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*Physique théorique.*

# On the relationship between the reversibility of dynamics and balance conditions

by

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ABSTRACT. — The relationship between the reversibility of dynamics and balance condition is studied. In particular we show that the semigroup dynamics with Detailed Balance condition is one step closer to reversible dynamics than the semigroup dynamics alone (theorem 2). Further (theorem 3) we prove that the description of dynamics, given in terms of GNS Hilbert space, has direct physical interpretation only for reversible time evolution. Models and some conclusions are given. The work is done in the algebraic formulation of statistical mechanics.

RÉSUMÉ. — On étudie la relation entre la réversibilité de la dynamique et la condition de bilan détaillé. On montre en particulier que la dynamique définie par un semi-groupe avec la condition de bilan détaillé est plus proche de la dynamique réversible que la dynamique définie seulement par un semi-groupe (Théorème 2). En outre, on prouve (Théorème 3) que la description de la dynamique en termes de l'espace de Hilbert provenant de la construction de GNS a une interprétation physique directe seulement pour une évolution temporelle réversible. Le problème est traité dans la formulation algébrique de la Mécanique Statistique.

## 0. INTRODUCTION

The principle of detailed balance is a property of the microscopic dynamics of the systems. Well understood in a classical context, its formulation for the quantum mechanical regime has recently been accomplished [1] [2] [3].

On the other hand there is a suggestion (see [4] [5]) that the principle of detailed balance is a link between equilibrium systems and certain non-equilibrium systems allowing the extension of certain equilibrium techniques to steady states far from equilibrium. Following this thought we wish to investigate the consequences of the assumed detailed balance condition. The paper is organized in the following way:

Section 1 gives preliminaries. Models explaining our ideas are described in Section 2. In order to clarify the nature of the assumed balance condition, the question of reversibility of dynamics is studied in Section 3. Also, as a by-product, some properties of the induced dynamics are established. Finally, in the last section, we show that the modular dynamics, i. e. the dynamics of KMS states, exhibits the detailed balance condition.

## 1. PRELIMINARIES

Let $\mathscr{A}$ be a W*-algebra with the faithful normal state $\omega$ and let $(\mathscr{H}, \Pi, \Omega)$ be the GNS construction associated with $\omega$. A dynamical semigroup of $\mathscr{A}$ is a weakly continuous map $\tau : [0, \infty) \to \mathscr{L}(\mathscr{A})$ into the bounded positive unital maps on $\mathscr{A}$ such that

$$(1) \qquad \begin{aligned} \tau_t \tau_s &= \tau_{t+s} \qquad t, s \in [0, \infty) \\ \tau_0 &= id_{\mathscr{A}} \end{aligned}$$

Assume that $\omega$ is $\tau_t$-invariant, i. e. $\omega \circ \tau_t = \omega$. Then, the dynamical semigroup $\tau_t$ on $\mathscr{A}$ induces another semigroup, $\hat{\tau}_t$, on the Hilbert space $\mathscr{H}$. $\hat{\tau}_t$ has the following properties: $\hat{\tau}_t$ is a weakly continuous one parameter semigroup, uniformly bounded with respect to the time $t$ and $\hat{\tau}_t\Omega = \Omega$, $t \geqslant 0$. In the sequel, $\hat{\tau}_t$ will be called a dynamical semigroup on the Hilbert space $\mathscr{H}$.

As $\Omega$ is a cyclic and separating vector for $\Pi(\mathscr{A})$, there exists a modular operator $\Delta$ and also a modular conjugation $j_m$ associated to the pair $(\Pi(\mathscr{A}), \Omega)$ by the Tomita-Takesaki theory. The natural positive cone in $\mathscr{H}$ is denoted by P.

By $\sigma$ we will denote a reversing operation of $\mathscr{A}$ which is an antilinear map $\sigma : \mathscr{A} \to \mathscr{A}$ such that

$$(2) \qquad \sigma^2 = id_{\mathscr{A}}$$

$$(3) \qquad \sigma(A^*) = \sigma(A)^* \qquad A \in \mathscr{A}$$

$$(4) \qquad \sigma(AB + BA) = \sigma(A)\sigma(B) + \sigma(B)\sigma(A) \qquad A, B \in \mathscr{A}$$

Let $\mathscr{J}$ be a conjugation on $\mathscr{H}$, i. e. an antilinear map $\mathscr{J} : \mathscr{H} \to \mathscr{H}$

with $\mathbb{1} = \mathscr{J}^2 = \mathscr{J}^*\mathscr{J}$. A densely defined linear operator A on $\mathscr{H}$ is called $\mathscr{J}$-self-adjoint with respect to conjugation $\mathscr{J}$ whenever

(5) $$\mathscr{J}A\mathscr{J} = A^*$$

(see [6], section 22, p. 76).

The following definition was introduced in [3]:

DÉFINITION 1. — *Let $\{ \tau_t ; t \geqslant 0 \}$ and $\sigma$ be a dynamical semigroup and a reversing operation on $\mathscr{A}$ respectively. A normal faithful state $\omega$ of $\mathscr{A}$ satisfies the detailed balance condition with respect to $\{ \tau_t \}$ and $\sigma$ whenever*

(6) $$\omega \circ \tau_t = \omega \qquad t \geqslant 0$$

(7) $$\omega(A^*\tau_t(B)) = \omega(\sigma(B^*)\tau_t \circ \sigma(A)) \qquad A, B \in \mathscr{A} ; t \geqslant 0$$

(8) $$\omega(\sigma(AB)) = \omega(\sigma(A)\sigma(B)) \qquad A, B \in \mathscr{A} .$$

Moreover in the above cited paper the following result was proved:

THEOREM 1. — *Let $\omega$ be a faithful state of a W*-algebra $\mathscr{A}$ and let $(\mathscr{H}, \Pi, \Omega)$ be the GNS construction associated to $\omega$. There is a bijection between:*

*i) the pairs $(\{ \tau_t \}, \sigma)$ such that $\omega$ satisfies the detailed balance condition with respect to $\{ \tau_t \}$ and $\sigma$ and where $\{ \tau_t \}$ is a dynamical semigroup of $\mathscr{A}$ and $\sigma$ is a reversing operation of $\mathscr{A}$.*

*ii) the pairs $(\{ \hat{\tau}_t \}, \mathscr{J})$ of a dynamical semigroup $\{ \hat{\tau}_t \}$ on $\mathscr{H}$ and a conjugation $\mathscr{J}$ of $\mathscr{H}$, such that:*

(9) $$\hat{\tau}_t\Omega = \Omega \qquad t \geqslant 0$$

(10) $$\hat{\tau}_t P \subseteq P$$

(11) $$\hat{\tau}_t \quad \text{strongly commutes with } \Delta \ (^1)$$

(12) $$\mathscr{J}\Omega = \Omega$$

(13) $$\mathscr{J}\Pi(\mathscr{A}^+)\Omega \subseteq \Pi(\mathscr{A}^+)\Omega$$
$$(\mathscr{A}^+ = \{ A \in \mathscr{A} ; A \geqslant 0 \})$$

(14) $$\hat{\tau}_t^* = \mathscr{J}\hat{\tau}_t\mathscr{J} \qquad t \geqslant 0 .$$

*Specifically $\hat{\tau}_t$ and $\mathscr{J}$ are given in terms of $\tau_t$ and $\sigma$ by :*

(15) $$\hat{\tau}_t\Pi(A)\Omega = \Pi(\tau_t(A))\Omega \qquad A \in \mathscr{A}, t \geqslant 0 .$$

(16) $$\mathscr{J}\Pi(A)\Omega = \Pi(\sigma(A))\Omega \qquad A \in \mathscr{A} .$$

*Remark.* — To avoid confusion we emphasize that $\mathscr{J}$ satisfies (13), therefore in general the conjugation $\mathscr{J}$ is not the modular conjugation $j_m$.

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(1) i. e. commutes with all spectral projectors of $\Delta$ (equivalently with all $e^{i\lambda\Delta}$ $\lambda \in \mathbb{R}$, or with all bounded functions of $\Delta$).

## 2. MODELS

Take $\langle \mathcal{H}, (.,.) \rangle$ to be a complex Hilbert space (for all details of algebraic description of Quantum Statistical Mechanics we recommend the book of O. Bratteli, D. W. Robinson [7]).

Denote by $\{ W(f) ; f \in \mathcal{H} \}$ the family of Weyl operators. In particular one has:

$$(17) \qquad \begin{aligned} W(f)W(g) &= \exp \{ -i/2 \, \mathrm{im}\,(f, g) \} W(f+g) \\ W(f)^* &= W(-f) \end{aligned}$$

The C*-algebra generated by the Weyl operators is called CCR-algebra. Describe the one-particle time evolution by the one-parameter unitary group $U_t = e^{iHt}$ on $\mathcal{H}$ (H-denotes one-particle hamiltonian). Define $\sigma'(W(f)) = W(Kf)$ where K is a time reversing operator on $\mathcal{H}$, i. e. K is an antiunitary operator on $\mathcal{H}$ such that $KHK^* = H$. Then there is an antilinear Jordan automorphism $\sigma$ such that $\sigma(W(f)) = \sigma'(W(f))$. In the sequel, $\sigma$ will be taken as the reversing operation. Now, for simplicity, let us put some restrictions.

Take $\mathcal{H} = \mathbb{C}$ ($\mathbb{C}$ denotes the complex numbers, so we work in one dimension). Then, one can take the one-particle evolution as $z \mapsto z_t = e^{i\omega_0 t}$ for some $\omega_0 > 0$.

The time reversing operation is defined as:

$$\sigma(W(e^{i\omega_0 t}z)) = W(e^{-i\omega_0 t}\bar{z}) = W(\bar{z}_t)$$

($\bar{z}$ denotes the complex conjugate of $z$). Consider the Weyl operators $W(f)$ as unitary operators on the Hilbert space $\mathcal{K}$ and take the weak closure of CCR-algebra, $\mathcal{M}$. Then each normal state $\nu$ on $\mathcal{M}$ has the following form:

$$\nu(A) = \mathrm{Tr}\, \rho_\nu A \qquad A \in \mathcal{M}$$

where $\rho_\nu$-denotes the corresponding density matrix. In particular $\omega(A) = \mathrm{Tr}\, \rho A$ with the density matrix strictly positive where for simplicity of the notation we put $\rho_\omega = \rho$. We now introduce a different time evolution $\tau_t$ which we will interpret as describing the diffusion of a quantum particle (see [8]). Define

$$(18) \qquad \tau_t(W(z)) = W(e^{-\lambda t}z) \exp \{ a \, | z |^2 (1 - e^{-2\lambda t}) \}$$

where $\lambda$, $a$ are fixed positive constants and $t \geqslant 0$. One can show that the above equality gives a well-defined dynamical semigroup.

Put $\phi(z) = \mathrm{Tr}\, \rho W(z)$. Then the detailed balance condition is equivalent to the following formula for the generating functional $\phi(z)$:

$$(19) \qquad \phi(e^{-\lambda t}\bar{z})\phi(z')\phi(-z+e^{-\lambda t}z') = \phi(\bar{z})\phi(e^{-\lambda t}z')\phi(-\bar{z}'+e^{-\lambda t}\bar{z})$$

for $z, z' \in \mathbb{C}$.

Note that the Ansatz $\phi(z) = \exp\{ - c\,|\,z\,|^2 \}$, $c$ an arbitrary positive constant, solves the equality (19). Denote by $\nu_c$ the corresponding states. On the other hand, the reasonable condidate for the KMS equilibrium state with respect to the evolution $z \mapsto z_t$ has the following generating functional (compare [7], vol. II, § 5.2.5)

$$\phi_\beta(W(z)) = \exp\left\{ - \frac{1}{4} Q\,|\,z\,|^2 \right\} \tag{20}$$

where $Q = \coth\left(\frac{1}{2}\beta\omega_0\right)$ and $\beta$ denotes the inverse temperature. Note the following inequality: $Q \geqslant 1$.

Therefore, even for such a simple model, there are dynamics $\tau_t$ and states $\nu_c$ such that the detailed balance condition is satisfies for $\nu_c$ with respect to $\tau_t$ and the states $\nu_c$ are not always KMS states (in general the constant $c$ does not depend on $\beta$). Makinḡ a slight generalization in the above example (e. g. put $\mathscr{K} = \mathbb{C}^2$) it is easy to get other non-trivial hamiltonian and semigroup dynamics for which the detailed balance is satisfied.

## 3. PROPERTIES OF INDUCED DYNAMICS

Quantum Mechanics offers two equivalent descriptions of dynamics, one describes the motion of observables, i. e. operators on the Hilbert space (Heisenberg picture), the second one describes the evolution of states, i. e. vectors of Hilbert space (Schrödinger picture). At first sight, there is a similar scheme for the semigroup dynamics on algebras if one assumes in addition the detailed balance condition (cf. theorem 1). On the other hand, examination of the relationship between semigroups $\tau_t$ and $\hat{\tau}_t$ implies that $\hat{\tau}_t$ has the standard properties of evolution of vectors on the Hilbert space $\mathscr{H}$, i. e. $\|\,\hat{\tau}_t\xi\,\| = \|\,\xi\,\|$, only if the dynamics $\hat{\tau}_t$ is a reversible one (cf. theorem 3).

But in order to make clear the nature of the detailed balance, let us start with the following definition and its consequences.

DEFINITION 2. — (*The strong balance condition*). *Let $\mathscr{A}$ be a von Neumann algebra, with $\{ \tau_t \}_{t\geq 0}$ a dynamical semigroup of $\mathscr{A}$, and $\sigma$ a reversing operation of $\mathscr{A}$.*

*A normal faithful state $\omega$ of $\mathscr{A}$ satisfies the strong balance condition with respect to $\{ \tau_t \}$ and $\sigma$ whenever*

$$\omega \circ \tau_t = \omega \qquad t \geqslant 0 \tag{21}$$

$$\omega(A^*\tau_t(B^*)C) = \omega(\tau_t \circ \sigma(C^*)\sigma(B)\tau_t \circ \sigma(A)) \qquad A, B, C \in \mathscr{A}, \quad t \geqslant 0 \tag{22}$$

$$\omega(\sigma(ABC)) = \omega(\sigma(A)\sigma(B)\sigma(C)) \qquad A, B, C \in \mathscr{A} \tag{23}$$

THEOREM 2. — *Let $\mathcal{A}$, $\{ \tau_t \}$, $\sigma$ and $\omega$ be as in the Definition 2, i. e. $\omega$ satisfies the strong balance condition with respect to $\{ \tau_t \}$ and $\sigma$. Then:*

*i) $\omega$ satisfies the detailed balance condition with respect to $\{ \tau_t \}$ and $\sigma$.*

*ii) $\{ \tau_t \}$ can be extended to a one parameter group of automorphisms of $\mathcal{A}$.*

*Proof.* — (*i*) Making $C = \mathbb{1}$ in (22) and (23) and taking into account the definition 1, the detailed balance condition follows.

*ii*) Let $\{ \hat{\tau}_t \}$ be the dynamical semigroup on $\mathcal{H}$ induced by $\{ \tau_t \}$ and $\mathcal{J}$ the conjugation of $\mathcal{H}$ induced by $\sigma$, as in Theorem 1. From Theorem 1 we have:

$$(24)\quad \begin{aligned}(\Pi(A)\Omega, \hat{\tau}_t\Pi(B)\hat{\tau}_t^*\Pi(C)\Omega &= (\hat{\tau}_t^*\Pi(A)\Omega, \Pi(B)\hat{\tau}_t^*\Pi(C)\Omega)\\ &= (\mathcal{J}\hat{\tau}_t\mathcal{J}\Pi(A)\Omega, \Pi(B)\mathcal{J}\hat{\tau}_t\mathcal{J}\Pi(C)\Omega)\\ &= \omega(\sigma\circ\tau_t\circ\sigma(A^*)B\sigma\circ\tau_t\circ\sigma(C))\\ &= \omega(\tau_t\circ\sigma(C^*)\sigma(B^*)\tau_t\circ\sigma(A))\\ &= \omega(A^*\tau_t(B)C) = (\Pi(A)\Omega, \Pi(\tau_t(B))\Pi(C)\Omega)\end{aligned}$$

where we used (22), first with $t = 0$ to yield the property:

$$(25)\qquad \omega(A'B'C') = \omega(\sigma(C'^*)\sigma(B'^*)\sigma(A'^*)) \qquad A'\, B', C' \in \mathcal{A}$$

and then as it stands with B replaced by B*. From this we conclude by density that:

$$(26)\qquad \hat{\tau}_t\Pi(B)\hat{\tau}_t^* = \Pi(\tau_t(B)) \qquad B \in \mathcal{A}$$

whence with $B = \mathbb{1}$, the isometric character of $\hat{\tau}_t^*$

$$(27)\qquad \hat{\tau}_t\hat{\tau}_t^* = \mathbb{1} \qquad t \geqslant 0$$

Let us denote by $\hat{\delta}_0$ the infinitesimal generator of $\hat{\tau}_t^*$. Differentiation of (27) at $t = 0$ implies that $\hat{\delta}_0$ is a skew symmetric operator, i. e.

$$(28)\qquad (\hat{\delta}_0 f, g) = -(f, \hat{\delta}_0 g)$$

for $f, g \in \mathcal{D}(\hat{\delta}_0)$([2]). Let us put $\hat{\delta} = i\hat{\delta}_0$.

Recall the following fact: If $T_t$, $t \geqslant 0$, is a weakly continuous semigroup of operators in Hilbert space with infinitesimal generator A, then $T_t^*$, $t \geqslant 0$, is a weakly continuous semigroup with infinitesimal generator A*. Using the above mentioned result and the $\mathcal{J}$-self-adjoint property of the $\hat{\tau}_t^*$-semigroup one can infer that the symmetric operator $\hat{\delta}$ is also a self-adjoint one. Therefore there is an extension of $\hat{\tau}_t^*$ to a one-parameter group of unitaries and subsequently, $\tau_t$ has an extension to a one-parameter group of automorphisms on $\mathcal{A}$.

([2]) $\mathcal{D}(\hat{\delta}_0)$ is the domain of $\hat{\delta}_0$.

Note, the conditions on dynamics expressed by means of two point correlations functions have a direct interpretation in terms of transition probabilities. This feature is very attractive for us, especially because the original Pauli detailed balance condition is given as the limitation of the transition probabilities. Restricting ourselves to the conditions on two-point correlation functions the typical results in this direction will be:

Assume the detailed balance condition for the dynamical semigroup $\tau_t$ with respect to the stationary state $\omega(A) = (\Omega, \Pi(A)\Omega)$. Moreover, let the following equality be valid:

$$\omega(\tau_t \circ \sigma(A)\tau_t \circ \sigma(A^*)) = \omega(A^*A) \tag{29}$$

Then, after simple calculations one gets: $\tau_t \circ \sigma \circ \tau_t \circ \sigma(A^*) = A^*$ for any $A \in \mathscr{A}$. So $\sigma \circ \tau_t \circ \sigma$ gives the reversal evolution with respect to the original one.

Now we want to study the consequences of the Detailed Balance Condition

THEOREM 3. — *Let $\mathscr{A}$, $\{ \tau_t \}$, $\sigma$ and $\omega$ be as in Definition 1, i. e. $\omega$ satisfies the detailed balance condition with respect to $\{ \tau_t \}$ and $\sigma$. $\{ \hat{\tau}_t \}$ denotes the dynamical semigroup on the Hilbert space $\mathscr{H}$ as in Theorem 1. Moreover let $\{ \hat{\tau}_t \}$ satisfy:*

*i) either*

$$(\hat{\tau}_t\xi, \hat{\tau}_t\xi) = (\xi, \xi) \tag{30}$$

*for all $\xi \in P$, $t \geqslant 0$*

*ii) or*

$$\tau_t(A^*A) \geqslant \tau_t(A)^*\tau_t(A) \tag{31}$$

*that is $\tau_t$ is a strongly positive semigroup, and*

$$(\hat{\tau}_t\zeta, \hat{\tau}\zeta) = (\zeta, \zeta) \tag{32}$$

*for all $\zeta \in V_0$ ($V_0$ denotes the weak closure of the set of vectors $\{ \Pi(A)\Omega ; A \in \mathscr{A}, A \geqslant 0 \}$), $t \geqslant 0$.*

*Then one can extend the semigroup $\tau_t$ to a one-parameter group of automorphisms of $\mathscr{A}$.*

*Proof.* — Assume the condition (*i*). Then

$$(\xi, (\mathbb{1} - \hat{\tau}_t^* \hat{\tau}_t)\xi) = 0 \tag{33}$$

for all $\xi \in P$.
Therefore

$$(j_m\xi', (\mathbb{1} - \hat{\tau}_t^* \hat{\tau}_t)j_m\xi) + (\xi, (\mathbb{1} - \hat{\tau}_t^* \hat{\tau}_t)\xi') = 0 \qquad \xi, \xi' \in P \tag{34}$$

where $j_m$ denotes the modular conjugation and we have used the Tomita-Takesaki theory. But then

$$j_m(\mathbb{1} - \hat{\tau}_t^* \hat{\tau}_t)j_m + (\mathbb{1} - \hat{\tau}_t^* \hat{\tau}_t) = 0 \tag{35}$$

On the other hand the argument given in the proof of lemma 1 (see [3], see also lemma 2, in [9]) implies that the modular conjugation $j_m$ commutes with $\hat{\tau}_t$. Therefore one has immediately: $\mathbb{1} = \hat{\tau}_t^* \hat{\tau}_t$, $t \geqslant 0$. The rest follows by straightforward repetition of the argument given in the end of the proof of theorem 2.

Assume (*ii*). Recall (Araki, [10]) that

$$V_0 = \Delta^{-\frac{1}{4}}\{ P \cap \mathscr{D}(\Delta^{-\frac{1}{4}})\}$$

Consequently, this result with the condition (*ii*) implies

$$(\hat{\tau}_t\Delta^{-\frac{1}{4}}\zeta,\ \hat{\tau}_t\Delta^{-\frac{1}{4}}\zeta) = (\Delta^{-\frac{1}{4}}\zeta,\ \Delta^{-\frac{1}{4}}\zeta) \tag{36}$$

for $\zeta \in P \cap \mathscr{D}(\Delta^{-\frac{1}{4}})$.

Hence we have

$$(\zeta,\ \Delta^{-\frac{1}{4}}(\mathbb{1} - \hat{\tau}_t^* \hat{\tau}_t)\Delta^{-\frac{1}{4}}\zeta) = 0 \qquad \zeta \in P \cap \mathscr{D}(\Delta^{-\frac{1}{2}}) \tag{37}$$

where we have used theorem 1 and Tomita-Takesaki theory. By repetition of the arguments given in the first part of the proof one has

$$j_m\Delta^{-\frac{1}{4}}(\mathbb{1} - \hat{\tau}_t^* \hat{\tau}_t)\Delta^{-\frac{1}{4}}j_m + \Delta^{-\frac{1}{4}}(\mathbb{1} - \hat{\tau}_t^* \hat{\tau}_t)\Delta^{-\frac{1}{4}} = 0 \tag{38}$$

Since $\tau_t$ is a strongly positive mapping, $\hat{\tau}_t$ is a contractive mapping and consequently $\mathbb{1} - \hat{\tau}_t^* \hat{\tau}_t$ is a positive operator. This implies that (38) is the sum of two positive operators.

Therefore (38) implies that $\hat{\tau}_t$ is a one parameter semigroup of isometries and the rest follows as in the first part of the proof of the theorem.

Now we want to comment on the above results. First of all, it is rather surprising that the auxiliary assumption of isometry on the cone (P or $V_0$) is such a strong condition (compare also the result of Demoen and Vanheuverzwijn [11]). Thus, is worth to point out the nature of the assumptions used in Theorem 3. Let us consider $\{ \Pi(A)\Omega,\ A \in \mathscr{A}^+ \}$. Then the fact that $(\hat{\tau}_t\zeta,\ \hat{\tau}_t\zeta) = (\zeta, \zeta)$ for $\zeta \in \{ \Pi(\mathscr{A}^+)\Omega \}$ is equivalent to the condition $\omega(\tau_t\{ A, B \}) = \omega(\{ \tau_t(A), \tau_t(B) \})$ (3) for $A, B \in \mathscr{A}$. Clearly, the above equality is a considerably weaker restriction on the evolution than the demand that the dynamics be a group of automorphisms. The condition (*i*) has a less natural interpretation than that given for condition (*ii*).

Next, as a by-product of our considerations, it seems to be impossible

(3) $\{ A, B \} = AB + BA$.

to establish the same scheme for semigroup evolutions $\tau_t$ and $\hat{\tau}_t$ as the scheme one has in Quantum Mechanics. The conditions (*i*) or (*ii*) causes the evolution $\hat{\tau}_t$ to be a group. On the other hand note that the lack of these conditions make it impossible to say that $\hat{\tau}_t$ maps directly states into states, so, in order to get the correct Schrödinger picture from $\hat{\tau}_t$ it is necessary first to describe $\{ \tau_t \}$ mappings and subsequently the dual family $\{ \tau_t^* \}$ on $\mathscr{A}^*$. Clearly the discussed scheme is always possible for the reversible dynamics (with the state $\omega$ $\tau_t$-invariant). Finally we wish to mention that although there is a unique correspondence between normal states of a von Neumann algebra and vectors from P (or $V_0$), in general, it is not convenient to use the above result for the description of the Schrödinger picture in the Hilbert space language. The main reason is that the mapping given by this correspondence is not a linear one.

## 4. REMARKS

Firstly we want to show in this section that the Detailed Balance condition is satisfied for the modular dynamics. Specifically one can state:

LEMMA. — *Let $\mathscr{J}$ be a conjugation on the Hilbert space $\mathscr{H}$ such that $\mathscr{J}\mathscr{A}^+\Omega \subseteq \mathscr{A}^+\Omega$, where $\mathscr{A}$ is a von Neumann algebra with cyclic and separating vector $\Omega$. Then $\mathscr{J}$ commutes strongly with $\Delta$.*

*Remark.* — Note that $\mathscr{J}$ is a bounded *antilinear* operator. So, we will say that $\mathscr{J}$ commutes strongly with an unbounded self adjoint operator A if the following equality $e^{itA}\mathscr{J} = \mathscr{J}e^{-itA}$ holds for all $t \in \mathbb{R}$.

*Proof.* — It is enough to repeat, with the obvious modifications for antilinear operator, the arguments leading to lemma 2 in [*9*].

COROLLARY. — *Assume that for a system described by a von Neumann algebra $\mathscr{A}$ with cyclic and separating vector $\Omega$ there is a reversing operation $\sigma$. Then, the modular dynamics is a $\mathscr{J}$-self-adjoint group.*

*Proof.* —

$$\mathscr{J}\Delta^{it}\mathscr{J} = \mathscr{J}\int e^{it\lambda}dE(\lambda)\mathscr{J} = \int e^{-it\lambda}\mathscr{J}dE(\lambda)\mathscr{J} = \int e^{-it\lambda}dE(\lambda) = (\Delta^{it})^* .$$

Note that the above corollary sugests that the $\mathscr{J}$-self-adjointness is a nearly trivial property of the candidate for the equilibrium (i. e. modular) dynamics. Moreover this result reflects our feeling that detailed balance should be a feature of equilibrium systems. On the other hand the application of this condition to a dynamical semigroup gives the similar scheme of description of dynamics as one has for reversible evolution in Quantum Mechanics. However, we would like to point out that there exists a full analogy between the two schemes only if the dynamics is reversible. Furthermore, the nature of the balance conditions (cf. theorems 2 and 3) is to force the semigroup description to be close to the group-description of time evolution.

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