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Large time, small coupling behaviour of a quantum particle in a random field

by

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SUMMARY. — For a quantum mechanical particle in a suitable random field we prove that all finite-dimensional distributions of extensive observables converge in the Van Hove limit to the corresponding distributions of a classical Poisson process. In the classical limit this process converges to a diffusion.

RÉSUMÉ. — On démontre, pour une particule quantique dans un potentiel aléatoire convenable, que toutes les distributions de dimension finie d'observables extensives convergent dans la limite de Van Hove vers les distributions correspondantes d'un processus de Poisson classique. Dans la limite classique, ce processus converge vers un processus de diffusion.

1. INTRODUCTION

The motion of a classical or quantum system in a random environment is expected to converge to a Markov process under suitable scaling limits, usually involving large time scales and small coupling. Results in this direction appear in the literature under various headings, e. g. homogeneization, method of average, diffusion limit, and require in general detailed specifications as to what constitutes a random environment and which are the observable quantities to be studied.

Formal results and applications can be found, e. g., in [*1*]. The subject has also been considered in the mathematical literature; general results can be found in [2], where one considers the case in which the interactions with the random environment are *a priori* assumed to be weakly correlated in time. These results can be viewed as a version of the central limit theorem for a class of dynamical systems.

A more physical setting is one in which the random force field is assumed to have weak correlations in space, to reproduce the properties of a rapidly fluctuating environment. In this case, the fact that successive interactions are weakly correlated in time becomes part of the proof (and indeed usually the most difficult part); once this is established, the results follow modulo (often very substantial) technical details. This more physical setting is beyond the reach of the general results in [2], and proofs require the development of specific techniques.

For example in [*3*] it is proved, under suitable but rather weak assumptions, that the velocity process of a classical particle moving in a random force field converges weakly to a diffusion process in the limit in which the force field becomes (locally) infinitesimal and the time scale is chosen indefinitely large (Van Hove limit).

Here we prove a similar result for the motion of a quantum particle in a random potential field.

As in the classical case, convergence will hold only for a restricted class of observables, in particular for bounded continuous functions of momentum.

Results in this direction are contained in a germinal paper by L. Van Hove [*4*]. Important steps and proofs are in [*5*], [*6*]. The limiting process is here a Poisson process, with transition amplitudes depending on Planck's constant $\hbar$.

It is a rather obvious question to inquire whether the results of [*3*] for the classical case can be recovered in the classical limit. This is indeed the case; in the last section of this paper we shall briefly indicate the way in which a proof is given. A full proof will be contained in [*10*].

It should be noted that the results we present here for the quantum case are obtained under conditions on the force field which are stronger than those of [*3*]. The results themselves are moreover weaker than their classical counterpart, in so far as we only prove convergence of all finite-dimensional distributions rather than convergence of processes. The assumptions are stronger both because the force field is taken to be potential – this seems unavoidable in Quantum Mechanics – and to admit moments of all orders, with suitable bounds in terms of the moment of order two. Some restrictions on the bounds can be lifted by more accurate estimates, but our method of proof does not exploit enough the details of the quantum mechanical evolution on the space-time scale characteristic of the problem. In the classical case, many estimates depend on a rather detailed description of « most » trajectories; that the Poisson process of the quantum case converges as a process to the diffusion of the classical case suggests that also in the quantum-mechanical setting it should be possible to have a better control of the « motion of the wave packet » for most configurations of the force field.

It seems however that the main drawback of the method presented here, both in terms of assumption needed and of results which can be obtained, is to be found in the fact that we are able to use only a very modest amount of probabilistic techniques. In particular we lack the inequalities for conditional expectations and the resulting tightness of a suitable family of probability measures, which are the main tool in the analysis given in [*3*], for the classical case.

A better strategy of proof could come from a more probabilistic approach to the quantum-mechanical case, for instance a formulation of the motion of a quantum particle in a potential field in terms of integrals over suitable functionals of a Poisson process, as developed in [*8*]. In this case, it is conceivable that a « small » set of trajectories will give the dominant contribution in the Van Hove limit, and that the techniques developed by Donsker and Varadhan [*9*] could put to use here.

We are indebted to Ph. Combe for some very suggestive discussions on this possibility.

The content of this paper is as follows.

In this section 2 we give some further qualitative comments and the description of the quantum mechanical evolution of a suitable class of observables in a properly defined random potential field.

In section 3 we provide motivations and describe the limit Markov process.

In section 4 we begin the proof of convergence of the averaged dynamics when the potential is a Gaussian random field and outline the strategy; further technical details and the completion of the proof are given in section 5.

In section 6 we outline the proof of convergence for all finite-dimensional distributions. We also outline how the proofs can be extended to cover the case of random potential fields which are not gaussian. In section 7 we prove that the Markov process described in section 3, converges, when $\hbar \to 0$, to the diffusion process of the classical case.

2. QUANTUM EVOLUTION IN A RANDOM HOMOGENEOUS POTENTIAL FIELD

Let $V(x)$ be the potential field. The motion of a quantum particle is described by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = (-\hbar^2 \Delta + V)\psi \tag{2.1}$$

when $\Delta \equiv \sum_{1}^{3} \frac{\partial^2}{\partial x_i^2}$, $\psi \in L^2(\mathbb{R}^3) \cap D(-\hbar^2\Delta + V)$, and for convenience we have taken units of mass such that $m = 1/2$.

We shall consider only motion in $\mathbb{R}^3$, although all the results we state also hold $\mathbb{R}^n$, $n > 3$, and in fact some of the proofs in § 4, 5 become simpler. Some crucial estimates in § 4, 5 fail instead for $n = 1$ or 2, as will be apparent in the sequel. The result could still be true for $n = 2$, which is somewhat a borderline case, but the method of proof we present here fails in this case. Since $\hbar$ plays no role until § 7, we shall set $\hbar = 1$ until then.

In order that (2.1) provide a unitary evolution in $L^2(\mathbb{R}^3)$ it is sufficient by Stone's theorem, that $-\Delta + V$ be self-adjoint. If this is the case, let $U(t)$ be the corresponding one-parameter group of unitary operators; one has $U(t) = \exp i(-\Delta + V)t$.

Let B be a symmetric bounded linear operator on $L^2(\mathbb{R}^3)$, i. e. a quantum mechanical observable; its time evolution, in the Heisemberg representation (which we shall adopt) is given by

$$B(t) = U(t)BU(-t) \tag{2.2}$$

Let $R^3 \in \underline{a} \leftrightarrow V(\underline{a})$, $(V(\underline{a})\psi)(\underline{x}) \equiv \psi(\underline{x} - \underline{a})$ be the standard representation of the group of space-translations.

We shall denote by $\mathscr{A}_0$ the linear span over the complex numbers of the observables which commute with $V(\underline{a})$ for all $\underline{a} \in R^3$. $\mathscr{A}_0$ is easily seen to be a commutative C* algebra, which can be identified via Fourier transform with $L^\infty(R^3)$, the algebra of essentially bounded functions on $\mathbb{R}^3$.

Indeed, if $A \in \mathscr{A}_0$, one has

$$(\widetilde{A\psi})(\underline{p}) = A(\underline{p})\tilde{\psi}(\underline{p})$$

for some function $A(\underline{p}) \in L^\infty(\mathbb{R}^3)$. Here $\tilde{\psi}$ is the Fourier transform of ψ. We shall call this the Fourier representation of A_0. Denote by $C_0(R^3)$ the class of continuous functions which vanish at ∞; $C_0(\mathbb{R}^3)$ is a subalgebra of L^∞, closed in the supremum norm. Let $\mathscr{A}$ be the subalgebra of $\mathscr{A}_0$ which has $\mathbb{C}_0(\mathbb{R}^3)$ as representative in the Fourier representation; $\mathscr{A}$ is then closed in the norm topology. The observables for which we shall prove limit theorems are the symmetric elements of $\mathscr{A}$.

We recall now briefly the definition of a random field. Let Ω be a probability space, with generic point ω, endowed with the measure μ. Let S be a linear subspace of $\mathbb{C}(R^3; R)$ (continuous functions from R^3 to R) and let $S \ni f \to V_f(\omega)$ be a linear map from S to the linear space of random variable (μ-measurable functions over Ω).

Formally, one writes

$$V_f(\omega) = \int f(x)V(x, \omega)dx$$

where for $a.a.\omega\, V(x, \omega)$ is a (generalized) function of x. In favourable cases, $V(x, \omega)$ will be for each x a random variable. For a measurable and integrable function on Ω, we define

$$E(\mathcal{G}) \equiv \int \mathcal{G}(\omega)\mu(d\omega)$$

We require that V be stationary and ergodic.

The field $V(\underline{x})$ is stationary if

$$E(V(\underline{x}_1, \cdot) \dots V(\underline{x}_n, \cdot)) = E(V(\underline{x}_1 + \underline{a}, \cdot) \dots V(\underline{x}_n + \underline{a}, \cdot)) \qquad (2.3)$$

for all $\underline{a} \in \mathbb{R}^3$, as elements of $(S^*)^n$, S^* being the dual of S.

One can choose Ω in such a way that there exists a representation of $\mathbf{R}^3$ by unitary operators $T_{\underline{a}}$ on $L^2(\Omega, \mu)$ such that

$$T_{\underline{a}} V(\underline{x}, \omega) T_{\underline{a}} = V(\underline{x} + a, \omega).$$

Ergodicity implies that every measurable function of the random field V, which is invariant under T_a $\forall \underline{a}$, differs from a constant function only on a set of μ-measure zero.

If $E(V(\underline{x}_1) \dots (V(\underline{x}_n))$ exist as continuous functions, then for each $\underline{x} \in \mathbb{R}^3$, $V(\underline{x})$ is a random variable, and one can choose a modification (on a set of zero measure) of $V(\underline{x})$ such that the resulting field is jointly measurable in x and ω.

These conditions are in particular met if $V(x)$ is a centered (= mean zero) Gaussian field of covariance $\mathcal{G}(\xi)$, where G is continuous. One has then of course

$$E(V_f) = 0 \qquad \forall f \in S$$

$$E(V_f V_g) = \int \bar{f}(x) g(y) G(x - y) d^3x d^3y$$

We shall state our results and give proofs only in the case in which $V(x)$ is a Gaussian random field. As will become apparent in the course of the proofs, the results can be extended to more general random fields, provided one has suitable *a priori* bounds on the moments of V.

On the Gaussian random field V we shall make the assumption.

ASSUMPTION A. — $\qquad G \in L^1, \tilde{G} \in L^1 \qquad (2.4)$

If $|g|_1$ is the L^1 norm of g, we shall use the notation

$$\|G\| = \max\{|G|_1, |\tilde{G}|_1\}.$$

Having thus set our notation, we begin constructing the evolution of the observables in $\mathcal{A}$ under the influence of the random potential field V.

In the Gaussian case, it is not difficult to prove that there exists a set Ω_0 of measure one, such that, if $\omega \in \Omega_0$, $H(\omega) \equiv -\Delta + V(x, \omega)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$. This is sufficient to define a dynamics for $a.a.\omega$. We shall however be interested in regularity properties of the average dynamics. To obtain these, we choose to approximate first $V(x, \omega)$ and to define the dynamics by a limiting procedure. A natural choice would be

$$V_\varepsilon^{(1)} \equiv V(x) \exp(\varepsilon V(x)) \tag{2.5}$$

For all x, ω, $V_\varepsilon^{(1)}(x, \omega)$ is bounded below, uniformly in x, ω. If $\exp(\varepsilon V(x)) \in L^1(\Omega)$, it follows from the individual ergodic theorem that for $a.a.\omega$, $V_\varepsilon^{(1)}(\cdot, \omega) \in L^2_{\text{loc}}(\mathbb{R}^3)$, the space of functions which are in L^2 when restricted to any bounded subset of $\mathbb{R}^3$.

Therefore there exists a set $\Omega_1 \subset \Omega$, $\mu(\Omega_1) = 1$, such that, if $\omega \in \Omega_1$, $-\Delta + V_\varepsilon^{(1)}(\cdot, \omega)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$. Moreover

$$D(-\Delta) \cap D(V_\varepsilon^{(1)}(x))$$

is a core for $-\Delta + V_\varepsilon^{(1)}$, and $\exp i(-\Delta + V_\varepsilon^{(1)}(\cdot, \omega))t$ is strongly continuous in t for all $\omega \in \Omega$, and strongly measurable in ω for all t (this can be proved, e. g., using the Trotter product formula, since the pointwise limit of measurable functions is itself measurable).

While (2.5) is in many ways a natural approximation, it requires much machinery to prove that, for all $t \in R$, $\omega \in \Omega_1$, the limit $\varepsilon \to 0$ exists as a unitary operator.

We shall choose therefore the following approximate random field

$$V_\varepsilon(x) \equiv V(x) e^{-\varepsilon V^2(x)} \tag{2.6}$$

Since $V(x, \omega)$ is jointly measurable in (x, ω), so is $V_\varepsilon(x, \omega)$. Moreover by construction, $V(x, \omega)$ is bounded uniformly in (x, ω).

We will prove

LEMMA. — For each t, λ there is a set $\overline{\Omega} \subset \Omega$, $\mu(\overline{\Omega})' = 1$, and a sequence ε_n, $\varepsilon_n \to 0$ when $n \to \infty$, such that, if $\omega \in \overline{\Omega}$, strong $\lim\limits_{n\to\infty} \exp \{ i(-\Delta + \lambda V_{\varepsilon_n})t \}$ exists. Call $U_\lambda(t)(\omega)$ this limit.

Then $U_\lambda(t)(\omega)$ is unitary for all $\omega \in \overline{\Omega}$ and μ-measurable. □

Proof. — We shall prove that, for every $\psi \in L^2(\mathbb{R}^3)$, one has

$$\lim_{\varepsilon,\varepsilon'\to 0} E(\| (\exp\{ i(-\Delta + \lambda V_\varepsilon)t \} - \exp\{ i(-\Delta + \lambda V_{\varepsilon'})t \})\psi \|^2) = 0 \tag{2.7}$$

Assuming for the moment the validity of (2.7), we complete the proof of the Lemma.

Let $\{ \psi_i \}$ be a denumerable basis in $L^2(\mathbb{R}^3)$. From (2.7),

$$\exp\{ i(-\Delta+\lambda V_\varepsilon)t \} \psi_K$$

is, for each $K = 1, 2, \ldots$ a Cauchy sequence in $L^2(\mathbb{R}^3 \times \Omega, \nu)$, where $\nu = \mu_l \times \mu$ and μ_l is Lebesgue's measure on $\mathbb{R}^3$. It then follows that, for each $K \in Z^+$ there is a set Ω_K, $\mu(\Omega_K) = 1$, and a subsequence $\varepsilon_n^{(K)} \downarrow 0$, such that, for all $\omega \in \Omega_K$, $\exp\{ i(-\Delta + \lambda V_{\varepsilon_n^{(K)}})t \} \psi_K$ converges in $L^2(\mathbb{R}^3)$.

Let $\overline{\Omega} \equiv \bigcap_{K \in Z^+} \Omega_K$. Therefore $\mu(\overline{\Omega}) = 1$.

One can choose a sequence $\varepsilon_n \downarrow 0$ such that, if $\omega \in \overline{\Omega}$,

$$\exp\{ i(-\Delta + \lambda V_{\varepsilon_n})t \} \psi_K$$

converges in $L^2(\Omega)$, for all $K \in Z^+$ (the sequence $\{ \varepsilon_m \}$ is a subsequence of each $\{ \varepsilon_n^{(K)} \}$).

Since $\exp\{ i(-\Delta + \lambda V_\varepsilon)t \}$ is norm-bounded uniformly in ε,

$$\exp\{ i(-\Delta + \lambda V_{\varepsilon_m})t \} \psi$$

converges in $L^2(\Omega)$ for all $\psi \in L^2(\Omega)$, as can be seen approximating ψ with finite linear combinations of the ψ_K's. Let $\chi_\psi(\omega)$ be the limit point. From (2.7) it follows that $\psi \to \chi_\psi(\omega)$ is linear and bounded, and in fact $\| \chi_\psi(\omega) \| = \| \psi \|$ (all norms being $L^2(\mathbb{R}^3)$ norms) since the unit sphere is closed under sequential strong convergence.

Therefore for each t, λ there exists a set $\overline{\Omega} \subset \Omega$, $\mu(\overline{\Omega}) = 1$ (the set $\overline{\Omega}$ depends in general on t, λ) such that, if $\omega \in \overline{\Omega}$, there exists a unitary operator $U_{\lambda,t}(\omega)$ which is the strong limit of $\exp\{ i(-\Delta + \lambda V_{\varepsilon_n}) \} t$. Measurability of $U_{\lambda,t}(\omega)$ follows since it is the pointwise limit of measurable functions.

It remains therefore to prove (2.7), which in turn is equivalent to

$$\lim_{\varepsilon,\varepsilon' \to 0} E(\| \psi \|^2 - \operatorname{Re} (e^{i(-\Delta + \lambda V_\varepsilon)t}\psi, e^{i(-\Delta + \lambda V_{\varepsilon'})t}\psi)) = 0 \qquad (2.7')$$

We shall use the following identity,

$$e^{i\Delta t} e^{i(-\Delta + \lambda V_\varepsilon)} = I + \sum_{n=1}^{\infty} i^n \lambda^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n V_\varepsilon(t_1) \ldots V_\varepsilon(t_n) \qquad (2.8)$$

where the series is norm convergent for all $\varepsilon > 0$ and $\omega \in \Omega$, uniformly in ω.

We shall refer to (2.8) as « Dyson series ». Notice that the left-hand of (2.8) satisfies the differential equation

$$\frac{d}{dt} [\exp(i\Delta t) \cdot \exp\{ i(-\Delta + \lambda V_\varepsilon)t \}] = \lambda V_\varepsilon(t) [\exp(i\Delta t) \cdot \exp\{ i(-\Delta + \lambda V_\varepsilon)t \}] \qquad (2.9)$$

where $V_\varepsilon(t) = e^{i\Delta t} V_\varepsilon e^{-i\Delta t}$.

The series (2.8) is obtained by iterating the integrated version of (2.9), also called Duhamel's formula, or « variations of constants ».

Substituting (2.8) in (2.7′) one sees that one must study the limit when ε, $\varepsilon' \to 0$ of

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} i^{n-m} \lambda^{n+m} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \int_0^t d\tau_1 \ldots \int_0^{\tau_{m-1}} d\tau_m E(\psi, V_{\varepsilon'}(\tau_m) \ldots V_{\varepsilon'}(\tau_1) V_\varepsilon(t_1) \ldots V_\varepsilon(t_n)\psi) \qquad (2.10)$$

(the exchange of the summation over m, n and integration over t_i, τ_j is legitimate in view of the boundedness of the $V_\varepsilon(\sigma)$'s).

The integrand can be expressed as a formal series

$$\sum_{k_i=0}^{\infty}\sum_{h_m=0}^{\infty} \varepsilon^{\sum_{i=1}^{n} K_i} \varepsilon'^{\sum_{j=1}^{m} h_j} \frac{(-1)^{\sum_{i=1}^{n} K_i + \sum_{j=1}^{m} K_j}}{K_1! \ldots h_m!} E(\psi, V^{2h_m+1}(\tau_m) \ldots V^{2K_n+1}(t_n)\psi) \quad (2.11)$$

Let $\varepsilon_0 = \max(\varepsilon, \varepsilon')$. Using the properties of the Gaussian integrals, we shall prove that $\exists \bar{\varepsilon}_0$ such that, for $\varepsilon_0 < \bar{\varepsilon}_0$, the series (2.11) is absolutely convergent, uniformly in ε_0. From this result (2.7′) follows, since it is easily verified that (2.10) has no terms of order zero in ε_0.

In particular, one has

$$E(V(x_1)^{2K_1+1} \ldots V(x_n)^{2K_n+1} = \sum_{\text{Pair}} \mathscr{G}^{K_{11}}(0)\mathscr{G}^{K_{12}}(x_1 - x_2) \ldots \mathscr{G}^{K_{nn}}(0) \quad (2.12)$$

where $1 + 2K_i = 2K_{ii} + \sum_{j \neq i} K_{ij}$ for all $i = 1 \ldots$, $\mathscr{G}(x_i - x_j) = E(V(x_i)V(x_j))$, and the sum is over all unordered pairing of the points $\{ x_i \}$, each of which is taken with multiplicity $2K_i + 1$. In (2.12), $K_{mn} \in Z^+$ is the number of the times the point x_m is paired with the point x_n.

By carrying out explicitly all calculations one verifies that each integrand in the serie (2.10) gives a contribution which is bounded in absolute value by $\| A \|^2 \| \tilde{\mathscr{G}} \|_1^{\sum_{i=1}^{n} K_i + \sum_{j=1}^{m} h_j + \frac{m+n}{2}} \| \psi \|^2$ independently of $t_1 \ldots \tau_m$. The integration over $t_1 \ldots \tau_m$ provides for each such term a factor $t^{n+m}(n!)^{-1}(m!)^{-1}$.

To prove absolute convergence of (2.11), uniformly in $0 \leqslant \varepsilon_0 \leqslant \bar{\varepsilon}_0$, it is therefore sufficient to prove absolute convergence of the series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |\lambda|^{n+m} \frac{2^{n+m}}{n!m!} t^{n+m} \sum_{K_1=0}^{\infty} \ldots \sum_{h_m=0}^{\infty} \frac{(\bar{\varepsilon}_0)^{\sum_{i=1}^{n} K_i + \sum_{j=1}^{m} h_j}}{K_1! \ldots h_m!} \| \tilde{\mathscr{G}} \|^{\sum_{i=1}^{n} K_i + \sum_{j=1}^{m} h_j + \frac{m+n}{2}} \cdot N_{\{K_i, h_j\}} \quad (2.13)$$

where $N_{\{K_i, h_j\}}$ represents the number of pairing among $n + m$ points x_i, $i = 1 \ldots n + m$, each taken with multiplicity K_i, and we have set $K_{n+j} = h_j$, $j = 1 \ldots m$.

To evaluate N, it is easier to count pairings in a somewhat different way.

Let K_{ij} $i, j = 1 \ldots n + m$ be the number of times the pair (i, j) appears in the pairing. Obviously

$$2K_i + 1 = 2K_{ii} + \sum_{j \neq i} K_{ij}. \tag{2.14}$$

The number of pairing is then

$$\prod_{i=1}^{n+m} \frac{(2K_i + 1)!}{(2K_{ii})! \prod_{j \neq i} K_{ij}!} \cdot \prod_i (2K_{ii} - 1)!! \prod_{i \neq j} K_{ij}! \tag{2.15}$$

We rewrite then (2.13) as

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2 \mid \lambda \mid t \,\|\, \tilde{\mathcal{G}} \,\|_1^{1/2})^{n+m}}{n!m!} \sum_{i,j=1}^{n+m} \sum_{K_{ii}=0}^{\infty} \sum_{\substack{K_{ij}=0 \\ i \neq j}}^{\infty} \prod_i \frac{(2K_i + 1)!}{(2K_{ii})! \left(\prod_{j \neq i} (K_{ij})! \right)^{1/2}} \frac{(2K_{ii} - 1)!!}{K_i!} \bar{\varepsilon}_0^{K_i} \,\|\, \tilde{\mathcal{G}} \,\|_1^{K_i} \tag{2.16}$$

We now use the fact that, if (2.13) holds, then $\dfrac{(2K_i + 1)!}{(2K_{ii})! \prod_{j \neq i} (K_{ij})!} > 1$,

and repeatedly Schwartz' inequality to dominate the series in (2.16) by

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2 \mid \lambda \mid t \,\|\, \tilde{\mathcal{G}} \,\|_1^{1/2})^{n+m}}{n!m!} \prod_i \sum_{K_{ii}=0}^{\infty} \sum_{\substack{K_{ij}=0 \\ i \neq j}}^{\infty} \frac{(2K_i + 1)!}{(2K_{ii})! \prod_{j \neq i} K_{ij}!} \bar{\varepsilon}_0^{K_i} \,\|\, \tilde{\mathcal{G}} \,\|_1^{K_i} \frac{(2K_{ii} - 1)!!}{(2K_{ii}!)^{1/2}} \cdot \frac{((2K_i + 1)!)^{1/2}}{K_i!} \tag{2.17}$$

where $2K_i + 1 = 2K_{ii} + K_{ij}$.

Now, $(2K_i + 1)! \leqslant 2^{K_i+1}((K_i + 1)!)^2$, and $(2K_{ii})! \geqslant ((2K_{ii} - 1)!!)^2$, and moreover

$$\sum_{\substack{K_{ii}, K_{ij}=0 \\ 2K_{ii}+K_{ij}=K_i}}^{\infty} \frac{(2K_i + 1)!}{(2K_{ii})! \prod_{j \neq i} K_{ij}!} = 2^{2K_i+1}$$

Therefore (2.17) is dominated by

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2 \mid \lambda \mid t \,\|\, \tilde{\mathcal{G}} \,\|_1^{1/2})^{m+n}}{n!m!} \sum_{K_i=0}^{\infty} \bar{\varepsilon}_0^{K_i} \,\|\, \tilde{\mathcal{G}} \,\|_1^{K_i} \cdot 2^{2K_i+1+\frac{1}{2}(K_i+1)} (K_i + 1)^{m+n} \tag{2.18}$$

which is absolutely convergent uniformly in λ, t over bounded sets, for $\bar{\varepsilon}_0 < 2^{-5/2}(\|\tilde{\mathcal{G}}\|_1)^{-1}$. □

We are mostly interested in the random evolution of quantum observables. This is again defined by a limiting procedure.

Let

$$A_{\varepsilon,\lambda}(t, \omega) \equiv \exp [i(-\Delta + \lambda V_\varepsilon)t] A \exp [-i(-\Delta + \lambda V_\varepsilon)t] \quad (2.19)$$

We have

THEOREM 1. — For each t, λ there is a set $\underline{\Omega} \subset \Omega$, $\mu(\underline{\Omega}) = 1$, and a sequence $\varepsilon_n \downarrow 0$ such that, if $\omega \in \underline{\Omega}$ and $A \in \mathscr{A}$, strong $\lim_{n\to\infty} A_{\varepsilon_n,\lambda}(t, \omega)$ exists. Call $A_\lambda(t)(\omega)$ the limit. Then $A \to A_\lambda(t)(\omega)$ is for each ω a unitary isomorphism of $\mathscr{A}$ with a subalgebra of $B(L^2(\mathbb{R}^3))$. Moreover $A_\lambda(t)(\omega)$ is weakly μ-measurable for each t, λ and, for all $\psi \in L^2(\mathbb{R}^3)$,

$$E(\psi, A_\lambda(t)\psi) = \lim_{n\to\infty} (\psi, A_{\varepsilon_n,\lambda}(t)\psi). \quad (2.20)$$

Proof. — A part from measurability and (2.20), all remaining statements of Theorem 1 follow from Lemma 1, and moreover one has

$$A_\lambda(t)(\omega) = (U_\lambda(t) A U_\lambda^{-1}(t))(\omega).$$

To prove the remaining two statements, it suffices to prove that, for any given $A \in \mathscr{A}$, $\psi \in L^2(\mathbb{R}^3)$, λ, $t \in R$.

$$\lim_{\varepsilon,\varepsilon' \to 0} E \| (A_{\varepsilon,\lambda}(t) - A_{\varepsilon',\lambda}(t))\psi \|^2 = 0 \quad (2.21)$$

Indeed, from the strong convergence of $A_{\varepsilon,\lambda}(t, \omega)$ to $A_\lambda(t, \omega)$ for $\omega \in \underline{\Omega}$, and Schwartz's inequality, one concludes that for every $\psi \in L^2(\mathbb{R}^3)$ $(\psi, A_{\varepsilon_n,\lambda}(t)\psi)$ converges in $L^1(\Omega, \mu)$ to $(\psi, A_\lambda(t)\psi)$ and this implies measurability and (2.20).

The proof of (2.21) follows the same lines as the proof of (2.7). One starts from the Dyson series, obtained by iterating the integral version of the equation

$$\frac{d\underset{\sim}{A}_{\varepsilon,\lambda}(t)}{\alpha t} = i\lambda [V_\varepsilon(t), \underset{\sim}{A}_{\varepsilon,\lambda}(t)]$$

where $\underset{\sim}{A}_{\varepsilon,\lambda}(t) \equiv e^{i\Delta t} A_{\varepsilon,\lambda}(t) e^{-i\Delta t}$.

The Dyson series for observables is

$$\underset{\sim}{A}_\varepsilon(t) = A + \sum_{n=1}^{\infty} i^n \lambda^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n [V_\varepsilon(t_1), \ldots [V_\varepsilon(t_n), A] \ldots] \quad (2.22)$$

a norm convergent series in view of the definition of V_ε. The proof of (2.21) given (2.22) follows then the same steps as the proof of (2.7) in Lemma 1, and we shall not repeat the details here. □

Remark. — Notice that Theorem 1 and its proof provide also an explicit formula for $E(\psi, A_\lambda(t)\psi)$. One has indeed

$$E(\psi, e^{i\Delta t}A_\lambda(t)e^{-i\Delta t}\psi) = \sum_{n=1}^{\infty} i^n\lambda^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n E((\psi, [V(t_1), \ldots [V(t_n), A] \ldots]\psi) \quad (2.23)$$

From the proof of theorem 1 it also follows

COROLLARY. — For every choice of $t_1, \ldots, t_n, \lambda$ there is a set Ω', $\mu(\Omega') = 1$, and a sequence $\varepsilon_n \downarrow 0$, such that if $\omega \in \Omega'$ and $A^{(i)} \in \mathscr{A}$ $i = 1 \ldots n$, the strong limit of $A^{(1)}_{\varepsilon_m,\lambda}(t_1) \circ \ldots \circ A^{(n)}_{\varepsilon_m,\lambda}(t_n)$ exists and coincides with

$$A^{(1)}_\lambda(t_1) \circ \ldots \circ A^{(n)}_\lambda(t_n).$$

Moreover the latter operator is weakly measurable in ω, and

$$E((\psi, A^{(1)}_\lambda(t_1) \ldots A^{(n)}_\lambda(t_n)\psi)) = \lim_{m\to\infty} (\psi, A^{(1)}_{\varepsilon_m,\lambda}(t_1) \ldots A^{(n)}_{\varepsilon_m,\lambda}(t_n)\psi) \quad (2.24)$$

for all $\psi \in L^2(\mathbb{R}^3)$. □

Remark. — Since the $A_{\varepsilon,\lambda}(t, \omega)$ are uniformly bounded for $\varepsilon \geqslant 0$ and weakly measurable, the operators $\overline{A}_{\varepsilon,\lambda}(t) \equiv E(A_{\varepsilon,\lambda}(t))$ and $\overline{A}_\lambda(t) \equiv E(A_\lambda(t))$ are well defined and belong to $\mathscr{A}_0$ since the process is stationary.

Similarly, $E(A^{(1)}_\lambda(t_1) \circ \ldots \circ A^{(n)}_\lambda(t_n)) \in \mathscr{A}_0$, but it is of course different from $\overline{A}^{(1)}_\lambda(t_1) \ldots \overline{A}^{(n)}_\lambda(t_n)$.

We now prove that all these operators are in fact in $\mathscr{A}$. Indeed one has

LEMMA 2. — For all choices of $\{ A^{(i)} \}$ and of $\{ t_i \}$, $E(A^{(1)}_\lambda(t_1) \ldots A^{(n)}_\lambda(t_n)) \in \mathscr{A}$, and is jointly continuous in the t_k's. In particular the average dynamics $A \to \overline{A}_\lambda(t)$ is defined in $\mathscr{A}$ and continuous in t. □

Proof. — We shall give the proof only for $n = 1$. A part from notational complications, there is no difficulty in extending the proof to the general case.

From (2.22) one obtains in the Fourier representation

$$\overline{A}_{\varepsilon,\lambda}(t)(p) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \sum_{h=0}^{n} \sum_{\text{perm}}' \int \exp\Big(i \sum_{K=1}^{n-1} p_K(x_{K+1} - x_K) - i \sum_{K=1}^{n-1} p_K^2(t_{i_K} - t_{i_{K-1}}) - ip^2(t_{i_1} - t_{i_n}) \} \Big)$$
$$\cdot A(p_h)E(V_\varepsilon(x_1) \ldots V_\varepsilon(x_n)) \prod_{i=1}^{n} d^3x_i \prod_{i=1}^{n} d^3p_i \quad (2.25)$$

where $p_0 = p_n = p$, and the third sum is over all permutations $K \to i_K$ such that $t_{i_K} \geq t_{i_{K+1}}$ if $K \leq h$ and $t_{i_K} \leq t_{i_{K+1}}$ if $K > h$.

The estimates given in the proof of Lemma 1 can then be applied to (2.25) (there is an extra factor 2^n in the estimates, coming from $\sum'_{\text{perm}}$; indeed this sum corresponds to summing over the 2^n terms in the multiple commutator which appears in (2.22)) and they are uniform in $p \in \mathbb{R}^3$.

Therefore $\overline{A}_{\varepsilon,\lambda}(t)$ converges to $\overline{A}_\lambda(t)$ in norm (recall that for each $\omega \in \Omega_{\lambda,t}$ one had only strong convergence) and it remains to prove that $\overline{A}_{\varepsilon,\lambda}(t) \in \mathscr{A}$.

By the same estimates as above, the series (2.25) is absolutely convergent uniformly in p, and therefore it suffices to prove that each term belongs to $\mathscr{A}$.

Each term in (2.25) with $h \neq 0$, n can be put in the form

$$B(p) = \int K(p - p')A(p')d^3p' \tag{2.26}$$

where $K \in L^1 \cap \mathbb{C}$ (and depends parametrically on λ, t), while the terms with $h = 0$ or n (and therefore $p_h = p$) are of the form

$$K_1(p)A(p) \tag{2.27}$$

where K_1 is a bounded continuous function of p.

Since $A \in \mathscr{A}$, clearly $K_1 A \in \mathscr{A}$.

To prove that $B(p) \in \mathscr{A}$, notice first that $B(p)$ is continuous, since $K \in L^1$. Indeed, for all $\underline{\varepsilon}$, $|\underline{\varepsilon}| > 0$ and $N > 0$ one has

$$| B(\underline{p} + \underline{\varepsilon}) - B(\underline{p}) | \leq \left| \int_{|\underline{p}' - \underline{p}| < N} K(\underline{p} - \underline{p}') [A(\underline{p}') - A(\underline{p}' - \underline{\varepsilon})] d^3p' \right| + 2 \left| \int_{|\underline{p}' - \underline{p}| < N} K(\underline{p} - \underline{p}') d^3p' \right| \cdot \| A \|$$

and this expression can be made arbitrarily small by first choosing N sufficiently large and then $|\underline{\varepsilon}|$ sufficiently small, using the continuity of $A(p)$. To prove that $\lim_{|\underline{p}| \to \infty} B(\underline{p}) = 0$, notice that, since $A \in \mathscr{A}$, given $\varepsilon > 0$ there exists M_ε such that $| A(\underline{p}) | < \dfrac{\varepsilon}{2 | K |_1}$ if $| \underline{p} | > M_\varepsilon$, where $| K |_1$ is the L^2-norm of K.

Also, since $K \in L^1$, $\exists N_\varepsilon$ such that $\displaystyle\int_{|\underline{z}| > N_\varepsilon} K(\underline{z}) d^3z < \frac{\varepsilon}{2 \| A \|'}$ where $\| A \| = \sup_{\underline{p}} | A(\underline{p}) |$.

One has then, for $| \underline{p} | > N_\varepsilon + M_\varepsilon$

$$| B(\underline{p}) | = \left| \int (\underline{p} - \underline{p}') K(\underline{p}') d^3p' \right| \leq \left| \int_{|p'| < N_\varepsilon} A(\underline{p} - \underline{p}') K(\underline{p}') d^3p' \right| + \frac{\varepsilon}{2} \leq \varepsilon$$

since, if $| \underline{p}' | < N_\varepsilon$ and $| \underline{p} | > N_\varepsilon + N_{\varepsilon'}$, then $| \underline{p} - \underline{p}' | > M_\varepsilon$. □

It follows from Lemma 2 that $A \to A_\lambda(t)$ is a linear continuous map of $\mathscr{A}$ into itself. It is given explicitely in (2.23) as a norm-convergent power serier in λ, but it is in general not differentiable in t; even for those $A \in \mathscr{A}$ for which differentiability can be proved, no simple equation will be satisfied by $\overline{A}_\lambda(t)$.

We shall however prove in § 4 that $\overline{A}_\lambda(\tau/\lambda^2)$ converges, when $\lambda \to 0$, to a Markov semi-group with continuous parameter τ.

3. SOME PROPERTIES OF THE EVOLUTION IN THE LIMIT $t = \tau/\lambda^2$, $\lambda \to 0$

We shall study in § 4 the limit $t = \tau/\lambda^2$, $\lambda \to 0$ of the averaged dynamics and of all correlation functions.

Here we provide some motivation to indicate which is the limit to be expected, and we study the convergence of a sequence of Markov processes T_λ somewhat related to the averaged dynamics.

A part from giving some hints at the mechanism which will be put at work in § 4, we give here also some estimates which will be of use in the sequel.

As a preliminary we shall study the operator

$$\hat{A}_\lambda(t) \equiv e^{i\Delta t} U_\lambda(t) A U_\lambda^{-1}(t) e^{-i\Delta t}$$

Obviously

$$E(\hat{A}_\lambda(t)) = E(A_\lambda(t))$$

Formally, $\hat{A}_\lambda(\tau/\lambda^2)$ satisfies the equation

$$i\frac{d}{d\tau}\hat{A}_\lambda(t) = \frac{1}{\lambda^2}[H_0, \hat{A}_\lambda(t)] + \left[\frac{1}{\lambda}V, \hat{A}_\lambda\right] \tag{3.1}$$

where $H_0 \equiv -\Delta$.

This relation is only suggestive, since we have not proved that there are $\omega \in \Omega$ for which $\hat{A}_\lambda(t, \omega)$ is differentiable, even if only weakly.

We write formally

$$\hat{A}_\lambda(\tau/\lambda^2) = A^{(0)}(\tau) + \lambda A^{(1)}(\tau) + \dots$$

and substitute in (3.1), equating terms of the same order in λ. This leads to

$$\begin{aligned} &[H_0, A^{(0)}(\tau)] = 0 \\ &[H_0, A^{(1)}(\tau)] + [V, A^{(0)}(\tau)] = 0 \\ &i\frac{d}{d\tau}A^{(0)} = [H_0, A^{(2)}] + [V, A^{(1)}] \end{aligned} \tag{3.2}$$

The first relation in (3.2) is compatible with $A^{(0)}(t, \omega) \in \mathscr{A}_0$ for all $\omega \to \Omega_1$, although it does not imply it. Due to the ergodicity of the process, one has then $A^{(0)}(t, \omega) = E(A^{(0)}(t))$ for *a.a.ω*.

From the second relation in (3.2) we conclude then, at a formal level

$$A^{(1)}(\tau) = i\int_0^\infty [V(s), A^{(0)}(\tau)]ds \tag{3.3}$$

(since $A^{(n)}(0) = 0 \ \forall n \geqslant 1$).

Equation (3.3) is understood in the sense that, for all $\tilde{\varphi}$ for which the integral converges for $a.a.p$,

$$(A^{(1)}(\tau, \omega)\varphi)(p) = \frac{1}{(2\pi)^3}\int \frac{1}{\underline{p}^2 - \underline{p}'^2}(V(p, p'; \omega)A^{(0)}(\tau)(p) - A^{(0)}(\tau)(p'))d^3p' \tag{3.3'}$$

where $\tilde{\psi}$ is the Fourier transform of $\psi \in L^2(\mathbb{R}^3)$ and $\tilde{V}(p, p'; \omega)$ is defined by $(\varphi, V(x, \omega)\psi) = \int \bar{\tilde{\varphi}}(p)\tilde{\psi}(p')e^{i(p-p')x}\tilde{V}(p, p'; \omega)$ for $\tilde{\varphi}, \tilde{\psi} \in C_0^\infty$.

We now substitute (3.3) in the third relation in (3.2), again formally since we do not control the domains of the generators involved, to obtain

$$i\frac{d}{d\tau}A^{(0)}(\tau) = [H_0, A^{(2)}(\tau)] + i\left[\underset{\sim}{V}(\tau), \int_0^\infty [V(\sigma), A_0(\tau)]d\sigma\right] \tag{3.4}$$

We integrate over Ω, and use the fact that, due to the stationarity of the process, $E([H_0, A^{(2)}(\tau)]) = 0$, and moreover $A^{(0)}(t, \omega)$ is independent of ω on a set of measure one. One obtains

$$\frac{d}{d\tau}E(A^0(\tau)) = \int_0^\infty E[V(\tau), [V(\sigma), E(A^{(0)}(\tau))]]d\sigma = (\mathscr{L} \cdot A^{(0)}(\tau)) \tag{3.6}$$

where

$$(\mathscr{L}A)(p) = \left(\frac{1}{2\pi}\right)^3 \int \delta(p^2 - p'^2)\tilde{\mathscr{G}}(p - p')(A(p') - A(p))d^3p' \tag{3.6}$$

with $\mathscr{G}(\xi) \equiv E(V(\xi)V(0))$.

Therefore,

$$\lim_{\lambda \to 0} E(A_\lambda(\tau/\lambda^2) = \exp\{-\mathscr{L}\tau\} \cdot A \equiv T_\tau A \tag{3.7}$$

Remark. — This heuristic argument would suggest that a stronger result should be expected, namely that there is a set $\Omega_1 \subset \Omega$, $\mu(\Omega_1) = 1$, such that

$$A_\lambda(\tau/\lambda^2, \omega) - e^{-\mathscr{L}\tau} \cdot A \quad \underset{\lambda \to 0}{\longrightarrow} 0 \tag{3.7'}$$

for all $\omega \in \Omega_1$.

We do not know if (3.7′) holds in a weak sense, with Ω_1 depending on τ and on the vectors in $L^2(\mathbb{R}^3)$ which enter in the definition of weak convergence. Certainly (3.7′) cannot hold in a strong sense, since it would contradict the result we establish for

$$E(A_\lambda(\tau/\lambda^2)B_\lambda(\tau_1/\lambda^2), \qquad \tau > \tau_1 .$$

The derivation of (3.7) given above is at best heuristic, as evident from the remark above. Still, (3.7) is correct, as we shall prove in § 4.

We shall do so by approximating $E(A_\lambda(\tau/\lambda^2)$ by the solution $\underline{A}_\lambda(\tau)$ of a suitable linear equation.

In § 4 we shall prove that $E(A_\lambda(\tau/\lambda^2)) - \underline{A}_\lambda(\tau)$ converges to zero in the topology of $\mathscr{A}$ when $\lambda \to 0$. Here we shall define $\underline{A}_\lambda(\tau)$ and prove that, again in the topology of $\mathscr{A}$, $\underline{A}_\lambda(\tau)$ converges to $e^{-\mathscr{L}\tau}A$ when $\lambda \to 0$.

We begin by noting that

$$E(A_\lambda(\tau/\lambda^2)) = E(\underset{\sim}{A}_\lambda(\tau/\lambda^2))$$

when $\underset{\sim}{A}_\lambda(\tau/\lambda^2) = \exp(i\Delta \cdot \tau/\lambda^2)A_\lambda(\tau/\lambda^2)\exp(-i\Delta\tau/\lambda^2)$ and that, according to (2.23)

$$E(\underset{\sim}{A}_\lambda(\tau/\lambda^2)) = A + \lambda^2 \int_0^{\tau/\lambda^2} d\sigma \int_\sigma^\sigma d\nu E[V(\sigma), [V(\nu), E(A_\lambda(\nu))]] + D_\lambda(\tau) \quad (3.8)$$

where

$$D_\lambda(\tau) = \lambda^2 \int_0^{\tau/\lambda^2} d\sigma \int_0^\sigma d\nu E([V(\sigma), [V(\nu), (A_\lambda(\nu) - E(A_\lambda(\nu)))]]) \quad (3.9)$$

From (3.9) one should expect $D_\lambda(\tau) \to 0$ when $\lambda \to 0$, due to some mild mixing properties of the process and the local decay of $e^{i\Delta t}Ve^{-i\Delta t}(x)$ for large t. Indeed, this is what is proved in the classical case, using *a priori* estimates for conditional expectations and some information on the properties of « most » classical trajectories. We shall prove in § 4 that $D_\lambda(\tau) \to 0$ in the topology of $\mathscr{A}$. Motivated by this, we define $\underline{A}_\lambda(\tau)$ to be the (unique) solution of

$$\underline{A}_\lambda(\tau) = A + \lambda^2 \int_0^{\tau/\lambda^2} d\nu \int_0^\nu d\sigma E[V(\sigma), [V(\nu), \underline{A}_\lambda(\lambda^2\nu)]] \quad (3.10)$$

Notice that $\underline{A}_\lambda(\tau)$ is « sure », i. e. it does not depend on ω.

Eq. (3.10) can be solved by iteration, which provides a norm convergent series for all λ, τ. The solution is therefore unique, and this proves that $\underline{A}_\lambda(\tau) \in \mathscr{A}_0$ since, if $\underline{A}^{(1)}(\tau)$ is a solution, so $V_{\underline{a}}\underline{A}_\lambda^{(1)}(\tau)V_{\underline{a}}$ for all $\underline{a} \in R^3$.

In the Fourier representation one has explicitely

$$\underline{A}_\lambda(\tau)(\underline{p}) = A(\underline{p}) + \frac{1}{\lambda^2}\frac{1}{(2\pi)^3}\int d^3K \int_0^\tau ds \int_0^s d\sigma \Big[e^{i(p^2-K^2)\left(\frac{s-\sigma}{\lambda^2}\right)} + e^{i(K^2-p^2)\left(\frac{s-\sigma}{\lambda^2}\right)}\Big]\overline{\mathscr{G}}(\underline{p} - \underline{K}) \cdot (\underline{A}_\lambda(\sigma)(\underline{p}) - \underline{A}_\lambda(\sigma)(\underline{K})) \quad (3.11)$$

Let $\mathscr{L}$ be defined as in (3.6) and let $\exp - \mathscr{L}t$ be the semi-group it generates (the existence of $\exp(-\mathscr{L}t)$ is part of the proof of the next lemma).

One has then

LEMMA 3. — For all $A \in \mathscr{A}$ and $\tau \to 0$, $\underline{A}(\tau) - e^{-\mathscr{L}\tau}$. A converges to zero, when $\lambda \to 0$, in the topology of $\mathscr{A}$. □

Proof. — From (3.11) it is easily seen that $|A(t)| < e^{\gamma_0 t}$ for some $\gamma_0 \geqslant 0$. We write (3.11) in integro-differential form, taking the derivative with respect to τ, and then take Laplace transform.

For $\mu > \gamma_0$, define

$$\underline{A}_{\lambda;\mu} \equiv \int_0^\infty e^{-\mu t}\underline{A}_\lambda(t)dt$$

From (3.11) one has

$$A(\underline{p}) + \mu\underline{A}_{\lambda;\mu}(\underline{p}) = (\mathscr{G}_{\lambda;\mu})(\underline{p}) \tag{3.13}$$

with

$$(\mathscr{G}_{\lambda;\mu}f)(\underline{p}) \equiv \frac{1}{(2\pi)^3}\int d^3K\tilde{\mathscr{G}}(\underline{p}-\underline{K})(f(\underline{K})-f(\underline{p}))\frac{\lambda^2\mu}{\lambda^4\mu^2+(\mu^2-K^2)^2} \tag{3.14}$$

As a linear operator on $\mathbb{C}_0(\mathbb{R}^3)$, $\mathscr{G}_{\lambda;\mu}$ is accretive for all $\mu > 0$. Indeed, if $f \in C_0(\mathbb{R}^3)$, let $\underline{p}_0$ be a point at which $|f|$ reaches its maximum (all our function spaces are real).

Let l_f be the element of $C_0^*(\mathbb{R}^3)$ defined by $l_f(g) = g(\underline{p}_0)f(\underline{p}_0)$. We prove that $l_f(\mathscr{G}f) \leqslant 0$. Indeed, notice that $f(\underline{p}_0) = 0 \Rightarrow f \equiv 0$; assume then $f(\underline{p}_0) \neq 0$. Then

$$l_f(\mathscr{G}f) = f(\underline{p}_0)\int d^3K\tilde{\mathscr{G}}(\underline{p} - \underline{K})(f(\underline{K}) - f(\underline{p}_0))\frac{\lambda^2\mu}{\lambda^4\mu^2 + (K^2 - p_0^2)^2}$$

Since, by definition of $\underline{p}_0$, $f(\underline{p}_0)(f(\underline{K}) - f(\underline{p}_0)) \leqslant 0 \ \forall \underline{K} \in \mathbb{R}^3$, one has indeed $l_f(\mathscr{G}f) \leqslant 0 \ \forall f \in C_0(\mathbb{R}^3)$.

Since $\mathscr{G}$ is bounded, range $(\mathscr{G} + \lambda_0) \equiv \mathbb{C}_0(\mathbb{R}^3)$ for λ_0 sufficiently large. We conclude that $\mathscr{G}_{\lambda;\mu}$ is accretive for every $\mu > 0$. Therefore its spectrum is contained in $\{\xi \mid \text{Re}\,\xi \leqslant 0\}$ and we can define, for all $\mu > 0$,

$$\underline{\underline{A}}_{\lambda;\mu}(p) \equiv ((\mathscr{G}_{\lambda;\mu} - \mu \cdot I)^{-1} \cdot A)(p) \tag{3.15}$$

Clearly $\underline{\underline{A}}_{\lambda;\mu} = \underline{A}_{\lambda;\mu}$ at least for $\mu > \gamma_0$; since $\underline{A}_{\lambda;\mu}$ is analytic for Re $\mu \geqslant 0$, it follows that $\underline{A}_\lambda(t)$ has a Laplace transform for $\mu > 0$, and $\underline{A}_{\lambda;\mu} = \underline{\underline{A}}_{\lambda,\mu}$ $\forall\mu > 0$. Let $A_{(\mu)}(p)$ be defined by $A_{(\mu)}(p) \equiv (\mathscr{L} - \mu \cdot I)^{-1}A(p)$; since $\mathscr{L}$ is accretive (the proof is given as above) $A_{(\mu)}(p)$ is well defined for all μ, Re $\mu > 0$, and is in fact the Laplace transform of $e^{+\mathscr{L}t}A$.

From (3.15),.(3.14)

$$(\mu \cdot I - \mathscr{G}_{\lambda;\mu})(\underline{\underline{A}}_{\lambda,\mu} - A_{(\mu)})(p) = (\underline{\mathscr{G}}_{\lambda,\mu} - \mathscr{L})A_{(\mu)}(p) \tag{3.16}$$

The right-hand side converges to zero in the topology of $\mathscr{A}$; since, for $\mu > 0$, $(\mu \cdot I - \mathscr{G}_{\lambda,\mu})$ is bounded away from zero uniformly in $\lambda \geqslant 0$, we conclude $\underline{\underline{A}}_{\lambda,\mu} - A_{(\mu)} \underset{\lambda \to 0}{\longrightarrow} 0$ for all $\mu > 0$, uniformly in $0 < \delta < \mu < N$.

From this it follows that $\hat{A}_\lambda(t) - e^{\mathscr{L}t}A \underset{\lambda \to 0}{\longrightarrow} 0$ in the same topology. □

Some information on the structure of the limit semi-group $T_t \equiv e^{\mathscr{L}t}$ is provided by the following

LEMMA 4. — The semi-group T_t is a contraction semi-group on $\mathscr{A}$. It is reduced by each ball $B_a^{(3)} : \{ p \mid |p| < a \}$ and defines on each sphere $S_a^{(3)} : \{ \underline{p} \mid |p| = a \}$ a contraction semi-group. On each sphere, the constant function is left invariant by T_t.

Proof. — We have already proved that $\mathscr{L}$ is accretive. Therefore T_t is a contraction semi-group. Next, we notice that, if $f \in C_0(\mathbb{R}^3)$ has support contained in $\{ p \mid a < |p| < b \}$ for some a, $b > 0$, then $T_t f$ has support contained in the same set. This is evident from the definition of $\mathscr{L}$, and the fact that $e^{t\mathscr{L}} f$ is given by a norm-convergent series in t.

Consider now on $C(S_1^3)$ (continuous functions on the sphere of radius one in $\mathbb{R}^3$) the family of operators

$$(\mathscr{L}_a h)(\hat{p}) \equiv a \int_{S_1^3} d\hat{K} \tilde{\mathscr{G}}_a(\hat{K}, \hat{p})(h(\hat{K}) - h(\hat{p})) \tag{3.17}$$

where $a > 0$ and $\tilde{\mathscr{G}}_a(\hat{K}, \hat{p}) \equiv \tilde{\mathscr{G}}(a\hat{K} - a\hat{p})$.

One verifies easily that, for all $f \in C_0(\mathbb{R}^3)$

$$(\mathscr{L} f)(p) = |p| \cdot (\mathscr{L}_{|p|} f_{|p|})(\hat{p}) \tag{3.18}$$

where $f_a(\hat{p}) \equiv f(a\hat{p})$.

The operators $\mathscr{L}_a$ are accretive for every $a > 0$. Let $T_t^{(a)}$ be the associated semi-group. From (3.18) one verifies that

$$T_t = \int_{\oplus} da T_t^{(a)} \tag{3.19}$$

in the natural decomposition of $C_0(\mathbb{R}^3 \backslash 0)$ as a subspace of $\int_{\oplus} da C(S_a^{(3)})$.

Constant functions on $S_a^{(3)}$ are left invariant by $T_t^{(a)}$ since $\mathscr{L}_a \cdot 1 = 0$. □

Assume now that $\mathscr{G}_a$ satisfies for every $a > 0$ a strong form of Doeblin's condition, i. e. for every $\hat{p} \in S_a^{(3)}$, $B \subset S_a^{(3)}$, $\int_B \tilde{\mathscr{G}}_a^{(n)}(\hat{p}, \hat{K}) d\hat{K} \neq 0$ for some $n \in Z^+$ (here $\tilde{\mathscr{G}}_a^n(\hat{p}, \hat{K}) = \int \tilde{\mathscr{G}}_a(\hat{p}, \hat{K}_1) \dots \tilde{\mathscr{G}}_a(\hat{K}_{n-1}, \hat{K}) d\hat{K}_1 \dots d\hat{K}_{n-1}$). A sufficient condition for this to hold is that $\mathscr{G}(\underline{\xi})$ has compact support, or exponential decay.

One has then

COROLLARY. — Under the condition stated above, the constant function on $S_a^{(3)}$ is a global attractor for the semi-group T_t^a. Given any function

$g \in C_0$, $T_t g$ converges when $t \to \infty$ towards $\bar{g}$, where $\bar{g}$ is a function only of $|\underline{p}|$, and $\bar{g}(|p|) = \int g(\underline{K})\delta(|K| - |p|)d^3K$.

Proof. — Under the stated conditions, 1 is the only eigenvector of $T_t^{(a)}$ to the eigenvalue zero. The corollary follows then from standard properties of contraction semi-groups. □

Remark. — The physical description behind Lemma 4 is that the random force field, in the limit in which $\lambda \to 0$, does not alter appreciably the energy of the particle, even if it acts for a time of order λ^{-2}. On such a long time scale however the momentum of the particle undergoes changes, in such a way that, on a still longer time scale ($\tau \to \infty$), its distribution becomes uniform on the mass shell.

4. CONVERGENCE TO THE MARKOV LIMIT: CONVERGENCE OF DYNAMICS

Let $A_\lambda(t)(\omega)$ be defined as in Theorem 1; in this section and in the following we shall study the limit of

$$E(A_\lambda^{(1)}(\tau_1/\lambda^2) \dots A_\lambda^{(n)}(\tau_n/\lambda^2))$$

when $\lambda \to 0$, when $A^{(i)} \in \mathscr{A}$ $i = 1 \dots n$, and $\tau_1 \geqslant \tau_2 \geqslant \dots \geqslant \tau_n$. In this section we consider the case $n = 1$, and prove

$$\lim_{\lambda \to 0} E(A_\lambda(\tau/\lambda^2)) = T_\tau \cdot A \tag{4.1}$$

where T_τ is defined as in (3.7), (3.6).
This will be the content of Theorem 2.

Before stating the theorem precisely, we perform some preliminary operations and prove two lemmas which will be used in the proof of Theorem 2.

From (3.8), (3.10) it follows that, setting

$$B_\lambda(\tau) \equiv E(A_\lambda(\tau/\lambda^2)) - \underline{A}_\lambda(\tau)$$

$$B_\lambda(\tau) = \lambda^2 \int_0^{\tau/\lambda^2} d\sigma \int_0^\sigma d\nu E([V(\sigma), [V(\nu), B_\lambda(\lambda^2\nu)]]) + D_\lambda(\tau) \tag{4.2}$$

where $D_\lambda(\tau)$ is defined as in (3.9)

$$D_\lambda(\tau) \equiv \lambda^2 \int_0^{\tau/\lambda^2} d\sigma \int_0^\sigma d\nu E([V(\sigma), [V(\nu), A_\lambda(\nu) - E(A_\lambda(\nu))]]) \tag{4.3}$$

If one can prove that $D_\lambda(\tau) \to 0$ when $\lambda \to 0$, then (4.1) follows from Lemma 3 and an application of Gronwall's inequality.

If one had for the present quantum-mechanical setting the definitions and *a priori* estimates, one has in the corresponding classical situation, one could provide a « probabilistic » proof of (4.3). Since we lack at the moment these instruments in the quantum case, our proof will be analytical, and based on the Dyson expansion given in § 2.

We believe that a more straightforward probabilistic proof exists, but we have been unable to find it. See however the remarks at the end of § 1.

In view of the structure of the integrands of the terms of the Dyson series (see e. g. (3.11), and in particular of the occurence of terms of the form $\exp i \{ \Sigma K_m^2(\tau_{l_m} - \tau_{l_{m-1}}) \}$, coming from the e^{iH_0t} in $V(t) \equiv e^{-iH_0t}Ve^{iH_0t}$, the following two lemmas will be important in the proof of (4.1).

LEMMA 5. — Let $\mathcal{G} \in L_1(R^\nu)$, $\tilde{\mathcal{G}} \in L_1(R^\nu)$, and set $\| \mathcal{G} \| = \max(|\mathcal{G}|_1, |\tilde{\mathcal{G}}|_1)$. Let Q be any symmetric N × N matrix, and for $0 \leqslant M \leqslant N$ let $Q^{(M)}$ be any of the M × M matrices obtained by deleting in Q, N − M rows and the corresponding columns, with the convention $Q^{(0)} \equiv 1$, $Q^{(N)} \equiv Q$. Then there exists a constant $\mathbb{C} > 0$, which depends on ν but not on N, such that, for all $0 \leqslant M \leqslant N$

$$\int e^{i(p,Qp)}\tilde{\mathcal{G}}(p_1) \dots \tilde{\mathcal{G}}(p_M) \prod^{N} dp_i \leqslant \mathbb{C}^N \| \mathcal{G} \|^N \cdot (\det Q^{(M)})^{-\nu/2} \quad (4.4)$$

when $(p, Qp) \equiv \sum_{i,j=1}^{N} Q_{ij}(\underline{p}_i \cdot \underline{p}_j)$, $\underline{p}_i \in \mathbb{R}^\nu$. □

Remark. — It is easy to see, considering the case N = 1, and $Q = t \cdot I$, that the statement of Lemma 5 is closely related to the local decay of wave packets in Quantum Mechanics (see e. g. [7]).

In the case N > 1, some information is contained also about the relation between spreading of the wave packet and motion of its « center of mass »; since this relation is also relevant to define multiple scattering, our proof probably contains in a hidden form an estimate on the probability of time-correlations due to multiple scattering. A better understanding of this relation could probably be used to improve effectively on the analysis given in this §, and to make direct connection with the proofs which have been given in [3], for the classical case. It should then also bring about an understanding of the deep reasons for the existence of the classical limit. We hope to come back to these points in the future.

Proof of Lemma 5. — We begin the proof by considering the case N = 1, $\nu = 1$.

We must in this case estimate

$$\int \exp(i\lambda K^2)\tilde{\mathcal{G}}(K)dK \equiv S(\lambda) \quad (4.6)$$

Obviously $|S(\lambda)| \leqslant |\tilde{\mathcal{G}}|_1$ for all $\lambda \in R$. Also, since $\tilde{\mathcal{G}} \in L_1$, $S(\lambda)$ is the boundary value, when $\operatorname{Im} z \downarrow 0$, of the function

$$\int \exp(izK^2)\tilde{\mathcal{G}}(K)dK \equiv S(z)$$

defined and analytic in $\{ z \mid \operatorname{Im} z > 0 \}$.

Therefore

$$S(\lambda) = \lim_{\varepsilon \to 0_+} \int \exp(i\lambda K^2 - \varepsilon K^2)\tilde{\mathcal{G}}(K)dK$$
$$= \lim_{\varepsilon \to 0_+} \int \exp(i\lambda K^2 - \varepsilon K^2)dK \int dx \mathcal{G}(x) e^{iKx} \quad (4.7)$$

Since $e^{-\varepsilon K^2} \in L_1(R)$ for every $\varepsilon > 0$, and $\mathcal{G} \in L_1$, we can use Fubini's theorem and conclude

$$S(\lambda) = \lim_{\varepsilon \to 0_+} \int dx \mathcal{G}(x) \int dK e^{iKx + i\lambda K^2 - \varepsilon K^2}$$
$$= \lim_{\varepsilon \to 0_+} \int dx \mathcal{G}(x) e^{\frac{-ix^2}{4(\lambda + i\varepsilon)}} \int dK \exp i\left(\sqrt{\lambda + i\varepsilon}K + \frac{1}{2\sqrt{\lambda + i\varepsilon}}x\right)^2$$

where we denote by $\sqrt{\lambda + i\varepsilon}$ the square root of $\lambda + i\varepsilon$ which has non-negative real part.

For all $\varepsilon > 0$,

$$\left| \exp \frac{-ix^2}{4(\lambda + i\varepsilon)} \right| \leqslant \left| \exp \frac{-i\lambda - \varepsilon}{4(\lambda^2 + \varepsilon^2)} x^2 \right| \leqslant 1 \qquad \forall x .$$

Therefore

$$S(\lambda)| \leqslant \overline{\lim_{\varepsilon \to 0}} \int dx \,|\mathcal{G}(x)| \left| \int dK \exp\left\{ i\left(\sqrt{\lambda + i\varepsilon}K + \frac{x}{2\sqrt{\lambda + i\varepsilon}}\right)^2 \right\} \right|$$

For all $\varepsilon > 0$, $x \in R$, $\exp i\left(\sqrt{\lambda + i\varepsilon}K + \frac{1}{2\sqrt{\lambda + i\varepsilon}}x\right)^2 \in L_1(dK)$. The integral on K can be computed explicitly, to give

$$\int dK \exp\left\{ i\left(\sqrt{\lambda + i\varepsilon}K + \frac{x}{2\sqrt{\lambda + i\varepsilon}}\right)^2 \right\} = \pi^{1/2} \left| \frac{\varepsilon + i\lambda}{\varepsilon^2 + \lambda^2} \right|^{1/2}$$

We conclude that, for $\lambda \neq 0$

$$|S(\lambda)| \leqslant \sqrt{\pi}\, |\mathcal{G}|_1 \cdot \lim_{\varepsilon \downarrow 0} \left| \frac{\varepsilon + i\lambda}{\varepsilon^2 + \lambda^2} \right|^{1/2} = \sqrt{\pi}\, |\mathcal{G}|_1 \lambda^{-1/2} \qquad (4.8)$$

For $\lambda = 0$, inequality (4.8) is obviously satisfied.

One has therefore

$$\left| \int e^{iK^2\lambda} \tilde{\mathcal{G}}(K) dK \right| \leqslant \sqrt{\pi} \,|\, \mathcal{G} \,|_1 \lambda^{-1/2}$$

$$\left| \int e^{iK^2\lambda} \tilde{\mathcal{G}}(K) dK \right| \leqslant |\, \tilde{\mathcal{G}} \,|_1 \tag{4.9'}$$

and therefore

$$\left| \int e^{iK^2\lambda} \tilde{\mathcal{G}}(K) dK \right| \leqslant \sqrt{\pi} \,||\, \mathcal{G} \,||\, \min(1, \lambda^{-1/2})$$

We have thereby proved Lemma 5 in the case $N = 1$, $\nu = 1$. The case $N = 1$, ν arbitrary follows immediately. It suffices to notice that

$$\int d^\nu K \exp\left\{ i \left(\sqrt{\lambda + i\varepsilon}\, \underline{K} + \frac{1}{2\sqrt{\lambda + i\varepsilon}} \underline{x} \right)^2 \right\}$$
$$= \prod_{s=1}^{\nu} \int dK_s \exp\left\{ i \left(\sqrt{\lambda + i\varepsilon}\, K_s + \frac{1}{2\sqrt{\lambda + i\varepsilon}} x_s \right)^2 \right\} \tag{4.10}$$

We now prove Lemma 5 for arbitrary N, ν.

Since Q is symmetric, one can find an orthogonal transformation T and real number $\lambda_1 \dots \lambda_N$ such that

$$\int e^{i(\underline{p}Q\underline{p})} \tilde{\mathcal{G}}(\underline{p}_1) \dots \tilde{\mathcal{G}}(\underline{p}_N) \prod_1^N \overset{\nu}{d} p_i = \int e^{i \sum_{m=1}^{N} \lambda_m \underline{K}_m^2} \tilde{\mathcal{G}}(\underline{l}_1(\underline{K})) \dots \tilde{\mathcal{G}}(\underline{l}_N(\underline{K})) \prod_1^N \overset{\nu}{d} K_i$$

where $\underline{l}_m(\underline{K}) = \sum_{s=1}^{N} T_{ms} \underline{K}_s$.

Since by assumption the product $\tilde{\mathcal{G}}(\underline{l}_1) \dots \tilde{\mathcal{G}}(\underline{l}_N)$ is an element of $L^1(R^{N\nu})$ and its norm is $||\, \tilde{\mathcal{G}} \,||_1^N$ since T is orthogonal,

$$\int e^{i(\underline{p}Q\underline{p})} \tilde{\mathcal{G}}(p_1) \dots \tilde{\mathcal{G}}(p_n) \pi \overset{\nu}{d} p_i \leqslant (\sqrt{\pi})^{\frac{N\nu}{2}} \,||\, \tilde{\mathcal{G}} \,||_1^N \,|\det Q\,|^{-\nu/2} \tag{4.11}$$

$\left(\text{recall that } |\det Q\,| = \prod_{i=1}^{N} |\,\lambda_i\,| \right)$.

Let now $Q^{(M)}$ be the $M \times M$ matrix defined by $Q_{ij}^{(M)} = Q_{ij}$ if $i, j = 1 \dots M$, $Q_{(N-M)}$ be the $(N - M) \times (N - M)$ matrix defined by $Q_{(N-M)ij} = Q_{ij}$ if $i, j = M + 1, \dots, N$, $Q_{(N-M)}^{(M)}$ be the $M \times (N - M)$ matrix defined by $Q_{(N-M)ij}^{(M)} = Q_{ij}$ if $i = 1 \dots M$
$j = M + 1 \dots N$.

Then

$$\int e^{i(\underline{p}Q\underline{p})}\tilde{\mathscr{G}}(p_1) \dots \tilde{\mathscr{G}}(p_N) \prod_{i=1}^{N} d^{\nu}p_i$$
$$= \int d^{\nu}p_{M+1} \dots d^{\nu}p_N \tilde{\mathscr{G}}(p_{M+1}) \dots \tilde{\mathscr{G}}(p_N) \exp \{ i(\underline{p}Q^{N-M}\underline{p}) \}$$
$$\cdot \int d^{\nu}p_1 \dots d^{\nu}p_M \exp \{ i[(\underline{p}Q^{(M)}\underline{p})$$
$$+ (\underline{p}Q^{(M)}_{(N-M)}\underline{p}) + (\underline{p}(Q^{(M)}_{(N-M)})^t\underline{p})] \} \tilde{\mathscr{G}}(p_1) \dots \tilde{\mathscr{G}}(p_M) \quad (4.12)$$

After a linear change of variables $\underline{p}_i \to \underline{p}_i + \underline{l}_i(\{\underline{p}\})$ $i = 1 \dots M$, where $\underline{l}_i$ are suitable linear homogeneous functions of the $\underline{p}_j$, $j > M$, and using translation invariance of Lebesgue measure, one can apply (4.11) to obtain

$$\left| \int d^{\nu}p_1 \dots d^{\nu}p_M \exp \{ i(\underline{p}, [Q^M + Q^{(M)}_{(N-M)} + (Q^{(M)}_{(N-M)})^t]\underline{p}) \} \tilde{\mathscr{G}}(\underline{p}_1) \dots \tilde{\mathscr{G}}(\underline{p}_M) \right|$$
$$\leqslant (\sqrt{\pi})^{\frac{M\nu}{2}} \| \mathscr{G} \|_1^M | \det Q^{(M)} |^{-\nu/2} \quad (4.13)$$

uniformly in $\underline{p}_{M+1}, \dots, \underline{p}_N$.

Since $\int | \tilde{\mathscr{G}}(\underline{p}_{M+1}) \dots \tilde{\mathscr{G}}(\underline{p}_N) | \prod_{i=M+1}^{N} d^{\nu}p_i = | \tilde{\mathscr{G}} |_1^{N-M}$, one has from (4.12)

$$\left| \int e^{i(\underline{p}Q\underline{p})}\tilde{\mathscr{G}}(\underline{p}_1) \dots \tilde{\mathscr{G}}(\underline{p}_N) \prod_{i=1}^{N} d^{\nu}p_i \right| \leqslant (\sqrt{\pi})^{\frac{N\nu}{2}} \| \mathscr{G} \|^N | \det Q^{(M)} |^{-\nu/2}$$

for all $0 \leqslant M \leqslant N$, where $Q^{(M)}$ is defined in the statement of the theorem (the $N - M$ rows which are deleted can always be defined to be the last ones). □

The proof of Lemma 5 applies verbatim to provide

LEMMA 5'. — Let notation and assumptions be as in Lemma 4. Let moreover $F_i \in L^{\infty}(R^{\nu})$ $i = 1 \dots N_1$, $N_1 \leqslant N$. Let $\underline{l}_i(\underline{p})$ $i = 1 \dots N_1$ be any linear homogeneous combination of the $\underline{p}_i$'s. Then, for all $0 \leqslant M \leqslant N$

$$| \int e^{i(\underline{p}Q\underline{p})}\tilde{\mathscr{G}}(p_1) \dots \tilde{\mathscr{G}}(p_N)F(\underline{l}_1(\underline{p})) \dots F_{N_1}(\underline{l}_{N_1}(\underline{p}))d^{\nu}p_N \dots d^{\nu}p_N |$$
$$\leqslant \mathbb{C}^N | F_1 |_{\infty} \dots | F_{N_1} |_{\infty} \| \mathscr{G} \|^N | \det Q^{(M)} |^{-\nu/2} \quad \square \quad (4.14)$$

Finally, since for every $0 \leqslant \alpha \leqslant 1$ and $a, b > 0$ one has $\min(a, b) \leqslant a^{\alpha}b^{1-\alpha}$, one has the following corollary

COROLLARY. — With the notation of Lemma 5′, and $Q_1^{(M_1)}$, $Q_2^{(M_2)}$ any two reduced matrices, one has

$$\left| \int e^{i(\underline{p}Q\underline{p})}\tilde{\mathcal{G}}(\underline{p}_1) \dots \tilde{\mathcal{G}}(\underline{p}_N) \prod_{i=1}^{N^1} F_i(\underline{l}_i(\underline{p})) \prod_{i=1}^{N} d^\nu p_i \right| \leqslant \mathbb{C}^N \prod_{i=1}^{N^1} |F_i|_\infty \|\mathcal{G}\|^N |\det Q^{(M_1)}|^{-\frac{\alpha\nu}{2}} |\det Q^{(M_2)}|^{-\frac{(1-\alpha)\nu}{2}} \quad (4.15)$$

for any $0 \leqslant \alpha \leqslant 1$.

In particular, choosing $M_1 = N$, $M_2 = 0$

$$\left| \int e^{i(\underline{p}Q\underline{p})} \prod_{i=1}^{N} \tilde{\mathcal{G}}(\underline{p}_i) \prod_{i=1}^{N^1} F_i\left(\underline{l}_i(\underline{p}) \prod_{i=1}^{N} d^\nu p_i\right) \right| \leqslant \mathbb{C}^N \prod_{i=1}^{N^1} |F_i|_\infty \|\mathcal{G}\|^N |\det Q|^{-\frac{\alpha\nu}{2}} \quad (4.16)$$

for all $0 \leqslant \alpha \leqslant 1$. □

We turn now to the proof that $D_\lambda(\tau) \to 0$ when $\lambda \to 0$. We use in (4.3) the Dyson series (3.11), to obtain

$$D_\lambda(t) = \sum_{n=2}^{\infty} \lambda^{2n} \int_0^{t/\lambda^2} dt_1 \dots \int_0^{t_{2n-1}} dt_{2n} E'[V(t_1) \dots [V(t_{2n}), A] \dots]. \quad (4.17)$$

where E′ stands for integration over the Gaussian random field, with the prescription that all terms containing $E(V(t_1) \cdot V(t_2))$ as a factor should be dropped.

It will be useful to have a graphical description of (4.17); this will give a convenient bookkeeping prescription. We shall give it here for the case in which V is a Gaussian random field; it can be extended to cover the non-Gaussian case, provided V has suitable mixing condition (*See* § 6). There are 2^{2n} terms in the n^{th} summand in (4.17) (a factor 2 for each commutator); each of them by the rules of Gaussian integration is expressed as the sum of $(2n-1)!!$ terms (number of possible pairings; we neglect for the moment the prescription which distinguishes E′ from E).

To each of these $2^{2n} \cdot (2n-1)!!$ terms we associate a graph, obtained with the following prescription.

Suppose one is considering the term

$$V(t_{i_1}) \dots V(t_{i_m}) A V(t_{i_{m+1}}) \dots V(t_{i_{2n}})$$

Mark on the real line $2n$ points, and label them $t_{i_1} \dots t_{i_{2n}}$, from left to the right.

Connect now pairwise with arcs γ_i in the upper half plane, $i = 1 \dots n$, the labeled points. There are $(2n-1)!!$ ways of doing so; each of the graphs so obtained corresponds in a natural way to one of the $2^n(2n-1)!!$ terms in (4.8).

We shall also agree to put a mark $*$ in correspondence to the position of the observable A in the term considered. If, e. g., one is considering the term $V(t_1)V(t_3)AV(t_4)V(t_2)$, one of the possible graphs will be

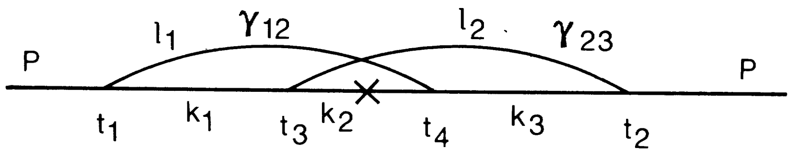

FIG. 1.

It is obvious, from the multi-commutator structure, that on the right of mark $*$, the times appear in increasing chronological order (from left to right), while they are in decreasing order on the left of $*$.

We shall use the following notation: $\gamma_{i,j}$ is the arc connecting t_i with t_j (so that $\gamma_{i,j} = \gamma_{j,i})\sigma_K$ is the interval $(t_{i_K}, t_{i_{K+1}})$, of length $|\sigma_K|$; $s(\gamma)$ (the shadow of γ) is the interval between the vertices of γ; $|\Gamma|$, the order of Γ, is the number of arcs in Γ.

With this notation, the prescription Σ' in the definition of $D_\lambda(\tau)$ is the following: the graph Γ cannot contain the arc γ_{12}.

Consider now a graph Γ of order n. Let C_Γ be the corresponding term in $D_\lambda(\tau)$. One has

$$C_\Gamma(p) = \pm\lambda^{2n} \int_{I^n_{\tau\lambda^2}} d^n t \int d^3K_1 \dots \int d^3K_{2n-1} \exp\left\{ i \sum_{m=1}^{2n-1} K^2_m(t_{i_m} - t_{i_{m-1}}) \right\} \cdot \prod_1^n \tilde{\mathcal{G}}(l_i(K)) \prod_1^{n-1} \delta(L_i(K))A(L(K)) \quad (4.18)$$

where $l_i \in R^3$, $L_i \in R^3$, $L \in R^3$ are linear combinations of the K's and we have used the notation

$$I^n_c \equiv \{ \underline{t} \,|\, C \geqslant t_1 \geqslant \dots \geqslant t_{2n} \geqslant 0 \}, \overset{n}{dt} = dt_1 \dots dt_n$$

It is convenient to regard K_m as the momentum associated to the σ_K, and l_i as the momentum associated to the arc γ_i, while L is the momentum of the interval which contains the mark $*$. Then the l_i and L_i are completely determined by the prescription that momentum be conserved at each vertex of the graph, and p be the momentum associated to the external lines.

For example, in the graph of Fig. 1 one would have

$$\begin{aligned} l_1(K) &= p - K_1 - K_2 + K_3 \\ l_2(K) &= K_1 - K_2 - K_3 + p \\ L_1(K) &= p - K_1 + K_1 - K_3 \\ L(K) &= K_2. \end{aligned} \tag{4.19}$$

With the change of variables

$$l_i(K) = u_i \qquad i = 1 \dots n,$$

and performing the integration over K_m, $m = n + 1, \dots, 2n - 1$, one verifies that C_Γ has the form

$$\int e^{i(uQu)} \tilde{\mathscr{G}}(u_1) \dots \tilde{\mathscr{G}}(u_n) A(L(u)) d^3u_1 \dots d^3u_n \tag{4.20}$$

i. e. a form suitable for the application of Lemma 5.

The matrix Q in (4.20) can be obtained according to the following prescription (*see* also [*6*]).

For the given graph Γ, order the arcs of Γ according to the order of their left vertices.

Then one has

$$\begin{aligned} Q_{ij} &= \sum_{K \in \mathscr{M}_{ij}} |\sigma_K| \\ \mathscr{M}_{ij} &\equiv \{ K \mid \sigma_K \subset s(\gamma_i) \cap s(\gamma_j) \} \end{aligned} \tag{4.21}$$

Also, if $\Gamma^{(M)}$ is a graph obtained from Γ deleting N − M arcs, the corresponding matrix $Q^{(M)}$ can be obtained from Q by deleting the corresponding rows and columns.

We now return to (4.2), (4.3) and write them in a form which is convenient for our later estimates. We shall then give a « graphical interpretation » of the new setting.

Writing B_λ for $B_\lambda(\tau)$, and with the agreement that from now on convergence is in $C(0, T; \mathscr{A})$, we rewrite (4.2) as

$$B_\lambda = K_\lambda \cdot B_\lambda + D_\lambda \tag{4.2'}$$

where $K_\lambda : C(0, T; \mathscr{A}) \to C(0, T; \mathscr{A})$ is defined by the first term on the right hand side of (4.2).

Similarly, we write now

$$(V_\lambda \circ A)(s) = \lambda^{-1} \int_0^{s\lambda^{-2}} [\hat{V}_\lambda(\sigma), A(\sigma)] d\sigma \tag{4.22}$$

where as usual $\hat{V}_\lambda(\sigma) = e^{i\sigma/\lambda^2 H_0} V e^{-i\sigma/\lambda^2 H_0}$ so that K_λ, as defined in (4.2'), can also be written

$$K_\lambda \cdot A = E(V_\lambda \circ V_\lambda \circ A) \quad \text{or, in short, } K_\lambda = E(V_\lambda \circ V_\lambda \circ)$$

Define now $$Z_\lambda \equiv V_\lambda \circ V_\lambda \circ - E(V_\lambda \circ V_\lambda \circ)\,. \tag{4.23}$$

One has then, for all $A \in B(H)$,

$$A_\lambda - E(A_\lambda) = V \circ A + Z \circ A \tag{4.24}$$

$$D_\lambda = E(V_\lambda \circ V_\lambda \circ (A_\lambda - E(A_\lambda))) \tag{4.25}$$

Using repeatedly (4.2′), (4.24), (4.25), one has

$$B_\lambda = K_\lambda \circ B_\lambda + \sum_{n=1}^{\infty} W^\lambda_{(n)} \circ E(A_\lambda) \tag{4.26}$$

where $$W^\lambda_{(n)} \circ A = E(K_\lambda \circ Z^n_\lambda \circ A) \qquad \text{for} \quad A \in \mathscr{A} \tag{4.27}$$

(we have used the fact that odd moments vanish).

Writing

$$W_\lambda \equiv \sum_{n=1}^{\infty} W^\lambda_{(n)} \tag{4.28}$$

as an operator on $C([0, T], \mathscr{A})$, one has finally

$$E(A_\lambda) - \underline{A}_\lambda = U_\lambda \cdot W_\lambda E(A_\lambda) \tag{4.29}$$

where $U_\lambda f \equiv \sum_{n=1}^{\infty} K^n_\lambda \cdot f$ and therefore

$$E(A_\lambda) - \underline{A}_\lambda = (1 - U_\lambda W_\lambda)^{-1} U_\lambda W_\lambda \underline{A}_\lambda \tag{4.30}$$

Since U_λ is bounded uniformly in λ for $0 \leqslant \lambda \leqslant \lambda_0$, and $\| \underline{A}_\lambda \| \leqslant \mathbb{C}$ $\forall \lambda : 0 \leqslant \lambda \leqslant \lambda_0$ (here $\| \underline{A}_\lambda \| \equiv \sup_{0 \leqslant t \leqslant \tau} \| \underline{A}_\lambda(t) \|$), we are left to prove that $W_\lambda \to 0$ when $\lambda \to 0$. Indeed, for λ sufficiently small, $1 - U_\lambda W_\lambda$ is invertible, and (4.30) allows then to conclude that $E(A_\lambda) - \underline{A}_\lambda \to 0$ when $\lambda \to 0$.

We shall prove that $W_\lambda \to 0$ by first performing a suitable resummation in (4.28), to write

$$W_\lambda = \sum_{n=1}^{\infty} \tilde{W}^\lambda_{(n)} \tag{4.31}$$

where the $\tilde{W}^\lambda_{(n)}$ will be defined presently, and then proving convergence to zero of each $\tilde{W}^\lambda_{(n)}$ and uniform convergence of the series in (4.31) for $0 \leqslant \lambda \leqslant \lambda_0$.

The graphical representation of the operator $W^{(\lambda)}_n$ will be useful. It is straightforward to see that, from the definition given, $(W^{(\lambda)}_n \circ A)(p)$ is the contribution from graphs of order $2n$, such that for no integer $m < n$

the first $2m$ vertices (in chronological order) are disjoint from the remaining $2(n - m)$. We call such graphs « tight ».

As a preliminary step we prove

LEMMA 6. — The contribution from a tight graph vanishes in the limit $\lambda \to 0$. □

Proof. — For any graph Γ of order n which is tight, let $\Gamma^{(M)}$ be a reduced subgraph with the following property: for all $0 \leqslant \varepsilon \leqslant 1/2$ one has

$$(\det Q)^{-1+\varepsilon}(\det Q^M)^{-1/2-\varepsilon} \in L^1(I^n_{\tau/\lambda^2}) \tag{4.32}$$

where Q^M is the matrix associated to the subgraph $\Gamma^{(M)}$.

Such subgraphs always exist for a tight graph; it suffices, e. g., to consider the subgraph composed of only one arc, which either intersects or graph-shadows some other arc in Γ.

From (4.14) one concludes that the contribution C_Γ from Γ can be estimated by noticing that

$$\int e^{i(pQp)}\tilde{\mathscr{G}}(p_1) \ldots \tilde{\mathscr{G}}(p_n)A(l(\underline{p}))d^3p_1 \ldots d^3p_n \leqslant 2\mathbb{C}^n\lambda^{2n} \| A \| \mathscr{G}^n \,|\det Q|^{-\frac{3}{2}\alpha} |\det Q^M|^{-\frac{3}{2}(1-\alpha)} \tag{4.33}$$

for all $0 \leqslant \alpha \leqslant 1$.

By (4.32), the r. h. side $\in L^1(I^n_{\tau/\lambda^2})$ for $1/3 < \alpha < 2/3$. Therefore

$$|C_\Gamma| \leqslant 2\mathbb{C}^n\lambda^{2n} \| A \| \, \| \mathscr{G} \|^n \int_{I^n_{\tau}/\lambda^2} |\det Q|^{-\frac{3}{2}\alpha} |\det Q^M|^{-3/2(1-\alpha)} d^n t \tag{4.34}$$

Remark now that Q is a homogeneous polynomial of order n, while Q^M is homogeneous of order M ($M \geqslant 1$).

Scaling $t_i \equiv \lambda^{-2}\tau \cdot s_i$ one has then

$$|\mathbb{C}_\Gamma| \leqslant 2\mathbb{C}^n \| A \| \, \| \mathscr{G} \|^n \tau^{2n-\frac{3}{2}n\alpha-\frac{3}{2}M(1-\alpha)} \lambda^{2n-4n+3n\alpha+3M(1-\alpha)} \cdot \int_{I_1} |(\det Q)|^{-3/2\alpha} |(\det Q^M)|^{-3/2(1-\alpha)2} d^n_s \tag{4.35}$$

Choosing α such that $2/3 > \alpha > \sup\left(1/3, \frac{2}{3}\left(1 - \frac{M}{n}\right)\right)$ the conclusion of Lemma 6 follows. □

There are at most $(2n - 1)!!$ tight graphs of order n contributing to W^λ_n. From (4.35) the proof of convergence of W^λ to zero, when $\lambda \to 0$, is achieved if one can prove that

A : one can choose Γ^M so that M/n remains bounded away from zero when $n \to \infty$,

B : the integral is dominated by $C^n \cdot (n!)^{-1}$ for some $C > 0$.

Indeed, if this is the case, one can choose α, $1/3 < \alpha < 2/3$, independent of Γ, so that the sum of all contributions of tight graphs of order n vanish as $\lambda^{+\beta(\alpha)n}$ for some $\beta(\alpha) > 0$.

It is intuitively clear that, the more « complicated » (i. e. interlocked) is a graph, the bigger one can choose M/n. « Very simple » graphs would instead lead to small values of M even for large values of n.

Consider, e. g., the graph Γ_n

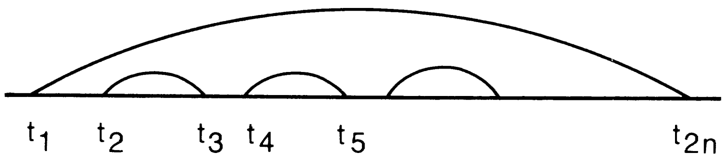

FIG. 2.

This is a tight graph, but it is straightforward to prove that $M = 1$ independently of n.

Indeed, the only subgraph $\Gamma^{(M)}$ such that

$$(\det Q)^{-1-\varepsilon}(\det Q^M)^{-1/2-\varepsilon} \in L_1$$

is the one composed only of the arc $\gamma_{1,2n}$.

We shall prove however that the only « dangerous » graphs are the ones which contain one or more arcs ot the type $\gamma_{K,K+1}$ for some $1 \leqslant K \leqslant N$, which are disjoint from the remaining arcs in Γ.

The resummation indicated before (4.31) will then consists in a resummation over all graphs which differ only for the presence of one or more such arcs.

Before giving the proof, we pause to provide a physical interpretation. An arc $\gamma_{i,i+1}$ represents a « rescattering process », i. e. a process in which there are correlations between two interactions which occur successively in time. It is known that rescattering is the source of most difficulties in the treatment of diffusion in a random medium.

It is also known in applied mathematics and physics how one should try to circumvent this difficulty: one should describe the process in terms of an « effective » propagation rather than « free propagation » between successive interactions. Rescattering goes then in the definition of « effective propagation ».

Technically, such effective propagation is described by a resummation over a set of graphs.

Our approach will be much in this spirit. As we shall see, we shall be able to give an estimate of the absolute value of the ratio between effective and free propagator, uniform in $0 \leqslant \lambda \leqslant \lambda_0$ and in τ over compact sets.

While no such bound will be found on the phase, the uniform bound on the absolute value will suffice, through Lemma 5′, to prove convergence of W_λ to zero, when $\lambda \to 0$, uniformly in τ over compacts of R^+.

Prompted by this physical interpretation, we introduce a few more notation.

5. ANALYSIS OF GRAPHS

Let $\left\lceil \frac{n}{2} \right\rceil$ be the smallest integer which is not smaller than $\frac{n}{2}$.

Let Γ be a graph. We say that $\gamma \in \Gamma$ is disconnected if it does not shadow or intersect any other arc of Γ.

We say that $\gamma_{i,j} \in \Gamma$ is a rescattering arc if $\gamma_{i,j}$ is disconnected and $j = i \pm 1$. We say that Γ is rescattering-free if it contains no rescattering arc.

If Γ is of order n and rescattering-free, we shall construct a class $\tilde{M}$ of subgraphs Γ^M, $M \in \tilde{M}$, all of order $\geqslant \left\lceil \frac{n}{2} \right\rceil$ and such that

$$\inf_{M \in \tilde{M}} (\det Q)^{-1+\alpha} (\det Q^M)^{-1/2-\beta} \in L^1(I^n_{\tau/\lambda^2}) \tag{5.1}$$

for all $0 \leqslant \beta \leqslant \alpha \leqslant 1/2$.

Let Γ be rescattering free, and $\gamma_{K_1,K_2} \in \Gamma$ be disconnected. Then $\exists K_0$, $K_0 \neq K_1, K_2$, such that $K_0 \in s(\gamma_{K_1,K_2})$.

Therefore the submanifold of I^n_{τ/λ^2} defined by $t_{K_2} - t_{K_1} = 0$ has co-dimension at least two, so that $(t_{K_2} - t_{K_1})^{-1+\delta}$ is integrable over I^n_{τ/λ^2} for $\delta > -1$.

In its dependence on $t_{K_2} - t_{K_1}$, det Q has the form

$$|\det Q| = |t_{K_2} - t_{K_1}| \, |t_{K_2} - t_{K_1} + \beta_Q(t)|$$

where β_Q is a polynomial in the remaining t_m's.

Therefore, if $\Gamma^{(M)}$ is constructed in such a way as to contain $\gamma_{K_1K_2}$ and a subset of the other links, and if in an open domain $D \subset R^{2n-2}$ the manifold of zeros of β_Q intersects transversally the manifold of zeros of $\beta_Q M$, then

$$(\det Q)^{-1+\alpha} (\det Q^M)^{-1/2-\beta} \in L_1(D \times I^{(2)})$$

for $0 \leqslant \beta \leqslant \alpha \leqslant 1/2$, where $I^{(2)} \equiv \{ 0 \leqslant t_{K_1} \leqslant t_{K_2} \leqslant \tau/\lambda^2 \}$.

On the other hand, suppose $\{ \gamma_i \dots \gamma_s \}$ are the arcs of a cluster (a maximal connected subgraph of Γ) with $s > 1$, and let $t_{K_1} \geqslant \dots \geqslant t_{K_{2s}}$ be the corresponding vertices. Det Q is a polynomial in $t_{K_1}, \dots, t_{K_{2s}}$; let $\Sigma_{\{\gamma_1 \dots \gamma_s\}}$ be its manifold of zeroes, which of course depend parametrically (in fact, rationally) on the remaining t's, which we denote by $t_{h_1} \dots t_{h_{2n-2s}}$.

Let $M \equiv \{ \gamma'_1 \dots \gamma'_p \}$ be a subset of $\{ \gamma_1 \dots \gamma_p \}$, $p = \left\lceil \frac{s}{2} \right\rceil$ and consider

the graph $\Gamma^{(M)}$ obtained from Γ deleting the arcs γ_m which are not in M.
Denote D^M the subset of $\tau/\lambda^2 \geqslant t_{h_1} \geqslant t_{h_{2n-2s}} \geqslant 0$ such that

$$\Sigma_{\{\gamma_1 \dots \gamma_s\}} \quad \text{and} \quad \Sigma_{\{\gamma'_1 \dots \gamma'_p\}} \quad \text{intersect transversally}.$$

Then $\bigcup_{M \in \tilde{M}} D^M \equiv \{ \tau/\lambda^2 \geqslant t_{h_1} \geqslant \dots \geqslant t_{h_{2n-2s}} \geqslant 0 \}$, where $\tilde{M}$ is the collection of subsets of $\{ \gamma_1 \dots \gamma_s \}$ of order $\left[\frac{s}{2}\right]$.

Therefore

$$\inf_{M \in \tilde{M}} (\det Q)^{-1+\alpha} (\det Q^{(M)})^{\frac{1}{2}-\beta} \in L^1(I^n_{\tau/\lambda^2})$$

if it is in L^1 over $\bigcup_{M \in \tilde{M}} D^M$ when considered as a function of the variables $t_{h_1} \dots t_{h_{2n-2s}}$.

Repeating inductively this procedure, first for those arcs which are disconnected from the rest of Γ, and then for the clusters of order $s > 1$, one concludes that there exists a collection $\tilde{M}$ of subgraphs Γ^M, $M \in \tilde{M}$, of order $\geqslant \left[\frac{n}{2}\right]$, such that

$$\inf_{M \in \tilde{M}} (\det Q)^{-1+\alpha} (\det Q^M)^{-\frac{1}{2}-\beta} \in L^1(I^n_{\tau/\lambda^2}) \tag{5.2}$$

for $0 \leqslant \beta \leqslant \alpha \leqslant 1/2$.

From (4.14), setting $\tau_0 = \sup(1, \tau)$ and taking $\lambda \leqslant 1$, by suitable rescaling and using the fact that $\det Q$ and $\det Q^{(M)}$ are homogeneous polynomials in the variables t_i, one has

$$| C_\Gamma | \leqslant 2C^n \| A \| \| \mathscr{G} \|^n \tau_0^{n\left(2-(1-\alpha)-\frac{1}{2}\left(\frac{1}{2}+\beta\right)\right)} \lambda^{n\left(-2+2(1-\alpha)+\frac{1}{2}+\beta\right)}$$
$$\cdot \int_{I_1^n} \inf_{M \in \tilde{M}} | \det Q |^{-1+\alpha} | \det Q^M |^{-\frac{1}{2}-\beta} d^n s_i \tag{5.3}$$

when $0 \leqslant \beta \leqslant \alpha \leqslant 1/2$.

We must now give a bound on the contribution to W_λ coming from all tight rescattering-free graphs of order n. Since there are at most $(2n - 1)!!$ such graphs, it will suffice to prove that the integral in (5.3) is dominated by $\frac{1}{n!} \mathbb{C}_0^n$ for some $\mathbb{C}_0 > 0$.

Since the integration domain is triangular, one may in fact have expected a factor $(2n!)^{-1}$. This is not so, since the integrand is *not* integrable over all permutations of I_1. This is due in particular to the fact that it contains a factor $(t_{i_K} - t_{i_{K+1}})^{-3/2+(\alpha-\beta)}$ for each disconnected arc.

From (5.3) it is seen that we must require $\frac{1}{2} + \beta > 2\alpha$, i. e.

$$\alpha - \beta < \frac{1}{2}\left(\frac{1}{2} - \beta\right) < 1/3 \quad \text{so that} \quad -\frac{3}{2} + (\alpha - \beta) < -\frac{5}{4};$$

the factor considered is integrable in I_1 only because the graph is, by assumption, rescattering free, and therefore $\exists K_0$ such that $t_{i_K} \geqslant t_{i_{K_0}} \geqslant t_{i_{K+1}}$.

It is however easy to see that integrability will still be guaranteed in all sectors $\pi(I)$ obtained from I by permutations which are such that the end points of a disconnected arc are permuted with the end points of another disconnected arc. The graph Γ will be rescattering-free also in these permuted sector.

It is easy to verify that there are at least $n!$ such permutations.

It remains to give a bound on the integral on each such permuted sector.

By repeatedly using the inequality

$$|a + b| \geqslant a^{\gamma} b^{1-\gamma}$$

and the properties of $\tilde{M}$, one can prove that each sector $\pi(I)$ can be partitioned in at most 2^{2n} subsets $\chi_i^{(\pi)}$, $\bigcup_i \chi_i^{(\pi)} \equiv \pi(I)$, such that in χ_i^{π}

$$\inf_{M \in \tilde{M}} |\det Q|^{-1+\alpha} |\det Q^M|^{-1/2+\beta} \leqslant \prod_{K=1}^{2n} |l_K(\underline{t})|^{-\gamma_K}$$

where $\gamma_K \leqslant \sup\left\{1 - \frac{2}{3}(\alpha - \beta),\ 1 - \alpha,\ \frac{1}{2} + \beta\right\}$ and the $l_K(\underline{t})$ can be chosen to be independent linear combinations of the $t_i's$, $i = 1 \ldots 2n$.

The Jacobian of the transformation $\underline{t} \to \underline{l}$ is $\leqslant 4^{2n}$.

Therefore

$$\int_I \inf_{M \in \tilde{M}} |\det Q|^{-1+\alpha} |\det Q^M|^{-1/2+\beta}$$
$$\leqslant \frac{1}{n!} \int_{\cup \pi(I)} \sup_{\pi} \inf_{M \in \tilde{M}} |\det Q|^{-1+\alpha} |\det Q^M|^{-\frac{1}{2}+\beta}$$
$$\leqslant \frac{1}{n!} 2^{2n} \int \prod_{K=1}^{2n} l_K(\underline{t})^{-\gamma_K} \prod_{i=1}^{2n} dt_i \leqslant \frac{1}{n!} 2^{2n} \cdot 4^{2n} \prod_{K=1}^{2n} \int_0^1 \sigma^{-\gamma_K} d\sigma$$

Take now $\alpha = \frac{1}{4}$, $\beta = \frac{1}{8}$. Then $\gamma_K \leqslant \frac{11}{12} \forall K$, so that

$$\int_I \inf_{M \in \tilde{M}} |\det Q|^{-3/4} |\det Q^M|^{-5/8} \leqslant \frac{1}{n!} 8^{2n} \cdot 12^{2n} \cdot 4^{2n}$$

From (5.3) one has then that $\tilde{\mathbb{C}}_n$, the contribution to W_λ from all tight, rescattering-free graphs, is bounded by

$$| \tilde{\mathbb{C}}_n | \leqslant 2C_0^n \| A \| \| \mathcal{G} \|^n \tau_0^{\frac{7}{8}n} \lambda^{n/8} \tag{5.4}$$

for C_0 sufficiently large.

Consider now the « rescattering » graphs, i. e. those graphs which are not rescattering-free.

As a set, they can be obtained introducing rescattering arcs in rescattering-free graphs.

For each Γ, let Γ^0 be the (rescattering-free) graph obtained from Γ by dropping all rescattering arcs. We call Γ^0 the skeleton of Γ, and Γ an ornament of Γ^0. Of course, we allow $\Gamma = \Gamma^0$, i. e. a skeleton is one of its own ornaments.

Denote by $\tilde{\Gamma}^0$ the set of all ornaments of the skeleton Γ^0. Let Σ^0 be the collection of all skeletons. Then $\bigcup_{\Gamma^0 \in \Sigma^0} \tilde{\Gamma}^0$ is the collection of all graphs, each counted once. For each Γ^0, let C_{Γ^0} be the contribution to W_λ coming from the graph Γ^0, and $C_{\tilde{\Gamma}^0}$ the sum of the contributions to W_λ of all ornaments of Γ^0.

We shall give an estimate, uniform in τ over compacts and in $0 \leqslant \lambda \leqslant 1$ of $| C_{\tilde{\tilde{\Gamma}}^0} |$ in terms of $| C_{\Gamma^0} |$.

We shall then use this estimate, together with the previous result on skeleton graphs, to prove convergence of W_λ to zero, when $\lambda \to 0$, uniformly in τ over compacts of R^+. Let Γ^0 be a skeleton, with vertices $t_{i_1} \dots t_{i_{2n}}$, and with a given set of arcs.

An ornament Γ of Γ^0, if $\Gamma \neq \Gamma^0$, is obtained by inserting rescattering arcs in one or more of the intervals σ_K of $\tilde{\Gamma}^0$. As we have already remarked, if $s_2^{(K)} \dots s_{2m_K}^{(K)}$ are the vertices of the m_K arcs inserted in σ_K, one has

$$t_{i_K} \geqslant s_p^{(K)} \geqslant s_{p+1}^{(K)} \geqslant t_{i_{K+1}} \quad \text{for all } p \quad \text{if } t_{i_K} \geqslant t_{i_{K+1}}$$
$$t_{i_K} \leqslant s_p^{(K)} \leqslant s_{p+1}^{(K)} \leqslant t_{i_{K+1}} \quad \text{for all } p \quad \text{if } t_{i_K} \leqslant t_{i_{K+1}}$$

This is a consequence of the commutator structure of the terms which we represent graphically.

For the same reason, the mark * which we have used to keep track of the multiplication by A, is never covered by a rescattering arc.

Consider now the contribution C_Γ to W_λ coming from a graph Γ which is an ornament of Γ^0 obtained by adding m_K rescattering arcs in σ_K, $K = 1 \dots 2n - 1$.

Call $\xi_n^{(K)}$, $n = 1 \dots 2m_L + 1$ the momenta of the intervals of Γ contained in the interval σ_K of Γ^0. Call $\sigma_i^{(K)}$, $i = 1 \ldots 2m_K$, the vertices of $\Gamma\backslash\Gamma^0$ contained in σ_K, and denote by p_K the momentum of σ_K in Γ^0.

Then, according to (4.18), the expression which defines C_Γ differs from the one which defines C_{Γ^0} by the presence of the extra integration variables $\xi_i^{(K)}$, and of the corresponding δ-functions and factors of the type $\exp(iK^2(\sigma - \sigma'))\tilde{\mathcal{G}}(l_i(K))$. Due to the rescattering nature of the arcs added, the new δ-functions imply that $\xi_{2n-1}^{(K)} = p_K$, $n = 1, \ldots, m_K + 1$. But then the exponential part of the integrand takes the form

$$\exp\left\{ i \sum_{K=1}^{2n-1} p_K^2(t_{i_K} - t_{i_{K+1}}) + i \sum_{K=1}^{2n-1} \sum_{s=1}^{m_K} (\xi_{2s}^K)^2(\sigma_{2s-1}^K - \sigma_{2s}^K) \right\}$$

Moreover, the $l_j^{(K)}(\xi)$ which appear in (4.18) are linear functions of the $\xi_m^{(K)}$'s and of the p^K's, with coefficients which depend only on K.

From this analysis, one concludes that, if C_{Γ^0} is given as in (4.18), then, after setting $t_i = \tau_i\lambda^{-2}$

$$C_\Gamma(\underline{p}) = \lambda^{-2n} \int_0^\tau d\tau_1 \ldots \int_0^{\tau_{2n-1}} d\tau_{2n} \int d^3p_1 \ldots \int d^3p_{2n-1}$$
$$\exp\left\{ i \sum_{m-1}^{2n-1} p_m^2(\tau_{i_m} - \tau_{i_{m+1}})\lambda^{-2} \right\}$$
$$\cdot \prod_{i=1}^{n} (l_i(p)) \prod_{i=1}^{n-1} \delta(L_i(\underline{p}))A(L(\underline{p})) \prod_{K=1}^{n} B_{(K)}^{(\Gamma|\Gamma^0)}(\tau_{i_K} - \tau_{i_{K-1}}, \underline{p}) \quad (5.5)$$

where

$$B_{(K)}^{(\Gamma|\Gamma^0)}(\tau_{i_K} - \tau_{i_{K+1}}, \underline{p})$$
$$= (-1)^m\lambda^{-2m} \int_{I_{\tau_{i_K},\tau_{i_{K+1}}}^{\Gamma|\Gamma^0}} \prod_{K=1}^{m_K} \exp\{ i(p_K^2 - (\xi_{2i}^K)^2)(\sigma_{2i}^K - \sigma_{2i+1}^K)\lambda^{-2} \}$$
$$\cdot \xi\tilde{\mathcal{G}}(\xi_{2i}^K - p_K)\xi d\sigma^K \xi d\xi^K \prod_{i=1}^{m_K} \quad (5.6)$$

and $I_{\tau_{i_K},\tau_{i_{K+1}}}^{\Gamma|\Gamma^0}$ is a subset of $\tau_{i_K} \leqslant \sigma_1 \leqslant \ldots \leqslant \sigma_{2m_K} \leqslant \tau_{i_{K+1}}$ if $\tau_{i_K} \leqslant \tau_{i_{K+1}}$, or of $\tau_{i_K} \geqslant \sigma_1 \geqslant \ldots \geqslant \sigma_{2m_K} \geqslant \tau_{i_{K+1}}$ if $\tau_{i_K} \geqslant \tau_{i_{K+1}}$, determined by $\Gamma \mid \Gamma^0$ (recall that, while s_{2i} is the chronological successor of s_{2i-1} for all i, it need not be true that s_{2i+1} be the successor of s_{2i}).

Notice now that the integrand in (5.6) is symmetric under permutations of the values of the index i. Therefore

$$\mathrm{B}_{(\mathrm{K})}^{(\Gamma|\Gamma^0)}(\tau_{i_\mathrm{K}} - \tau_{i_{\mathrm{K}+1}} ; \underline{p}) = \frac{(-1)^m \lambda^{-2m}}{m!} \int_{\tilde{\mathrm{I}}^{\Gamma|\Gamma^0}_{\tau_{i_\mathrm{K}} - \tau_{i_{\mathrm{K}+1}}}} \prod_{i=1}^{m_\mathrm{K}} \exp\{ i(p_\mathrm{K}^2 - (\xi_{2i}^\mathrm{K})^2)(\sigma_{2i}^\mathrm{K} - \sigma_{2i+1}^\mathrm{K})\lambda^{-2} \} \cdot \prod_i \tilde{\mathscr{G}}(\xi_{2i}^\mathrm{K} - p_\mathrm{K}) \prod_i d\sigma_i^\mathrm{K} \prod_i d\xi_i^\mathrm{K} \tag{5.7}$$

where $\tilde{\mathrm{I}}^{\Gamma|\Gamma^0}_{\tau_{i_\mathrm{K}}, \tau_{i_{\mathrm{K}+1}}}$ is the union of the $\mathrm{I}^{\Gamma|\Gamma^0}_{\tau_{i_\mathrm{K}}, \tau_{i_{\mathrm{K}+1}}}$ over all permutations π of the index set $1 \dots m_\mathrm{K}$, such that

$$\tau_{i_\mathrm{K}} \leqslant \sigma_{2\pi(1)-1} \leqslant \sigma_{2\pi(1)} \leqslant \dots \leqslant \sigma_{2\pi(m_\mathrm{K})} \leqslant \tau_{i_{\mathrm{K}+1}}$$

if $\tau_{i_\mathrm{K}} \leqslant \tau_{i_{\mathrm{K}+1}}$, and a corresponding expression if $\tau_{i_\mathrm{K}} \geqslant \tau_{i_{\mathrm{K}+1}}$.
It follows that

$$\prod_\mathrm{K} \mathrm{B}_{(\mathrm{K})}^{\Gamma|\Gamma^0}(\tau_{i_\mathrm{K}} - \tau_{i_{\mathrm{K}+1}}, \underline{p}_\mathrm{K})$$

is the product of the terms of order $m_1 \dots m_{2n-1}$ in the expansion of the exponentials in

$$\prod_\mathrm{K} \mathrm{D}_\mathrm{K}(\tau_{i_\mathrm{K}} - \tau_{i_{\mathrm{K}+1}}, \underline{p}_\mathrm{K}) \tag{5.8}$$

where

$$\mathrm{D}_\mathrm{K}(\tau_{i_\mathrm{K}} - \tau_{i_{\mathrm{K}+1}}, \underline{p}_\mathrm{K}) = \exp\left(-\frac{1}{\lambda^2} \int_0^{\tau_{i_\mathrm{K}} - \tau_{i_{\mathrm{K}+1}}} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int d^3h \tilde{\mathscr{G}}(p_\mathrm{K} - h) e^{i(p_\mathrm{K}^2 - h^2)\sigma/\lambda^2} \right) \tag{5.9}$$

Recalling that Γ was obtained adding m_K rescattering arcs in the K^{th} interval of Γ^0, we conclude that, if

$$\mathrm{C}_{\Gamma^0}(p) = \pm \lambda^{-2n} \int_0^\tau d\tau_1 \dots \int_0^{\tau_{2n-1}} d\tau_{2n} \dots \int d^3p_1 \dots \int d^3p_{2n-1} \exp\left\{ i \sum_{m=1}^{2n-1} p_m^2(t_{i_m} - t_{i_{m+1}}) \right\} \cdot \prod_1^n \tilde{\mathscr{G}}(l_i(p)) \prod_1^{n-1} \delta(\mathrm{L}_i(p))\mathrm{A}(\mathrm{L}(p)) \tag{5.10}$$

then $C_{\tilde{\Gamma}^0}(p)$ is given by an expression which differs from (5.10) only by the additional factor

$$\prod_{m=1}^{2n-1} D_m(\tau_{i_m} - \tau_{i_{m+1}})(p_m) \tag{5.11}$$

where $D_m(\tau_{i_m} - \tau_{i_{m+1}})$ is given by (5.9).

It is now evident from (4.22) and Lemma 5′ that, in order to extend to $C_{\tilde{\Gamma}^0}$ the estimates given for C_{Γ^0}, it suffices to prove that there is a constant $\mathbb{C} > 0$ such that, uniformly in v over compacts of R^+ and $0 \leqslant \lambda \leqslant 1$

$$\| D_K(v, \cdot) \|_\infty < e^{\mathbb{C}v} \tag{5.12}$$

where $\| B(\cdot) \|_\infty \equiv \sup_{p \in R^3} | B(p) |$.

In turn, (5.12) will follow from the estimate

$$\sup_{p \in R^3} \left| \lambda^{-2} \operatorname{Re} \int_0^\sigma d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int d^3h \tilde{\mathcal{G}}(\underline{p} - \underline{h}) e^{i(p^2 - h^2)\sigma_2 \lambda^{-2}} \right| < \mathbb{C}_1 \sigma \tag{5.13}$$

uniformly in σ over compacts of R^+ and in λ in $[0, 1]$.

We shall now prove that, under the assumptions made on $\tilde{\mathcal{G}}$, (5.13) holds for a suitably chosen constant $\mathbb{C}_1$.

Setting $\sigma_2 = s\lambda^2$, we must study

$$\sup_{p \in R^3} \left| \int_0^\sigma d\sigma_1 \int_{-\sigma_1/\lambda^2}^{\sigma_1/\lambda^2} ds \int d^3h \tilde{\mathcal{G}}(\underline{p} - \underline{h}) e^{i(p^2 - h^2)s} \right| \tag{5.14}$$

Since $G \in L_1$ by assumption, $\tilde{\mathcal{G}}$ is continuous, moreover, again by assumption on G, $K(\underline{p}) \to 0$ when $|\underline{p}| \to \infty$, where $K(p) \equiv \int \tilde{\mathcal{G}}(\underline{p} - \underline{h})\delta(p^2 - h^2 d^3h$.

Therefore

$$\lim_{\lambda \to 0} \int_{-a/\lambda^2}^{a/\lambda^2} ds \int d^3h \tilde{\mathcal{G}}(\underline{p} - \underline{h}) e^{i(p^2 - h^2)s} = \int d^3h \tilde{\mathcal{G}}(\underline{p} - \underline{h}) \delta(p^2 - h^2) \tag{5.15}$$

holds uniformly in $p \in \mathbb{R}^3$ and in a for $a \geqslant a_0 \lambda^{3/2}$. To evaluate (5.14), we divide the integration over σ_1 in the two intervals $0 \leqslant \sigma_1 \leqslant a_0 \lambda^{3/2}$, $a_0 \lambda^{3/2} \leqslant \sigma_1 \leqslant \sigma$. If $\sigma_1 \leqslant a_0 \lambda^{3/2}$, one has

$$\left| \int_{-\sigma_1 \lambda^2}^{\sigma_1 \lambda^2} ds \int d^3h \tilde{\mathcal{G}}(\underline{p} - \underline{h}) e^{i(p^2 - h^2)s} \right| \leqslant 2a_0 \lambda^{-1/2} \| \tilde{\mathcal{G}} \|_1 \tag{5.16}$$

Therefore

$$\left| \int_0^{a_0 \lambda^{3/2}} d\sigma_1 \int_{-\sigma_1/\lambda^2}^{\sigma_1/\lambda^2} ds \int d^3h \tilde{\mathcal{G}}(\underline{p} - \underline{h}) e^{i(p^2 - h^2)s} \right| \leqslant 2a_0^2 \lambda \| \tilde{\mathcal{G}} \|_1 \xrightarrow[\lambda \to 0]{} 0 \tag{5.17}$$

uniformly in $\underline{p} \in R^3$.

On the other hand, in view of (5.15)

$$\lim_{\lambda\to 0}\int_{a_0\lambda^{3/2}}^{\sigma} d\sigma_1 \int_{-\sigma_1\lambda^{-2}}^{\sigma_1\lambda^{-2}} ds \int d^3h\tilde{\mathcal{G}}(\underline{p}-\underline{h})e^{i(p^2-h^2)s} = \sigma\int d^3h\tilde{\mathcal{G}}(\underline{p}-\underline{h})\delta(p^2-h^2) \quad (5.18)$$

the limit being uniform for $\underline{p}\in \mathrm{R}^3$.
Let

$$\mathbb{C}_2 \equiv \sup_{p\in \mathrm{R}3}\int d^3h\tilde{\mathcal{G}}(\underline{p}-\underline{h})\delta(p^2-h^2) \quad (5.19)$$

and notice that, for $\lambda > \lambda_0$, $\exists \mathrm{C}_3(\lambda_0)$ such that

$$\left|\sup_{p\in\mathrm{R}^3}\int_0^{\sigma} d\sigma_1 \int_{-\sigma_1\lambda^{-2}}^{\sigma_1\lambda^{-2}} ds \int d^3h\mathcal{G}(\underline{p}-\underline{h})e^{i(p^2-h^2)s}\right| \leqslant \mathrm{C}_3(\lambda_0)\cdot\sigma^2\cdot\mathbb{C}_2 \quad (5.20)$$

From (5.17), (5.18), (5.20) one concludes that (5.19) holds uniformly in σ over any compact K for a suitably chosen $\mathbb{C}_1 > 0$ (of course $\mathbb{C}_1$ depends on K).

From (5.4), Lemma 5′, (5.12) and the estimate (5.2) we conclude that the contribution to W_λ from all tight graphs is dominated by the series

$$\sum_{n\geqslant 1} 2^{n+1}(\sqrt{\pi})^{3n}\,\|\mathcal{G}\|^n e^{n\mathrm{C}_1\tau_0}\,\|\mathrm{A}\|\,\tau_0^{\frac{n}{4}(5-3\alpha)}\lambda^{\frac{n}{2}(3\alpha-1)}\overline{\mathbb{C}}_\alpha^n \quad (5.21)$$

for $1/3 < \alpha < 2/3$. We shall take $\alpha = 1/2$.

For $\lambda < \lambda_0(\mathcal{G}, \mathbb{C}_1, \overline{\mathbb{C}}_{1/2}, \tau_0)$ this series is absolutely convergent, and therefore the series converges to zero when $\lambda \to 0$, since each term converges to zero in that limit.

The series (5.21) dominates $|\mathrm{W}^\lambda|$; therefore $\mathrm{W}^\lambda \to 0$ when $\lambda \to 0$, uniformly in τ over compacts.

From (4.31) and the uniform boundedness of U_λ we conclude that

$$\mathrm{E}(\mathrm{A}_\lambda) - \underline{\mathrm{A}}_\lambda \underset{\lambda\to 0}{\longrightarrow} 0$$

uniformly in τ over compacts of R^+.

We summarize this analysis in

THEOREM 2. — Under the assumptions $\mathcal{G}\in \mathrm{L}_1$, $\tilde{\mathcal{G}}\in\mathrm{L}_1$,

$$\sup_{p\in\mathrm{R}^3}\int \mathcal{G}(\underline{p}-\underline{\mathrm{K}})\delta(p^2-\mathrm{K}^2)d^3\mathrm{K} < \infty\,,$$

one has

$$\lim_{\lambda\to 0}\mathrm{E}(\mathrm{A}_\lambda(\tau/\lambda^2))(p) = (\mathrm{T}_\tau\cdot\mathrm{A})(p)$$

in the sup norm, uniformly in τ over compacts of $\mathbb{R}^+$, where T_τ is defined in (3.6), (3.7). □

Before closing this section, we would like to make a short comment on estimate (5.13). It is easy to see that no such estimate would be available ont he imaginary part of the integral, i. e. on the phase of the factors D_K, when $\lambda \to 0$. This fact is very closely connected with the problems one would encounter if one were to study the Van Hove limit for the evolution of states rather than of observables (this added difficulty is sometimes called Casimir effect). With $\psi \in L^2(\mathbb{R}^3)$, one could for example study

$$E(e^{iH_0\tau/\lambda^2}U_\lambda(\tau/\lambda^2)\psi) \tag{5.22}$$

in the limit $\lambda \to 0$.

For $\lambda \neq 0$, (5.22) is well defined, in view of § 2.

One can use the Dyson expansion in the study of (5.22) and attempt a proof along lines similar to the ones above.

A graphical analysis can still be done, but a major difficulty comes here from rescattering graphs, mostly because one cannot control uniformly the phase in the integral (5.13).

6. CONVERGENCE TO THE MARKOV LIMIT: CONVERGENCE OF CORRELATIONS

We have seen, in § 4 and § 5, that the average of the dynamics for translation-invariant observables converges in norm to a Markov semi-group, uniformly in time over compacts.

It is therefore natural to expect that all correlations between observables at different times converge to the correlations given by the Markov semi-group.

We prove this result in the present section. We have

THEOREM 3. — Let the conditions on V be as in Th. 2, $A_\lambda(t)(\omega)$ be defined as in Th. 1, and $T_\tau \cdot A$ as in (3.6), (3.7).

Let $A^{(1)}, \ldots, A^{(N)}$ be arbitrary elements of $\mathscr{A}$, and choose $\tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_N$.

Then, in the norm of $\mathscr{A}$

$$\lim_{\lambda\to 0} E(A_\lambda^{(1)}(\tau_1/\lambda^2) \ldots A_\lambda^{(N)}(\tau_N/\lambda^2)) = T_{\tau_N} \cdot A^{(N)} \cdot T_{\tau_{N-1}-\tau_N} \cdot A^{(N-1)} \ldots T_{\tau_1\tau_2}A^{(1)} \tag{6.1}$$

uniformly in τ_i over compacts. □

Let $U_\lambda(t)(\omega)$ be the unitary group which implements the isomorphism $A \mapsto A_\lambda(t)(\omega)$.

Recall that $U_\lambda t)(\omega)$ is, for $\omega \in \overline{\Omega}_{\lambda,t}$, $\mu(\overline{\Omega}_{\lambda,t}) = 1$, the strong limit along the sequence $\varepsilon_n \downarrow 0$ of $\exp(iH_\lambda^{(n)}t)$, where $H_\lambda^{(n)} \equiv -\Delta + \lambda V_{\varepsilon_n} \equiv H_0 + \lambda V_{\varepsilon_n}$.

If H is self-adjoint, and $A^{(i)}$ are bounded operators which commute with e^{iH_0t}, the following identity is readily verified:

$$e^{-iH_0t_1}(e^{iHt_1}A^{(1)}e^{-iHt_1}e^{iHt_2}A^{(2)}e^{-iHt_2}\dots e^{iHt_N}A^{(N)}e^{-iHt_N})e^{iH_0t_1}$$
$$= e^{-iH_0t_N}e^{iHt_1-t_N\cdot t_N}(e^{-iH_0(t_1-t_N)}e^{iH(t_1-t_N)}A^{(1)}$$
$$\dots A^{(N-1)}e^{iH(t_{N-1}-t_N)}e^{iH_0(t_{N-1}-t_N)})\cdot A^{(N)}e^{-iHt_1-t_N\cdot t_N}e^{iH_0t_N} \quad (6.2)$$

where $H^s \equiv e^{-iH_0s}He^{iH_0s}$.

We substitute $H \to H_\lambda^{(n)} \equiv H_0 + \lambda V_{\varepsilon_n}$ in (6.2), and take the limit $n \to \infty$. This can be done because strong convergence is guaranteed by Lemma 1 and there are only a finite number of t_i's involved. One obtains an identity of the form (6.2), with the substitution

$$e^{-iH^{\sigma}\cdot t} \to U_\lambda^\sigma(t)$$

where $U_\lambda^\sigma(t) \equiv e^{iH_0\sigma}U_\lambda(t)e^{iH_0\sigma}$.

Iterating (6.2), and taking the average one has

$$E(A_\lambda^{(1)}(t_1)\dots A_\lambda^{(N)}(t_n))$$
$$=E\{e^{-iH_0t_N}U_\lambda^{t_1-t_N}(t_N)e^{-iH_0(t_{N-1}-t_N)}U_\lambda^{t_1-t_{N-1}}(t_{N-1}-t_N)$$
$$e^{-iH_0(t_{N-2}-t_{N-1})}U_\lambda^{t_1-t_{N-2}}(t_{N-2}-t_{N-1})$$
$$\dots e^{-iH_0(t_1-t_2)}U_\lambda(t_1-t_2)A^{(1)}U_\lambda(t_2-t_1)e^{iH_0(t_1-t_2)}A^{(2)}$$
$$\dots A^{(N)}U_\lambda^{t_1-t_N}(-t_N)e^{iH_0t_N}\} \quad (6.3)$$

Recall now that, from Theorem 2, uniformly in ρ, v and uniformly in σ over compacts one has, in norm convergence

$$\lim_{\lambda\to 0} E(e^{-iH_0v/\lambda^2}U_\lambda^\rho(\sigma/\lambda^2)AU_\lambda^\rho(-\sigma/\lambda^2)e^{iH_0v/\lambda^2}) = T_\sigma \cdot A \quad (6.4)$$

In fact, the l. h. side of (6.4) is independent of ρ, v due to the stationary character of the random field V.

It follows then that, setting $t_K \equiv \tau_K \cdot \lambda^{-2}$,

$$\lim_{\lambda\to 0} E\{e^{-iH_0t_N}U_\lambda^{t_1-t_N}(t_N)E\{e^{-iH_0(t_1-t_N)}U_\lambda^{t_1-t_{N-1}}(t_{N-1}-t_N)$$
$$E\{\dots E\{e^{iH_0(t_2-t_1)}U_\lambda(t_1-t_2)A^{(1)}U_\lambda(t_2-t_1)e^{-iH_0(t_2-t_1)}\}A^{(2)}$$
$$\cdot U_\lambda^{t_1-t_2}(t_3-t_2)e^{-iH_0(t_2-t_3)}\}A^{(3)}\dots\}A^{(N)}U_\lambda^{t_1-t_N}(-t_N)e^{iH_0t_N}\}$$
$$=T_{\tau_N}\cdot A^{(N)}\dots T_{\tau_1-\tau_2}A^{(1)} \quad (6.5)$$

To prove Theorem 3 we must then prove that the difference between the left-hand side of (6.5) and the right-hand side of (6.3) vanishes, when $\lambda \to 0$, in the norm of $\mathscr{A}$, uniformly over compact sets in the τ_i's.

Once again, we use Dyson's expansion for $e^{-iH_0t}U_\lambda(t)$ and a representation in terms of diagrams.

It is not difficult to verify that the r. h. side of (6.3) can be written in the form

$$\sum_{j=1}^{N}\sum_{n_j=0}^{\infty}(-i)^{\sum_{j=1}^{N} n_j}\lambda^{2\sum_{j=1}^{N} n_j}\int_{I_{\{\tau_1\ldots\tau_N\}}}\prod_{j,K_j} dt_{K_j}^{(j)} E\{[V(t_1)^{(1)}, \ldots [V(t_{n_1}^{(1)}), [V(t_1^{(2)}),$$
$$\ldots [V(t_{n_N}^{(N)}), A^{(N)}]_{n_N} \ldots]A^{(N-1)} \ldots], A^{(2)}]_{n_2} \ldots], A^{(1)}]_{n_1} \ldots]\} \quad (6.6)$$

where $I_{\{\tau_1\ldots\tau_N\}}$ is defined by

$$\lambda^{-2}\tau_j \geqslant t_1^{(j)} \geqslant \ldots \geqslant t_K^{(j)} \geqslant t_{K+1}^{(j)} \geqslant \ldots \geqslant \lambda^{-2}\tau_{j+i} \qquad j = 1 \ldots N-1$$
$$\lambda^{-2}\tau_N \geqslant \ldots \geqslant t_K^{(N)} \geqslant t_{K+1}^{(N)} \geqslant \ldots \geqslant 0$$

In deriving (5.6) one has used the fact that, by definition,

$$e^{iH_0 s}V(s_1)e^{-iH_0 s} = V(s_1 - s).$$

Expansion (5.6) admits a diagramatic analysis, using the rules of Gaussian integration, much in the same way as described in § 4.

The difference between the l. h. side of (5.5) and the r. h. side of (5.3) can then be estimated, using a formula analogous to (4.31), in terms of the contribution of tight graphs. Indeed, (5.6) differs from (4.17) only because of the larger number of $A^{(i)}$ present, as can be seen by relabelling the times $t_{K_j}^{(j)}$. Conversely, (4.17) can be regarded as particular case of (5.6), obtained setting $A^{(i)} = I$, $i = 2 \ldots N$, $A^{(1)} = A$. (Strictly speaking, this is not correct, since $\mathscr{A}$ does not contain the identity. However, there is no difficulty in extending the results of § 4, 5 and of the present § to cover the case $A \in \underset{\sim}{\mathscr{A}}$, where $\underset{\sim}{\mathscr{A}}$ is the smallest C* algebra which contains the identity and $\mathscr{A}$.)

In view of Lemma 5′, the presence of some extra factors $A^{(i)}$ in (4.7) causes only very minor modifications in the estimates needed. In particular, the analysis of § 5 in terms of rescattering-free graphs and of their ornaments carries over without modifications to the present context. In particular notice that, due to the form of (6.6) and to the structure of the set $I_{\{\tau_1\ldots\tau_N\}}$, no symbol $*$ corresponding to one of the $A^{(K)}$ can be shadowed by a rescattering link.

Since there are only very minor but tedious modifications to the analysis performed in § 4 and § 5, we omit here the details. □

Remark. — The same result would have been obtained if one had studied the limit, when $\lambda \to 0$, of

$$E(A_\lambda^{(K)}(\tau_N/\lambda^2) \ldots A_\lambda^{(1)}(\tau_1/\lambda^2) \quad (6.7)$$

when $\tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_N$.

Indeed, (6.7) is the adjoint of the l. h. side of (6.1) if the $A^{(i)}$ are self-adjoint, as an element of $\mathscr{A}$, while $T_{\tau_N} \cdot A^{(N)} \ldots T_{\tau_1-\tau_2}A^{(1)}$ is self-adjoint.

It is perhaps interesting to remark also that the limit would have been different, and in general not connected with a Markov semi-group, if one had considered (6.7) with an ordering of $\tau_1 \dots \tau_N$ which is neither chronological nor anti-chronological.

Technically, this comes about because in this case there would be an overlap between the ranges of the $t_k^{(j)}$ in (6.6) for different values of j.

This fact allows for a larger number of rescattering graphs, and also for the fact that some of the $*$ marks could be now shadowed by a rescattering arc. All these terms are more difficult to control.

Of course, more detailed assumptions about the $A^{(i)}$ could improve the situation, as can be seen from the following very trivial consideration: if the $A^{(i)}$ which violate chronological (or anti-chronological) order are replaced by I (the unit element in $\mathscr{A}_0$), then Theorem 3 holds.

More generally, one can see that a necessary condition on $A^{(i)}$, if τ_i is neither the successor nor the precessor of τ_{i+1}, e. g. if $\tau_i > \tau_{i+1}$, $\tau_i > \tau_{i-1}$, is that $T_{\tau_{i-1}-\tau_i} \cdot A^{(i)}$ be well defined (notice that here $\tau_{i-1} - \tau_i < 0$).

We shall not pursue further here this point.

Before closing this section we want to remark briefly on the assumptions made and on the method of proof.

A) The assumption that V be Gaussian is not crucial and has been made only to keep the level of formal complexity within reasonable limits.

Our results, i. e. Theorems 1, 2 and 3 are valid under the following much weaker (and perhaps natural) conditions: Every correlation function

$$W(x_1 \dots x_n) \equiv E(V(x_1) \dots V(x_n))$$

can be written in the form

$$W(x_1 \dots x_n) = \sum_{\mathscr{I}} U^{(n_1)}(x_1^{(1)} \dots x_{n_1}^{(1)}) \dots U^{(n_K)}(x_1^{(K)} \dots x_{n_K}^{(K)}) \quad (6.8)$$

when $\sum_{s=1}^{K} n_s = n$, $\mathscr{I}$ is the collection of all partitions of the set $\{ x_1 \dots x_n \}$ in subsets, the « connected components » $U^{(p)}$ are translation invariant, i. e. $U^{(m)}(x \dots x_m) = U^{(m)}(x_1 - x_2, \dots, x_{m-1} - x_m)$, and moreover

$$U^{(p)}(\xi_1 \dots \xi_p) \in L^1(\mathbb{R}^{3p}), \ \tilde{U}^{(p)}(K_1 \dots K_p) \in L^1(\mathbb{R}^{3p}) \quad (6.9)$$

The functions $U^{(p)}$ are called often Ursell functions in statistical mechanics, and (6.8) is referred to as Ursell expansion of the correlation W.

The $U^{(p)}$ are obtained from the W using recursively (6.8); they exist and have property (6.9) under very mild assumptions on the random field V, and reflect a suitable « mixing » property. We still assume $E(V(x))=0$, so that $U^2(x-y) = E(V(x)V(y))$. At the next step one obtains, e. g.

$$U^{(3)}(x_1 - x_2, x_2 - x_3, x_3 - x_4) = W(x_1 \dots x_4) - W(x_1, x_2)W(x_3 x_4) - W(x_1 x_3)W(x_2 x_4) - W(x_1 x_4)W(x_2 x_3)$$

The Gaussian random field is characterized by $U^{(p)} = 0$ for $p > 2$.

The analysis of § 5 can be repeated using (6.8) instead of the rules of Gaussian integration (which of course coincide with (6.8) when V is Gaussian).

One will now have a larger number of graphs; an end point can belong to more than one link. For example, one can represent $E_c(V(t_1) \dots V(t_4))$ with the graph

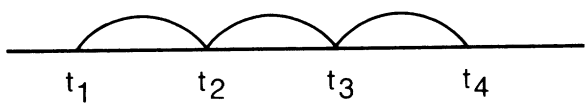

FIG. 3.

The definition of tight, rescattering-free, rescattering additions, ornaments is the same as before.

Due to (6.9), there will be in general fewer δ-functions in expression such as (4.18), and therefore the matrix Q in (4.22) will be of order $\geqslant n$, and one will have some $\tilde{U}$ instead of some of the factors $\tilde{\mathcal{G}}$.

Det Q will then be a polynomial in the t_i's of order $\geqslant n$. The scaling done in (5.3) will produce now, for the graph of order n, a factor $\lambda^{\frac{1}{8}(n+K)}$, $K \geqslant 0$ (depending on the graph); the estimates on the integral still hold, with an extra factor $\mathbb{C}_2^K$.

Estimating, as done in § 5, the number of graphs of order n one proves then that Th. 2 (and Th. 3) hold if there is a constant $a > 0$ such that

$$\| U^{(p)} \| < a^p \| \mathcal{G} \|^{p/2} \tag{6.10}$$

where $\| F \| = \sup \{ \| F |_1, \| \tilde{F} \|_1 \}$ and $\mathcal{G}(x-y) = U^{(2)}(x-y) = E(V(x)V(y))$.

From the proof of Theorem 1 it is not difficult to see that also Th. 1 holds under condition (6.10), and in fact under the weaker condition $\| U^{(p)} \|_1 < a^p \| \mathcal{G} \|_1^{p/2}$.

We shall not detail here the simple but very tedious proof.

B) The assumption that V admits moments of all orders, so that $W(x_1 \dots x_N)$ is defined for all $N \in Z^+$, is most probably an artefact of our method of proof, based on estimates on the full Dyson expansion.

From (4.32) one feels that it should be possible to provide *a priori* estimates, of the martingale type, using some form of Gronwall's inequality and the mixing properties of the random field V, without assuming the existence of momenta of all orders. This is indeed what can be done in the case of a classical particle in the random potential V, because there one has, for $\lambda > 0$, a genuine Markov process in $(\underline{x}, \underline{v})$.

We have been unable to find such martingale-type estimates, although the results of § 5 strongly suggest their existence. It is very plausible that steps in this direction could be taken using the description of the evolution of quantum observables through Poisson point processes advocated e. g. in [8]. We shall not pursue here this point.

7. CONVERGENCE TO THE CLASSICAL LIMIT

In [3], it is shown that, under rather mild conditions, the momentum process of a classical particle in a random force field converges, in the Van Hove limit, to a diffusion process. If the force field is conservative, this diffusion is reduced by each sphere $S_a^2 \subset R^3$, $a > 0$ (here

$$S_a^3 \equiv \{ \underline{p} \in \mathbb{R}^3 \mid |\underline{p}| = a \}).$$

It is therefore natural to inquire whether there is a limit, when $\hbar \to 0$, of the Markov process described in § 3, and in particular whether this limit is precisely the diffusion associated to the classical case.

In this § we shall give a positive answer to both questions.

We begin by noticing that, when $\hbar \neq 1$, the natural representation of $\mathscr{A}$ by $\mathbb{C}(\mathbb{R}^3)$ is through the Fourier representation of $L^2(\mathbb{R}^3)$ given by

$$L^2(\mathbb{R}^3) \ni \varphi \to \tilde{\varphi} \equiv (\hbar)^{-3/2} \int \exp\left(\frac{i\underline{p} \cdot \underline{x}}{\hbar}\right) \cdot \varphi(x) d^3x \qquad (7.1)$$

This comes from the fact that one is interested in a representation in which the observable $\mathbb{P}_i$ (the i$^{\text{th}}$ component of the momentum) is represented by multiplication by the coordinate function p_i.

The map (7.1) has this property, since

$$\mathbb{P}_K = -i\frac{\partial}{\partial x_K} \qquad K = 1, 2, 3$$

The evolution of the observable A is given by

$$A \mapsto A(t) \equiv \exp\left(\frac{itH}{\hbar}\right) A \exp\left(\frac{-itH}{\hbar}\right)$$

where $H = -\hbar^2\Delta + \lambda V$.

Taking these explicit dependences on $\hbar$ into account, one obtains, for the limiting semi-group studied in § 3, the expression

$$T_{t;\hbar} = \exp(t\mathscr{L}_\hbar)$$

where

$$(\mathscr{L}_\hbar A)(p) = \hbar^{-2}\int_{-\infty}^{\infty} dt \int d\overset{3}{x} \int d\overset{3}{y} \int d\overset{3}{K} \int d^3p' \mathscr{G}(x-y) \exp\left\{ i\left(\frac{p+p'}{\hbar} - K\right)(x-y) + i\left(\frac{p-p'}{\hbar}\right)(x+y) + \frac{it}{\hbar}(p^2 - \hbar^2 K^2)\right\}(A(p') - A(p)) \quad (7.2)$$

Setting $l = K - \hbar^{-1}p$ one has

$$(\mathscr{L}_\hbar A)(p) = \frac{1}{2}\hbar^{-2}\int d^3l \tilde{\mathscr{G}}(l)\delta\left(\underline{l}\cdot\underline{p} + \frac{\hbar l^2}{2}\right)(A(p + \hbar l) - A(p)) \quad (7.3)$$

Since $\mathscr{L}_\hbar$ defines a Markov semi-group, and therefore a closed Dirichlet form, it is natural to seek convergence when $\hbar \to 0$ in the sense of Dirichlet forms. This will also imply convergence of semi-groups. For a direct proof of strong semi-group convergence, see e. g. [*10*].

The Dirichlet form associated to $\mathscr{L}$ is

$$\varepsilon_\hbar(f, g) \equiv \int \bar{f}(p)(\mathscr{L}_\hbar g)(p)d^3p \quad (7.4)$$

with domain

$$D_\hbar = \{ f \mid f \in L^2(\mathbb{R}^3),\ \varepsilon_\hbar(f, f) < \infty \}$$

We now remark that $\tilde{\mathscr{G}}(l)$ is symmetric (and real) since

$$E(V(x)V(y)) = E(V(y)V(x)) = \overline{E(V(x)V(y))}\,.$$

Therefore (7.4) can be written

$$\varepsilon_\hbar(f, g) = -\frac{1}{4\hbar^2}\int d^3p \int d^3l (f(p + \hbar l) - f(p))(g(p + \hbar l) - g(p)) \cdot \tilde{\mathscr{G}}(l)\delta\left(\underline{l}\cdot\underline{p} + \frac{\hbar}{2}l^2\right) \quad (7.5)$$

In view of the fact that $\tilde{\mathscr{G}} \in L_1$, and under a further assumption on $\mathscr{G}$ which guarantees among other things the existence of the integral (7.7) below, we expect that $\varepsilon_\hbar$ converge, in the sense of forms, when $\hbar \to 0$, to

$$\varepsilon_0(f\,;g) = -\frac{1}{4}\int \sum_{i,j=1}^{3} A_{ij}(p)\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_i} \quad (7.6)$$

where

$$A_{ij}(p) = \frac{1}{4}\int \tilde{\mathscr{G}}(l) l_i l_j \delta(\underline{p}\cdot\underline{l})d^3l \quad (7.7)$$

The (closed) Dirichlet form ε_0 is associated to a diffusion process with generator $\mathscr{L}$ defined by

$$(\mathscr{L}f)(p) = \sum_{ij} \left(\frac{\partial}{\partial p_i} A_{ij}(p) \frac{\partial}{\partial p_j} \right) f(p) \tag{7.8}$$

which is precisely the generators of the diffusion one obtain in the Van Hove limit for the momentum process of a classical particle in the random force field which has $-$ V as potential.

The formal convergence of $\varepsilon_\hbar$ to ε is obvious.

To give an actual proof of convergence, notice that the Dirichlet form ε_0 is reduced by each sphere $S^3_{\mathbb{C}}$, $\mathbb{C} > 0$, and defines there a diffusion semigroup which has as generator the formal reduction of $\mathscr{L}$ to $S^3_{\mathbb{C}}$.

Also $\varepsilon_\hbar$ is reduced by each $S^3_{\mathbb{C}}$; denote $\varepsilon_\hbar^{\mathbb{C}}$ such reduction.

For all smooth functions f, g on $S^3_{\mathbb{C}}$ one has then

$$\varepsilon_\hbar^{\mathbb{C}}(f, g) = -\frac{1}{4}\hbar^{-5}\int_{S^3_c} d\hat{p} \int_{S^3_c} d\hat{p}(f(\hat{p}') - f(\hat{p}))(g(\hat{p}') - g(\hat{p}))\tilde{\mathscr{G}}\left(\frac{\hat{p}' - \hat{p}}{\hbar}\right)$$

where $\hat{p}$, $\hat{p}'$ are vectors of lenght C.

For $\hat{p} \in S^3_c$, let the set $\mathscr{M}^c_{p,\hbar}$ be defined by

$$\mathscr{M}^c_{p,\hbar} \equiv \{ l \in \mathrm{R}^3 \mid \hat{p} + \hbar \underline{l} \in S^3_c \}$$

Then

$$\varepsilon_\hbar^{\mathbb{C}}(f, g) = -\frac{1}{4}\int_{S^3_c} d\hat{p} \int_{\mathscr{M}^c_{p,\hbar}} dl \frac{f(\hat{p}+\hbar l) - f(\hat{p})}{|\underline{l}|} \frac{g(\hat{p}+\hbar l) - g(\hat{p})}{|\underline{l}|} |\underline{l}|^2 \tilde{\mathscr{G}}(l) \tag{7.9}$$

Notice that, uniformly in compacts of $\mathbb{R}^3$, the manifold $\mathscr{M}^c_{p,\hbar}$ tends, as $\hbar \to 0$, to the manifold $\mathscr{N}_p$ defined by

$$\mathscr{N}_p \equiv \{ l \in \mathbb{R}^3 \mid \underline{l} \cdot \underline{p} = 0 \}$$

Let f be a differentiable function on S^3_c. Then, for $p, p' \in S^3_c$,

$$\frac{|f(p) - f(p')|}{|p - p'|} < \gamma_f,$$

with $\gamma_f > 0$.

Assume further that there exists a function $\mathscr{G}_0(\rho)$, $\rho \geqslant 0$, such that

$$\mathscr{G}_0(|\underline{l}|) \geqslant \mathscr{G}(\underline{l}) \qquad \forall \underline{l} \in \mathbb{R}^3 \tag{7.10}$$

and such that

$$\mathscr{G}_0 \in L_1(\mathbb{R}^3) \tag{7.11}$$

(this is of course the case if $\tilde{\mathscr{G}}(l)$ is rotationally-invariant). Then $\varepsilon^c_\hbar(f, g) < K\gamma_f\gamma_g$ uniformly in $\hbar \geqslant 0$.

Moreover

$$\varepsilon^c_\hbar(f, g) \to \varepsilon^c_0(f, g) \tag{7.12}$$

where

$$\varepsilon_0^c(f, g) = -\frac{1}{4C^3}\int_{S_1^3} d\hat{p} \int d^3l(\nabla f_c \cdot \hat{l})(\nabla g_c \cdot \hat{l}) \mid l \mid^2 \tilde{\mathscr{G}}(l)\delta(l \cdot p) \qquad (7.14)$$

with $f_c(\hat{\mathrm{K}}) \equiv f(c\hat{\mathrm{K}})$ for $\hat{k} \in \mathrm{S}_1^3$.

Since $\mathbb{C}^1$ functions are a core for ε_0^c, it follows that $\mathscr{L}$ converges to $\mathscr{L}_0$ in strong resolvent sense, and therefore $\mathrm{T}_{t,\hbar} \to \exp \mathscr{L}_0 t$ strongly.

We summarize these results in

THEOREM 4. — Under the assumptions (7.10) and (7.11) the Markov semi-group described in § 3 converges strongly, when $\hbar \to 0$, to the diffusion semi-group which describes the Van Hove limit for a classical particle.

Proof. — We have already proved strong convergence on each sphere S_c^3; convergence is in fact uniform, as easily checked, for c over compacts in $(0, \infty)$.

Consider in $\mathrm{L}^2(\mathrm{R}^3)$ the set of functions of the form

$$f(\underline{p}) = f_1(\mid p \mid)f_2(\hat{p}), \; f_i \in \mathbb{C}^1 \qquad (7.14)$$

where f_1 has compact support in $(0, \infty)$.

Since all semi-groups involved are reduced by each sphere S_c^3, and convergence holds strongly on each sphere, uniformly for c over compact of $(0, \infty)$, it follows that strong convergence holds for all functions of the form (7.14).

We now remark that $\mathscr{L}_0$ is non-positive, and so are $\mathscr{L}_\hbar$.

Since the set (7.14) is dense in $\mathrm{L}^2(\mathbb{R}^3)$, strong convergence of $\mathrm{T}_{t,\hbar}$ to $\exp \mathscr{L}_0 t$ on $\mathrm{L}^2(\mathbb{R}^3)$ follows. □

Remark. — One should notice that we have taken the limits in the following order: first the Von Hove limit $\lambda \to 0$, then the classical limit $\hbar \to 0$.

In some sense, Theorem 4 shows that one obtains the same result by taking the limits in the reverse order. But it should be observed that in this case the classical limit $\hbar \to 0$ is taken in a somewhat different sense, e. g. using coherent states, and not directly on extensive observables.

It is natural to inquire about limits along other directions in the λ, $\hbar$ plane; we have so far no definite results on this problem.

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