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Section A :
Physique théorique.

A stochastic scheme for constructing solutions of the Schrödinger equations

by

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(Elworthy's class (*)).

ABSTRACT. — Stochastic differential equations on fibre bundles are used to suggest path integral solutions for certain Schrödinger equations. Three examples are discussed in detail: motion in curved spaces, motion in an external magnetic field considered as a gauge field, and multiply-connected configuration spaces.

I. A NEW POINT OF VIEW

The lagrangian-hamiltonian formalism provides reliable methods for studying classical systems. The path integral formalism could, in principle, serve the same purpose for quantum systems: choose a Lagrangian L, find out the Schrödinger equation satisfied by the path integral

$$« \int \exp\left(\frac{i}{\hbar} S(x)\right) \mathscr{D}x »,$$

(*) C.De Witt-Morette, K.D. Elworthy, B. L. Nelson (**) and G. S. Sammelman. Class work done during a course on stochastic differential equations and a course on fiber bundles given by K. D. Elworthy (on leave from the University of Warwick) in the Department of Physics of the University of Texas at Austin 1978-1979.

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and read off the hamiltonian operator $\widehat{\mathrm{H}}$. It works well for simple systems, but it has been plagued by ambiguities as soon as the system becomes more complex: curved configuration spaces, presence of gauge fields, constraints, etc. The theory of stochastic processes on fibre bundles offers a new approach to these vexing problems. In particular it enables one to decide on the « short time » propagator needed for the time slicing approach to path integration on riemannian manifolds.

Recall first the relationship between stochastic processes and the path integral solutions of diffusion equations, *i. e.* recall the Feynman-Kac formula. Given a system of stochastic differential equations

$$\begin{cases} dx(t, \omega) = \mathrm{X}(x(t, \omega))dz(t, \omega) & \text{with} \quad x(t_0, \omega) = x_0 \\ dg(t, \omega) = \langle \mathrm{A}_0(x(t, \omega)), dz(t, \omega) \rangle g(t, \omega) + \mathrm{V}(x(t, \omega))g(t, \omega)dt & \\ & \text{with} \quad g(t_0, \omega) = 1 \end{cases} \tag{1}$$

where we have used the following notation: $(\Omega, \mathscr{F}, \gamma)$ is a probability space, $\omega \in \Omega$. The measure γ on Ω is the Wiener measure. z is a brownian motion on $\mathbb{R}^n$, defined for the time interval $\mathrm{T} = [t_0, t]$, $z : \mathrm{T} \times \Omega \to \mathbb{R}^n$. The explosion time will be assumed to be infinite for simplicity. $\mathrm{X}(x(t, \omega))$ is a linear map from $\mathbb{R}^n$ into $\mathbb{R}^m$, $\mathrm{X} : \mathbb{R}^m \to \mathrm{L}(\mathbb{R}^n ; \mathbb{R}^m)$. V is a scalar potential, $\mathrm{V} : \mathbb{R}^m \to \mathbb{R}^m$.

$\mathrm{A} = \mathrm{XA}_0$ where $\mathrm{A}_0 : \mathbb{R}^m \to \mathbb{R}^n$ and A is a vector field which maps $\mathbb{R}^m$ into $\mathbb{R}^n$.

We can write

$$g(t, \omega) = \exp\left(\oint_{\mathrm{T}} \langle \mathrm{A}_0(x(s, \omega)), dz(s, \omega) \rangle + \int_{\mathrm{T}} \mathrm{V}(x(s, \omega))ds \right)$$

where $\oint$ is the Stratanovich integral, *i. e.*

$$\oint \mathrm{A}_0 dz = (It\widehat{o}) \int \mathrm{A}_0 dz + \frac{1}{2} \int | \mathrm{A}_0 |^2 dt.$$

Let f be a differentiable function on $\mathbb{R}^m$; let

$$\mathrm{F}(x_0, t) \equiv \mathrm{E}(f(x(t))g(t)) \equiv \int_\Omega d\gamma(\omega) f(x(t, \omega))g(t, \omega)$$

be the expectation value of $f(x(t, \omega))g(t, \omega)$ for the process x starting at x_0 at t_0. Note that g is not a function of $x(t, \omega)$ but a function of x, hence as far as we are concerned now, a function of t and ω. Under reasonable conditions $\mathrm{F}(x_0, t)$ satisfies the diffusion equation

$$\begin{cases} \dfrac{\mathrm{F}}{\partial t} = \mathscr{A}\mathrm{F} \\ \mathscr{A}\mathrm{F} = \dfrac{1}{2} \displaystyle\sum_{i=1}^{n} \mathrm{X}_i^\alpha(x_0)\mathrm{X}_i^\beta(x_0)\partial^2\mathrm{F}/\partial x_0^\alpha x_0^\beta + \mathrm{A}^\alpha(x_0)\partial\mathrm{F}/\partial x_0^\alpha + \mathrm{V}(x_0)\mathrm{F} \\ \mathrm{F}(x_0, t_0) = f(x_0) \end{cases} \tag{2}$$

The sum over the Greek indices, not written explicitly, runs from 1 to m.

The sequence « Stochastic differential equation—expectation value of an arbitrary function of the stochastic process—diffusion equation » is the prototype of our approach. But we start with stochastic processes on fibre bundles, (1) an approach also pursued by Eells and Malliavin, (2) and we go one step further than the prototype, namely, we compute the « WKB approximation » of the path integral to read off the Lagrangian of the system.

The choice of fibre bundle is dictated by the physical system under consideration. The three cases treated here are:

i) The frame bundle for systems with riemannian configuration spaces;

ii) The U(1) bundle over $\mathbb{R}^3$ for a nonrelativistic particle in an electromagnetic field;

iii) Multiply connected configuration spaces.

Other cases being investigated are:

iv) The SU(2) bundle over $\mathbb{R}^3$;

v) Bundles over riemannian manifolds;

vi) The Spin (4) $\times$ U(1) bundle over Minkowski space for a Dirac particle in an electromagnetic field. In this case the result is only formal;

vii) Differential generators with potentials and additional drifts (3) generalizing equation (2). This includes (4) the case of the de Rham-Hodge laplacian on differential forms $\Delta = \frac{1}{2}$ trace $\nabla^2 + K$ where the vector bundle map $K : TM \to TM$ comes from the Ricci tensor.

We are obviously interested in applications in quantum mechanics, but our prototype is in diffusion theory. We scale $\omega \in \Omega$, *i. e.* we map Ω into Ω by $S : \omega \to \mu\sqrt{s}\omega$ where $\mu = \sqrt{\hbar/m}$ and s is a « parameter » not otherwise defined. The final results are rigorous for $s \in \mathbb{R}^+$ and formal for $s = i$.

We will use two closely related results from the theory of diffusion on fibre bundles.

II. BASIC THEOREMS

Consider a smooth vector bundle $p : B \to M$ with fibre F^n with $F = \mathbb{C}$ or $\mathbb{R}$ and structure group G. Let $\Pi : G(B) \to M$ be the associated principal

(1) Equations (1) can already be thought of as defined on a fibre bundle, namely the product bundle $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$.

(2) [J. Eells and P. Malliavin, *Diffusion process in riemannian bundles*, 1972-1973, unpublished].

(3) The *potential* V in $\mathscr{A}$ is a « *vertical* » *drift* in the stochastic equation, the *drift* A in $\mathscr{A}$ is a « *vertical* » *noise* in the stochastic equation.

(4) [Airault].

G-bundle. Then any $u_0 \in G(B)$ can be considered as an admissible map [Steenrod] $u_0 : F \to B_{x_0}$, $x_0 = \Pi(u_0)$, $B_{x_0} = p^{-1}(x_0)$.

Suppose $\Pi : G(B) \to M$ has a connection. This determines

$$\tilde{X}(u) : T_{\Pi(u)}M \to T_u G(B) \quad \text{for each} \quad u \in G(B)$$

such that $\tilde{X}(u)e$ is horizontal and $T\Pi \circ \tilde{X}(u)e = e$, for all $e \in T_{\Pi(u)}M$.

The following theorems are essentially two special cases of proposition 20.B in [Elworthy, 1978]. They come from Itô's formula; case *i)* deals with an arbitrary vector bundle over $\mathbb{R}^n$, case *ii)*, which is better known ([5]), deals with the frame bundle over a riemannian manifold.

i) Let $M = \mathbb{R}^n$. Then $\tilde{X}(u) : \mathbb{R}^n \to T_u G(B)$. Let $z : [0, \infty) \times \Omega \to \mathbb{R}^n$ be a brownian motion on $\mathbb{R}^n$; and for fixed $u_0 \in \Pi^{-1}(x_0)$, let $u : [0, \infty) \times \Omega \to G(B)$ satisfy

$$du = \tilde{X}(u)dz \text{ (Stratanovich sense ([6])) with } \qquad u(0, \omega) = u_0 \text{ a. s.} \qquad (3)$$

For simplicity we assume non explosion. Note that $\Pi(u(s, \omega)) = x_0 + z(s, \omega)$ a. s. since $T\Pi \circ \tilde{X}(u)\, e = e$. Essentially $u(s, \omega)$ is the horizontal lift of $x_0 + z(s, \omega)$.

THEOREM. — Let $\varphi : B \to F$ be a linear form, *i. e.* $\varphi \mid B_x : B_x \to F$ linearly for each $x \in M$. Set $v(t, \omega) = u(t, \omega)u_0^{-1}v_0$ for $v_0 \in p^{-1}(x_0)$. Then

$$\varphi(v(t, \omega)) = \varphi(v_0) + \int_0^t \nabla_{dz(s)}\varphi(v(s, \omega)) + \frac{1}{2}\int_0^t \Delta\varphi(v(s, \omega))ds$$

where ∇_e is the covariant derivative along e for $e \in \mathbb{R}^n$ with respect to the connection on G(B) and the flat connection of $\mathbb{R}^n$ and $\Delta = \sum_{i=1}^{n} \nabla_{e_i}\nabla_{e_i}$ where $\{ e_i \}$ is an orthonormal basis for $\mathbb{R}^n$. When $E\varphi(v(t))$ exists we can set $P_t\varphi = E\varphi(v(t))$. We get then a semigroup with differential generator $\frac{1}{2}\Delta$.

For the case when the bundle is furnished with a metric (G a subset of either the orthonormal group or the unitary group), the expectation of $\varphi(v(t, \omega))$ exists if φ is bounded. If the metric is Lorentzian, the conditions under which the expectation of $\varphi(v(t, \omega))$ exists are more complex. In any of these cases we can define φ by

$$\varphi(v(t, \omega)) = \langle \hat{\varphi}(x(t, \omega)), v(t, \omega) \rangle_{x(t,\omega)}$$

([5]) [Eells, Elworthy].

([6]) See [Clark] or [Elworthy].

where $\widehat{\varphi}$ is a section of the bundle and the scalar product $\langle , \rangle_{x(t,\omega)}$ is taken with respect to the metric on the fibre $B_{x(t,\omega)}$. Then $P_t\varphi$ is defined by

$$P_t\varphi(v_0) = \langle P_t\widehat{\varphi}(x_0), v_0 \rangle_{x_0} = E \langle \widehat{\varphi}(x(t)), v(t) \rangle_{x(t)}$$
$$= \langle Eu_0 u(t)^{-1}\widehat{\varphi}(x(t)), v_0 \rangle_{x_0}.$$

The second and fourth members of this equation show that the corresponding semigroup defined on sections $\widehat{\varphi}$ is given by

$$P_t\widehat{\varphi}(x_0) = u_0 E u(t)^{-1}\widehat{\varphi}(x(t)).$$

ii) Let M be an n-dimensional riemannian manifold, let G(M) be the frame bundle on M. If we have a smooth connection σ on G(M), we can define the horizontal lift of a smooth path x on M by the ordinary differential equation

$$\frac{du}{dt} = \tilde{X}(u(t))\frac{dx}{dt} \qquad \text{with} \qquad u(t_0) = u_0 \qquad \text{and} \qquad \tilde{X} \text{ defined as before.}$$

Define $X : G(M) \to L(\mathbb{R}^n ; TG(M))$ by $X(u(t)) = \tilde{X}(u(t))u(t)$ then

$$\frac{du}{dt} = X(u(t))u^{-1}(t)\frac{dx}{dt}.$$

For a brownian motion $x(t, \omega)$ on M it is convenient to define x as the projection $\Pi \circ u$ of a process u on G(M); the u-process can be thought of as the « horizontal lift of x » but will be defined as the solution of the (Stratanovich) stochastic differential equation for $u : [0, \infty) \times \Omega \to G(M)$, given $u_0 \in G(M)$:

$$du(t, \omega) = X(u(t, \omega))dz(t, \omega) \qquad \text{with} \qquad u(0, \omega) = u_0$$

We will assume that such a (non explosive) solution exists. This is true by results of S. T. Yau (7) whenever the Ricci curvature of M is bounded below.

THEOREM 2. — The expectation value $F(x_0, t)$ of $f(\Pi u(t, \omega))$ satisfies the equation $\frac{\partial F}{\partial t} = \frac{1}{2}\Delta F$ where Δ is the laplacian on the riemannian manifold M.

(7) [Yau].

III. EXAMPLES: STOCHASTIC FRAMES AND STOCHASTIC PHASES

1) The Schrödinger equation in curved spaces (vertical drift).

Let M be a riemannian manifold and G(M) its frame bundle. Consider the following stochastic differential equation for $(u(t, \omega), v(t, \omega)) \in G(M) \times R$

$$\begin{cases} du = X(u(t))\mu\sqrt{s}dz & u(t_0, \omega) = u_0,\ \mu = \sqrt{\hbar/m} \\ dv = \dfrac{1}{\mu^2 s}\dfrac{1}{m}V(\Pi u(t))v(t)dt & v(t_0, \omega) = 1 \end{cases} \tag{4}$$

where $X(u(t)) = \tilde{X}(u(t))u(t)$ is the natural inverse of the canonical one form, defined in paragraph II. *ii*) and where the vertical drift $V : M \to \mathbb{R}$ is bounded above.

Let $S : \omega \mapsto \sqrt{s}\omega$ and let γ_s be the image of γ under S. By a straightforward application of the Itô formula, it can be shown that the expectation value $\psi(x_0, t)$ of $\psi_0(\Pi u(t))v(t)$ namely

$$\psi(x_0, t) = \int_\Omega d\gamma_s(\omega) \exp\left(\frac{1}{\mu^2 s}\frac{1}{m}\int_T V(\Pi u(s))ds\right)\psi_0(\Pi u(t)) \tag{5}$$

satisfies the Schrödinger equation

$$\frac{\partial\psi}{\partial t} = \frac{1}{2}\mu^2 s\Delta\psi + \frac{1}{\mu^2 s}\frac{V}{m}\psi \qquad \text{with} \qquad \psi(x_0, t_0) = \psi_0(x_0). \tag{6}$$

In quantum mechanics the Feynman-Kac formula (5) is usually written as an integral over the space Ω_+ of paths ω_+ vanishing at t_0. It is easily obtained from (4) by mapping $\omega \mapsto \omega_+$ with $\omega_+(s) = \omega(t + t_0 - s)$. We can read off the hamiltonian operator from equation (6).

The WKB approximation of (5) has been computed when ψ_0 is of the form (9)

$$\psi_0(x_0) = \exp(-S_0(x_0)/m\mu^2 s)T(x_0) \tag{7}$$

where T is an arbitrary well behaved function on M which does not depend on μ nor s. We can choose T to be of compact support.

S_0 is taken to be the initial value of the solution S of the Hamilton-Jacobi equation: $S(t_0, x_0) = S_0(x_0)$. In flat space, we could choose

$$\nabla S_0(x_0) = p_0$$

(8) In this section, we follow the probabilists' notation and write $z(t)$, $u(t)$, ..., for $z(t, \omega)$, $u(t, \omega)$, ..., immediately after these quantities have been introduced.

(9) [Truman].

for all x_0 and the initial wave function (7) would be a plane wave of momentum p_0. Let Z be the classical path on M such that $Z(t) = x$, and $\dot{Z}(t_0)$ is such that $\nabla_\alpha S_0(Z(t_0)) = mg_{\alpha\beta}(Z(t_0))\dot{Z}^\beta(t_0)$.

Assume $Z(t)$ to be within focal distance of $Z(t_0)$. The result [10] of a long calculation is

$$\psi_{\mathrm{WKB}}(x, t) = (\det_{\alpha,\beta} \partial Z^\alpha(t_0)/\partial Z^b(t))^{1/2} \exp\left(\frac{-1}{s\hbar}\bar{S}(t, x)\right)T(Z(t_0)) \tag{8}$$

where the action function $\bar{S}$ is the general solution of the Hamilton-Jacobi equation of the system with Cauchy data S_0 at t_0

$$\bar{S}(t, x) = S_0(Z(t_0)) + \int_T \left(\frac{m}{2}\|\dot{Z}(t)\|^2 - V(Z(t))\right)dt \tag{9}$$

We can read off the lagrangian from equation (9).

The expression for ψ_{WKB} is not unexpected but it is gratifying to obtain it by expanding (4) which is a rather difficult path integral to evaluate.

The piecewise linear approximation [11] of (5) :

When Feynman introduced the path integral formalism in quantum physics, he expressed the wave function $\psi(x, t)$ as the limit when k tends to infinity of

$$I_k = \int_{\mathbb{R}^{nk}} \bar{K}(k\,; k-1)\bar{K}(k-1\,; k-2)\ldots\bar{K}(1\,; 0)\psi_0(x_0)dv(x_0)\ldots dv(x_{k-1}) \tag{10}$$

where $\bar{K}(m\,; n) \equiv \bar{K}(t_m, x_m\,; t_n, x_n)$, with $x_k = x$, $t_k = t$, and where $dv(x_j)$ is the riemannian volume element $\sqrt{g(x_j)}dx_j^1 \ldots dx_j^n$. If one chooses (10) as a starting point, one must

i) choose a simple $\bar{K}$ such that, K being the exact propagator,

$$K(j+1\,; j) = \bar{K}(j+1\,; j)(1 + 0(\Delta_j t)^2) \quad \text{with} \quad \Delta_j t = t_{j+1} - t_j\,;$$

ii) check that the limit of I_k exists.

If one chooses (4) as a starting point, one can obtain $\bar{K}$ by computing the piecewise linear approximation of (5). With $\bar{K}$ thus obtained, the limit of I_k when $s \in \mathbb{R}^+$ is not only known *to exist*, it is known *explicitly*: it is $\psi(x, t)$ given by equation (5). The piecewise linear approximation of (5) is obtained by replacing the brownian path z by its piecewise linear approximation z_Π for the partition $\Pi = (t_0, t_1, \ldots, t_k = t)$:

$$z_\Pi(t) = (t_{j+1} - t_j)^{-1}[(t_{j+1} - t)z(t_j) + (t - t_j)z(t_{j+1})].$$

Assume for simplicity that there is a unique geodesic joining any two points

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(10) See [Pinsky], [Elworthy, Truman] and [De Witt-Morette, Maheshwari, Nelson]
(11) [Elworthy, Truman].

(e. g. M simply connected with non-positive curvature). The piecewise linear approximation of (5) gives

$$\overline{K}(j+1\,;j) = \left(\frac{m}{2\pi s\hbar}\Delta_j t\right)^{n/2} \mathscr{D}(j+1\,;j) \exp\left(\frac{s}{\hbar}S(j+1\,;j)\right) \quad (11)$$

where $\Delta_j t = t_{j+1} - t_j$, and $\mathscr{D}(j+1\,;\,j)$ is the absolute value of the invariant Van Vleck determinant for the action $S(j+1\,;j)$ evaluated along the geodesic from $(x_j,\ t_j)$ to $(x_{j+1},\ t_{j+1})$:

$$\mathscr{D}(j+1\,;j) = (\Delta_j t)^{-n} g^{-1/2}(x_{j+1}) g^{-1/2}(x_j) \mid \det_{\alpha\beta} (\partial(\exp^{-1}_{x_{j+1}})^\alpha(x_j)/\partial x_j^\beta) \mid$$

with $g = \det_{\alpha\beta} g_{\alpha\beta}$.

It has been shown ([12]) that, $R_{\alpha\beta}$ being the Ricci tensor,

$$\mathscr{D}(j+1\,;j) = (\Delta_j t)^{-n}\left(1 + \frac{1}{6} R_{\alpha\beta}\Delta_j x^\alpha \Delta_j x^\beta + 0((\Delta_j x)^3)\right) \quad (12)$$

where $\Delta_j x = x_{j+1} - x_j$.

Hence the computation of the piecewise linear approximation of the Schrödinger equation (5) in curved space gives for $\overline{K}$ an expression different from the expressions previously proposed: in the following brief story of the piecewise approximation, the statements are purely formal even for s positive. All limits are to be qualified by the phrase « if they exist », and we take $s = i$. It would be interesting to have a mathematical discussion of some of these statements for the case of positive s.

In the early fifties, it was thought that the WKB approximation of the exact propagator K was a good choice for the following reasons:

i) It was obtained ([13]) in the flat case by requiring probability conservation (i. e. unitarity): if the L_2-norm of ψ_0 is unity, requiring the L_2-norm of ψ at time $t + \varepsilon$ to be unity and ignoring terms of order Δx^2 and $\Delta x \Delta t$ leads to $|\overline{K}|^2 = |K_{WKB}|^2$ where

$$K_{WKB}(j+1\,;j) = (m/2\pi i\hbar)^{n/2} \mathscr{D}^{1/2}(j+1\,;j) \exp\left(\frac{i}{\hbar} s(j+1\,;j)\right).$$

ii) Since $K = K_{WKB}(1 + 0(\Delta_j t)^2)$ for many physical systems ([14]) it was assumed that $\overline{K}$ could be taken equal to K_{WKB}.

However, when the configuration space is a riemannian manifold B. S. DeWitt ([15]) showed that for a system with Lagrangian

$$L = \frac{1}{2} m \mid \dot{q} \mid^2 - V(q) \quad (13)$$

([12]) Equation (6.38) in [B. S. DeWitt].
([13]) [Morette, 1951].
([14]) [Pauli, 1952].
([15]) [B. S. De Witt, 1957].

if one chooses $\overline{K} = K_{WKB}$, the limit of I_k, if it exists, should satisfy the Schrödinger equation

$$i\hbar\partial\psi_+/\partial t = \widehat{H}_+\psi_+ \qquad \text{with} \qquad \widehat{H}_+ = \widehat{H} + \frac{1}{12}\hbar^2 R$$

where R is the Riemann curvature scalar. It is often thought that the Correspondence Principle favors this choice. The original ([16]) remark said only « The quantum theory that one arrives at by applying the Correspondence Principle via [choosing $\overline{K}$ to be the W. K. B. approximation] is determined not by the operator $\widehat{H}$ but by the operator $\widehat{H}_+$ ». Moreover, DeWitt also showed that if one chooses $\overline{K}$ equal to the simplest guess one can make, namely $\overline{K}(j+1,\ j) = (m/\Delta_j t 2\pi i\hbar)^{n/2} \exp\left(\frac{i}{\hbar} S(j+1\ ;\ j)\right)$, then for the same Lagrangian, the limit of I_k should satisfy

$$i\hbar\partial\psi_{++}/\partial t = \widehat{H}_{++} \qquad \text{with} \qquad \widehat{H}_{++} = \widehat{H} + \frac{1}{6}\hbar^2 R.$$

Both $\widehat{H}_+$ and $\widehat{H}_{++}$ agree with $\widehat{H}$ when $\hbar$ tends to zero, and are self adjoint when $\widehat{H}$ is self adjoint, i. e. both schemes are unitary.

If one had insisted that, for a system whose classical lagrangian and hamiltonian are L and H, the canonical pair (L, $\widehat{H}$) was to be preferred over the pairs (L, $\widehat{H}_+$) or (L, $\widehat{H}_{++}$) one would have said

$$\overline{K}(j+1\,;j) = (m/2\pi i\hbar\Delta_j t)^{n/2}\left(1 + \frac{1}{6} R_{\alpha\beta}\Delta_j x^\alpha \Delta_j x^\beta\right) \exp\left(\frac{i}{\hbar} S(j+1\,;j)\right) \quad (14)$$

i. e. an expression which agree with (11) for small $\Delta_j x$, but one would not have thought of (11) which looks wrong to anyone expecting $\overline{K}$ to be K_{WKB}.

Formally, $\overline{K}(j+1\ ;j)$ given by (14) or (11) can be replaced by

$$K_{WBK}(j+1\ ;j) \exp\left(\frac{1}{12} R_{\alpha\beta}\Delta_j x^\alpha \Delta_j x^\beta\right)$$

and in the Feynman integral (10) by

$$K_{WKB}(j+1\ ;j) \exp\left(\frac{i}{12}\frac{\hbar}{m}\Delta_j t\hbar R_{\alpha\beta}\right), \text{ i. e. } K_{WKB} \times \text{phase}.$$

In conclusion, the stochastic scheme presented in this paper is rigorous for the diffusion equation. It suggests that for L given by (13), the wave function (5) satisfies

$$i\hbar\partial\psi/\partial t = \widehat{H}\psi.$$

Its precise piecewise linear approximation gives $\overline{K}$ by equation (11). Since

([16]) [B. S. De Witt, p. 394].

the WKB approximation of the wave function is given by equation (8), the wave function satisfies the Principle of Correspondence in the following sense:

Let $\mathscr{C}_t : M \to M$ be the transformation generated by the flow of classical paths Z defined earlier. The probability of finding in a subset A of the configuration space M at time t the system with lagrangian L known to be in $\mathscr{C}_t^{-1}A$ at time t_0 tends to unity asymptotically when $\hbar$ tends to zero. Finally, since $\widehat{H}$ is self adjoint, ψ conserves the probability.

2) Particle in an electromagnetic field (vertical noise).

The electromagnetic potential on $\mathbb{R}^3$ can be defined as a connection on the trivial principal bundle $\Pi : G(B) \to \mathbb{R}^3$ with structure group $G = U(1)$. Consider the following special case of the stochastic differential equation for $(g(t, \omega), x(t, \omega)) \in U(1) \times \mathbb{R}^3$

$$\begin{cases} dg = -ig(t)\Gamma_\alpha(x(t))dx^\alpha(t) \text{ (Stratanovich)} & g(t_0) = 1 \\ dx = \mu\sqrt{s}dz & x(t_0) = x_0,\ \mu = \sqrt{h/m} \end{cases} \tag{12}$$

We choose the connection coefficient Γ_α such that the covariant derivative of the wave function is $\nabla_\alpha = D_\alpha + ieA_\alpha/\hbar c$, namely $\Gamma_\alpha = eA_\alpha/\hbar c$.

Let $S : \omega \to \sqrt{s}\omega$ and let γ_s be the image of γ under S. By a straightforward application of the basic theorem *ii*) it can be shown that the expectation value $\psi(x_0, t)$ of $\psi_0(x(t))g(t)^{-1}$, namely

$$\psi(x_0, t) = \int_\Omega d\gamma_s(\omega) \exp\left(+ i\oint_T \Gamma_\alpha(x(t))dx^\alpha(t)\right)\psi_0(x(t)) \tag{13}$$

satisfies the Schrödinger equation

$$\begin{cases} \partial\psi/\partial t = \widehat{H}\psi \\ \widehat{H} = \frac{1}{2}\mu^2 s(D_\alpha + ieA_\alpha(x_0)/\hbar c)^2 \\ \psi(x_0, t_0) = \psi_0(x_0) \end{cases} \tag{14}$$

Using Truman's method ([17]) the lagrangian of the system is readily obtained from equation (13): Choose the initial wave function to be ψ_0 given by equation (7), and make the change of variable of integration $\omega \mapsto y$ where y is defined by

$$x(t, \omega) \equiv x_0 + \mu\sqrt{s}z(t, \omega) \equiv x_0 + \mu\sqrt{s}\omega(t) = Z(t) + \mu\sqrt{s}y(t)$$

([17]) [Truman].

where Z is a smooth path on $\mathbb{R}^3$ such that $Z(t_0) = x_0$. Expand the integrand in powers of μ. The first term is of order μ^{-2}, it yields the lagrangian

$$L = \frac{m}{2} | \dot{Z} |^2 + \frac{e}{c} \langle A \cdot \dot{Z} \rangle \tag{15}$$

In this case, the natural factor ordering for the hamiltonian operator $\hat{H}$ follows from the natural choice of the stochastic equation on the natural fiber bundle for the physical system under consideration. The stochastic scheme can be said to give a canonical relationship between the hamiltonian operator (14) and the lagrangian (15).

3) Multiply connected configuration spaces.

The basic theorem 1 gives the path integral solution of the Schrödinger equation for a system whose configuration space is multiply connected. It has been shown [18] that the propagator for such a system is a linear combination of path integrals, each computed over a space of homotopic paths; the coefficients of this linear combination form a unitary representation of the fundamental group. There are many applications of this theorem: system of indistinguishable particles [18], particles with spin [19], electrons in a lattice [20], superconductors [21], etc. Multiply connected spaces in field theory are currently receiving a great deal of attention [22].

It is natural to apply the basic theorem 1 to this problem since the universal covering is a principal bundle. Indeed let M be a multiply connected space and let $\tilde{M}$ be its universal covering. By definition $\pi : \tilde{M} \to M = \tilde{M}/G$ where G is a properly discontinuous discrete group of automorphisms of M, isomorphic to the fundamental homotopy group $\pi_1(M)$.

The covering space $\tilde{M}$, considered as a principle bundle, Prin G, has a unique connection: a horizontal lift u of a curve x in the base space is just the uniquely defined lift $\tilde{x}$ of x to $\tilde{M}$ with a given $\tilde{x}(t_0)$. A representation a of G determines a weakly associated vector bundle B. The natural connection on $\tilde{M}$ determines parallel transport in B. Let G_0 be the structural group of B and Prin G_0 the principal bundle associated to B.

From the group homomorphism

$$a : G \to G_0$$

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(18) [Laidlaw, DeWitt]. For a derivation of this result via the universal covering see [Laidlaw], [Dowker].

(19) [Schulman, 1968]. This example can hardly be called an « application » of the theorem, since it was worked out before the theorem was stated, and indeed the example which led to the theorem.

(20) [Schulman, 1969].

(21) [Petry].

(22) [B. S. De Witt, Hart, Isham], [Avis, Isham].

we obtain a map $\tilde{a}$: Prin G $=$ $\tilde{\mathrm{M}} \to$ Prin G_0 which is fibre preserving, and equivariant

$$a(\tilde{x}g) = \tilde{a}(\tilde{x})a(g) \quad , \quad \tilde{x} \in \tilde{\mathrm{M}},\ g \in \mathrm{G}$$

Let ψ be a section of the B-bundle

$$\psi : \mathrm{M} \to \mathrm{B}$$

The parallel transport of $\psi(x(t))$ from $x(t)$ to $x(t_0)$ along the curve $x(s)$ is

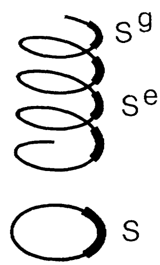
$$\tau^t_{t_0}\psi(x(t)) = \tilde{a}(\tilde{x}(t_0))[\tilde{a}(\tilde{x}(t))]^{-1}\psi(x(t))$$

abbreviated to

$$\tau^t_{t_0}\psi(x(t)) = \tilde{x}(t_0)\tilde{x}(t)^{-1}\psi(x(t))$$

Suppose for simplicity that ψ has support in some small, or at least simply connected, subset S of M. Our integration can be taken over the space $\Omega_{x_0\mathrm{S}}$ of paths beginning at x_0 and ending in S. This space decomposes into homotopy classes which can be parametrized, non uniquely, by G

$$\Omega_{x_0\mathrm{S}} = \bigcup_{g \in \mathrm{G}} \Omega^g_{x_0\mathrm{S}}$$



$$\mathrm{S}^g = \{\tilde{\mathrm{X}}(t) : \mathrm{X} \in \Omega^g_{x_0,\mathrm{S}}\}$$

in such a way that if $x_1 \in \Omega^{g_1}_{x_0\mathrm{S}}$ and $x_2 \in \Omega^{g_2}_{x_0\mathrm{S}}$ with $x_1(t) = x_2(t)$, then the lifts with the same initial point $\tilde{x}_0$ satisfy

$$\tilde{x}_1(t) = \tilde{x}_2(t)g_2^{-1}g_1.$$

Then

$$\mathrm{P}_t\psi(x_0) = \sum_{g \in \mathrm{G}} \int_{\Omega^g_{x_0\mathrm{S}}} d\gamma(\omega)\tilde{x}(t_0)\tilde{x}(t)^{-1}\psi(x(t))$$

Since there exists $\tilde{x}_e \in \Omega^e_{x_0\mathrm{S}}$ such that $\tilde{x}_1(t) = \tilde{x}_e(t)g_1$ we can rewrite this equation

$$\mathrm{P}_t\psi(x_0) = \sum_{g \in \mathrm{G}} \int_{\Omega^e_{x_0\mathrm{S}}} d\gamma(\omega)x(t_0)g^{-1}x(t)^{-1}\psi(x(t))$$

Alternatively, as with more general fibre bundles, we could lift ψ to a differentiable map $\tilde{\psi}$ on $\tilde{M}$ with values in the typical fibre F of B such that

$$\tilde{\psi}(\tilde{x}) = [\tilde{a}(\tilde{x})]^{-1}\psi(x) = u^{-1}\psi(x)$$

It follows that

$$\tilde{\psi}(\tilde{x}g) = a(g)^{-1}\tilde{\psi}(\tilde{x}), \qquad g \in G$$

Thus the space of sections ψ is replaced by the space of F-valued maps $\tilde{\psi}$ satisfying this last equation. The semigroup $\tilde{\psi} \to P_t\tilde{\psi}$ is equivariant and so determines a semigroup $\psi \to P_t\psi$ on M.

This situation is discussed in detail in Dowker ([23]).

EXAMPLE 1. — The wave function of a particle on a circle. The covering space of S^1 is $\mathbb{R}$, the structural group G is $(\mathbb{Z}, +)$. The nature of ψ determines the typical fiber F of B. Choose $F = \mathbb{C}$. Then $G_0 = U(1)$ and

$$a : \mathbb{Z} \to U(1) \quad \text{by} \quad n \to \exp(in\alpha) \quad \text{with} \quad \alpha \in \mathbb{R}$$

$$P_t\psi(x_0) = \sum_{n \in Z} \int_{\Omega^0_{x_0}S} d\gamma(\omega)\tilde{x}(t_0)e^{-in\alpha}\tilde{x}(t)^{-1}\psi(x(t))$$

EXAMPLE 2. — Electromagnetism with zero field (e. g. the Bohm-Aharanov effect), or more generally a gauge field with zero curvature over a nonsimply connected region M of $\mathbb{R}^n$. If the curvature of a connection on the bundle is zero, then through each u_0 there is an embedded covering space $\tilde{M}$ of M whose group is the holonomy group G of the connection. In this situation $\tilde{M}$ may not be the universal covering space and G will only be a quotient group of the fundamental group of M. Since any horizontal lift through u_0 with respect to the given connection is a lift to $\tilde{M}$, the more general gauge theory reduces to the covering space theory.

IV. FINAL REMARKS

Some of these results can be derived without working with the fiber bundle ([24]). Starting from a stochastic process on a fiber bundle has the following advantages:

i) It applies to a wide class of system.

([23]) [Dowker].

([24]) See for instance [Pinsky] for diffusions on riemannian manifolds. See [McLaughlin, Schulman], [Garczynski], [B. Nelson, Sheeks], and [Simon] for a particle in an electromagnetic field.

ii) It gives simple answers to such problems as:
parallel transport along a brownian path;
piecewise linear approximation on riemannian manifolds;
canonical relationship between lagrangian and hamiltonian *operator*.
iii) It is cast in a framework which guarantees gauge invariance.

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