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All linear representations of the Poincaré group up to dimension 8

by

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ABSTRACT. — All representations of the universal covering of the Poincaré group on a complex vector space of dimension 8 or less, including the incompletely reducible representations, are determined explicitly. Considering two representations conjugate by a fixed linear transformation as the same, there are only a finite and small number of indecomposable such representations in each such dimension. Their invariant sesquilinear forms, and their extensions to discrete symmetries, scale transformations, and to the conformal group are also determined. Certain of the results and methods are applicable to other semi-direct product groups.

RÉSUMÉ. — Toutes les représentations complexes de dimension $\leqslant 8$ du groupe de Poincaré, y compris celles qui ne sont pas complètement réducibles, sont déterminées explicitement. Deux représentations étant considérées identiques si l'une est conjuguée de l'autre par une transformation linéaire fixe, on trouve qu'il existe un nombre fini de telles représentations indécomposables. Leurs formes sesquilinaires invariantes, et leurs extensions à des symmétries discrètes, aux homothéties, et au groupe conforme sont également déterminées. Certains résultats et les méthodes utilisées sont applicables à l'étude d'autres produits semi-directs.

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1. INTRODUCTION

Group representations which are indecomposable but not completely reducible have been suggested for use in the modeling of unstable particles and interactions ([*1*] [*2*] [*3*] [*5*] [*9*] [*10*]). However, a classification of finite-dimensional such representations for some (necessarily) non-semi-simple Lie group, or even a determination of all representations in spaces of the lowest possible few dimensions of some familiar semi-direct product group (e. g., the affine isometry group for a euclidean or pseudo-euclidean vector space) appears nonexistent in the mathematical literature. In view of long-standing questions in the classification and representation of solvable Lie groups, such problems seem difficult, and for a given group one might reasonably expect a continuum of inequivalent such representations in a given sufficiently large dimension.

Yet surprisingly, the present results (in summary, Theorem 3) suggest quite the reverse for certain Lie groups whose maximal solvable subgroup is abelian, and indicate considerable unicity in the representation theory of a physically fundamental and mathematically prototypical case, namely the Poincaré group, whose universal (double) cover is denoted $\tilde{\mathbf{P}}$. It is shown here that except for a relatively unfamiliar eight-dimensional representation and its dual, complex conjugate, and anti-dual, the remaining few representations of $\tilde{\mathbf{P}}$ of equal or smaller dimension are all well known, and are obtained by restricting the adjoint or coadjoint representations of $\tilde{\mathbf{P}}$ (or $\tilde{\mathbf{P}}$ extended by scale transformations, denoted $\tilde{\mathbf{P}}^e$; cf. section 5) to invariant subspaces, or obtained by restricting the defining representations of SU(2,2) and 0(2,4) to subgroups locally isomorphic to $\tilde{\mathbf{P}}$.

The motivation for this work is not only or even, primarily group-theoretical, however, but comes largely from joint work [*6*] with I. E. Segal, developing an elementary particle theory on the « universal » space-time $\tilde{\mathbf{M}}$, known in one form having additional structure as the « Einstein univers ». The isotropy group or « little group » of the full 15-dimensional causal group acting on $\tilde{\mathbf{M}}$ is isomorphic to $\tilde{\mathbf{P}}$. Therefore, causally covariant fields over $\tilde{\mathbf{M}}$ are naturally induced from representations of $\tilde{\mathbf{P}}^e$, and species of this usual type of finite-component fields thereby correspond in a $1-1$ manner to finite-dimensional representations of $\tilde{\mathbf{P}}^e$; cf. [*7*] and [*8*] for further discussion.

2. UPPER TRIANGULAR FORMS

We will need a general structure theorem for finite-dimensional representations of $\tilde{\mathbf{P}}$, which is also applicable to rather general groups. The essential content of Theorem 1 is that given any finite-dimensional representation ρ of a Lie algebra $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{n}] = \mathfrak{n}$, where $\mathfrak{n}$ is the maximal solvable ideal of $\mathfrak{g}$, then

$$\bigcap_{X \in \mathfrak{n}} \ker \rho(X)$$

is nonzero. This conclusion follows entirely from Cor. 2 on p. 45 of (4) (which is also cited and applied to $\tilde{\mathbf{P}}$ in [5]); the following brief alternative proof avoids the use of enveloping algebras and extensive preliminaries.

Given a real or complex vector space V, V* will denote the space of all complex-valued linear functionals on V.

THEOREM 1. — Let $\mathfrak{g}$ be any real finite-dimensional Lie algebra such that $[\mathfrak{g}, \mathfrak{n}] = \mathfrak{n}$, where $\mathfrak{n}$ is the maximal solvable ideal of $\mathfrak{g}$. Let $\mathfrak{h}$ be any semisimple subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{n}$. Then, given any finite-dimensional representation ρ of $\mathfrak{g}$ in a complex vector space V, there exists a direct sum decomposition

$$V = V_1 + \ldots + V_n$$

such that the V_j are invariant and irreducible under $\rho(\mathfrak{h})$, and such that

$$\rho(\mathfrak{n})V_j \subseteq \sum_{l<j} V_l \qquad \text{for} \quad j = 1, \ldots, n.$$

REMARK 1. — The hypothesis $[\mathfrak{g}, \mathfrak{n}] = \mathfrak{n}$ of the Theorem is easily seen to be equivalent to the following: the only element of $\mathfrak{n}^*$ that is invariant under the coadjoint action of the adjoint group of $\mathfrak{g}$ is 0.

Proof. — The proof is by induction on the dimension of V. It suffices to find a nonzero subspace W of V that is invariant under $\rho(\mathfrak{h})$ and such that $\rho(\mathfrak{n})W = \{0\}$. For then such a subspace W which is irreducible under $\rho(\mathfrak{h})$, and a complementary $\rho(\mathfrak{h})$-invariant subspace U, may be found. The inductive hypothesis may then be applied to the representation of $\mathfrak{g}$ on $V/W \cong U$.

Let G be the simply connected Lie group corresponding to $\mathfrak{g}$; let R denote the representation of G corresponding to ρ. By Lie's theorem ([1], p. 201), there exists a common eigenvector v for all $\rho(X)$ ($X \in \mathfrak{n}$), i. e., there exists $\lambda \in \mathfrak{n}^*$ such that

$$\rho(X)v = \lambda(X)v \tag{2.1}$$

for all $X \in \mathfrak{n}$. Now for all $g \in G$,

$$R(g)\rho(X)R(g^{-1}) = \rho(\mathrm{Ad}(g)X),$$

so that

$$\rho(\mathrm{Ad}(g)X)R(g)v = \lambda(X)R(g)v \tag{2.2}$$

for all $X \in \mathfrak{n}$. Thus $R(g)v$ is also an eigenvector for $\rho(X)$ with eigenvalue $\lambda(\mathrm{Ad}(g^{-1})X)$. We will show next that

$$\lambda(\mathrm{Ad}(g^{-1})X) = \lambda(X) \tag{2.3}$$

for all $X \in \mathfrak{n}$ and $g \in G$; this will imply $\lambda = 0$ by hypothesis (cf. Remark 1), so that the span of all vectors $R(g)v$ $(g \in G)$ will provide a subspace W of the type sought.

To show (2.3), i. e., that $\lambda \in \mathfrak{n}^*$ is invariant under G, argue as follows. Note first that there exist only finitely many $f \in \mathfrak{n}^*$ such that an equation of the form

$$\rho(X)w = f(X)w \tag{2.4}$$

holds for all $X \in \mathfrak{n}$ and some nonzero $w \in V$. To see this, take a basis $\{\beta_j\}$ of V such that the matrix entries for all $\rho(X)$ $(X \in \mathfrak{n})$ are 0 below the diagonal, which exists again by Lie's theorem. Then if (2.4) holds for some f and w where $w = \Sigma_j a_j \beta_j$ $(a_j \in C)$, then there exists a unique k such that $a_k \neq 0$ and $a_j = 0$ for all $j > k$. It follows easily from (2.4) that $f(X)$ must equal the k-th diagonal matrix element of $\rho(X)$; there are only finitely many such diagonal entries.

Now each element of the set

$$\{ \lambda(\mathrm{Ad}(g^{-1}).) : g \in G \}$$

satisfies (2.4) by (2.2); thus this set must be finite. By continuity of the coadjoint action and connectivity of G, all $\lambda(\mathrm{Ad}(g).)$ are equal, i. e. λ is G-invariant, as claimed.

3. INTERACTING PAIRS OF REPRESENTATIONS OF SL(2, C)

This section considers the representations of $\tilde{\mathbf{P}}$ where the number n of summands in Theorem 1 is two, and finds all such representations of dimension 8 or less. It will be seen in the next section (Lemma 3.1) that this case is fundamental for the general case.

To proceed further we need a concrete form for $\tilde{\mathbf{P}}$; this will be the semidirect product of SL(2, C) and $\mathbf{H}(2)$ = all 2×2 hermitian matrices, written SL(2, C) $\tilde{\times}$ $\mathbf{H}(2)$, as in [*6*], section 2.1. The group multiplication is then

$$(3.1) \quad (L \tilde{\times} F)(L' \tilde{\times} F') = LL' \tilde{\times} (F + LF'L^*) \quad (L, L' \in SL(2, C),\ F, F' \in \mathbf{H}(2))$$

(* denoting hermitian conjugate); it is obtained from the following group action of $\tilde{\mathbf{P}}$ on $\mathbf{H}(2)$:

$$L \tilde{\times} F : H \to F + LHL^* \quad (L \in SL(2, C),\ H, F \in \mathbf{H}(2)).$$

Elements of the Lie algebra of $\tilde{\mathbf{P}}$ will be written $A \tilde{+} F$, where $A \in sl(2, C)$ = traceless 2 × 2 matrices and $F \in \mathbf{H}(2)$. The derived commutation relations are then

(3.2) $$[A \tilde{+} F, A' \tilde{+} F'] = [A, A'] \tilde{+} (AF' + F'A^* - A'F - FA'^*)(A, A' \in sl(2, C),\ F, F' \in \mathbf{H}(2)).$$

Let $\{ \mathbf{D}(j_+, j_-) \}$ denote a fixed set of representatives for the irreducible finite-dimensional representations of SL(2, C), which arise in our application of Theorem 1. In this standard parametrization the « spins » $j_\pm$ are in the range 0, 1/2, 1, ..., and the *dimension* of $\mathbf{D}(j_+, j_-)$ (dimension of the representation space) is $(2j_+ + 1)(2j_- + 1)$. The labeling is such that $\mathbf{D}(j, 0)(\mathbf{D}(0, j))$ is a holomorphic (resp. anti-holomorphic) representation. Some further terminology will be especially convenient.

DÉFINITION 1. — Two irreducible representations R_1 and R_2 of SL(2, C) on the vector spaces V_1 and V_2 are said to *interact* if there exists a linear transformation α, not identically zero, from $\mathbf{H}(2)$ to the linear transformations from V_2 to V_1, such that the mapping R from $\tilde{\mathbf{P}}$ to $GL(V_1 \oplus V_2)$, determined by

$$R(L \tilde{\times} F) = R(I \tilde{\times} F)R(L \tilde{\times} 0),$$

where

$$R(L \tilde{\times} 0)v = \begin{cases} R_1(L)v & \text{if } v \in V_1 \\ R_2(L)v & \text{if } v \in V_2 \end{cases}$$

and

$$R(I \tilde{\times} F)v = \begin{cases} 0 & \text{if } v \in V_1 \\ \alpha(F)v & \text{if } v \in V_2 \end{cases}$$

($L \in SL(2, C)$, $F \in \mathbf{H}(2)$) is a representation of $\tilde{\mathbf{P}}$. In this case, R_1 and R_2 are said to be an *interacting pair*. The *interaction* of R_1 and R_3 is further said to be *uniquely determined* if any two such representations R are linearly conjugate (under a similarity transformation).

REMARK 2. — In the usual matrix form,

$$R(L \tilde{\times} F) = \begin{pmatrix} R_1(L) & \alpha(F)R_2(L) \\ 0 & R_2(L) \end{pmatrix}.$$

REMARK 3. — The definitions are actually symmetric in R_1 and R_2, by formation of the representation dual (contragredient) to R. (Recall that all the irreducible finite-dimensional representations of SL(2, C) are self-dual.) Also, clearly R_1 and R_2 interact if and only if any linearly equivalent pair of representations R'_1 and R'_2 also interact.

By Theorem 1, two irreducible representations R_1 and R_2 interact if and only if there exists an indecomposable representations of $\tilde{\mathbf{P}}$ whose restriction to $\{ L \tilde{\times} 0 : L \in SL(2, C) \}$ is equivalent to $R_1 \oplus R_2$.

THEOREM 2. — Among the complete list of irreducible representations $\mathbf{D}(j_+, j_-)$ of SL(2, C) ($j_+, j_- = 0, 1/2, 1, \ldots$), the only interacting pairs whose dimensions sum to 8 or less are:

$$\mathbf{D}(1/2, 0) \text{ and } \mathbf{D}(0, 1/2), \qquad \mathbf{D}(0, 0) \text{ and } \mathbf{D}(1/2, 1/2),$$
$$\mathbf{D}(1, 0) \text{ and } \mathbf{D}(1/2, 1/2), \qquad \mathbf{D}(1/2, 0) \text{ and } \mathbf{D}(1, 1/2),$$

and the additional complex conjugate pairs,

$$\mathbf{D}(0, 1) \text{ and } \mathbf{D}(1/2, 1/2), \qquad \mathbf{D}(0, 1/2) \text{ and } \mathbf{D}(1/2, 1),$$

In each case the interaction is uniquely determined. (The corresponding representations are given in Theorem 3 of section 4.)

Proof. — A series of lemmas that exclude various possible interactions will first be required.

LEMMA 2.1. — Two given irreducible representations R_1 and R_2 of SL(2, C) interact if and only if there exists a nonvanishing α as in Definition 1 such that

$$R_1(L)\alpha(F) = \alpha(LFL^*)R_2(L) \tag{3.3}$$

for all $L \in SL(2, C)$, $F \in \mathbf{H}(2)$, or equivalently

$$dR_1(A)\alpha(F) - \alpha(F)dR_2(A) = \alpha(AF + FA^*) \tag{3.4}$$

for all $A \in sl(2, C)$, $F \in \mathbf{H}(2)$.

Proof. — The first equation follows from the multiplication rule (3.1) and Remark 2. The second reflects the commutation relations (3.2).

LEMME 2.2. — Suppose that the irreducible representations R_1 and R_2 interact. Then if the map α, in the notation of Lemma 2.1, is unique up to a scalar factor, it follows that the interaction of R_1 and R_2 is uniquely determined. If R_1 is inequivalent to R_2, then the converse is also true.

Proof. — The first statement is clear. Conversely, suppose that R_1 is inequivalent to R_2, and take any two representations R and R′ of $\tilde{\mathbf{P}}$ that are determined as in Definition 1 by α, resp. α'. Then, any equivalence between R and R′ must commute with all $R_1(L) \oplus R_2(L)$ ($L \in SL(2, C)$), hence has the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}$, $\lambda, \lambda' \in C$. It follows that α' and α differ only by a scalar factor.

LEMMA 2.3. — If the representations $\mathbf{D}(j_+, j_-)$ and $\mathbf{D}(l_+, l_-)$ interact, then at least one of

$$j_+ + j_-, \qquad j_+ + j_- - 1, \ldots, \qquad |j_+ - j_-|$$

equals one of

$$l_+ + l_-, \qquad l_+ + l_- - 1, \ldots; \qquad | l_+ - l_- | .$$

If in addition these two lists have only one member in common, then the interaction of $\mathbf{D}(j_+, j_-)$ and $\mathbf{D}(l_+, l_-)$ is uniquely determined.

Proof. — Assuming the first stated hypothesis, it follows that $\alpha(\mathrm{I}) \neq 0$, since if $\alpha(\mathrm{I}) = 0$ then $\alpha(\mathrm{LL}^*) = 0$ for all $\mathrm{L} \in \mathrm{SL}(2, \mathrm{C})$ by Lemma 2.1; such LL^* span $\mathbf{H}(2)$, implying the vanishing of α, a contradiction. Thus $\alpha(\mathrm{I})$ is an intertwining operator for the restrictions of R_1 and R_2 to $\mathrm{SU}(2) = \{ \mathrm{L}: \mathrm{LL}^* = \mathrm{I} \}$, again by (3.3). The first conclusion then follows from Schur's lemma and the well-known decomposition of the $\mathbf{D}(j_+, j_-)$ upon restriction to SU(2).

Since the summands of this decomposition are of multiplicity 1, the second hypothesis implies that the intertwining operator $\alpha(\mathrm{I})$ is uniquely determined up to a scalar factor. α is then determined on a spanning set in $\mathbf{H}(2)$ (the LL^*) by the following equation obtained from (3.3):

$$\alpha(\mathrm{LL}^*) = \mathrm{R}_1(\mathrm{L})\alpha(\mathrm{I})\mathrm{R}_2(\mathrm{L})^{-1} .$$

The conclusion then follows from Lemma 2.2.

LEMMA 2.4. — (1) No $\mathbf{D}(j, 0)$ (resp. $\mathbf{D}(0, j)$) interacts with any $\mathbf{D}(l, 0)$ (resp. $\mathbf{D}(0, l)$).

(2) No $\mathbf{D}(j, 0)$ interacts with any $\mathbf{D}(0, l)$ where $j \neq 1$.

(3) $\mathbf{D}(l, 0)$ interacts with $\mathbf{D}(0, l)$ if and only if $l = 1/2$.

Proof. — Part (2) and the case $j \neq 1$ of part (1) follow from Lemma 2.3. In the case of $\mathbf{D}(j, 0)$ with $\mathbf{D}(j, 0)$, $\alpha(\mathrm{I})$ is clearly constrained to be a scalar, say c. But then by Lemma 2.1,

$$c\mathbf{D}(j, 0)(\mathrm{L}) = \alpha(\mathrm{LL}^*)\mathbf{D}(j, 0)(\mathrm{L}) ,$$

or simply $c = \alpha(\mathrm{LL}^*)$, for all $\mathrm{L} \in \mathrm{SL}(2, \mathrm{C})$. This clearly implies $c = 0$, hence $\alpha = 0$.

To prove (3), we may assume that $\mathbf{D}(l, 0)$ and $\mathbf{D}(0, l)$ are holomorphic, resp. anti-holomorphic extensions of a representation of SU(2) of spin 1 on a space V. By Lemma 2.1, we must show that only if $l = 1/2$ does there exist a nonvanishing linear map α from $\mathbf{H}(2)$ to gl(V) such that

$$\mathrm{V}_1(\mathrm{A})\alpha(\mathrm{F}) - \alpha(\mathrm{F})\mathrm{V}_2(\mathrm{A}) = \alpha(\mathrm{AF} + \mathrm{FA}^*) \qquad (3.5)$$

for all $\mathrm{A} \in \mathrm{sl}(2, \mathrm{C})$ and $\mathrm{F} \in \mathbf{H}(2)$, where V_1 and V_2 are the Lie algebra representations corresponding to $\mathbf{D}(l, 0)$ and $\mathbf{D}(0, l)$. As in the proof of

Lemma 2.3, $\alpha(\mathrm{I})$ must be a scalar, and without loss of generality equals 1, say. Taking $A = \sigma_1$, $F = I$ in (3.5), then

$$\begin{aligned}\alpha(2\sigma_1) &= V_1(\sigma_1) - V_2(\sigma_1)\\ &= iV_1(-i\sigma_1) + iV_2(-i\sigma_1)\\ &= 2iV_1(-i\sigma_1),\end{aligned}$$

so $\alpha(\sigma_1) = iV_1(-i\sigma_1)$. ($\sigma_1, \sigma_2, \sigma_3$ are the usual Pauli matrices.) But then substitution into (3.5) with $A = F = \sigma_1$ gives

$$\begin{aligned}2 = \alpha(\mathrm{I}) &= V_1(\sigma_1)iV_1(-i\sigma_1) - iV_1(-i\sigma_1)V_2(\sigma_1)\\ &= iV_1(-i\sigma_1)iV_1(-i\sigma_1) - iV_1(-i\sigma_1)(-i)V_1(-i\sigma_1)\\ &= -2[V_1(-i\sigma_1)]^2.\end{aligned}$$

Thus $V_1\left(-\frac{i}{2}\sigma_1\right)^2 = -\frac{1}{4}$, and similarly for σ_2 and σ_3. Thus

$$\Sigma_{j=1}^{3} V_1\left(-\frac{j}{2}\sigma_j\right)^2 = -3/4.$$

But the left hand side of this equation equals $-l(l+1)$ in general, where l is the spin of the representation; thus $l = \frac{1}{2}$.

Conversely, the interaction of **D**(1/2, 0) with **D**(0, 1/2) is listed in Theorem 3).

LEMMA 2.5. — The trivial representation of the subgroup $\{ L \tilde{\times} 0 : L \in SL(2, C) \}$ of $\tilde{\mathbf{P}}$ on any finite-dimensional space cannot be nontrivially extended to a representation of $\tilde{\mathbf{P}}$.

Proof. — By Lemma 2.1, any representation R of $\tilde{\mathbf{P}}$ extending a trivial representation of the homogeneous Lorentz group would satisfy

$$R(I \tilde{\times} F) = F(I \tilde{\times} LFL^*)$$

for all $L \in SL(2, C)$, $F \in \mathbf{H}(2)$, thus also (differentiating)

$$0 = dR(0 \tilde{+} (AF + FA^*)) \qquad (A \in sl(2, C),\ F \in \mathbf{H}(2)).$$

But such $AF + FA^*$ span $\mathbf{H}(2)$, hence R is trivial on the translations.

Proof of Theorem 2. — No pair of irreducible representation of SL(2, C) whose dimensions sum to 3 or less can interact, by Lemmas 2.3 and 2.5. In the case of such dimensions summing to 4, the possibilities of interaction of **D**(0, 0) and **D**(1, 0), **D**(1/2, 0) and **D**(1/2, 0), and their complex conjugates (henceforth abbreviated to cc.) are excluded by Lemma 2.3 and Lemma 2.4.(1), respectively. The interaction of **D**(1/2, 0) and D(0, 1/2) stated in Theorem 3 is easily verified, and in any case uniquely determined by Lemma 2.3.

In the case of dimensions summing to 5, interactions of **D**(1/2, 0) and **D**(1, 0), **D**(1/2, 0) and **D**(0, 1), **D**(3/2, 0) and **D**(0, 0), and cc. are excluded by Lemma 2.3. On the other hand, the existence of the representations of $\tilde{\mathbf{P}}$ listed in Theorem 3 restricting to **D**(1/2, 1/2) ⊕ **D**(0, 0) under SL(2, C) is easily verified; by Lemma 2.3 this interaction is also uniquely determined.

Now consider two irreducible representations of SL(2, C) whose dimensions sum to 6. If the dimensions of the summands are 1 and 5, or 2 and 4, then interaction is excluded by Lemma 2.3. Finally, interaction of **D**(1, 0) with **D**(1, 0) or **D**(0, 1) with **D**(0, 1) is excluded by Lemma 2.4 (1); exclusion of the pair { **D**(1, 0), **D**(0, 1) } is by Lemma 2.4.(3).

Consider next the cases of dimensions summing to 7. The cases of dimensions 6 plus 1, 5 plus 2, and the cases of 4 plus 3 where the representation of dimension 4 is either **D**(3/2, 0) or **D**(0, 3/2), are excluded again by Lemma 2.3. The remaining possibilities, namely, interaction of **D**(1/2, 1/2) with **D**(1, 0) and **D**(0, 1), which are listed in Theorem 3, are unique by Lemma 2.3; they are easily verified to be representations of $\tilde{\mathbf{P}}$.

Finally, of the cases of dimensions summing to 8, the cases of 1 plus 7 and 3 plus 5 are excluded by Lemma 2.3. Concerning the cases of dimensions 4 plus 4, by Lemma 2.3 and Lemma 2.4., (1) and (3), it suffices to rule out the interaction of **D**(1/2, 1/2) with itself by a special argument. It is convenient to realize the infinitesimal representation ρ, corresponding to **D**(1/2, 1/2), on **H**(2) in a standard way:

$$\rho(\mathrm{A}): \mathrm{H} \to \mathrm{AH} + \mathrm{HA}^*$$

for all $\mathrm{A} \in \mathrm{sl}(2, \mathrm{C})$, $\mathrm{H} \in \mathbf{H}(2)$. If now

$$(3.6)\quad \rho(\mathrm{A})\alpha(\mathrm{F}) - \alpha(\mathrm{F})\rho(\mathrm{A}) = \alpha(\mathrm{AF} + \mathrm{FA}^*) \quad (\mathrm{A} \in \mathrm{sl}(2, \mathrm{C}),\ \mathrm{F} \in \mathbf{H}(2)),$$

obtained from equation (3.4), then it follows that

$$\alpha(\mathrm{I})(\mathrm{F}) = z\mathrm{F} + w\sigma_2\overline{\mathrm{F}}\sigma_2$$

for constants z and w, by application of Schur's lemma to ρ restricted to su(2). The substitutions $\mathrm{A} = \sigma_3$, $\mathrm{F} = \mathrm{I}$ in (3.6) then lead to

$$\alpha(\sigma_3)\mathrm{F} = w\sigma_3\sigma_2\overline{\mathrm{F}}\sigma_2 + w\sigma_2\overline{\mathrm{F}}\sigma_2\sigma_3 \qquad (\mathrm{F} \in \mathbf{H}(2)).$$

But then resubstitution into (3.6) ($\mathrm{A} = \mathrm{F} = \sigma_3$) yields

$$4w\sigma_2\overline{\mathrm{F}}\sigma_2 + 4w\sigma_1\overline{\mathrm{F}}\sigma_1 = 2z\mathrm{F} + 2w\sigma_2\overline{\mathrm{F}}\sigma_2 \quad (\mathrm{F} \in \mathbf{H}(2)).$$

For $\mathrm{F} = \mathrm{I}$ this is $8w = 2z + 2w$ or $z = 3w$; when $\mathrm{F} = \sigma_3$ obtain

$$-4w\sigma_3 - 4w\sigma_3 = 2z\sigma_3 - 2w\sigma_3$$

or $-3w = z$. Thus $z = w = 0$ so $\alpha = 0$.

The remaining 8-dimensional cases are those where the dimensions of the summands are 2 and 6; by Lemma 2.3 only the possible interactions of $\mathbf{D}(1/2, 0)$ and cc. with $\mathbf{D}(1, 1/2)$ and cc. need consideration. The concrete form given in Theorem 3 for the infinitesimal representation corresponding to $\mathbf{D}(1, 1/2)$ was constructed as follows. Note that

$$-\frac{i}{2}\sigma_1 \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad -\frac{i}{2}\sigma_2 \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$-\frac{i}{2}\sigma_3 \to \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and of course $-\frac{i}{2}\sigma_j \to -\frac{i}{2}\sigma_j$ ($j = 1, 2, 3$) define representations of the Lie algebra su(2). The 6×6 matrices given for $d\mathrm{R}_8^2(\mathrm{A} \tilde{+} 0)$ ($\mathrm{A} \in \mathrm{sl}(2, \mathrm{C})$) determine the Kronecker (tensor) product of these two representations, extended i-linearly to sl(2, C) on the first (left) factor and i-anti-linearly on the second factor.

The argument which excludes interaction of $\mathbf{D}(1/2, 0)$ with $\mathrm{D}(1/2, 1)$ (and thus also the complex conjugate pair) is similar to that above excluding self-interaction of $\mathbf{D}(1/2, 1/2)$. $\alpha(\mathrm{I})$ is first determined up to a scalar factor, as previously, by restriction of the representations to SU(2). Then substitution of $\mathrm{F} = \mathrm{I}$ and $\mathrm{A} = \sigma_3$ into (3.4) determines $\alpha(\sigma_3)$; but then resubstitution into (3.4) ($\mathrm{A} = \mathrm{F} = \sigma_3$) leads to a contradiction unless $\alpha(\mathrm{I}) = 0$, or $\alpha = 0$. Further details for this case are omitted.

4. LOW DIMENSIONAL INDECOMPOSABLE REPRESENTATIONS OF $\tilde{\mathrm{P}}$

A finite-dimensional group representation is defined to be *indecomposable* if it cannot be written as the direct sum of two other representations of smaller dimensions.

The plan of this section is similar to that of the previous one. First the main result is stated; then several lemmas are derived providing sufficient conditions of a general nature for decomposability of certain representations of $\tilde{\mathbf{P}}$; finally, Theorem 3 is proven.

The notation $\mathbf{S}(2)$ will be used for the space of all symmetric complex 2×2 matrices.

THEOREM 3. — The following is a list of mutually inequivalent representatives of all of the (complex-linear) equivalence classes of indecomposable representations of $\tilde{\mathbf{P}}$, of dimension 8 or less, that are nontrivial on the translation subgroup of $\tilde{\mathbf{P}}$.

In dimension 4 :

$$R_4 : L \tilde{\times} F \to \begin{pmatrix} L & \frac{i}{2} F L^{*-1} \\ 0 & L^{*-1} \end{pmatrix}$$

and its contragredient $\hat{R}_4$, or equivalently complex conjugate representation ((8)).

In dimension 5 : The representation R_5 of $\tilde{\mathbf{P}}$ on $\mathbf{H}(2) \oplus R^1$, -- determined by

$$R_5(L \tilde{\times} 0)(H, x) = (LHL^*, x)$$

and

$$R_5(I \tilde{\times} F)(H, x) = (xF, 0) \ (L \in SL(2, C), x \in R^1, H, F \in \mathbf{H}(2)),$$

-- and its dual $\hat{R}_5$ on $R^1 \oplus \mathbf{H}(2)$, defined by

$$\hat{R}_5(L \tilde{\times} 0)(x, H) = (x, LHL^*)$$

and

$$\hat{R}_5(I \tilde{\times} F)(x, H) = (\mathrm{Tr}(H\sigma_2 \overline{F} \sigma_2), 0).$$

Both R_5 and $\hat{R}_5$ are real representations.

In dimension 6 : The real and self-contragredient representation R_6, defined on $R^1 \oplus \mathbf{H}(2) \oplus R^1$ by

$$R_6(L \tilde{\times} 0)(x, H, y) = (x, LHL^*, y)$$

and

$$R_6(I \tilde{\times} F)(x, H, y) = (\mathrm{Tr}(H\sigma_2 \overline{F} \sigma_2), \ Fy, 0)(x, y \in R^1, F, H \in \mathbf{H}(2)).$$

In dimension 7 : The representation R_7 on $V_7 = \mathbf{S}(2) \oplus (\mathbf{H}(2) + i\mathbf{H}(2))$ -- defined on $\mathbf{S}(2) \oplus \mathbf{H}(2)$ by

$$R_7(L \tilde{\times} 0)(S, H) = (LSL^T, LHL^*)$$

and

$$R_7(I \tilde{\times} F)(S, H) = (H\sigma_2 \overline{F} - F\sigma_2 \overline{H}, 0) \ (F, \ H \in \mathbf{H}(2), \ S \in \mathbf{S}(2), \ L \in SL(2, C)).$$

(^{T} denoting transpose) and extended to V_7 by complex-linearity --, and the dual $\hat{R}_7$, complex conjugate $\overline{R}_7$, and anti-dual representations $\hat{\overline{R}}_7$ (all inequivalent).

In dimension 8 : There are 8 such representations. First, the representation R_8^1 on $C \oplus \mathbf{S}(2) \oplus (\mathbf{H}(2) + i\mathbf{H}(2)) = V_8^1$-- defined on $R^1 \oplus \mathbf{S}(2) \oplus \mathbf{H}(2)$ by

$$R_8^1(L \tilde{\times} 0)(x, S, H) = (x, LST^T, LHL^*)$$

and

$$R_8^1(I \tilde{\times} F)(x, S, H)$$
$$= (\mathrm{Tr}(H\sigma_2 \overline{F} \sigma_2), H\sigma_2 \overline{F} - F\sigma_2 \overline{H}, 0) \ (x \in R^1, S \in \mathbf{S}(2), H, F \in \mathbf{H}(2))$$

and extended to V_8^1 by linearity--, and the representations dual, complex conjugate, and anti-dual to R_8^1, all mutually inequivalent. Secondly, there are four others, three being dual, complex conjugate, and anti-dual to the representation R_8^2 on the space $C^2 \oplus C^6$, whose corresponding infinitesimal representation dR_8^2 is defined as follows.

Let $v \in C^2$ and $w \in C^6$, regarded as column vectors. Then for

$$A = -\frac{i}{2}(x\sigma_1 + y\sigma_2 + z\sigma_3),$$

where x, y, and z are real,

$dR_8^2(A \tilde{+} 0)(v, w)$

$$= \left(Av, \begin{pmatrix} -\frac{i}{2}z & -z & y & -\frac{1}{2}y-\frac{i}{2}x & 0 & 0 \\ z & -\frac{i}{2}z & -x & 0 & -\frac{1}{2}y-\frac{i}{2}x & 0 \\ -y & x & -\frac{i}{2}z & 0 & 0 & -\frac{1}{2}y-\frac{i}{2}x \\ -\frac{i}{2}x-\frac{1}{2}y & 0 & 0 & \frac{i}{2}z & -z & y \\ 0 & -\frac{i}{2}x+\frac{1}{2}y & 0 & z & \frac{i}{2}z & -x \\ 0 & 0 & -\frac{i}{2}x+\frac{1}{2}y & -y & x & \frac{i}{2}z \end{pmatrix} w \right);$$

for $A = \frac{1}{2}(x\sigma_1 + y\sigma_2 + z\sigma_3)$ $(x, y, z$ real),

$dR_8^2(A \tilde{+} 0)(v, w)$

$$= \left(Av, \begin{pmatrix} -\frac{1}{2}z & -iz & iy & -\frac{1}{2}x+\frac{i}{2}y & 0 & 0 \\ iz & -\frac{1}{2}z & -ix & 0 & -\frac{1}{2}x+\frac{i}{2}y & 0 \\ -iy & ix & -\frac{1}{2}z & 0 & 0 & -\frac{1}{2}x+\frac{i}{2}y \\ -\frac{1}{2}x-\frac{i}{2}y & 0 & 0 & \frac{1}{2}z & -iz & iy \\ 0 & -\frac{1}{2}x-\frac{i}{2}y & 0 & iz & \frac{1}{2}z & -ix \\ 0 & 0 & -\frac{1}{2}x-\frac{i}{2}y & -iy & ix & \frac{1}{2}z \end{pmatrix} w \right);$$

and for $F = u + x\sigma_1 + y\sigma_2 + z\sigma_3$ $(x, y, z, u \text{ real})$,

$$dR_8^2(0 \tilde{+} F)(v, w) = \left(\begin{pmatrix} x+iy & y-ix & u+z & u-z & i(z-u) & x-iy \\ u+z & i(u+z) & -x-iy & x-iy & y+ix & z-u \end{pmatrix} w, 0\right).$$

LEMMA 3.1. — Let R_i be a finite-dimensional irreducible representation of SL(2, C) on V_i, for $i=1, \ldots, n$. Let V be the direct sum $V_1 + \ldots + V_n$. Define $R(L \tilde{\times} 0)$ $(L \in SL(2, C))$ on V by

$$R(L \tilde{\times} 0)v = R_i(L)v \qquad \text{for all} \quad v \in V_i.$$

For each pair (i, j) such that $1 \leqslant i < j \leqslant n$, let α_{ij} be a given linear mapping of $\mathbf{H}(2)$ into the linear transformations from V_j to V_i. For each $L \tilde{\times} F \in \tilde{\mathbf{P}}$, define

$$R(L \tilde{\times} F) = \exp(\rho(F))R(L \tilde{\times} 0),$$

where

$$\rho(F)v = \sum_{i<j} \alpha_{ij}(F)v \qquad \text{for} \quad v \in V_j.$$

Then R is a representation of $\tilde{\mathbf{P}}$ if and only if

(*i*) $\{ \rho(F) : F \in \mathbf{H}(2) \}$ is commutative, and

(*ii*) $R_i(L)\alpha_{ij}(F) = \alpha_{ij}(LFL^*)R_j(L)$ for all $L \in SL(2, C)$ and $F \in \mathbf{H}(2)$.

Proof. — By comparison with equation (3.1).

LEMMA 3.2. — Let R be a finite-dimensional representation of $\tilde{\mathbf{P}}$ on the vector space V. Let $V = V_1 + \ldots + V_n$ be a direct sum decomposition of V with the properties asserted in Theorem 1. Let R_i be the representation of SL(2, C) obtained by restricting $L \to R(L \tilde{\times} 0)$ to V_i. For each pair of indices (i, j) such that $1 \leqslant i < j \leqslant n$, let α_{ij} be the linear mapping from $\mathbf{H}(2)$ to the linear transformations from V_j to V_i such that for all $v \in V_j$ and $F \in \mathbf{H}(2)$,

$$R(I \tilde{\times} F) = \exp(\rho(F)) \qquad \text{and} \qquad \rho(F)v = \sum_{i<j} \alpha_{ij}(F)v.$$

Then if $\{ 1, \ldots, n \}$ is the disjoint union of two non empty subsets such that $\alpha_{ij} = 0$ whenever i and j belong to different subsets, then R is decomposable.

Proof. — Let A and B be the hypothesized subsets of $\{ 1, \ldots, n \}$. Then clearly $\Sigma_{j \in A} V_j$ and $\Sigma_{j \in B} V_j$ are complementary R-invariant subspaces of V.

LEMMA 3.3. — Let R be a finite-dimensional representation of $\tilde{\mathbf{P}}$ on the vector space V; let V_i, R_i, and α_{ij} be as in the statement of Lemma 3.2. suppose that there are indices k, l, m such that:

a) $1 \leqslant k < l < m \leqslant n$,

b) the R_i for $i=k, \ldots, l$ are all equivalent, and inequivalent to R_m,

c) $\alpha_{ij} = 0$ whenever $i < l, j = k, \ldots, l$, and $i < j$,

d) $\alpha_{ij} = 0$ whenever $i = k, \ldots, l, j > l$, and $j \neq m$, and

e) in the case that $\alpha_{im} \neq 0$ for all $i = k, \ldots, l$, it is assumed that the interaction of R_k and R_m (which exists by Lemma 3.1) is uniquely determined.

Then R is decomposable.

REMARK 4. — By formation of the representation contragredient to R, the following hypotheses, alternative to *a*)-*e*) above, are also sufficient to imply the decomposability of R.

a)′ $1 \leqslant m < k < l \leqslant n$,

b)′ the R_i for $i=k, \ldots, l$ are all equivalent, and in equivalent to R_m,

c)′ $\alpha_{ij} = 0$ whenever $i = k, \ldots, l, j > k$, and $i < i$,

d)′ $\alpha_{ij} = 0$ whenever $i < k, i \neq m$, *and* j $k, \ldots, l$, and

e) in the case that $\alpha_{mi} \neq 0$ for all $i = k, \ldots, l$, it is assumed that the interaction of R_k and R_m is uniquely determined.

Proof. — If $\alpha_{im} = 0$ for some i such that $k \leqslant i \leqslant l$, then it is easily checked that V_i and $\Sigma_{j \neq i} V_j$ are complementary R-invariant subspaces.

Now suppose $\alpha_{im} \neq 0$ for all $i = k, \ldots, l$. By *b*), there exists an isomorphism T of V_k with V_{k+1} such that

$$TR_k(L) = R_{k+1}(L)T$$

for all $L \in SL(2, C)$. By Lemmas 3.1 and 2.1, each of the triples (R_k, R_m, α_{km}), $(R_{k+1}, R_m, \alpha_{k+1m})$ determines a representation of $\tilde{P}$ which is nontrivial on the translations and of the kind considered in section 3. By *b*), *e*), and Lemma 2.2, it must be the case that

$$\alpha_{k+1m} = cT\alpha_{km}$$

for some nonzero constant c. Defining

$$W = \sum_{j<k} V_j + \{ x + cTx : x \in V_k \} + \sum_{j>k+1} V_j ,$$

it follows that $R(g)V_m \subseteq W$ for all $g \in \tilde{P}$, and in addition that W and V_{k+1} are nonzero complementary subspaces of V which are invariant under R.

Proof of Theorem 3. — By Theorems 1 and 2 it suffices to find those indecomposable representations of $\tilde{P}$ where the number n of irreducible summands, upon restriction to $\{ L \tilde{\times} 0 : L \in SL(2, C) \}$, is three or more.

By Lemmas 2.1 and 3.1, any one of the dimensions of such irreducible summands of such representations of $\tilde{\mathbf{P}}$ must also be the dimensions of members of interacting pairs (cf. Definition 1). These pairs of dimensions (with sum $\leqslant 8$) being

$$\{2,2\}, \qquad \{1,4\}, \qquad \{3,4\}, \qquad \{2,6\}, \tag{4.1}$$

it follows that there can be no such representations additional to those of Theorem 2 which have dimension 5 or less.

Consider next an indecomposable representation of dimension 6. By Lemma 3.2 and Theorem 2 the dimensions of its irreducible summands must clearly be either $2+2+2$ or $1+1+4$. Such representations of the former type, except those of the form R where

$$(4.2) \quad \mathrm{R}: \mathrm{L} \tilde{\times} 0 \to \begin{pmatrix} \mathrm{L} & 0 & 0 \\ 0 & \mathrm{L}^{*-1} & 0 \\ 0 & 0 & \mathrm{L} \end{pmatrix}, \quad d\mathrm{R}: 0 \tilde{+} \mathrm{F} \to \begin{pmatrix} 0 & \alpha_{12}(\mathrm{F}) & 0 \\ 0 & 0 & \alpha_{23}(\mathrm{F}) \\ 0 & 0 & 0 \end{pmatrix},$$

are excluded by Lemma 3.3 and Remark 4. If such a representation of the form (4.2) is indecomposable, then clearly neither α_{12} or α_{23} can vanish identically. By Theorem 2, $\alpha_{12}(\mathrm{F})$ and $\alpha_{23}(\mathrm{F})$ must, up to scalar factors, be equal to F, resp. $\sigma_2\overline{\mathrm{F}}\sigma_2$. But it is easily verified that

$$\begin{pmatrix} 0 & \mathrm{F} & 0 \\ 0 & 0 & \sigma_2\overline{\mathrm{F}}\sigma_2 \\ 0 & 0 & 0 \end{pmatrix} \qquad (\mathrm{F} \in \mathbf{H}(2))$$

is not a commutative family of matrices (cf. (*i*) of Lemma 3.1).

Of the putative latter type of representations, indecomposability of those of the form

$$\mathrm{L} \tilde{\times} \mathrm{F} \to \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & \mathbf{D}(1/2, 1/2)(\mathrm{L}) \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \mathbf{D}(1/2, 1/2)(\mathrm{L}) & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

is excluded by Lemma 3.3. The remaining possibility for an indecomposable representation, namely R where

$$\mathrm{R}: \mathrm{L} \tilde{\times} 0 \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{D}(1/2, 1/2)(\mathrm{L}) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d\mathrm{R}: 0 \tilde{+} \mathrm{F} \to \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$$

is uniquely determined up to equivalence by Lemma 3.1 and Theorem 2.

Consider next a possible 7-dimensional indecomposable representation of $\tilde{\mathbf{P}}$ with 3 or more subrepresentations irreducible under SL(2, C). Again by Lemma 3.2 and the list (4.1), these summands must have dimensions 1 or 4, but then all indecomposable such possibilities are excluded by Lemma 3.3.

Finally, consider an 8-dimensional indecomposable representation of $\tilde{P}$ with 3 or more irreducible sub-representations. Again, by Lemma 3.2 and (4.1), it follows that the possible decompositions under SL(2, C), according to dimensions of the summands, are (*i*) $1+1+1+1+4$, (*ii*) $2+2+2+2$, and (*iii*) $1+3+4$. Indecomposability in case (*i*) is excluded by Lemma 3.3, and likewise for all but the following possible instances (and their complex conjugates and contragredients) of (*ii*):

$$L \tilde{\times} F \to \begin{pmatrix} L & * & * & * \\ 0 & \bar{L} & * & * \\ 0 & 0 & L & * \\ 0 & 0 & 0 & \bar{L} \end{pmatrix}, \tag{4.3}$$

$$L \tilde{\times} F \to \begin{pmatrix} L & * & * & * \\ 0 & \bar{L} & * & * \\ 0 & 0 & \bar{L} & * \\ 0 & 0 & 0 & L \end{pmatrix}, \tag{4.4}$$

and

$$L \tilde{\times} F \to \begin{pmatrix} L & * & * & * \\ 0 & L & * & * \\ 0 & 0 & \bar{L} & * \\ 0 & 0 & 0 & \bar{L} \end{pmatrix}. \tag{4.5}$$

We will show that all possibilities for representations of the forms (4.3-5) are decomposable. Elements of the representation space C^8 are as usual regarded as column vectors.

CASE (4.3). — By Lemma 2.4, (1) and (3), the translation part must have the form

$$dR(0 \tilde{+} F) = \begin{pmatrix} 0 & A & 0 & D \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where A, B, C, and D are linear functions of F. If now $B \neq 0$, then A and C must be zero, because (as shown above) there is no six-dimensional representation R′ of $\tilde{P}$ such that

$$R'(L \tilde{\times} 0) = \begin{pmatrix} L & 0 & 0 \\ 0 & \bar{L} & 0 \\ 0 & 0 & L \end{pmatrix}, \qquad dR'(0 \tilde{+} F) = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix},$$

where α and β are both nonzero. But R is clearly decomposable if A = C = 0.

On the other hand, if B = 0 then observe that

$$\begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (x \in \mathbf{C}).$$

$= \Omega$, say, commutes with $R(L \tilde{\times} 0)$, and

$$\Omega d R(0 \tilde{\times} F)\Omega^{-1} = \begin{pmatrix} 0 & A & 0 & D+xC \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If C = 0, then R is clearly decomposable. If C $\neq$ 0, then by Theorem 2 D must be a scalar multiple of C; thus $\Omega R \Omega^{-1}$, hence R, is decomposable.

CASE (4.4). — Here the translation part must have the form

$$d R(0 \tilde{+} F) = \begin{pmatrix} 0 & A & B & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & D \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We may conjugate by

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

without disturbing the $R(L \tilde{\times} 0)$; then

$$\Omega d R(0 \tilde{+} F)\Omega^{-1} = \begin{pmatrix} 0 & A & B-xA & 0 \\ 0 & 0 & 0 & C+xD \\ 0 & 0 & 0 & D \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus if D $\neq$ 0, C + xD = 0 for some x, in which case $\left\{ \begin{pmatrix} * \\ 0 \\ * \\ * \end{pmatrix} \right\}$ is an invariant subspace for $\Omega R \Omega^{-1}$.

By the 6-dimensional result applied above in case (4.3), B $-x$A must also be 0, and the representation is decomposed.

If D = 0 *and* C = 0, then the representation R is clearly decomposed.

If $D=0$ but $C \neq 0$, then

$$\left\{\begin{pmatrix} * \\ * \\ 0 \\ * \end{pmatrix}\right\}$$

is an invariant subspace, and again $C \neq 0$ forces $A = 0$; it results that

$$\left\{\begin{pmatrix} * \\ 0 \\ * \\ 0 \end{pmatrix}\right\} \quad \text{and} \quad \left\{\begin{pmatrix} 0 \\ * \\ 0 \\ * \end{pmatrix}\right\}$$

are complementary invariant subspaces.

CASE (4.5). — The translations must have the form

$$d\mathrm{R}(0 \tilde{+} \mathrm{F}) = \begin{pmatrix} 0 & 0 & \mathrm{A} & \mathrm{B} \\ 0 & 0 & \mathrm{C} & \mathrm{D} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

By a suitable conjugation by $\begin{pmatrix} \multicolumn{2}{c}{\mathrm{Y}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$ where $\mathrm{Y} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$,

We can assume $A = 0$ in (4.6).

Secondly, note that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} d\mathrm{R}(0 \tilde{+} \mathrm{F}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \mathrm{B} \\ 0 & 0 & \mathrm{C} & x\mathrm{B}+\mathrm{D} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $B = 0$, then $\left\{\begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 0 \\ * \\ * \\ * \end{pmatrix}\right\}$ are invariant; if $B \neq 0$ then $xB + D = 0$ for some x and the representation is decomposed.

It remains to consider the case of (*iii*). The four-dimensional representation must be $\mathbf{D}(1/2, 1/2)$, and without loss of generality we may take the three-dimensional one equal to $\mathbf{D}(1,0)$. It remains to note by a short computation that R, where

$$R: L \tilde{\times} 0 \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{D}(1/2, 1/2)(L) & 0 \\ 0 & 0 & \mathbf{D}(1,0)(L) \end{pmatrix}.$$

$$dR: 0 \tilde{+} F \to \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}.$$

does not define a representation if α and β (each uniquely determined up to a scalar factor by Theorem 2 in any case) are both nonzero, since (*i*) of Lemma 3.1 is not satisfied.

The mutual inequivalence of the representations listed is clear. Noting that any $\tilde{\mathbf{P}}$-invariant subspace must be the sum of its intersections with the SL(2, C)-isotypic invariant subspaces (all of multiplicity 1 or 0 except for R_6), indecomposability also follows easily.

5. EXTENSIONS TO SCALE AND THE CONFORMAL GROUP

In the rest of this paper the possibilities are considered of extending the representations listed in Theorem 3 to certain physically relevant groups containing $\tilde{\mathbf{P}}$ as a subgroup. An « extension of a group representation » means here an extension in the usual sense of a given homomorphism of a group into the linear operators on some space, i. e., the representation space is unchanged. (The « real representations » R_5, $\hat{R}_5$, and R_n are assumed for this purpose already extended by complex-linearity to the complexified 5- and 6-dimensional spaces.)

The eleven-dimensional group $\tilde{\mathbf{P}}^e$ (double cover of the scale-extended Poincaré group) mentioned in section 1 is by definition the semi-direct product $(R^1 \times SL(2,C)) \tilde{\times} \mathbf{H}(2)$ (cf. (6), section 2.1) with the multiplication

$$((t \times L) \tilde{\times} F)((t' \times L') \tilde{\times} F')$$
$$= ((t+t') \times LL') \tilde{\times} (F + e^t LF'L^*)(t, t' \in R^1,\ L,\ L' \in SL(2,\ C),\ F,\ F' \in \mathbf{H}(2)).$$

$\tilde{\mathbf{P}}$ is regarded as a subgroup of $\tilde{\mathbf{P}}^e$ in the obvious way.

Elements of the Lie algebra of $\tilde{\mathbf{P}}^e$ will be written $(t, A) \tilde{+} F$ ($t \in R^1$, $A \in sl(2, C)$, $F \in \mathbf{H}(2)$); the commutation relations are

$$[(t, A) \tilde{+} F, (t', A') \tilde{+} F']$$
$$= (0, [A, A']) \tilde{+} (AF' + F'A^* - A'F - FA'^* + tF' - t'F)$$

Consistently with (6), we define the scale generator $\mathbf{S} = (1, 0) \tilde{+} 0$ in the

Lie algebra of $\tilde{\mathbf{P}}^e$. Thus $\mathbf{S}$ commutes with all $(t, A) \tilde{+} 0$ ($t \in R^1$, $A \in sl(2, C)$), and

$$[\mathbf{S}, X] = X \tag{5.1}$$

for all X in the subalgebra of the Lie algebra of $\tilde{\mathbf{P}}^e$ generating the translations.

Now given any finite-dimensional representation R of $\tilde{\mathbf{P}}^e$ such that, when restricted to $\{(0 \times L) \tilde{\times} 0 : L \in SL(2, C)\}$, the multiplicities of irreducible representations are at most 1, then the action of $dR(\mathbf{S})$ must be multiplication by a constant w_j on each irreducible subspace V_j. In addition, by (5.1) $w_i = w_j + 1$ whenever for some X in the translation subalgebra and $v \in V_j$, $dR(X)v$ has a nonzero component in V_i (as observed in (5)). These necessary conditions for extending a representation from $\tilde{\mathbf{P}}$ to $\tilde{\mathbf{P}}^e$ are also clearly sufficient, and suffice to so treat all the representations in Theorem 3 except for R_6. Summarizing these considerations and a simple computation in the case of R_6, we obtain

PROPOSITION 4. — The indecomposable representations of $\tilde{\mathbf{P}}$ listed in Theorem 3 all extend to $\tilde{\mathbf{P}}^e$. In all cases except that of R_6, the possible extensions are parametrized as above by a constant $w \in C$, equal to, say, the value of $dU(\mathbf{S})$ in some subspace irreducible under $\{(0 \times L) \tilde{\times} 0 : L \in SL(2, C)\}$.

The possible extensions R_6^e of R_6 to $\tilde{\mathbf{P}}^e$ are parametrized by two constants w and u, via

$$dR_6^e(\mathbf{S}) : (x, H, y) \to (x(w+2)+uy, (w+1)H, wy)$$
$$(x, y, u, w \in C, H \in \mathbf{H}(2) + i\mathbf{H}(2)).$$

It is well known that $\tilde{\mathbf{P}}^e$ is isomorphic to a subgroup of SU(2, 2), and that the latter group has two inequivalent fundamental four-dimensional representations [5]; the restriction of these to $\tilde{\mathbf{P}}$ are obviously equivalent to R_4 and $\hat{R}_4$. It is also not hard to check that the defining representation of 0(2, 4) on R^6 restricts to an indecomposable representation of a group isomorphic to $\tilde{\mathbf{P}}$/(two element center), which must then be equivalent to R_6. On the other hand, by Weyl's dimension formula, for example, the dimensions of irreducible representations of SU(2, 2) are 4, 6, 10, 15, ...; thus clearly only R_4, $\hat{R}_4$, and R_6 extend to SU(2, 2) in the sense earlier indicated.

6. DISCRETE SYMMETRIES AND INVARIANT HERMITIAN FORMS

In this last section the possible extensions of the representations of Theorem 3 to the discrete symmetries T, C, and P, and the invariant hermitian forms on the representation spaces, will be determined. The existence of the latter, and the possibility of extending a representation to C (according to the definition given here), will be seen to be closely related. The geometrical symmetries T and P will be considered first.

The space- and time- reversal symmetries determine a pair of outer automorphisms of $\tilde{\mathbf{P}}$, denoted π, resp. τ, which have the forms

$$\begin{aligned} \pi &: \mathrm{L} \tilde{\times} \mathrm{F} \to \mathrm{L}^{*-1} \tilde{\times} (\mathrm{Tr F} - \mathrm{F}) \\ \tau &: \mathrm{L} \tilde{\times} \mathrm{F} \to \mathrm{L}^{*-1} \tilde{\times} (\mathrm{F} - \mathrm{Tr\,F}) \end{aligned} \quad (\mathrm{L} \in \mathrm{SL}(2, \mathrm{C}), \mathrm{F} \in \mathbf{H}(2))$$

(Cf. [*6*], sections 2.3 and 7.2; note also $\mathrm{Tr\,F} = \mathrm{F} + \sigma_2 \overline{\mathrm{F}} \sigma_2$ for all $\mathrm{F} \in \mathbf{H}(2)$). The corresponding semi-direct product group is denoted here $\tilde{\mathbf{P}}^+$, or $\mathbf{F} \tilde{\times} \tilde{\mathbf{P}}$, $\mathbf{F}$ being a 4-element group $\{ e, \mathrm{T}, \mathrm{P}, \mathrm{TP} \}$, -- an abelian and non-cyclic group, $\mathbf{F}$ for « group ». The elements $\mathrm{T}, \mathrm{P} \in \tilde{\mathbf{P}}^+$ are defined so as to satisfy $\mathrm{T}g\mathrm{T}^{-1} = \tau(g)$, $\mathrm{P}g\mathrm{P}^{-1} = \pi(g)$ $(g \in \tilde{\mathbf{P}})$. $\tilde{\mathbf{P}}^+$ is also enlarged to the direct product $\tilde{\mathbf{P}}^+ \times \mathrm{Z}_2$, where the generator of the Z_2 group is denoted C.

DÉFINITION 2. — Consider a representation R of $\tilde{\mathbf{P}}$ on a finite-dimensional space V. R is said to *admit a* T (resp. *a*P) if there exists a nonsingular map R(T) (resp. R(P)) such that

$$\mathrm{R(T)\ anti\text{-}linear,\ R(T)R}(g)\mathrm{R(T)}^{-1} = \mathrm{R}(\tau(g)) \qquad (6.1)$$

(resp.

$$\mathrm{R(P)\ linear,\ R(P)R}(g)\mathrm{R(P)}^{-1} = \mathrm{R}(\pi(g)) \qquad (6.2))$$

for all $g \in \tilde{\mathbf{P}}$. An a priori more restrictive condition on R is that it admit a *projective extension to* $\tilde{\mathbf{P}}^+$, where such an extension R′ is determined by assignments for R(T) and R(P) satisfying (6.1-2), commuting within scalar factors, and having squares equal to constants.

R is said to *admit a* C if there exists an antilinear equivalence, denoted C, of R (acting on V) with its dual (acting canonically on V*). In the case R is self-dual (1), a *projective extension of* R *to* $\tilde{\mathbf{P}}^+ \times \mathrm{Z}_2$ is defined to be a projective extension in the usual sense subject to the constraints:

R(T), R(C), and R(P) commute within scalars and have squares equal to scalars, R(C) is anti-linear and commutes with all R(g) $(g \in \tilde{\mathbf{P}}^+)$, and R(T) and R(P) satisfy (6.1) and (6.2). (6.3)

It is easy to see that the direct sum of any representation of $\tilde{\mathbf{P}}$ with its anti-dual admits a C. The following determines which representations admit a T, C, or P, and some additional extensions.

PROPOSITION 5. — All the indecomposable representations listed in

(1) The conventional action of the Lorentz group on 4-component spinors, namely the direct sum of the 'Weyl representations' $\mathrm{L} \to \mathrm{L}$, $\mathrm{L} \to \overline{\mathrm{L}}$ $(\mathrm{L} \in \mathrm{SL}(2, \mathrm{C}))$, is self-dual.

Theorem 3 admit a T; only R_5, $\hat{R}_5$, and R_6 admit a P; only R_4, $\hat{R}_4$, and R_6 admit a C.

Each of the direct sums $R_4 \oplus R_4$, $R_7 \oplus \overline{R}_7$, $R_8^j \oplus \overline{R}_8^j$ ($j = 1, 2$) and their duals admit a P.

The self-dual representation R_6 admits a projective extension to $\tilde{\mathbf{P}}^+ \times Z_2$, defined on the real form $R^1 \oplus \mathbf{H}(2) \oplus R^1$ by the assignments:

$$R(T)(x, H, y) = (x, -\sigma_2 \overline{H} \sigma_2, y),$$
$$R(C)(x, H, y) = (x, H, y),$$

and

$$R(P)(x, H, y) = (x, \sigma_2 \overline{H} \sigma_2, y) \qquad (x, y \in R^1, H \in \mathbf{H}(2))$$

and then extended linearly for R(P) and anti-linearly for R(T) and R(C). Such R(T), R(C), and R(P) satisfying (6.3) are unique up to scalar factors.

Proof. — Essentially it suffices to note that $L \to L^{*-1}$ ($L \in SL(2, C)$) carries each representation $\mathbf{D}(j_+, j_-)$ into one equivalent to $\mathbf{D}(j_-, j_+)$, and that a representation admits a C if and only if it is equivalent to its anti-dual. It is straightforward to check that the discrete symmetries given for R_6 satisfy (6.3); the uniqueness follows from the easily checked fact that the only matrices commuting with all $R_6(g)$ ($g \in \tilde{\mathbf{P}}$) *and* that have constant squares are scalars. Q. E. D.

If a representation R of $\tilde{\mathbf{P}}$ on V preserves a nondegenerate sesquilinear form $\langle ., . \rangle$, then R is linearly equivalent to its anti-dual $\hat{\overline{R}}$, via the map $x \to \langle x, . \rangle$ carrying V into its anti-dual (space of anti-linear functionals), cf. (6), section 6.2. The only such representations in Theorem 3 equivalent to their anti-duals are R_4, $\hat{R}_4$, and R_6, by Proposition 5. Conversely, these representations clearly do preserve nondegenerate hermitian forms, namely the forms appearing in the defining representations of SU(2, 2) and 0(2, 4), noted in the previous section.

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