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Section A :
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The low energy expansion in nonrelativistic scattering theory

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ABSTRACT. — We study in detail the low energy behaviour of Schrödinger operators with particular attention to scattering theory. We exploit the fact that the low energy behaviour of $-\Delta + V(\underline{x})$ in $L^2(\mathbb{R}^3)$ is determined by the behaviour of the scaled Hamiltonian $-\Delta + \varepsilon^{-2}V(\underline{x}/\varepsilon)$ for $\varepsilon \to 0_+$, which in turn is given by point interactions. In particular we obtain analytic expansions in powers of ε of the scattering matrix and of the off-shell scattering amplitude. We also get Puiseux resp. Taylor expansions for energy eigenvalues and resonances. The results are largely independent of the shape of the interaction and correspond to expansions around the zero energy limit, which is expressed by suitable point interactions.

RÉSUMÉ. — On étudie en détail le comportement à basse énergie des opérateurs de Schrödinger, en particulier en relation avec la théorie de la diffusion. On exploite le fait que le comportement à basse énergie de $-\Delta + V(\underline{x})$ dans $L^2(\mathbb{R}^3)$ est déterminé par celui du Hamiltonien trans-

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formé d'échelle $-\Delta + \varepsilon^{-2}V(\underline{x}/\varepsilon)$ pour $\varepsilon \to 0_+$, qui correspond à une interaction ponctuelle. En particulier, on obtient des développements en puissance de ε pour l'opérateur de diffusion et les amplitudes de diffusion hors couche. On obtient aussi des développements de Puiseux ou de Taylor pour les niveaux d'énergie et les résonances. Les résultats sont largement indépendants de la forme de l'interaction et correspondent à des développements autour de la limite d'énergie nulle, qui correspond à des interactions ponctuelles convenables.

1. INTRODUCTION

The study of the low energy behaviour of non relativistic quantum mechanics has theoretical and practical importance, as recognized quite early, e. g. in nuclear physics (see references to 1935-36 work of L. Thomas and E. Fermi and the work in [*13*]).

In fact low energy quantum mechanical phenomena abound in atomic and molecular physics and chemistry (see e. g. the references in [*7*] [*28*] as well as in nuclear physics and solid state physics (see e. g. the references in [*5*] [*9*] [*16*] [*17*]). Scattering at low energies has received great attention in connection with partial wave analysis (see e. g. [*28*, ch. 11]), low energy phenomena in three particle systems (Efimov effect, see [*5*] and references therein), and propagation of cold neutrons in a cystal (e. g. [*9*] and references therein). The study of the low energy limit permits the exact computation, at the limit of zero energy, of quantities like scattering amplitudes and bound states energies, and in turn these computations serve, at least in principle, as a basis of a perturbation expansion around the zero energy limit.

E. g. in the zero energy limit the scattering is as simple as possible and has universal, geometrical features, largely independent of the particular structure of the system. This is captured by expressions like « effective range », « scattering length approximation », « shape independent approximation », which have a large literature (see e. g. [*28*]). In particular the zero energy scattering is well described by a point interaction of suitable strength (in the case of point interactions the scattering length approximation is exact by construction). The study of a non relativistic particle interacting with a δ-interaction has been done per se in a large number of papers, see [*1*]-[*4*] [*8*]-[*17*] [*39*] [*41*]-[*43*]. Recently methods of non standard analysis [*3*] have permitted the extension to the study of one particle many (infinitely many included) center problems [*12*] [*15*]-[*17*] [*23*] [*39*] [*41*] [*43*]. Moreover the proof of the convergence of the resolvent of a one particle-many center problem with local interaction to the zero energy limit [*4*], as well as an expansion for the energy eigenvalues and resonances have been obtained [*6*]. For the case of three particle systems some (at the moment less detailed) results have also been obtained: δ-interactions have been studied in connection with the deuteron problem (see e. g. [*13*] and references therein) and the connection with the zero energy limit of systems with smooth interaction has been studied mathematically (in connection also with the Efimov effect) in [*5*].

The present paper is devoted to the detailed study of the low energy limit of scattering in two particle non relativistic quantum mechanics. Let us mention some preceding work on this particular problem. Much work had been done in classical references going back to the late forties and early fifties, in the work of R. Jost and N. Levinson, in the radial symmetric case, see e. g. [*7*] [*28*]. A recent extension of some of this work to the non central case, centered around an extension of Levinson's theorem on the relation of the phase shift at zero energy and the number of eigenvalues, has been obtained by Bollé-Osborn, Buslaev, Dreyfus, Newton, and Wollenberg, see the references in [*29*]. Asymptotic results for low energy of the resolvent, the spectral density and the on shell scattering operator have been obtained by Jensen and Kato [*21*], using methods of weighted Sobolev spaces. Besides obtaining related results by completely different means, namely by the method of scaling and under different assumptions, we also obtain detailed results on the off-shell scattering matrix, giving for all quantities involved the explicit computation of the lower order coefficients in the analytic (resp. asymptotic) expansions around the zero energy limit. Our method relies on the physical intuition that small momenta (small energies) correspond to large distances which are most easily attained by scaling $\underline{x} \to \underline{x}/\varepsilon$, $\varepsilon \to 0_+$, which can be unitarily implemented in the Hilbert space $L^2(\mathbb{R}^3)$. Our results depend on a case distinction according to possible zero energy resonances and eigenvalues of $H = -\Delta + V$. For general references concerning the discussion of resonances we refer to [*6*] [*20*] [*31*] [*37*]. The limit situations where eigenvalues approach zero as the coupling constant approaches a « critical value » is of direct relevance to our work and has been studied recently by Klaus and Simon [*24*] [*25*].

Let us now shortly summarize the content of our paper. In section 2 we introduce the basic Hamiltonian $H(\varepsilon) = -\Delta + \lambda(\varepsilon)V(\underline{x})$ (λ analytic in $[0, 1]$, $\lambda(0_+)=1$) and the scaled one $H_\varepsilon = -\Delta + \lambda(\varepsilon)\varepsilon^{-2}V(\underline{x}/\varepsilon)$, as well as the associated scattering amplitudes. We point out that the expansion of the latters for small ε is given essentially by the one of $\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}$, where $u(\underline{x}) = |V(\underline{x})|^{\frac{1}{2}} \operatorname{sign} V(\underline{x})$, $v(\underline{x}) = |V(\underline{x})|^{\frac{1}{2}}$, and $G_k = (-\Delta - k^2)^{-1}$, $\operatorname{Im} k > 0$. We then recall the discussion of the poles of $(uG_0v + 1)^{-1}$, distinguishing four basic cases, and we discuss the limit of H_ε as $\varepsilon \to 0_+$ extending the results of [*4*].

In section 3 we discuss the detailed expansion in powers of ε of the above quantity $\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}$ which gives the scattering amplitude for H_ε. In some cases, which we specify, one has an analytic expansion and in some other cases one has a Laurent expansion with a simple pole in ε.

In section 4 we discuss the asymptotic behaviour at low energy of the off shell scattering amplitude, the scattering operator as well as the energy eigenvalues and resonances associated with the operator

$$H(\varepsilon) = -\Delta + \lambda(\varepsilon)V(\underline{x}) \qquad \text{for } \varepsilon \to 0_+ .$$

The extension of this work to Coulomb and many center systems is in progress.

2. NOTATIONS AND BASIC FACTS

Let λ denote a real and analytic function on $[0, 1]$ with $\lambda(0_+) = 1$ and define U_ε to be the unitary dilation group on $L^2(\mathbb{R}^3)$

$$(U_\varepsilon h)(\underline{x}) = \varepsilon^{-3/2}h(\underline{x}/\varepsilon), \qquad \varepsilon > 0, \qquad h \in L^2(\mathbb{R}^3). \tag{2.1}$$

Throughout this paper we assume V to be a real measurable function on $\mathbb{R}^3$ in the Rollnik class R i. e. such that

$$\| V \|_R^2 = (4\pi)^{-2} \int d^3x d^3y \frac{| V(\underline{x}) | \, | V(\underline{y}) |}{| \underline{x} - \underline{y} |^2} < \infty . \tag{2.2}$$

We shall use the notation $V_\varepsilon = \varepsilon^{-2}U_\varepsilon V U_\varepsilon^{-1}$, so that

$$V_\varepsilon(\underline{x}) = \varepsilon^{-2}V(\underline{x}/\varepsilon). \tag{2.3}$$

With these definitions we introduce the following operators in $L^2(\mathbb{R}^3)$

$$H = -\Delta + V, \tag{2.4}$$

$$H(\varepsilon) = -\Delta + \lambda(\varepsilon)V, \qquad 0 \leq \varepsilon \leq 1, \tag{2.5}$$

$$H_\varepsilon = \varepsilon^{-2}U_\varepsilon H(\varepsilon)U_\varepsilon^{-1} = -\Delta + \lambda(\varepsilon)V_\varepsilon, \qquad 0 < \varepsilon \leq 1 \tag{2.6}$$

where $-\Delta$ denotes the usual kinetic energy operator in $L^2(\mathbb{R}^3)$ and all Hamiltonians are defined in the sense of quadratic forms [22] [32] [35]. We also note that

$$(H_\varepsilon - k^2)^{-1} = \varepsilon^2 U_\varepsilon(H(\varepsilon) - (\varepsilon k)^2)^{-1}U_\varepsilon^{-1}, \qquad \operatorname{Im} k > 0. \tag{2.7}$$

(2.6) and (2.7) clearly indicate the connection between the spectra of $H(\varepsilon)$ and H_ε.

Using the notations

$$G_k = (-\Delta - k^2)^{-1}, \qquad \operatorname{Im} k > 0, \tag{2.8}$$

and

$$v(\underline{x}) = | V(\underline{x}) |^{1/2}, \qquad u(\underline{x}) = | V(\underline{x}) |^{1/2} \operatorname{sign} V(\underline{x}), \tag{2.9}$$

we obtain by iterating the resolvent equation

$$(\mathrm{H}(\varepsilon) - k^2)^{-1} = \mathrm{G}_k - \lambda(\varepsilon)\mathrm{G}_k v t(\varepsilon, k) u \mathrm{G}_k \tag{2.10}$$

where the t-matrix $t(\varepsilon, k)$ is given by

$$t(\varepsilon, k) = (\lambda(\varepsilon) u \mathrm{G}_k v + 1)^{-1} . \tag{2.11}$$

If in addition

$$e^{2a|\underline{x}|}\mathrm{V}(\underline{x}) \in \mathrm{R} \qquad \text{for some } a > 0 , \tag{2.12}$$

then the scattering amplitude $f(\varepsilon, \underline{p}, \underline{q}, k)$ corresponding to $\mathrm{H}(\varepsilon)$, defined by

$$f(\varepsilon, \underline{p}, \underline{q}, k) = -(4\pi)^{-1}\lambda(\varepsilon)(ve^{i\underline{p}\cdot}, t(\varepsilon, k)ue^{i\underline{q}\cdot}) \tag{2.13}$$

is analytic in $\underline{p}$, $\underline{q}$, k in the region $|\operatorname{Im} \underline{p}| < a$, $|\operatorname{Im} \underline{q}| < a$, $\operatorname{Im} k > -a$ [*18*] [*19*] [*27*] [*33*] [*35*] (except for those discrete values of k where $t(\varepsilon, k)$ does not exist).

From (2.7), (2.10), (2.11), and $\mathrm{U}_\varepsilon \mathrm{G}_k \mathrm{U}_\varepsilon^{-1} = \varepsilon^{-2}\mathrm{G}_{\varepsilon^{-1}k}$ we get

$$\begin{aligned}(\mathrm{H}_\varepsilon - k^2)^{-1} \\ = \mathrm{G}_k - \varepsilon^{-2}\lambda(\varepsilon)\mathrm{G}_k(\mathrm{U}_\varepsilon v \mathrm{U}_\varepsilon^{-1})\mathrm{U}_\varepsilon(\lambda(\varepsilon)u\mathrm{G}_{\varepsilon k}v + 1)^{-1}\mathrm{U}_\varepsilon^{-1}(\mathrm{U}_\varepsilon u \mathrm{U}_\varepsilon^{-1})\mathrm{G}_k\end{aligned} \tag{2.14}$$

and thus $t_\varepsilon(k)$, the t-matrix associated with H_ε defined by (2.10) with $\mathrm{H}(\varepsilon)$ replaced by H_ε, is given by

$$t_\varepsilon(k) = \mathrm{U}_\varepsilon t(\varepsilon, \varepsilon k)\mathrm{U}_\varepsilon^{-1} = \mathrm{U}_\varepsilon(\lambda(\varepsilon)u\mathrm{G}_{\varepsilon k}v + 1)^{-1}\mathrm{U}_\varepsilon^{-1} . \tag{2.15}$$

Under the condition (2.12) the scattering amplitude $f_\varepsilon(\underline{p}, \underline{q}, k)$ corresponding to H_ε reads

$$\begin{aligned} f_\varepsilon(\underline{p}, \underline{q}, k) \\ &= -(4\pi)^{-1}\varepsilon^{-2}\lambda(\varepsilon)(\mathrm{U}_\varepsilon v \mathrm{U}_\varepsilon^{-1}e^{i\underline{p}\cdot}, \mathrm{U}_\varepsilon(\lambda(\varepsilon)u\mathrm{G}_{\varepsilon k}v + 1)^{-1}\mathrm{U}_\varepsilon^{-1}\mathrm{U}_\varepsilon u \mathrm{U}_\varepsilon^{-1}e^{i\underline{q}\cdot}) \\ &= -(4\pi)^{-1}\lambda(\varepsilon)(ve^{i\varepsilon\underline{p}\cdot}, \varepsilon(\lambda(\varepsilon)u\mathrm{G}_{\varepsilon k}v + 1)^{-1}ue^{i\varepsilon\underline{q}\cdot}) & (2.16) \\ &= \varepsilon f(\varepsilon, \varepsilon\underline{p}, \varepsilon\underline{q}, \varepsilon k), \quad |\operatorname{Im}\underline{p}| < a, \quad |\operatorname{Im}\underline{q}| < a, \quad \operatorname{Im} k > -a . & (2.17)\end{aligned}$$

In order to have an expansion of $f_\varepsilon(\underline{p}, \underline{q}, k)$ at $\varepsilon = 0$ we thus need an expansion of $(\lambda(\varepsilon)u\mathrm{G}_{\varepsilon k}v + 1)^{-1}$. It turns out that the explicit form of the latter expansion strongly depends on the fact whether H has a zero energy resonance (bound state) or not. We recall here the definition of a zero energy resonance. If -1 is an eigenvalue of $u\mathrm{G}_0 v$ i. e.

$$u\mathrm{G}_0 v\phi_j = -\phi_j \quad \text{for some} \quad \phi_j \in \mathrm{L}^2(\mathbb{R}^3), \qquad j = 1, \ldots, \mathrm{N}, \tag{2.18}$$

we call the functions

$$\psi_j(\underline{x}) = (\mathrm{G}_0 v\phi_j)(\underline{x}), \qquad j = 1, \ldots, \mathrm{N} \tag{2.19}$$

(zero energy) resonance functions (cf. [*4*]) and note that

$$\mathrm{H}\psi_j = (-\Delta + \mathrm{V})\psi_j = 0, \qquad j = 1, \ldots, \mathrm{N}$$

in the sense of distributions (here it suffices $\mathrm{V} \in \mathrm{R}$). Since $u\mathrm{G}_0 v$ is Hilbert Schmidt [35] N is necessarily finite. In general ψ_j need not be in $L^2(\mathbb{R}^3)$.

With these definitions we now distinguish the following cases (see also [21] [24] [29]):

CASE I: There exist no resonance functions ψ_j or equivalently, -1 is not an eigenvalue of uG_0v.

CASE II: There exists only one resonance function ψ (i. e. -1 is a simple eigenvalue of uG_0v) and ψ is not in $L^2(\mathbb{R}^3)$.

CASE III: There exist $N \geq 1$ resonance functions ψ_j, $j = 1, \ldots, N$ which are all in $L^2(\mathbb{R}^3)$.

CASE IV: There exist $N \geq 2$ resonance functions ψ_j, $j = 1, \ldots, N$ and at least one of them is not in $L^2(\mathbb{R}^3)$.

Under suitable conditions on V there is a simple criterion which helps to decide whether a resonance function ψ is in $L^2(\mathbb{R}^3)$ or not:

PROPOSITION 2.1. — Assume $V \in L^1(\mathbb{R}^3) \cap R$ and let $\psi(\underline{x}) = (G_0 v\phi)(\underline{x})$ where $uG_0v\phi = -\phi$, $\phi \in L^2(\mathbb{R}^3)$. Then $\psi \in L^2_{\text{loc}}(\mathbb{R}^3)$. If in addition

$$|\underline{x}|V(|\underline{x}|) \in L^1(\mathbb{R}^3) \qquad \text{then} \qquad \psi \in L^2(\mathbb{R}^3)$$

is equivalent to

$$(v, \phi) = -\int d^3x V(\underline{x})\psi(\underline{x}) = 0. \tag{2.20}$$

Proof. — Following Newton [29] we decompose

$$\psi(\underline{x}) = (4\pi)^{-1}\int d^3y\, |\underline{x}-\underline{y}|^{-1} v(\underline{y})\phi(\underline{y}) = 4\pi\,|\underline{x}|^{-1}\int d^3y\, v(\underline{y})\phi(\underline{y}) + \psi_1(\underline{x}) \tag{2.21}$$

where

$$\psi_1(\underline{x}) = (4\pi)^{-1}\int d^3y \left(\frac{1}{|\underline{x}-\underline{y}|} - \frac{1}{|\underline{x}|}\right) v(\underline{y})\phi(\underline{y}). \tag{2.22}$$

From the fact that for any $r > 0$

$$\int_{|\underline{x}| \leq r} d^3x \left(\frac{1}{|\underline{x}-\underline{y}|} - \frac{1}{|\underline{x}|}\right)^2 = |\underline{y}|\int_0^{r/|\underline{y}|} d|\underline{x}| \int d\omega \left(\frac{|\underline{x}|}{|\underline{x}-\hat{\underline{y}}|} - 1\right)^2 \leq cr, \qquad \hat{\underline{y}} = \underline{y}/|\underline{y}|, \tag{2.23}$$

for some constant c, we immediately obtain local square integrability of ψ since

$$\int_{|\underline{x}| \leq r} d^3x\, |\psi_1(\underline{x})|^2 \leq (4\pi)^{-2} cr\, \|\phi\|^2\, \|V\|_1 < \infty. \tag{2.24}$$

On the other hand the assumption $|\underline{x}||V(\underline{x})| \in L^1(\mathbb{R}^3)$ and the estimate

$$\int d^3x \left(\frac{1}{|\underline{x}-\underline{y}|} - \frac{1}{|\underline{x}|} \right)^2 = |\underline{y}| \int_0^\infty d|\underline{x}| \int d\omega \left(\frac{|\underline{x}|}{|\underline{x}-\hat{\underline{y}}|} - 1 \right)^2 \leq \tilde{c}\,|\underline{y}|, \quad (2.25)$$

for some constant $\tilde{c}$, immediately yield

$$\| \psi_1 \|^2 \leq (4\pi)^{-2}\tilde{c}\, \| \phi \|^2 \int d^3y\, |\underline{y}|\, |V(\underline{y})| < \infty. \quad (2.26)$$

This shows that $\psi \in L^2(\mathbb{R}^3)$ if and only if

$$(v, \phi) = -\int d^3x V(\underline{x})\psi(\underline{x}) = 0. \quad (2.27)$$

REMARK 2.1. — *a*) The condition $|\underline{x}|V(\underline{x}) \in L^1(\mathbb{R}^3)$ is obviously too strong but it suffices for our purposes since we are interested in a strong fall off of V at infinity (see sections 3 and 4). For alternative conditions on V yielding the same result see [*24*] and [*29*].

b) Under the conditions of Proposition 2.1 we thus can always choose in case IV a particular set of linear combinations of the resonance functions such that $\int d^3x V(\underline{x})\psi_{j_0}(\underline{x}) \neq 0$ for some j_0 and $\int d^3x V(\underline{x})\psi_j(\underline{x}) = 0$ if $j = 1, \ldots, N, j \neq j_0$. Under these conditions in case III zero is an eigenvalue of H with multiplicity N whereas in case IV its multiplicity is always $N - 1$.

c) If V is spherically symmetric then, due to symmetry, $\int d^3x V(\underline{x})\psi(\underline{x}) = 0$ for any resonance function ψ that belongs to angular momentum ≥ 1. For a discussion that $\int d^3x V(\underline{x})\psi(\underline{x}) \neq 0$ for *s*-wave resonance functions (or for functions associated with the ground state absorption in the sense of [*24*] [*36*]) see [*24*] and [*29*].

We now briefly describe the connection between H_ε and point interactions. For a detailled discussion of this connection see [*4*] and for the theory of point interactions compare [*1*]-[*5*] [*9*]-[*17*] [*23*] [*39*] [*41*]-[*43*] and the references cited therein.

Let $-\Delta_\alpha$ denote the Hamiltonian with point interaction of parameter $\alpha \in \mathbb{R}$ centered at $\underline{x}=0$ i. e. $-\Delta_\alpha$ is the self-adjoint extension of $-\Delta|_{C_0^\infty(\mathbb{R}^3-\{0\})}$ given by the boundary condition

$$[-4\pi\alpha|\underline{x}|g(|\underline{x}|)+(\hat{\underline{x}}\cdot\nabla)(|\underline{x}|g(|\underline{x}|))]\big|_{|\underline{x}|=0}=0, \qquad \hat{\underline{x}}=\underline{x}/|\underline{x}|, \quad (2.28)$$

in the partial wave subspace corresponding to angular momentum zero. We also recall that

$$(-\Delta_\alpha - k^2)^{-1} = G_k - \left(\frac{ik}{4\pi} - \alpha \right)^{-1} (\overline{g_k}, .)g_k, \qquad g_k(\underline{x}) = \frac{e^{ik|\underline{x}|}}{4\pi|\underline{x}|}. \quad (2.29)$$

Then we have

THEOREM 2.1. — Let $V \in R$ in case I. In case II, and if $\lambda'(0_+) \neq 0$ in cases III and IV, assume in addition $(1 + |\underline{x}|)^2 V \in L^1(\mathbb{R}^3)$. Then H_ε converges to $-\Delta_\alpha$ in norm resolvent sense. If $\lambda'(0_+) = 0$ in cases III and IV and in addition $(1 + |\underline{x}|)^4 V \in L^1(\mathbb{R}^3)$ then H_ε converges to $-\Delta_\alpha$ in strong resolvent sense. Here α is given by (with $\tilde{\phi} = (\text{sign } V)\phi$)

$$\alpha = \begin{cases} \infty \text{ in case I,} \\ \dfrac{\lambda'(0_+)}{|(v, \phi)|^2/(\tilde{\phi}, \phi)} & \text{in case II,} \\ \infty \text{ in case III,} \\ \dfrac{\lambda'(0_+)}{\sum_{j=1}^{N} |(v, \phi_j)|^2/(\tilde{\phi}_j, \phi_j)} & \text{in case IV.} \end{cases} \tag{2.30}$$

In the case $\alpha = \infty$ one has $-\Delta_\alpha = -\Delta$.

We defer the proof of Theorem 2.1 to the end of section 3 and note that in cases III and IV if $\lambda'(0_+) = 0$ strong resolvent convergence can not be replaced by norm resolvent convergence in general (cf. Remark 4.2).

3. EXPANSION OF $\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}$

We first start with a preliminary result:

LEMMA 3.1. — Let $V \in L^1(\mathbb{R}^3) \cap R$. Then $(uG_0v + 1 + \varepsilon)^{-1}$ has a norm convergent Laurent expansion around $\varepsilon = 0$

$$(uG_0v + 1 + \varepsilon)^{-1} = \frac{P}{\varepsilon} + \sum_{m=0}^{\infty} (-\varepsilon)^m T^{m+1} \tag{3.1}$$

where P is the projector onto the N-dimensional eigenspace of uG_0v to the eigenvalue -1 and is given by

$$P = -(2\pi i)^{-1} \oint_\Gamma dz(uG_0v - z)^{-1} = \sum_{j=1}^{N} \frac{(\tilde{\phi}_j, .)\phi_j}{(\tilde{\phi}_j, \phi_j)}, \qquad N = \dim P, \tag{3.2}$$

where ϕ_j are the solutions of

$$uG_0v\phi_j = -\phi_j, \quad \tilde{\phi}_j = (\text{sign } V)\phi_j, \quad (\tilde{\phi}_j, \phi_l) = 0, \quad j \neq l, \quad j = 1, \ldots, N. \tag{3.3}$$

and T is a bounded operator given by

$$T=(2\pi i)^{-1}\oint_\Gamma dz(z+1)^{-1}(uG_0v-z)^{-1}=n\text{-}\lim_{\varepsilon\to 0_+}(uG_0v+1+\varepsilon)^{-1}(1-P). \quad (3.4)$$

Here Γ surrounds in the usual way [22] [34] only the isolated eigenvalue -1 of uG_0v counter clockwise (if uG_0v has no eigenvalue -1 then obviously $P = 0$ and $T = (uG_0v + 1)^{-1}$).

Proof. — From [22] we infer

$$(uG_0v + 1 + \varepsilon)^{-1} = \frac{P}{\varepsilon} + \sum_{m=1}^{N-1} \varepsilon^{-m-1}(-D)^m + \sum_{m=0}^{\infty}(-\varepsilon)^m T^{m+1} \quad (3.5)$$

where

$$D = (uG_0v + 1)P = -(2\pi i)^{-1}\oint_\Gamma dz(z+1)(uG_0v - z)^{-1} \quad (3.6)$$

and we only have to prove $D = 0$ and the second equality in (3.2). For that purpose we introduce the self-adjoint operator $G_0^{1/2}VG_0^{1/2}$ and note the following relation between eigenvectors of $G_0^{1/2}VG_0^{1/2}$ and uG_0v resp. vG_0u: Let, for some number λ and function $\Phi \in L^2(\mathbb{R}^3)$

$$uG_0v\Phi = \lambda\Phi, \qquad vG_0u\tilde\Phi = \lambda\tilde\Phi, \qquad \tilde\Phi = (\text{sign } V)\Phi, \quad (3.7)$$

then

$$\chi = G_0^{1/2}v\Phi = G_0^{1/2}u\tilde\Phi \quad (3.8)$$

fulfills

$$G_0^{1/2}VG_0^{1/2}\chi = \lambda\chi. \quad (3.9)$$

If $\lambda \neq 0$ then

$$(\tilde\Phi, \Phi) = |\lambda|^{-2}(vG_0u\tilde\Phi, uG_0v\Phi) = (\bar\lambda)^{-1}\|\chi\|^2. \quad (3.10)$$

Thus λ is real and sign $(\tilde\Phi, \Phi) =$ sign λ.

Conversely assume that for some $\chi_1 \in L^2(\mathbb{R}^3)$, $\lambda_1 \in \mathbb{R}$

$$G_0^{1/2}VG_0^{1/2}\chi_1 = \lambda_1\chi_1 \quad (3.11)$$

then

$$\Phi_1 = uG_0^{1/2}\chi_1, \qquad \tilde\Phi_1 = vG_0^{1/2}\chi_1 \quad (3.12)$$

fulfill

$$uG_0v\Phi_1 = \lambda_1\Phi_1, \qquad vG_0u\tilde\Phi_1 = \lambda_1\tilde\Phi_1. \quad (3.13)$$

If $\lambda_1 \neq 0$ then

$$\|\chi_1\|^2 = |\lambda_1|^{-2}(G_0^{1/2}VG_0^{1/2}\chi_1, G_0^{1/2}VG_0^{1/2}\chi_1) = \lambda_1^{-1}(\tilde\Phi_1, \Phi_1) \quad (3.14)$$

and thus

$$\text{sign } (\tilde\Phi_1, \Phi_1) = \text{sign } \lambda_1. \quad (3.15)$$

Since $G_0^{1/2}VG_0^{1/2}$ is self-adjoint and Hilbert Schmidt we have

$$h = \sum_{n=0}^{\infty} Q_{\lambda_n}h, \qquad \text{for all } h \in L^2(\mathbb{R}^3) \quad (3.16)$$

where Q_{λ_n} and λ_n are the eigenprojections and eigenvalues of $G_0^{1/2}VG_0^{1/2}$ respectively. Suppose $\lambda_0 = 0$. Then, for $n \neq 0$, Q_{λ_n} is of the type

$$Q_{\lambda_n} = \sum_{j=1}^{\dim Q_{\lambda_n}} \frac{(\chi_{nj}, \,.\,)\chi_{nj}}{\| \chi_{nj} \|^2}, \qquad (\chi_{nj}, \chi_{nl}) = 0 \quad \text{if} \quad j \neq l \tag{3.17}$$

and thus

$$g = uG_0^{1/2}h \tag{3.18}$$

fulfills

$$g = uG_0^{1/2}h = uG_0^{1/2}Q_{\lambda_0}h + \sum_{n=1}^{\infty} uG_0^{1/2}Q_{\lambda_n}h$$

$$= uG_0^{1/2}Q_{\lambda_0}h + \sum_{n=1}^{\infty} \sum_{j=1}^{\dim Q_{\lambda_n}} \frac{(\tilde{\Phi}_{nj}, g)\Phi_{nj}}{(\tilde{\Phi}_{nj}, \Phi_{nj})} \tag{3.19}$$

where

$$\Phi_{nj} = uG_0^{1/2}\chi_{nj}, \qquad \tilde{\Phi}_{nj} = vG_0^{1/2}\chi_{nj}, \qquad (\tilde{\Phi}_{nj}, \Phi_{nl}) = 0 \quad \text{if} \quad j \neq l. \tag{3.20}$$

If we define

$$P_{\lambda_n} = -(2\pi i)^{-1} \oint_{\Gamma_n} dz(uG_0v - z)^{-1}, \; n \geq 1 \quad (\Gamma_n \text{ encircles only } \lambda_n) \tag{3.21}$$

and

$$\tilde{P}_{\lambda_n} = \sum_{j=1}^{\dim Q_{\lambda_n}} \frac{(\tilde{\Phi}_{nj}, \,.\,)\Phi_{nj}}{(\tilde{\Phi}_{nj}, \Phi_{nj})}, \qquad n \geq 1, \tag{3.22}$$

then (3.19) obviously implies (with $\mathscr{R}(.)$ for range)

$$\tilde{P}_{\lambda_m}g = P_{\lambda_m}g = Q_{\lambda_m}g \qquad \text{for all} \qquad g \in \mathscr{R}(uG_0^{1/2}), \quad m \geq 1. \tag{3.23}$$

Now let $\eta \in \mathscr{R}(uG_0^{1/2})^{\perp}$ i. e.

$$(\eta, uG_0^{1/2}h) = 0 \qquad \text{for all } h \in L^2(\mathbb{R}^3) \tag{3.24}$$

yielding

$$G_0^{1/2}u\eta = 0 \tag{3.25}$$

and

$$vG_0u\eta = 0. \tag{3.26}$$

Since $u\eta \in L^1(\mathbb{R}^3)$, (3.25) implies $u\eta = 0$ by a distributional argument. But $u\eta = 0$ implies $v\eta = 0$ and thus

$$uG_0v\eta = 0. \tag{3.27}$$

(3.26) and (3.27) yield

$$\tilde{P}_{\lambda_m}\eta = P_{\lambda_m}\eta = 0 \qquad \text{for all } \eta \in \mathcal{R}(uG_0^{1/2})^{\perp}, \qquad m \geq 1 . \tag{3.28}$$

Since $\mathcal{R}(uG_0^{1/2}) \cup \mathcal{R}(uG_0^{1/2})^{\perp}$ is dense in $L^2(\mathbb{R}^3)$ we finally obtain

$$\tilde{P}_{\lambda_m} = P_{\lambda_m}, \qquad m \geq 1 \tag{3.29}$$

and hence the second equality in (3.2) is proved. But then

$$D = (uG_0v + 1)P = \sum_{j=1}^{N} \frac{(\tilde{\Phi}_j, .)}{(\tilde{\Phi}_j, \Phi_j)} (uG_0v + 1)\Phi_j = 0 \tag{3.30}$$

and the proof is finished.

Remark 3.1. — Lemma 3.1 shows that all results of [*4*] where the additional assumption $V \leq 0$ if $N \geq 2$ has been made (in that case uG_0v is self-adjoint and the above proof becomes superfluous) carry over to arbitrary potentials in $L^1(\mathbb{R}^3) \cap R$.

Before we state the main results of this section we introduce some further notations. If $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ then $\lambda(\varepsilon)uG_{\varepsilon k}v$ is analytic in ε and we expand

$$\lambda(\varepsilon)uG_{\varepsilon k}v = A + \varepsilon B + \varepsilon^2 C + \varepsilon^3 D + 0(\varepsilon^4) \tag{3.31}$$

where

$$A = uG_0v, \tag{3.32}$$

$$B = \lambda'(0_+)uG_0v + \frac{ik}{4\pi}(v, .)u, \tag{3.33}$$

$$C = \frac{\lambda''(0_+)}{2} uG_0v + \frac{ik}{4\pi}\lambda'(0_+)(v, .)u + k^2\hat{C} \tag{3.34}$$

$$(\hat{C}h)(\underline{x}) = -(8\pi)^{-1}\int d^3y u(\underline{x}) \,|\, \underline{x} - \underline{y} \,|\, v(\underline{y})h(\underline{y}), \qquad h \in L^2(\mathbb{R}^3) \tag{3.35}$$

and similar for D and the other coefficients in (3.31).

Remark 3.2. — If $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ then the singular continuous part of the spectrum of H (and of $H(\varepsilon)$ and H_ε) is empty and the bound states of H (and of $H(\varepsilon)$, H_ε) (in particular the positive bound states) are finite in number [*34*] [*35*]. This assumption also implies that $(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)$ is invertible for $\varepsilon > 0$ small enough.

In the following we shall discuss separately the cases I, IV defined in section 2. We have

THEOREM 3.1. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and assume case I (in this case $P = 0$, $T = (uG_0v + 1)^{-1}$). Then $\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}$ is analytic at $\varepsilon = 0$ and

$$\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1} = \varepsilon T - \varepsilon^2 TBT + 0(\varepsilon^3). \tag{3.36}$$

Proof. — Using (3.31) we get

$$\begin{aligned}
&\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1} \\
&\quad = \varepsilon[A + 1 + \varepsilon + \varepsilon(B - 1 + \varepsilon C + 0(\varepsilon^2))]^{-1} \\
&\quad = [1 + \varepsilon(A + 1 + \varepsilon)^{-1}(B - 1 + \varepsilon C + 0(\varepsilon^2))]^{-1}\varepsilon(A + 1 + \varepsilon)^{-1} \\
&\quad = [1 + \varepsilon(TB - T) + \varepsilon^2(TC - T^2B + T^2) + 0(\varepsilon^3)]^{-1}(\varepsilon T - \varepsilon^2T^2 + 0(\varepsilon^3)) \\
&\quad = \varepsilon T - \varepsilon^2 TBT + 0(\varepsilon^3).
\end{aligned}$$

Here the simplest case of Lemma 3.1 namely $(A+1+\varepsilon)^{-1} = T - \varepsilon T^2 + 0(\varepsilon^2)$ has been applied.

THEOREM 3.2. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and assume case II $\left(\text{i. e. } P = \frac{(\tilde{\phi},.)\phi}{(\tilde{\phi}, \phi)}, (v, \phi) \neq 0\right)$. Then $\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}$ is analytic at $\varepsilon = 0$ and

$$\begin{aligned}
&\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1} \\
&\quad = \left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-1}(\tilde{\phi},.)\phi + \varepsilon T \\
&\quad - \varepsilon\frac{ik}{4\pi}\left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-1}(\tilde{\phi}, u)(T^*v,.)\phi \\
&\quad - \varepsilon\frac{ik}{4\pi}\left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-1}(v, \phi)(\tilde{\phi},.)Tu \\
&\quad + \varepsilon\left(\frac{ik}{4\pi}\right)^2\left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-2}|(v, \phi)|^2(v, Tu)(\tilde{\phi},.)\phi \\
&\quad - \varepsilon\left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-2}(\tilde{\phi}, C\phi)(\tilde{\phi},.)\phi + 0(\varepsilon^2).
\end{aligned} \tag{3.37}$$

Proof. — From (3.31) and (3.1) we obtain

$$\begin{aligned}
&\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1} \\
&\quad = [1 + \varepsilon(A + 1 + \varepsilon)^{-1}(B - 1 + \varepsilon C + 0(\varepsilon^2))]^{-1}\varepsilon(A + 1 + \varepsilon)^{-1} \\
&\quad = [1 + PB - P + \varepsilon(PC + TB - T) + 0(\varepsilon^2)]^{-1}(P + \varepsilon T + 0(\varepsilon^2)) \\
&\quad = [1 - \varepsilon(1 + PB - P)^{-1}(PC + TB - T) + 0(\varepsilon^2)](1 + PB - P)^{-1}(P + \varepsilon T + 0(\varepsilon^2)).
\end{aligned} \tag{3.38}$$

Noting

$$TP = PT = 0, \tag{3.39}$$

$$[1 + PB - P]^{-1} = 1 + \left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-1}(1 + \lambda')(\tilde{\phi},.)\phi$$
$$- \frac{ik}{4\pi}\left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-1}(\tilde{\phi}, u)(v,.)\phi, \tag{3.40}$$

$$[1+PB-P]^{-1}P = \left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-1}(\tilde{\phi},.)\phi, \tag{3.41}$$

it is straightforward to get (3.37) from (3.38).

THEOREM 3.3. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and assume case III $\left(\text{so that } P = \sum_{j=1}^{N} \frac{(\tilde{\phi}_j,.)\phi_j}{(\tilde{\phi}_j, \phi_j)}, (v, \phi_j) = 0, j = 1, \ldots, N\right)$. If $\lambda'(0_+) \neq 0$ (case *a*) below) then $\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}$ is analytic at $\varepsilon = 0$ whereas if $\lambda'(0_+) = 0$ (case *b*) below) it has a Laurent expansion around $\varepsilon = 0$. More precisely, we have the following results:

a) $\lambda'(0_+) \neq 0$: In this case we have the Taylor expansion

$$\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}$$
$$= -(\lambda')^{-1}P + \varepsilon T - \varepsilon(\lambda')^{-2}PCP - \varepsilon^2 TBT + \varepsilon^2(\lambda')^{-1}PCT$$
$$- \varepsilon^2(\lambda')^{-1}[(1 - (1 + (\lambda')^{-1})P)(PC + TB - T)]^2 P$$
$$+ \varepsilon^2(\lambda')^{-1}(1 - (1 + (\lambda')^{-1})P)(PD + TC - T^2B)P + 0(\varepsilon^3). \tag{3.42}$$

b) $\lambda'(0_+) = 0$: In this case the Laurent expansion

$$\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1} = \frac{1}{\varepsilon}\left(\sum_{j,l=1}^{N}(\tilde{\phi}, C\phi)^{-1}_{jl}(\tilde{\phi}_l,.)\phi_j\right) + P + \left(\sum_{jl}\right)$$
$$- \left(\sum_{jl}\right)CP - \left(\sum_{jl}\right)C\left(\sum_{j'l'}\right) - \left(\sum_{jl}\right)D\left(\sum_{j'l'}\right)$$
$$+ \varepsilon T - \varepsilon\left(\sum_{jl}\right)CT - \varepsilon\left(\sum_{jl}\right)DP + \varepsilon\left(\sum_{jl}\right)C\left(\sum_{j'l'}\right)CP$$
$$+ \varepsilon\left(\sum_{jl}\right)D\left(\sum_{j'l'}\right)CP - \varepsilon\{1\}\left(\sum_{jl}\right) + 0(\varepsilon^2) \tag{3.43}$$

where the following abbreviations have been used

$$\left(\sum_{jl}\right) = \sum_{j,l=1}^{N}(\tilde{\phi}, C\phi)^{-1}_{jl}(\tilde{\phi}_l,.)\phi_j, \tag{3.44}$$

here $(\tilde{\phi}, C\phi)_{jl}^{-1}$ denotes the inverse matrix of $(\tilde{\phi}_j, C\phi_l)$,

$$\{1\} = 1 + TC - T(B-1) - T^2(B-1) - \Big(\sum_{jl}\Big)C + \Big(\sum_{jl}\Big)CT(B-1)$$
$$- \Big(\sum_{jl}\Big)CTC + \Big(\sum_{jl}\Big)CT^2(B-1) - \{2\}^2, \quad (3.45)$$

$$\{2\} = T(B-1) - 1 + \Big(\sum_{jl}\Big)C - \Big(\sum_{jl}\Big)CT(B-1) + \Big(\sum_{jl}\Big)D. \quad (3.46)$$

Proof. — Let us first consider case *a*) where $\lambda'(0_+) \neq 0$. We note that

$$\begin{aligned}\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v+1)^{-1} = \{&1-\varepsilon(1+PB-P)^{-1}(PC+TB-T)\\ &+\varepsilon^2[(1+PB-P)^{-1}(PC+TB-T)]^2\\ &-\varepsilon^2(1+PB-P)^{-1}(PD+TC-T^2B+T^2)+0(\varepsilon^3)\}.\\ &\cdot(1+PB-P)^{-1}(P+\varepsilon T-\varepsilon^2T^2+0(\varepsilon^3)) \quad (3.47)\end{aligned}$$

as in (3.38). Now (3.42) follows since by assumption we are in case III and therefore

$$(1 + PB - P)^{-1} = 1 + (1 + (\lambda')^{-1})P \quad (3.48)$$

and

$$(1 + PB - P)^{-1}P = -(\lambda')^{-1}P. \quad (3.49)$$

We go now over to the case *b*) where $\lambda'(0_+) = 0$. We start from

$$\begin{aligned}\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v+1)^{-1} = [1 + PB - P + \varepsilon + \varepsilon(PC + TB - T - 1)\\ +\varepsilon^2(PD+TC-T^2B+T^2)+0(\varepsilon^3)]^{-1}(P+\varepsilon T-\varepsilon^2T^2+0(\varepsilon^3)). \quad (3.50)\end{aligned}$$

Observing that in the present case we have

$$PB = 0 \quad (3.51)$$

and

$$(1 - P + \varepsilon)^{-1} = \frac{1}{\varepsilon}P + (1+\varepsilon)^{-1}(1-P) \quad (3.52)$$

we get

$$\begin{aligned}\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v+1)^{-1} = \{&1+(1-P+PC)^{-1}[\varepsilon(1+\varepsilon)^{-1}(P-1)+\varepsilon(1+\varepsilon)^{-1}T(B-1)\\ &+\varepsilon PD+\varepsilon^2TC-\varepsilon^2T^2(B-1)+0(\varepsilon^3)]\}^{-1}(1-P+PC)^{-1}\left[\frac{1}{\varepsilon}P+(1+\varepsilon)^{-1}(1-P)\right].\\ &\cdot(P+\varepsilon T-\varepsilon^2T^2+0(\varepsilon^3)). \quad (3.53)\end{aligned}$$

From

$$[1 - P + PC]^{-1} = 1 - \Big(\sum_{jl}\Big)(C-1), \quad (3.54)$$

$$[1 - P + PC]^{-1}P = \Big(\sum_{jl}\Big), \quad (3.55)$$

we finally obtain

$$\varepsilon(\lambda(\varepsilon)u\mathrm{G}_{\varepsilon k}v + 1)^{-1} = \{1-\varepsilon\{2\}-\varepsilon^2\{1\}+0(\varepsilon^3)\}\left[\frac{1}{\varepsilon}\Big(\sum_{jl}\Big)+(1-\varepsilon+\varepsilon^2)\right.$$

$$\left.-(1-\varepsilon+\varepsilon^2)\Big(\sum_{jl}\Big)\mathrm{C}+0(\varepsilon^3)\right](\mathrm{P}+\varepsilon\mathrm{T}-\varepsilon^2\mathrm{T}^2+0(\varepsilon^3)) \quad (3.56)$$

and, after multiplying term by term, (3.43) results.

THEOREM 3.4. — Let $e^{2a|\underline{x}|}\mathrm{V}(\underline{x}) \in \mathrm{R}$ for some $a > 0$ and assume case IV $\left(\text{in this case we have } \mathrm{P} = \sum_{j=1}^{\mathrm{N}} \frac{(\tilde{\phi}_j, .)\phi_j}{(\tilde{\phi}_j, \phi_j)}, (v, \phi_{j_0}) \neq 0 \text{ for some } j_0\right)$. If $\lambda'(0_+) \neq 0$ ((case a) below) then $\varepsilon(\lambda(\varepsilon)u\mathrm{G}_{\varepsilon k}v + 1)^{-1}$ is analytic at $\varepsilon = 0$ whereas if $\lambda'(0_+) = 0$ ((case b) below) it has a Laurent expansion around $\varepsilon = 0$. More precisely we have the following results:

<u>$a)$ $\lambda'(0_+) \neq 0$</u> : Here we have the Taylor expansion

$$\varepsilon(\lambda(\varepsilon)u\mathrm{G}_{\varepsilon k}v + 1)^{-1} = \sum_{j,l=1}^{\mathrm{N}} (\tilde{\phi}, \mathrm{B}\phi)_{jl}^{-1}(\tilde{\phi}_l, .)\phi_j + \varepsilon\mathrm{T} - \varepsilon\frac{ik}{4\pi}\left(\Big(\sum_{jl}{}'\Big)^* v, .\right)\mathrm{T}u$$

$$-\varepsilon\frac{ik}{4\pi}(\mathrm{T}^*v, .)\Big(\sum_{jl}{}'\Big)u + \varepsilon\left(\frac{ik}{4\pi}\right)^2(v, \mathrm{T}u)\left(\Big(\sum_{jl}{}'\Big)^* v, .\right)\Big(\sum_{jl}{}'\Big)u$$

$$-\varepsilon\Big(\sum_{jl}{}'\Big)\mathrm{C}\Big(\sum_{j'l'}{}'\Big) + 0(\varepsilon^2) \quad (3.57)$$

where the abbreviation

$$\Big(\sum_{jl}{}'\Big) = \sum_{j,l=1}^{\mathrm{N}} (\tilde{\phi}, \mathrm{B}\phi)_{jl}^{-1}(\tilde{\phi}_l, .)\phi_j \quad (3.58)$$

has been used and $(\tilde{\phi}, \mathrm{B}\phi)_{jl}^{-1}$ denotes the inverse matrix of $(\tilde{\phi}_j, \mathrm{B}\phi_l)$.

<u>$b)$ $\lambda'(0_+) = 0$</u> : Here we have the Laurent expansion

$$\varepsilon(\lambda(\varepsilon)u\mathrm{G}_{\varepsilon k}v+1)^{-1} = \{3\}\left[\frac{1}{\varepsilon}\mathrm{P} - \frac{1}{\varepsilon}(v, \mathrm{P}u)^{-1}(\mathrm{P}^*v, .)\mathrm{P}u + \frac{4\pi}{ik}(v, \mathrm{P}u)^{-2}(\mathrm{P}^*v, .)\mathrm{P}u\right]$$

$$-\{3\}\left[\frac{4\pi}{ik}(v, \mathrm{P}u)^{-2}((\mathrm{C}^*-1)\mathrm{P}^*v, .)\mathrm{P}u + (1-(v, \mathrm{P}u)^{-1}(v, .)\mathrm{P}u)(\mathrm{TB}-\mathrm{T}+\mathrm{P}-1)\right.$$

$$\left.+\mathrm{PD}-(v, \mathrm{P}u)^{-1}(\mathrm{D}^*\mathrm{P}^*v, .)\mathrm{P}u\right]\{3\}[\mathrm{P}-(v, \mathrm{P}u)^{-1}(\mathrm{P}^*v, .)\mathrm{P}u] + 0(\varepsilon) \quad (3.59)$$

where $\{3\}$ denotes the expression

$$\{3\} = \left\{ 1 + \left[\sum_{j',l'=1}^{N} (v, \phi_{j'})(\tilde{\phi}, C\phi)^{-1}_{j'l'}(\tilde{\phi}_{l'}, u) \right]^{-1} \cdot \right.$$

$$\left. \cdot \sum_{j,l=1}^{N} (\tilde{\phi}, C\phi)^{-1}_{jl}(\tilde{\phi}_l, u)((C^* - 1)P^*v,.)\phi_j \right\} \cdot \left\{ 1 - \sum_{j,l=1}^{N} (\tilde{\phi}, C\phi)^{-1}_{jl}((C^* - 1)\tilde{\phi}_l, .)\phi_j \right\}. \quad (3.60)$$

Proof. — *a*) $\lambda'(0_+) \neq 0$: In this case inserting

$$(1 + PB - P)^{-1} = 1 - \left(\sum_{jl}{}'\right)(B - 1), \quad (3.61)$$

$$(1 + PB - P)^{-1}P = \left(\sum_{jl}{}'\right) \quad (3.62)$$

into (3.38) we obtain (3.57).

b) $\lambda'(0_+) = 0$: Here we insert

$$(1 + PB - P + \varepsilon)^{-1} = \left[1 - \frac{ik}{4\pi}\left(\varepsilon + \frac{ik}{4\pi}(v, Pu)\right)^{-1}(v,.)Pu \right] \left[\frac{1}{\varepsilon}P + (1+\varepsilon)^{-1}(1 - P) \right] \quad (3.63)$$

into (3.50) and get

$$\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1} = \{1 + \varepsilon[1 + P(C-1) - (v, Pu)^{-1}((C^* - 1)P^*v, .)Pu]^{-1} \cdot$$
$$\cdot \left[\frac{4\pi}{ik}(v, Pu)^{-2}(P^*v, .)Pu + (1 - (v, Pu)^{-1}(v, .)Pu)(1 - P) \right] [PC + TB - T - 1]$$
$$+ \varepsilon[1 + P(C-1) - (v, Pu)^{-1}((C^* - 1)P^*v, .)Pu]^{-1}[PD - (v, Pu)^{-1}(D^*P^*v, .)Pu]$$
$$+ 0(\varepsilon^2)\}^{-1} \cdot$$
$$\cdot [1 + P(C-1) - (v, Pu)^{-1}((C^* - 1)P^*v, .)Pu]^{-1} \cdot$$
$$\cdot (1 + PB - P + \varepsilon)^{-1}(P + \varepsilon T + 0(\varepsilon^2)). \quad (3.64)$$

From

$$[1 + P(C - 1) - (v, Pu)^{-1}((C^* - 1)P^*v, .)Pu]^{-1} = \{3\} \quad (3.65)$$

(cf. (3.60)) we finally obtain (3.59).

REMARK 3.3. — Although some of these expressions for $\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}$ are complicated the computation of the scattering amplitude in all of these cases is almost trivial (see the next section).

We finally present the

Proof of Theorem 2.1. — Using the resolvent equation we get

$$(H_\varepsilon - k^2)^{-1} - G_k = -\lambda(\varepsilon)G_kV_\varepsilon G_k + \varepsilon^{-2}\lambda^2(\varepsilon)G_kU_\varepsilon VG_kVU_\varepsilon^{-1}G_k$$
$$- \varepsilon^{-2}\lambda^3(\varepsilon)G_kU_\varepsilon VG_{\varepsilon k}v(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}uG_{\varepsilon k}VU_\varepsilon^{-1}G_k. \quad (3.66)$$

The first two terms on the right hand side of (3.66) tend to zero as $\varepsilon \to 0_+$ in norm [4], so we only have to study the third term. Following the notation in [4] we have

$$-\varepsilon^{-2}\lambda^3(\varepsilon)G_kU_\varepsilon VG_{\varepsilon k}v(\lambda(\varepsilon)uG_{\varepsilon k}v+1)^{-1}uG_{\varepsilon k}VU_\varepsilon^{-1}G_k = -\lambda^3(\varepsilon)A_\varepsilon B_\varepsilon D_\varepsilon B_\varepsilon C_\varepsilon \tag{3.67}$$

where A_ε, B_ε, C_ε are Hilbert Schmidt operators with integral kernels

$$A_\varepsilon(\underline{x}, \underline{y}) = g_k(\underline{x} - \varepsilon\underline{y})v(\underline{y}), \tag{3.68}$$

$$B_\varepsilon(\underline{x}, \underline{y}) = u(\underline{x})g_{\varepsilon k}(\underline{x} - \underline{y})v(\underline{y}), \tag{3.69}$$

$$C_\varepsilon(\underline{x}, \underline{y}) = u(\underline{x})g_k(\varepsilon\underline{x} - \underline{y}), \tag{3.70}$$

and

$$D_\varepsilon = \varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}. \tag{3.71}$$

In cases I and II and if $\lambda'(0_+) \neq 0$ in cases III and IV D_ε converges to some operator D_0 in norm as $\varepsilon \to 0_+$ (cf. (3.36), (3.37), (3.42), and (3.57)). Also B_ε tends to $B_0 = uG_0v$ in norm, A_ε converges weakly to $A_0 = (v, .)g_k$, and C_ε converges strongly to $C_0 = (\overline{g_k}, .)u$ in the limit $\varepsilon \to 0_+$ [4].

But from

$$\| A_\varepsilon \|_{H-S} = \| A_0 \|_{H-S}, \qquad \| C_\varepsilon \|_{H-S} = \| C_0 \|_{H-S} \tag{3.72}$$

we infer that actually A_ε and C_ε tend to A_0 and C_0 in Hilbert Schmidt norm (see [38] and the references cited there). Thus (3.67) converges to $-A_0B_0D_0B_0C_0$ in norm, in the cases considered. From the explicit form of D_0 we then get norm resolvent convergence and (2.30). If $\lambda'(0_+) = 0$ in cases III and IV we have to proceed in a different way since D_ε has no limit as $\varepsilon \to 0_+$. We only discuss case IV (case III is similar). From (3.59) we infer

$$D_\varepsilon = \frac{1}{\varepsilon}D_{-1} + D_0 + O(\varepsilon) \tag{3.73}$$

where

$$D_{-1}u = D^*_{-1}v = 0 \tag{3.74}$$

and

$$(v, D_0u) = 4\pi/ik. \tag{3.75}$$

With $f, g \in C_0^\infty(\mathbb{R}^3 - \{0\})$ we thus obtain

$$-\lambda^3(\varepsilon)(f, A_\varepsilon B_\varepsilon D_\varepsilon B_\varepsilon C_\varepsilon g) = -\lambda^3(\varepsilon)\frac{1}{\varepsilon}(f, A_\varepsilon B_\varepsilon D_{-1}B_\varepsilon C_\varepsilon g)$$
$$- \lambda^3(\varepsilon)(f, A_\varepsilon B_\varepsilon D_0 B_\varepsilon C_\varepsilon g) + O(\varepsilon), \tag{3.76}$$

where we used the assumption that $(1 + |\underline{x}|)^4V(\underline{x}) \in L^1(\mathbb{R}^3)$ to control the remainder. Now

$$-\lambda^3(\varepsilon)(f, A_\varepsilon B_\varepsilon D_0 B_\varepsilon C_\varepsilon g) \underset{\varepsilon\to 0_+}{\longrightarrow} -(A_0^*f, B_0D_0B_0C_0g) = -(A_0^*f, D_0C_0g)$$
$$= -(f, g_k)(v, D_0u)(g_k, g)$$
$$= -(4\pi/ik)(f, g_k)(g_k, g) \tag{3.77}$$

where g_k is defined by (2.29), and therefore the first term on the right hand side of (3.76) remains to be discussed. From the expansion

$$B_\varepsilon = B_0 + \varepsilon B_1 + 0(\varepsilon^2)$$

and (3.74) we get

$$\begin{aligned} -\lambda^3(\varepsilon)\frac{1}{\varepsilon}(f, A_\varepsilon B_\varepsilon D_{-1} B_\varepsilon C_\varepsilon g) &= -\frac{1}{\varepsilon}(f, A_\varepsilon B_0 D_{-1} B_0 C_\varepsilon g) + 0(\varepsilon) \\ &= -\frac{1}{\varepsilon}(f, A_\varepsilon D_{-1} C_\varepsilon g) + 0(\varepsilon) \qquad (3.78) \end{aligned}$$

where we used that $B_0 D_{-1} B_0 = D_{-1}$. Since f and g have support bounded away from zero we can expand the matrix element in (3.78) to get

$$\begin{aligned} -\frac{1}{\varepsilon}(f, A_\varepsilon D_{-1} C_\varepsilon g) &= -\frac{1}{\varepsilon}(f, g_k)(v, D_{-1}u)(g_k, g) \\ &- (f, g_k)\sum_{r=1}^{3}\int d^3x d^3x' D_{-1}(\underline{x}, \underline{x}')v(\underline{x})u(\underline{x}')\underline{x}'_r((\nabla g_k)_r, g) \qquad (3.79) \\ &+ \sum_{r=1}^{3}(f, (\nabla g_k)_r)\int d^3x d^3x' \underline{x}_r v(\underline{x}) D_{-1}(\underline{x}, \underline{x}')u(\underline{x}')(g_k, g) + 0(\varepsilon) = 0(\varepsilon) \end{aligned}$$

by (3.74). Here $D_{-1}(\underline{x}, \underline{x}')$ denotes the kernel of D_{-1}. Since f and g were taken arbitrarily in $C_0^\infty(\mathbb{R}^3 - \{0\})$ we have proved

$$w - \lim_{\varepsilon \to 0_+} (H_\varepsilon - k^2)^{-1} = G_k - \frac{4\pi}{ik}(\overline{g_k}, \cdot)g_k, \ \text{Im}\, k > 0 \qquad (3.80)$$

and since the limit is the resolvent of a self-adjoint operator (namely the resolvent of $-\Delta_\alpha$ with $\alpha = 0$) strong resolvent convergence follows.

4. THE LOW ENERGY BEHAVIOUR OF THE SCATTERING AMPLITUDE AND SCATTERING MATRIX

We first recall that, from (2.17)

$$f_\varepsilon(\underline{p}, \underline{q}, k) = \varepsilon f(\varepsilon, \varepsilon\underline{p}, \varepsilon\underline{q}, \varepsilon k) = -(4\pi)^{-1}\lambda(\varepsilon)(ve^{i\varepsilon\underline{p}\cdot}, \varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}ue^{i\varepsilon\underline{q}\cdot}),$$
$$|\,\text{Im}\,\underline{p}\,| < a, \qquad |\,\text{Im}\,\underline{q}\,| < a, \qquad \text{Im}\, k > -a \quad (4.1)$$

under condition (2.12). To get the desired expansion of $f_\varepsilon(\underline{p}, \underline{q}, k)$ we thus only need to expand $\lambda(\varepsilon)$, $e^{i\varepsilon\underline{p}\cdot}$, $e^{i\varepsilon\underline{q}\cdot}$ and use our previous results on $\varepsilon(\lambda(\varepsilon)uG_{\varepsilon k}v + 1)^{-1}$. We state the results in a series of Lemmas, corresponding to the different cases I-IV discussed in section 2.

LEMMA 4.1. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and assume case I. Then

$$- 4\pi f_\varepsilon(\underline{p}, \underline{q}, k) = \varepsilon(v, Tu) - \varepsilon^2 \frac{ik}{4\pi}(v, Tu)^2 + \varepsilon^2\lambda'(v, T^2u) - i\varepsilon^2(\underline{p}.v, Tu) + i\varepsilon^2(T^*v, \underline{q}.u) + 0(\varepsilon^3). \quad (4.2)$$

Proof. — Insert (3.36) into (4.1) and expand $\lambda(\varepsilon)$ and both exponentials using the analyticity of all functions involved.

LEMMA 4.2. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and assume case II. Then

$$\begin{aligned} -4\pi f_\varepsilon(\underline{p}, \underline{q}, k) = &\left[\frac{ik}{4\pi} - \lambda'(\tilde{\phi}, \phi)|(v, \phi)|^{-2}\right]^{-1} + \varepsilon\lambda'\left[\frac{ik}{4\pi} - \lambda'(\tilde{\phi}, \phi)|(v, \phi)|^{-2}\right]^{-1} \\ &- \varepsilon\left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-2}|(v, \phi)|^2(\tilde{\phi}, C\phi) \\ &+ \varepsilon(\lambda')^2\left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-2}(\tilde{\phi}, \phi)^2(v, Tu) \\ &- i\varepsilon\left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-1}(\tilde{\phi}, u)(\underline{p}.v, \phi) \\ &+ i\varepsilon\left[\frac{ik}{4\pi}|(v, \phi)|^2 - \lambda'(\tilde{\phi}, \phi)\right]^{-1}(v, \phi)(\tilde{\phi}, \underline{q}.u) + 0(\varepsilon^2). \quad (4.3) \end{aligned}$$

Proof. — Insert (3.37) into (4.1) and expand as in Lemma 4.1.

LEMMA 4.3. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and assume case III. Then

$$\begin{aligned} - 4\pi f_\varepsilon(\underline{p}, \underline{q}, k) = &\ \varepsilon(v, Tu) - \varepsilon^2 \frac{ik}{4\pi}(v, Tu)^2 \\ &- \varepsilon^2\lambda'(v, T^2u) - \varepsilon^2(\lambda')^{-1}\sum_{j=1}^{N}\frac{(\underline{p}.v, \phi_j)(\tilde{\phi}_j, \underline{q}.u)}{(\tilde{\phi}_j, \phi_j)} \\ &- i\varepsilon^2(\underline{p}.v, Tu) + i\varepsilon^2(T^*v, \underline{q}.u) + 0(\varepsilon^3) \qquad \text{if} \quad \lambda'(0_+) \neq 0, \quad (4.4) \end{aligned}$$

$$- 4\pi f_\varepsilon(\underline{p}, \underline{q}, k) = \varepsilon(v, Tu) + \varepsilon\sum_{j,l=1}^{N}(\underline{p}.v, \phi_j)(\tilde{\phi}, C\phi)_{jl}^{-1}(\tilde{\phi}_l, \underline{q}.u) + 0(\varepsilon^2) \qquad \text{if} \quad \lambda'(0_+) = 0. \quad (4.5)$$

Proof. — Noting that $Pu = P^*v = 0$ in this case, one immediately obtains (4.4) and (4.5) after inserting (3.42) and (3.43) in (4.1).

LEMMA 4.4. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and assume case IV. Then

$$- 4\pi f_\varepsilon(\underline{p}, \underline{q}, k) = \left[\frac{ik}{4\pi} - \lambda'\left(\sum_{j=1}^{N} |(v, \phi_j)|^2/(\tilde{\phi}_j, \phi_j)\right)^{-1}\right]^{-1}$$

$$+ \varepsilon\lambda'\left[\frac{ik}{4\pi} - \lambda'\left(\sum_{j=1}^{N} |(v, \phi_j)|^2/(\tilde{\phi}_j, \phi_j)\right)^{-1}\right]^{-1}$$

$$+ \varepsilon(\lambda')^2(v, Tu)\left(\sum_{j=1}^{N} |(v, \phi_j)|^2/(\tilde{\phi}_j, \phi_j)\right)^{-2}\left[\frac{ik}{4\pi} - \lambda'\left(\sum_{j=1}^{N} |(v, \phi_j)|^2/(\tilde{\phi}_j, \phi_j)\right)^{-1}\right]^{-2}$$

$$- \varepsilon \sum_{j,l=1}^{N} \sum_{j',l'=1}^{N} (v, \phi_j)(\tilde{\phi}, B\phi)_{jl}^{-1}(\tilde{\phi}_l, C\phi_{j'})(\tilde{\phi}, B\phi)_{j'l'}^{-1}(\tilde{\phi}_{l'}, u)$$

$$- i\varepsilon \sum_{j,l=1}^{N} (\underline{p}.v, \phi_j)(\tilde{\phi}, B\phi)_{jl}^{-1}(\tilde{\phi}_l, u)$$

$$+ i\varepsilon \sum_{j,l=1}^{N} (v, \phi_j)(\tilde{\phi}, B\phi)_{jl}^{-1}(\tilde{\phi}_l, \underline{q}.u) + 0(\varepsilon^2) \quad \text{if} \quad \lambda'(0_+) \neq 0, \tag{4.6}$$

$$- 4\pi f_\varepsilon(\underline{p}, \underline{q}, k) = \frac{4\pi}{ik} + 0(\varepsilon) \quad \text{if} \quad \lambda'(0_+) = 0. \tag{4.7}$$

Proof. — From

$$\sum_{j,l=1}^{N} (v, \phi_j)(\tilde{\phi}, B\phi)_{jl}^{-1}(\tilde{\phi}_l, u) = \left[\frac{ik}{4\pi} - \lambda'\left(\sum_{j=1}^{N} |(v, \phi_j)|^2/(\tilde{\phi}_j, \phi_j)\right)^{-1}\right]^{-1} \tag{4.8}$$

(4.6) immediately follows from (3.57) and (4.1). (4.7) requires a somewhat lengthy but straightforward calculation using (3.59) and (4.1).

Lemmas 4.1-4.4 show that $f_\varepsilon(\underline{p}, \underline{q}, k)$ is analytic at $\varepsilon = 0$ in all the cases I-IV. Since we are actually interested in the low energy behaviour of $H = -\Delta + V$ we summarize the above results in the special case $\lambda(\varepsilon) = 1$. As before we state the results for the cases I-IV of section 2 separately:

THEOREM 4.1. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and denote by $f(\underline{p}, \underline{q}, k)$ the scattering amplitude associated with $H = -\Delta + V$

$$f(\underline{p}, \underline{q}, k) = -(4\pi)^{-1}(ve^{i\underline{p}\cdot}, (uG_k v + 1)^{-1}ue^{i\underline{q}\cdot}). \tag{4.9}$$

Then we have the low energy expansions

$$f(\varepsilon \underline{p}, \varepsilon \underline{q}, \varepsilon k) = -(4\pi)^{-1}(v, (uG_0 v + 1)^{-1}u) + i(4\pi)^{-2}(\varepsilon k)(v, (uG_0 v + 1)^{-1}u)^2 + (4\pi)^{-1} i(\varepsilon \underline{p} . v, (uG_0 v + 1)^{-1}u) - (4\pi)^{-1} i(v, (uG_0 v + 1)^{-1} \varepsilon \underline{q} . u) + 0(\varepsilon^2) \quad \text{in case I,} \quad (4.10)$$

$$f(\varepsilon \underline{p}, \varepsilon \underline{q}, \varepsilon k) = \frac{i}{\varepsilon k} - 4\pi \,|\, (v, \phi) \,|^{-2}(\tilde{\phi}, \hat{C}\phi) + k^{-1}(v, \phi)^{-1}(\underline{p} . v, \phi) - k^{-1}(\tilde{\phi}, u)^{-1}(\tilde{\phi}, \underline{q} . u) + 0(\varepsilon) \quad \text{in case II}, \quad (4.11)$$

$$f(\varepsilon \underline{p}, \varepsilon \underline{q}, \varepsilon k) = -(4\pi)^{-1}(v, Tu) - (4\pi)^{-1}k^{-2} \sum_{j,l=1}^{N} (\underline{p} . v, \phi_j)(\tilde{\phi}, \hat{C}\phi)_{jl}^{-1}(\tilde{\phi}_l, \underline{q} . u) + 0(\varepsilon) \quad \text{in case III}, \quad (4.12)$$

$$f(\varepsilon \underline{p}, \varepsilon \underline{q}, \varepsilon k) = \frac{i}{\varepsilon k} + 0(1) \quad \text{in case IV} \quad (4.13)$$

where $\hat{C}$ is defined by (3.35) and T is given by (3.4). Here ϕ is such that $\psi = G_0 v \phi$ is the zero energy resonance function of H in case II. Moreover $\psi_j = G_0 v \phi_j$ are the zero energy bound state functions of H in case III.

Having discussed the (off-shell) scattering amplitude $f(\varepsilon \underline{p}, \varepsilon \underline{q}, \varepsilon k)$ we now turn to the on-shell scattering matrix $S(k)$ associated with the pair $-\Delta$, H.

We recall that $S(k)$ is a unitary operator in $L^2(S^{(2)})$ ($S^{(2)}$ the unit sphere in $\mathbb{R}^3$) such that the integral kernel of $S(k) - 1$ (1 the unit operator in $L^2(S^{(2)})$) is proportional to k times the on-shell scattering amplitude

$$f(k, \underline{\omega}, \underline{\omega}') = f(\underline{p}, \underline{q}, k)\,|_{|\underline{p}| = |\underline{q}| = k}, \quad \underline{p}, \underline{q}, k \text{ real}, \quad \underline{\omega} = \underline{p}/|\underline{p}|, \quad \underline{\omega}' = \underline{q}/|\underline{q}|. \quad (4.14)$$

More precisely we have [*7*] [*21*] [*26*] [*33*]

$$(S(k)h)(\underline{\omega}) = h(\underline{\omega}) - (2\pi i)^{-1}k \int_{S^{(2)}} d\omega' f(k, \underline{\omega}, \underline{\omega}')h(\underline{\omega}'), \quad h \in L^2(S^{(2)}). \quad (4.15)$$

Hence taking $|\underline{p}| = |\underline{q}| = k$ in (4.10)-(4.13) we get the low energy expansion of $S(k)$:

THEOREM 4.2. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and assume case I. Then the scattering matrix $S(\varepsilon k)$ is analytic in ε with the Taylor expansion

$$S(\varepsilon k) = 1 + (2\pi i)^{-1}(\varepsilon k)(v, (uG_0 v + 1)^{-1}u)(Y_0, .)Y_0 - (8\pi^2)^{-1}(\varepsilon k)^2(v, (uG_0 v + 1)^{-1}u)^2(Y_0, .)Y_0 - (\varepsilon k)^2(Y_0, .)Y_1 + (\varepsilon k)^2(Y_1, .)Y_0 + 0((\varepsilon k)^3), \quad k > 0 \quad (4.16)$$

where

$$Y_0(\underline{\omega}) = (4\pi)^{-1/2} \quad \text{and} \quad Y_1(\underline{\omega}) = (4\pi^{3/2})^{-1} \int d^3x(\underline{\omega} \cdot \underline{x})v(\underline{x})((uG_0 v + 1)^{-1}u)(\underline{x}) \quad (4.17)$$

are spherical harmonics of degree zero and one.

Note that $Y_1(\underline{\omega}) = 0$ if V is spherically symmetric.

THEOREM 4.3. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and assume case II. Then the scattering matrix $S(\varepsilon k)$ is analytic in ε with the Taylor expansion

$$S(\varepsilon k) = 1 - 2(Y_0, .)Y_0 - 8\pi i(\varepsilon k) | (v, \phi) |^{-2}(\tilde{\phi}, \hat{C}\phi)(Y_0, .)Y_0 \\ + i(\varepsilon k)(v, \phi)^{-1}(Y_0, .)\hat{Y}_1 - i(\varepsilon k)(\tilde{\phi}, u)^{-1}(\hat{Y}_1, .)Y_0 + 0((\varepsilon k)^2), \quad k>0 \quad (4.18)$$

where $\hat{C}$ is given by (3.35) and

$$\hat{Y}_1(\underline{\omega}) = \pi^{-1/2}(\underline{\omega}.v, \phi) = -\pi^{-1/2}\int d^3x(\underline{\omega}\cdot\underline{x})V(\underline{x})\psi(\underline{x}). \quad (4.19)$$

THEOREM 4.4. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and assume case III. Then the scattering matrix $S(\varepsilon k)$ is analytic in ε and has the Taylor expansion

$$S(\varepsilon k) = 1 + (2\pi i)^{-1}(\varepsilon k)(v, Tu)(Y_0, .)Y_0 \\ + (8\pi i)^{-1}(\varepsilon k)\sum_{j,l=1}^{N}(\tilde{\phi}, \hat{C}\phi)_{jl}^{-1}(\hat{Y}_{1l}, .)\hat{Y}_{1j} + 0((\varepsilon k)^2), \quad k > 0 \quad (4.20)$$

where T is given by (3.4) and

$$\hat{Y}_{1j}(\underline{\omega}) = \pi^{-1/2}(\underline{\omega}.v, \phi_j) = -\pi^{-1/2}\int d^3x(\underline{\omega}\cdot\underline{x})V(\underline{x})\psi_j(\underline{x}). \quad (4.21)$$

If V is spherically symmetric then $\hat{Y}_{1j}(\underline{\omega}) \neq 0$ if ψ_j correspond to p-wave bound states.

THEOREM 4.5. — Let $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ and assume case IV. Then the scattering matrix $S(\varepsilon k)$ is analytic in ε and has the Taylor expansion

$$S(\varepsilon k) = 1 - 2(Y_0, .)Y_0 + 0(\varepsilon k), \quad k > 0. \quad (4.22)$$

REMARK 4.1. — (4.10), (4.13) and (4.20) are new. (4.16), (4.18) (in less explicit form), and (4.22) have been derived by Jensen and Kato [*21*] with the help of weighted Sobolev spaces. Since they use weaker conditions on V (but see Remark 4.4) they get asymptotic expansions of $S(k)$ instead of Taylor expansions at $k = 0$. All the above formulas generalize known results from the special case where V is spherically symmetric (see e. g. [*7*] [*28*] and the references therein) to the non central case.

We now discuss some of the physical consequences of these results:

i) In cases I and III $(4\pi)^{-1}(v, (uG_0v + 1)^{-1}u)$ resp. $(4\pi)^{-1}(v, Tu)$ are nothing else but the scattering lengths. Actually

$$(4\pi)^{-1}(v, Tu) = (4\pi)^{-1}(v, [1 - P]\Phi) \quad (4.23)$$

where Φ fulfills the inhomogeneous equation

$$\Phi = u - uG_0v\Phi \quad (4.24)$$

In case I (4.23) and (4.24) are obvious since the homogeneous equation $\tilde{\phi} = -vG_0u\tilde{\phi}$ has no solutions ($P = 0$). In case III there are solutions $\tilde{\phi}_j$, $j = 1, \ldots, N$ of $\tilde{\phi} = -vG_0u\tilde{\phi}$ but since u is orthogonal to all

$\tilde{\phi}_j$, $(\tilde{\phi}_j, u) = 0, j = 1, \ldots, N$ Fredholm's alternative applies again and we obtain

$$Tu = [1 - P]\Phi \tag{4.25}$$

and thus (4.23). Hence in case I scattering in the low energy limit (ε small) is independent of the detailed shape of the potential V and determined by the scattering length. This fact is clearly not confined to local potentials V but also holds if non local interactions are present (see [*30*]).

ii) Case III contains a striking fact. If the second term on the right-hand side of (4.12) is non vanishing the scattering cross section even at zero energy never becomes isotropic. For spherically symmetric potentials this quisotropy is known to occur precisely at zero energy *p*-wave bound states.

iii) In cases II and IV we recognize a fact which is also well known in the spherically symmetric case: if there exists a resonance at zero energy (in the sense that the associated wave function ψ is not in $L^2(\mathbb{R}^3)$) the scattering matrix converges to -1 in the subspace of angular momentum zero as $\varepsilon \to 0_+$. (If V is spherically symmetric, then zero energy resonances only occur in the *s*-wave and the corresponding phase shift $\delta_{l=0}(\varepsilon k)$ tends to $\pi/2$ implying $S_{l=0}(\varepsilon k) = e^{2i\delta_{l=0}(\varepsilon k)} \underset{\varepsilon \to 0_+}{\longrightarrow} -1$). In these two cases u is not orthogonal to at least one $\tilde{\phi}_{j_0}$ and hence (4.24) has no solution in $L^2(\mathbb{R}^3)$. As a consequence the scattering length becomes infinite.

Next we give a short description of complex poles in $f_\varepsilon(\underline{p}, \underline{q}, k)$. For a classification of these poles see [*40*], for recent discussions on resonances and threshold behaviour of eigenvalues see [*6*] [*20*] [*24*] [*25*] [*31*] [*36*] [*37*]. In order to deal with bound states and resonances of $H(\varepsilon) = -\Delta + \lambda(\varepsilon)V$ we consider the eigenvalue problem

$$\lambda(\varepsilon)uG_{\varepsilon k(\varepsilon)}v\Phi_j = -\Phi_j, \tag{4.26}$$

assuming

$$uG_0v\phi_j = -\phi_j, \qquad j = 1, \ldots, N.$$

Then, depending on the sign of $(\lambda(\varepsilon) - 1)$ near $\varepsilon = 0$, there are the following possibilities for $k(\varepsilon)$ ([*6*] [*24*] [*31*] [*36*]):

i) In case II $k(\varepsilon)$ is analytic at $\varepsilon = 0$ and

$$k(\varepsilon) = \frac{-4\pi i\lambda'(0_+)}{|(v, \phi)|^2/(\tilde{\phi}, \phi)} + O(\varepsilon). \tag{4.27}$$

Since $(\tilde{\phi}, \phi) < 0$, $\lambda'(0_+) > 0$ corresponds to a bound state and $\lambda'(0_+) < 0$ to a virtual state of $H(\varepsilon)$. Looking at the poles of $f_\varepsilon(\underline{p}, \underline{q}, k)$ in (4.3) we obtain $k(\varepsilon)$ to zeroth order in ε. In fact $k(0_+)$ corresponds precisely to the bound state or virtual state of $-\Delta_\alpha$ (the limit of H_ε as $\varepsilon \to 0_+$ in norm resolvent sense cf. (2.30)).

ii) In case III, $k_j(\varepsilon)$ is of the type

$$k_j(\varepsilon) = \frac{c_j\sqrt{-\lambda'(0_+)}}{\varepsilon^{1/2}} + 0(1), \qquad c_j > 0 \qquad \text{if} \qquad \lambda'(0_+) \neq 0. \quad (4.28)$$

For N = 1 we have

$$c = \frac{|8\pi(\tilde{\phi}, \phi)|^{1/2}}{\left| \int d^3x d^3y v(\underline{x})\overline{\phi(\underline{x})} \,|\underline{x} - \underline{y}|\, v(\underline{y})\phi(\underline{y}) \right|^{1/2}}. \quad (4.29)$$

If $\lambda'(0_+) > 0$ we get a bound state ($\text{Im}\, k_j > 0$, $\text{Re}\, k_j = 0$) and a virtual state ($\text{Im}\, k_j < 0$, $\text{Re}\, k_j = 0$) whereas if $\lambda'(0_+) < 0$ we obtain a resonance pair. Since H_ε tends to $-\Delta$ in norm resolvent sense $k_j(\varepsilon)$ tends to infinity as $\varepsilon \to 0_+$.

If $\lambda'(0_+) = 0$ and N = 1 we obtain

$$k(\varepsilon) = \frac{|4\pi(\tilde{\phi}, \phi)|^{1/2}\sqrt{-\lambda''(0_+)}}{\left| \int d^3x d^3y v(\underline{x})\overline{\phi(\underline{x})} \,|\underline{x} - \underline{y}|\, v(\underline{y})\phi(\underline{y}) \right|^{1/2}} + 0(\varepsilon), \quad \lambda'(0_+)=0. \quad (4.30)$$

If $\lambda'(0_+) = 0$ and $N > 1$ one has

$$k_j(\varepsilon) = \tilde{c}_j\sqrt{-\lambda''(0_+)} + 0(\varepsilon), \qquad \tilde{c}_j > 0, \qquad \lambda'(0_+) = 0 \qquad (4.31)$$

and (as in (4.30)) $k_j(0_+) = \tilde{c}_j\sqrt{-\lambda''(0_+)}$ are the poles of $f_\varepsilon(\underline{p}, \underline{q}, k)$ in (4.5) i. e. the solutions of

$$\det(\tilde{\phi}_j, \hat{C}\phi_l) = 0. \qquad (4.32)$$

iii) In case IV one of the branches $k_j(\varepsilon)$, say $k_{j_0}(\varepsilon)$, is analytic at $\varepsilon = 0$ and

$$k_{j_0}(\varepsilon) = \frac{-4\pi i\lambda'(0_+)}{\sum_{j=1}^{N} |(v, \phi_j)|^2/(\tilde{\phi}_j, \phi_j)} + 0(\varepsilon). \qquad (4.33)$$

Note that $k_{j_0}(0_+)$ is precisely the solution of

$$\det(\tilde{\phi}_j, B\phi_l) = 0 \qquad (4.34)$$

and hence corresponds to the pole of $f_\varepsilon(\underline{p}, \underline{q}, k)$ in (4.6). Similar to case II, $k_{j_0}(0_+)$ corresponds to the bound state or virtual state of $-\Delta_\alpha$ (see (2.30)). The remaining $k_j(\varepsilon)$, $j \neq j_0$, $j = 1, \ldots, N$ behave like (4.28) or (4.31) according to whether $\lambda'(0_+) \neq 0$ or $\lambda'(0_+) = 0$.

REMARK 4.2. — If $\lambda'(0_+) = 0$ and $\lambda''(0_+) > 0$ the results of *ii*) and *iii*) imply in particular that in these cases the convergence in Theorem 2.1 is not in the norm resolvent sense.

We now summarize the discussion concerning the low energy expansion of eigenvalues and resonances (for detailed expositions and proofs see [*6*]): we have functions $k(\lambda(\varepsilon)$ resp. $k_\varepsilon(\lambda(\varepsilon))$ which give the eigenvalues and resonances for $H(\varepsilon) = -\Delta + \lambda(\varepsilon)V(\underline{x})$ resp. $H_\varepsilon = -\Delta + \varepsilon^{-2}\lambda(\varepsilon)V(\underline{x}/\varepsilon)$ and are defined as the solutions of

$$D_2(1 + \lambda(\varepsilon)uG_k v) = 0 \text{ resp. } D_2(1 + \varepsilon^{-2}\lambda(\varepsilon)u(\underline{.}/\varepsilon)G_k v(\underline{.}/\varepsilon)) = 0\,,$$

where D_2 denotes the modified Fredholm determinant. From (2.6) we have

$$H_\varepsilon = \varepsilon^{-2}U_\varepsilon H(\varepsilon)U_\varepsilon^{-1}\,,$$

and therefore

$$k_\varepsilon(\lambda(\varepsilon)) = \varepsilon^{-1}k(\lambda(\varepsilon))\,. \tag{4.35}$$

We also note that $k_\varepsilon(\lambda(\varepsilon)) = k(\varepsilon)$, where $k(\varepsilon)$ satisfies (4.26). It is shown in [*6*] that the functions $k(\lambda(\varepsilon))$ have at most branch points of finite order as singularities and else are holomorphic in λ. Using then Puiseux resp. Taylor expansions for $k(\lambda(\varepsilon))$ and the above results (4.27)-(4.34) on $k(\varepsilon) = k_\varepsilon(\lambda(\varepsilon))$, together with formula (4.35) we obtain the following

THEOREM 4.6. — Let $V \in R$ with compact support and assume $\lambda(\varepsilon)$ is real analytic. Let $k(\lambda(\varepsilon))$ resp. $k_\varepsilon(\lambda(\varepsilon))$ be the functions giving the eigenvalues and resonances of $H(\varepsilon) = -\Delta + \lambda(\varepsilon)V(\underline{x})$ resp. $H_\varepsilon = -\Delta + \varepsilon^{-2}\lambda(\varepsilon)V(\underline{x}/\varepsilon)$, then we have

$$k_\varepsilon(\lambda(\varepsilon)) = \varepsilon^{-1}k(\lambda(\varepsilon))\,.$$

$k(\lambda(\varepsilon))$ is an analytic function of ε, except for possible branch points of finite order. As $\varepsilon \to 0_+$ we have the following low energy expansions:

1) If $k(\lambda(0_+)) \neq 0$ then

$$k_\varepsilon(\lambda(\varepsilon)) = \frac{k(\lambda(0_+))}{\varepsilon} + 0(1)\,. \tag{4.36}$$

2) If $k(\lambda(0_+)) = 0$ then *a*) in case II $k_\varepsilon(\lambda(\varepsilon))$ is analytic at $\varepsilon = 0$ and

$$k_\varepsilon(\lambda(\varepsilon)) = -\frac{4\pi i\lambda'(0_+)}{|(v, \phi)|^2/(\tilde{\phi}, \phi)} + 0(\varepsilon) \tag{4.37}$$

b) In case III with $\lambda'(0_+) \neq 0$, $k_\varepsilon(\lambda(\varepsilon))$ has N branches $k_{j,\varepsilon}(\lambda(\varepsilon)), j = 1, \ldots, N$ behaving as follows:

$$k_{j,\varepsilon}(\lambda(\varepsilon)) = \frac{c_j\sqrt{-\lambda'(0_+)}}{\varepsilon^{1/2}} + 0(1)\,, \qquad c_j > 0\,. \tag{4.38}$$

For N = 1 we have

$$c = \frac{|8\pi(\tilde{\phi}, \phi)|^{1/2}}{\left|\int d^3x d^3y v(\underline{x})\overline{\phi(\underline{x})}\,|\underline{x} - \underline{y}|\,v(\underline{y})\phi(\underline{y})\right|^{1/2}}\,. \tag{4.39}$$

If $\lambda'(0_+) = 0$ then we have for $N = 1$

$$k_\varepsilon(\lambda(\varepsilon)) = \frac{|4\pi(\tilde{\phi}, \phi)|^{1/2}\sqrt{-\lambda''(0_+)}}{\left|\int d^3x d^3y v(\underline{x})\overline{\phi(\underline{x})} |\underline{x} - \underline{y}| v(\underline{y})\phi(\underline{y})\right|^{1/2}} + O(\varepsilon) \tag{4.40}$$

and for $N > 1$ we have again N branches $k_{j,\varepsilon}(\lambda(\varepsilon))$ behaving like

$$k_{j,\varepsilon}(\lambda(\varepsilon)) = \tilde{c}_j\sqrt{-\lambda''(0_+)} + O(\varepsilon), \qquad \tilde{c}_j > 0. \tag{4.41}$$

c) In case IV one of the branches $k_{j_0,\varepsilon}(\lambda(\varepsilon))$ is analytic at $\varepsilon = 0$ and

$$k_{j_0,\varepsilon}(\lambda(\varepsilon)) = \frac{-4\pi i\lambda'(0_+)}{\sum_{j=1}^{N} |(v, \phi_j)|^2/(\tilde{\phi}_j, \phi_j)} + O(\varepsilon). \tag{4.42}$$

The remaining branches $k_{j,\varepsilon}(\lambda(\varepsilon))$ behave in case $\lambda'(0_+) \neq 0$ as in (4.38) and in case $\lambda'(0_+) = 0$ as in (4.41).

REMARK 4.3. — The formulas for the asymptotic behaviour of the energy eigenvalues and resonances in Theorem 4.6 are expansions in ε with the leading coefficients determined by suitable point interactions, as expected from the strong (resp. norm) resolvent convergence described in Theorem 2.1.

In conclusion the low energy behaviour of $-\Delta + V(\underline{x})$ is completely governed by expansions around the point interaction and is therefore largely independent of the special features of the interaction V.

We finally end with a remark concerning possible generalizations of our approach.

REMARK 4.4. — Throughout section 4 we used the strong condition $e^{2a|\underline{x}|}V(\underline{x}) \in R$ for some $a > 0$ implying exponential falloff of V at infinity. As a consequence we got strong results for $f_\varepsilon(\underline{p}, \underline{q}, k)$, $S(\varepsilon k)$, namely analyticity in ε at $\varepsilon = 0$. If we only assume $V \in R$ and $(1 + |\underline{x}|)^n V(\underline{x}) \in L^1(\mathbb{R}^3)$, $n > 2$ we obtain asymptotic expansions for $f_\varepsilon(\underline{p}, \underline{q}, k)$, $S(\varepsilon k)$ as $\varepsilon \to 0_+$, the order in ε up to which the expansion is valid depending on n. E. g. in case I if $n = 4$ we obtain Lemma 4.1, (4.10), and (4.16) with the remainder being $O(\varepsilon^3)$, $O(\varepsilon^2)$, and $O((\varepsilon k)^3)$ respectively. Corresponding results hold in all cases.

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