ANNALES DE L'I. H. P., SECTION A

S. KLAINERMAN
P. SARNAK

Explicit solutions of $\square u = 0$ on the Friedmann-Robertson-Walker space-times

Annales de l'I. H. P., section A, tome 35, nº 4 (1981), p. 253-257
<http://www.numdam.org/item?id=AIHPA_1981__35_4_253_0>

Ann. Inst. Henri Poincaré,
Vol. XXXV, n° 4, 1981, p. 253-257.

Section A:
Physique théorique.

Explicit Solutions of $\square u = 0$ on the Friedmann-Robertson-Walker Space-Times

by

S. KLAINERMAN and P. SARNAK
Courant Institute of Mathematical Sciences,
251, Mercer Street, New York, N. Y. 10012

1. In this note we calculate explicitly, in terms of weighted spherical means, the general solution to $\square u = 0$ on the « dust » models of the cosmological universe. Under the assumption of spatial homogeneity the line element of space-time takes the form

$$ds^2 = -dt^2 + S^2(t)d\sigma^2 \tag{1.1}$$

where $d\sigma^2$ is the line element for one of the 3-spheres, Euclidean 3-space or the 3 pseudo sphere (see Einstein [*1*]). The change of variable

$$\frac{d\tau}{dt} = \frac{1}{S(t)}$$

brings the line element (1.1) to the form

$$ds^2 = S^2(\tau)(d\tau^2 - d\sigma^2) \tag{1.2}$$

which is conformal to the Einstein static universe. In the dust model of Friedmann and more generally that of Robertson and Walker, the pressure and cosmological member are taken to be zero, and the function $S(\tau)$ is one of the following, depending on the sign of the spatial curvature K,

$$\begin{array}{lll} S = \cosh\tau - 1 & \text{if} & K = -1 \\ S = \tau^2 & \text{if} & K = 0 \\ S = 1 - \cos\tau & \text{if} & K = 1 \end{array} \tag{1.3}$$

(see Hawking and Ellis [*4*]).

The D'Alembertian $\square$ in the coordinates of (1.2) takes the form

$$\square u = -\frac{1}{S^2}u_{\tau\tau} - \frac{2\dot{S}}{S^3}u_\tau + \frac{1}{S^2}\Delta u \tag{1.4}$$

where Δ is the Laplace-Beltrami operator for $d\sigma^2$.

We will solve $\square u = 0$ for arbitrary initial data on the Cauchy surfaces $\tau = \tau_0 =$ constant for each of the above cases. We give details of the calculations for the pseudosphere model, $K = -1$.

2. Consider first the static universe, we will reduce the expanding universe to this case in Section 3.

We begin with the « Klein-Gordon » equation

$$u_{tt} - Lu + m^2 u = 0 \tag{2.1}$$

with initial data

$$u(x, 0) = 0, \qquad u_t(x, 0) = g(x),$$

where $Lu = \Delta u + u$, and $x = (x^1, x^2, x^3)$ is the space variable.

In geodesic polar coordinates about a point x the spatial line element reads

$$d\sigma^2 = dr^2 + \sinh^2 r d\theta^2 \tag{2.2}$$

where $d\theta$ is the line element for the sphere S^2. The Laplacian becomes

$$\frac{\partial^2}{\partial r^2} + 2\coth(r)\frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_s \tag{2.3}$$

Δ_s being the corresponding Laplacian on S^2.

Define spherical means of u about x by

$$M_u(x, r, t) = \frac{1}{4\pi(\sinh r)^2}\int_{S(x,r)} u(x', t)dA(x')$$

where $S(x, r)$ is the geodesic sphere of radius r about x, and dA its area element.

Extend $M_u(x, r, t)$ to negative values of r by $M_u(x, r, t) = M_u(x, -r, t)$.

A simple computation shows that (2.1) can be written as

$$\frac{\partial^2 M_u}{\partial t^2} = \left(\frac{\partial^2}{\partial r^2} + 2\coth(r)\frac{\partial}{\partial r} + 1 + m^2\right)M_u \tag{2.4}$$

with $M_u(x, r, 0) = 0$, $\dfrac{\partial M_u}{\partial t}(x, r, 0) = M_g(x; r)$.

Now if $v = v(r, t)$ satisfies (2.4) then $w(r, t) = \sinh(r)v(r, t)$ satisfies

$$\omega_{tt} - \omega_{rr} + m^2 w = 0; \tag{2.5}$$

that is, the classical one-dimensional telegraph equation. For given initial data

$$\omega(r, 0) = 0\,, \qquad \omega_t(r, 0) = \psi(r)\,,$$

the solution of (2.5) is

$$w(r, t) = \frac{1}{2}\int_{r-t}^{r+t} J_0(m\sqrt{t^2 - (r-s)^2})\psi(s)ds$$

where

$$J_0(z) = \frac{2}{\pi}\int_0^{\pi} \cos(z \sin\theta)d\theta.$$

Hence

$$M_u(x, r, t) = \frac{1}{2\sinh r}\int_{r-t}^{r+t} J_0(m\sqrt{t^2-(r-s)^2}) \sinh s M_g(s, x)ds.$$

Letting $r \to 0$, $M_u(x, r, t) \to u(x, t)$ so that

$$u(x, t) = \frac{1}{t}\frac{\partial}{\partial t}\int_0^t sJ_0(m\sqrt{t^2 - s^2}) \sinh s M_g(x, s)ds. \tag{2.6}$$

In the case of m being zero (2.6) simplifies to

$$u(x, t) = \sinh t M_g(x, t)\,. \tag{2.7}$$

This last case may also be solved by « progressing waves », see Lax-Phillips [5]. Notice that a distinct feature of the case $m = 0$ is the sharp propagation of signals.

Finally for $K = 0$ the general solution of (2.1) is well known while for $K = 1$ it can be written in terms of spherical harmonics.

3. Returning to (1.4), $\square u = 0$ *we see that for* $K = -1$ *this becomes*

$$u_{\tau\tau} + \frac{2\sinh\tau}{\cosh\tau - 1}u_\tau = \Delta u. \tag{3.1}$$

If $t = \tau/2$ this becomes

$$u_{tt} + 4\coth(t)u_t = 4\Delta u. \tag{3.2}$$

Now let

$$v(t, x) = \frac{1}{\sinh t}(\sinh^3(t)u)_t\,. \tag{3.3}$$

Notice that this last transformation is invertible.

$$u(t, x) = \frac{1}{(\sinh t)^3}\int_{t_0}^t (\sinh s)v(s, x)ds + u(t_0, x)\frac{(\sinh t_0)^3}{(\sinh t)^3}$$

A straightforward calculation shows that (3.2) becomes

$$v_{tt} = 4Lv$$

so that a final substitution $t' = 2t$ yields $v_{t't'} = Lv$ which is precisely (2.1) for $m = 0$! Using the explicit solution (2.7) found for (2.1) we may finally write out the complete solution (one needs to keep track of the initial conditions on the initial Cauchy surface).

If

$$u(\tau_0, x) = g(x) \quad \text{and} \quad u_\tau(\tau_0, x) = h(x) \tag{3.4}$$

are the initial data at $\tau = \tau_0$, then we let

$$\begin{aligned} \phi(x) &= (\cosh \tau_0 - 1)h(x) + \frac{3}{2} \sinh (\tau_0) g(x) \\ \psi(x) &= \sinh \tau_0 h(x) + 3 \cosh \tau_0 g(x) + 2 (\cosh \tau_0 - 1)\Delta g(x) \end{aligned} \tag{3.5}$$

The explicit solution of (3.1) in terms of weighted spherical means is:

$$\begin{aligned} &(\sinh (\tau/2))^3 u(\tau, x) \\ &= \frac{1}{4}\int_0^{\tau-\tau_0}\int_{S^2} \sinh\left(\frac{r+\tau_0}{2}\right) \sinh r \psi(r, \theta) d\omega(\theta) dr \\ &- \frac{1}{4}\int_0^{\tau-\tau_0}\int_{S^2} \cosh\left(\frac{r+\tau_0}{2}\right) \sinh r \phi(r, \theta) d\omega(\theta) dr \\ &+ \frac{1}{2} \sinh (\tau/2) \sinh (\tau - \tau_0) \int_{S^2} \phi(\tau - \tau_0, \theta) d\omega(\theta) + (\sinh (\tau_0/2))^3 g(x) \end{aligned} \tag{3.6}$$

where all integrals are in spherical polar coordinates about x.

For the Friedmann model (K = 0), the transformation corresponding to (3.3) is $v = \frac{1}{\tau}(\tau^3 u)_\tau$. This transformation has been used by Fritz John [3] in the method of descent for the wave equation in 5-dimensional Euclidian space.

The explicit solution $\Box u = 0$ with initial data (3.4) for the K = 0 model reads

$$\begin{aligned} \tau^3 u(\tau, x) &= \int_0^{\tau-\tau_0}\int_{|\theta|=1} \psi(x + \theta r)(\tau_0 + r) r d\theta dr \\ &- \int_0^{\tau-\tau_0}\int_{|\theta|=1} \phi(x + \theta r) r d\theta dr \\ &+ (\tau - \tau_0)\tau \int_{|\theta|=1} \phi(x + \theta(\tau - \tau_0)) d\theta + \tau_0^3 g(x) \end{aligned}$$

where

$$\begin{aligned} \phi(x) &= \tau_0^2 h(x) + 3\tau_0 g(x) \\ \psi(x) &= \tau_0^2 \Delta g(x) + \tau_0 h(x) + 3g(x). \end{aligned}$$

For the model K = 1 the equation $\square u = 0$ may be solved by the reduction

$$v = \frac{1}{\sin \tau}((\sin \tau)^3 u)_\tau$$

but of course this time one cannot express the solution in terms of spherical means.

Finally we remark that the conformally invariant equation

$$\left(\square + \frac{R}{6}\right)u = 0\,, \qquad \text{where} \qquad R = \text{scalar curvature}$$

may be solved explicitly for any Robertson-Walker metric (i. e. general $S(\tau)$) in (1.2) since the above operator is trivially related to the underlying static conformally invariant operator by

$$\left(\square + \frac{R}{6}\right)u = \frac{1}{S(\tau)^3}\left(\hat{\square} + \frac{\hat{R}}{6}\right)(S(\tau)u)$$

$\hat{}$ denoting the static metric. See for example Friedlander [2].

REFERENCES

[1] A. EINSTEIN, *The Meaning of Relativity.*

[2] F. FRIEDLANDER, *Wave Equations in Curved Space Time*, Cambridge Monographs.

[3] FRITZ JOHN, *Partial Differential Equations*, Springer-Verlag.

[4] S. HAWKING and G. ELLIS, *The Large Scale Structure of Space Time*, Cambridge Monographs.

[5] P. LAX and R. PHILLIPS, *Scattering Theory for Automorphic Functions*, Annals of Mathematical Studies, Princeton.

(Manuscrit reçu le 5 juin 1981)