. The substitutions

$$ds = 2(\delta\varepsilon)^{\frac{1}{2}} d\bar{s}, \qquad t = 2(\delta\varepsilon)^{\frac{1}{2}}\bar{t}, \qquad \rho = \left(\frac{\delta}{\varepsilon}\right)^{\frac{1}{2}} \tanh\frac{r}{2},$$

convert (8.29) into

$$\left.\begin{aligned} ds^2 &= \left(\frac{1-\phi}{1+\phi}\right)^2 dt^2 - (1+\phi)^4 \bar{f}^2(t) \{ dr^2 + \sinh^2 r d\Omega^2 \}, \\ \phi &= \left[\frac{2}{\lambda_0^2}\left(\frac{\bar{\alpha}}{\varepsilon}+\frac{\bar{\beta}}{\delta}\right)\cosh r - \frac{2}{\lambda_0^2}\left(\frac{\bar{\alpha}}{\varepsilon}-\frac{\bar{\beta}}{\delta}\right)\right]^{-\frac{1}{2}} \frac{1}{\bar{f}(t)}. \end{aligned}\right\} \tag{8.30}$$

The $\alpha = 0$ McV-metric obtained from (5.18) with $\beta = 1$ and the third column of Table I has the metric

$$ds^2 = \left(\frac{1-\Gamma(Q/S)}{1+\Gamma(Q/S)}\right)^2 dt^2 - \frac{K}{16\Gamma^2} S^2 \{ 1 + \Gamma(Q/S) \}^4 \{ dr^2 + \sinh^2 r d\Omega^2 \}$$

$$Q(r) = (C_1 + C_2 \cosh r)^{-\frac{1}{2}}.$$

It is immediately identifiable with (8.30) if

$$\bar{f}(t) = S(t), \qquad \Gamma = 1, \qquad K = 16,$$

$$C_1 = \frac{2}{\lambda_0^2}\left(\frac{\bar{\alpha}}{\varepsilon}+\frac{\bar{\beta}}{\delta}\right), \qquad C_2 = -\frac{2}{\lambda_0^2}\left(\frac{\bar{\alpha}}{\varepsilon}-\frac{\bar{\beta}}{\delta}\right).$$

The same method turns every metric (8.29) into an $\alpha = 0$ McV-metric obtained from (5.18) with $\beta = 1$. These metrics were listed under (A.31) of McV67 and so were available at the time Glass and Mashhoon were carrying out their investigations.

vi) Glass (1979).

This paper contains metrics whose coefficients depend on two arbitrary functions of the time and so do not convert into McV-metrics. However, Glass does give a metric which can be so converted. It refers to a uniform density solution and, with the aid of Glass' equations (47), (6), (5) and the definitions following (8), it may be written as

$$ds^2 = \mathrm{H}^{-2}(\bar{t})\left(\frac{\dot{g}(\bar{t})}{g(\bar{t})}\right)^2 \left\{\frac{1 + g\bar{r}}{1 - g\bar{r}}\right\}^2 d\bar{t}^2$$
$$- \frac{4}{9}\bar{k}^2(1 - g\bar{r})^4 \frac{1}{g^2}\left(\frac{d\bar{r}^2 + \bar{r}^2 d\Omega^2}{\bar{r}^4}\right)$$

where $\bar{k}$ is a constant, g has been written for Glass' c_2, and H, g are arbitrary functions of $\bar{t}$. The metric is simplified by the introduction of the coordinates t and r, where

$$dt = \frac{\dot{g}}{g}\frac{d\bar{t}}{\mathrm{H}(\bar{t})}, \qquad r = \frac{1}{\bar{r}}$$

and it becomes

$$ds^2 = \left(\frac{1+g/r}{1-g/r}\right)^2 dt^2 - \frac{4\bar{k}^2}{9}\left(1 - \frac{g}{r}\right)^4 \frac{1}{g^2}\{ dr^2 + r^2 d\Omega^2 \}, \qquad (8.31)$$

g now being regarded as a function of t.

The $\alpha = 0$ McV-metric obtained from (5.18) with $\beta = 1$ and the second column of Table I is

$$ds^2 = \left(\frac{1-\Gamma(\mathrm{Q/S})}{1+\Gamma(\mathrm{Q/S})}\right)^2 dt^2 - \frac{\mathrm{K}}{16\Gamma^2}\mathrm{S}^2\{ 1 + \Gamma(\mathrm{Q/S}) \}^4(dr^2 + r^2 d\Omega^2),$$
$$\mathrm{Q} = (\mathrm{C}_1 + \mathrm{C}_2 r^2)^{-\frac{1}{2}}.$$

It is identifiable with (8.31) if

$$\mathrm{C}_1 = 0, \qquad \Gamma = \mathrm{C}_2^{\frac{1}{2}}, \qquad \mathrm{S} = -\frac{1}{g}, \qquad \frac{\mathrm{K}}{\mathrm{C}_2} = \frac{64}{9}\bar{k}^2.$$

This McV-metric is the same as the McV33 metric with zero space-curvature ($k = 0$) as may be seen from McV66, equation (8), which shows that the density of the material represented by the space-time is uniform.

vii) Kramer, Stephani, MacCallum & Herlt (1980).

These authors favour the method of Kustaanheimo and Qvist (1948) which, in its various forms, can produce metrics whose coefficients depend on more than one arbitrary function of the time in contrast to the single function of a McV-metric (Thompson and Whitrow 1967-1968; Bondi 1969). Kramer *et al.* state that « To the authors' knowledge all explicitly known shearfree and expanding spherically symmetric perfect fluid solutions are contained in the class » defined by assuming in (7.3) that

$$F(x) = (ax^2 + bx + c)^{-5/2} \tag{8.32}$$

where a, b, c are arbitrary constants. They also remark on the « surprising » $-5/2$ power; indeed one may wonder if Kustaanheimo and Qvist would have hit on this particular power had not the metrics of McV33 been available on which to try their method. Be that as it may, it is certainly true that most of the McV-metrics of Part I that have $\beta^2 = 1$ correspond to functions $F(x)$ of the form (8.32). But it is also possible to find relatively simple examples which correspond to functions $F(x)$ involving even odder powers than $-5/2$. Consider those metrics whose (y, η) functions are given by (4.12) while their (f, Q) functions are taken from the second column of Table I. The metrics are therefore of the form (7.1) with $(\bar{r}, \bar{t})$ identified as (r, t). Moreover $\beta^2 = 1$ means by (3.15) that the permissible values of α are 1 and $-1/6$. Also, with $x = r^2$,

$$Q = (C_1 + C_2 x)^{-n}, \qquad n = \frac{1}{7} \text{ or } \frac{6}{7},$$

and therefore

$$\frac{\partial z}{\partial x} = -\frac{nC_2}{C_1 + C_2 x}, \qquad \frac{\partial^2 z}{\partial x^2} = \frac{1}{n}\left(\frac{\partial z}{\partial x}\right)^2.$$

By (7.3)

$$F(x) = L_{xx}/L^2, \qquad L = e^{-\eta/2}S^{-1},$$

and therefore

$$F(x) = e^{\eta}S\left(\frac{\partial z}{\partial x}\right)^2 \left\{ \frac{\partial^2 e^{-\eta/2}}{\partial x^2} + \frac{1}{n}\frac{\partial e^{-\eta/2}}{\partial x} \right\}.$$

Since all McV-metrics imply that

$$\frac{1}{2}\eta_z = 1 - y,$$

it follows that

$$F(x) = e^{\eta/2}S\left(\frac{\partial z}{\partial x}\right)^2 \left\{ (y-1)^2 + y_z + \frac{1}{n}(y-1) \right\}.$$

The expression for y in (4.12) is

$$\alpha(1 + 2\Gamma e^{\alpha z})^{-1}(1 + \Gamma e^{\alpha z})^{-1}$$

and, after considerable calculation, it is found that

$$\left.\begin{aligned} F(x) &= -\frac{24}{49}K^{\frac{1}{2}}C_2^{\frac{2}{7}}(C_1 + C_2 x)^{-15/7}, \qquad \alpha = 1\,; \\ F(x) &= -\frac{24}{49}K^{\frac{1}{2}}C_2^{\frac{2}{7}}(C_1 + C_2 x)^{-20/7}, \qquad \alpha = -\frac{1}{6}. \end{aligned}\right\} \tag{8.33}$$

When F(x) is calculated for the solution (4.11), in which the constant g_3 involved in the definition of the elliptic function is not zero, the same powers of $C_1 + C_2x$ as in (8.33) are found. It is, of course, again assumed that $\beta^2 = 1$ and that Table I, second column, applies. The expression for F(x) which would arise in a McV-metric that has $\beta^2 \neq 1$ has been left for later investigation.

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