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Applications of the characteristic identity for GL(N)

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RÉSUMÉ. — Dans cet article on montre que la matrice $(a^i{}_j)$, où $\{ a^i{}_j \mid 1 \leqslant i, j \leqslant n \}$ est la base canonique de l'algèbre de Lie de GL(n), possède de nombreuses propriétés des matrices à coefficient numérique. On obtient des définitions appropriées de l'inverse et du déterminant de cette matrice, et l'on montre que les éléments de la matrice inverse $(a^i{}_j)^{-1}$ engendrent une algèbre de Lie. On donne aussi une définition pour l'espace des « opérateurs vectoriels » pour GL(n), et l'on trouve des conditions sous lesquelles les composantes d'un « opérateur vectoriel » sont permutables. Puisque la matrice $(a^i{}_j)$ satisfait une « identité caractéristique », on en déduit une construction explicite des « opérateurs de projection de Young ». Cette construction entraîne les invariants de GL(n).

ABSTRACT. — In this article one shows that the matrix $(a^i{}_j)$, where $\{ a^i{}_j \mid 1 \leqslant i, j \leqslant n \}$ are the canonical generators of the group GL(n), possesses many of the properties of matrices with numerical entries. One obtains appropriate definitions for the inverse and determinant of this matrix, and it is shown that the elements of the inverse matrix $(a^i{}_j)^{-1}$ generate a Lie algebra. One gives also a definition for the space of « vector operators » for GL(n) and one finds some conditions under which the components of a « vector operator » are commutable. Since the matrix $(a^i{}_j)$ satisfies a « characteristic identity » one thereby deduces an explicit construction of Young's Projection operators. This construction involves only the invariants of GL(n).

1. INTRODUCTION

The generators of the group GL(n) satisfy the commutation relations

$$[a^i{}_j, a^k{}_l] = \delta^k{}_j a^i{}_l - \delta^i{}_l a^k{}_j .$$

The fundamental invariants σ_r of GL(n), defined by

$$\sigma_1 = a^i{}_i, \quad \sigma_2 = a^i{}_j a^j{}_i, \quad \sigma_3 = a^i{}_j a^j{}_k a^k{}_i,$$

etc., are Casimir operators (i. e. commute with all the elements of the Lie algebra). Therefore the centre of the universal enveloping algebra U_n of GL(n) is $Z_n = F[\sigma_1, \ldots, \sigma_n]$ where $F[x_1, \ldots, x_n]$ denotes the ring of polynomials over the underlying field F (usually F = R or C) in determinates $x_1, \ldots, x_n$.

Associated with GL(n) is its fundamental matrix a whose (i, j) entry is the generator $a^i{}_j$;

viz.
$$a = \begin{pmatrix} a^1{}_1 & a^1{}_2 & \ldots & a^1{}_n \\ a^2{}_1 & a^2{}_2 & \ldots & a^2{}_n \\ a^n{}_1 & a^n{}_2 & \ldots & a^n{}_n \end{pmatrix}$$

Recently it has been shown by Carey, Cant and O'Brien [1] that the matrix a of GL(n) satisfies a polynomial identity $m(a) = 0$, where $m(x)$ is a unique monic polynomial of degree n whose coefficients lie in the centre Z_n of U_n. We call $m(x)$ the GL(n) characteristic polynomial.

It is one of our aims to show that $m(x)$ is the minimum polynomial of GL(n); that is if a satisfies any other polynomial identity $p(a) = 0$ over Z_n, then $m(x)$ divides $p(x)$.

Green [2] has shown that on a finite dimensional representation of GL(n) with highest weight $(\lambda_1, \ldots, \lambda_n)$ (where the λ_i must be intergers satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$) that the characteristic identity can be written in split form

$$\prod_{r=1}^{n} (a - b_r) = 0 \tag{1}$$

where the b_r take the constant values $b_r = \lambda_r + n - r$. The roots b_r are related to the GL(n) invariants σ_r by equations of the form

$$\begin{aligned} \sigma_1 &= \sum_{r=1}^{n} (b_r - n + r) \\ \sigma_2 &= \sum_{r=1}^{n} (b_r - n + r)(b_r + 1 - r) \end{aligned} \tag{2}$$

etc., (see Green [3] for a more general expression).

In this paper we regard the operators b_r as solutions to the equations (2). Therefore, in general, the explicit form of the b_r in terms of the invariants σ_k requires the solution of algebraic equations of degree less than or equal to n. On an irreducible representation of GL(n) the invariants σ_k reduce to a possibly complex multiple of the identity and therefore the b_r may be defined as a multiple of the identity within that representation. On such representations a unique solution to the equations (2) may be chosen so that the real part of the eigenvalues β_r of the b_r are in descending order; i. e.

$$\operatorname{Re}\beta_1 \geqslant \operatorname{Re}\beta_2 \geqslant \ldots \geqslant \operatorname{Re}\beta_n \geqslant 0 .$$

Thus on a finite dimensional irreducible representation of GL(n) with highest weight $(\lambda_1, \ldots, \lambda_n)$ the unique solutions to the equations (2) are $\beta_r = \lambda_r + n - r$.

As an example let us consider GL(2). The explicit solutions to the equations

$$\sigma_1 = (b_1 - 1) + b_2$$
$$\sigma_2 = (b_1 - 1)b_1 + b_2(b_2 - 1)$$

are

$$b_1 = \frac{1}{2}\sigma_1 + \frac{1}{2}[1 + (2\sigma_2 - \sigma_1{}^2 + 1)^{\frac{1}{2}}]$$

$$b_2 = \frac{1}{2}\sigma_1 + \frac{1}{2}[1 - (2\sigma_2 - \sigma_1{}^2 + 1)^{\frac{1}{2}}] .$$

Since the invariants σ_1 and σ_2 take constant complex values γ_1 and γ_2 respectively on irreducible representations of GL(n) we may interpret the operator $(2\sigma_2 - \sigma_1{}^2 + 1)^{\frac{1}{2}}$ as that operator which takes the constant value $+ (2\gamma_2 - \gamma_1{}^2 + 1)^{\frac{1}{2}}$ (the positive square root in the right hand side of the complex plane).

The split GL(2) identity is therefore

$$\left[a - \frac{1}{2}\sigma_1 - \frac{1}{2} - (2\sigma_2 - \sigma_1{}^2 + 1)^{\frac{1}{2}}\right]\left[a - \frac{1}{2}\sigma_1 - \frac{1}{2} + (2\sigma_2 - \sigma_1{}^2 + 1)^{\frac{1}{2}}\right] = 0$$

and the polynomial identity is

$$a^2 - (\sigma_1 + 1)a + \frac{1}{2}(\sigma_1{}^2 + \sigma_1 - \sigma_2) = 0 .$$

Since the operators b_r are solutions to polynomial equations involving only the σ_k, we see that they must commute with the GL(n) generators; i. e. $[b_r, a^i{}_j] = 0$.

It is convenient to extend the centre Z_n of U_n to include the operators b_r. Thus, instead of working in U_n, we work in the extended enveloping algebra

$$\bar{U}_n = \bar{Z}_n \otimes_{Z_n} U_n$$

where $\bar{Z}_n = F(b_1, \ldots, b_n)$ and $F(x_1, \ldots, x_n)$ is the field of quotients for $F(x_1, \ldots, x_n]$.

From the equations (2) we see that $Z_n \subseteq \bar{Z}_n$ and we may write the GL(n) characteristic identity in its split form (1). Hence, working in $\bar{U}_n$, we may write the characteristic polynomial as

$$m(x) = \sum_{r=1}^{n} (x - b_r) \in \bar{Z}_n[x] .$$

This result is analogous to the classical Cayley-Hamilton theorem but now the roots b_r are no longer scalars but are operators which commute with the GL(n) generators and satisfy the equations (2).

2. VECTOR AND CONTRAGREDIENT VECTOR OPERATORS

We define a GL(n) vector operator ψ as an operator with n components ψ^i which satisfy

$$[a^i{}_j, \psi^k] = \delta^k{}_j \psi^i . \tag{3}$$

We define an rth rank tensor operator T as an operator with components $T^{pq\ldots s}$ (r superscripts), $1 \leqslant p, q, \ldots, s \leqslant n$, which satisfy

$$[a^i{}_j, T^{pq\ldots s}] = \delta^{iq\ldots s} + \delta^q{}_j T^{pi\ldots s} + \ldots + \delta^s{}_j T^{pq\ldots i} .$$

In an analogous way we define a contragredient vector operator ϕ as an operator with n components ϕ_i which satisfy

$$[a^i{}_j, \phi_k] = - \delta^i{}_k \phi_j .$$

Similarly we define an rth rank contragredient tensor T with components $T_{pq\ldots s}$.

Since the most obvious examples of GL(n) vector operators lie in the enveloping algebra U_{n+1} of GL($n + 1$) (e. g. the vector operator ψ with components $\psi^i = a^i{}_{n+1}$) we shall now consider GL($n + 1$) and its characteristic identity.

According to Humphreys [*4*] U_{n+1} is a free U_n module with free basis consisting of monomials

$$x^i x^j, \ldots, x^k \qquad 1 \leqslant i, j, \ldots, k \leqslant 2n + 1$$

where

$$x^i = a^i{}_{n+1} \qquad i = 1, \ldots, n + 1$$

$$x^{n+1+i} = a^{n+1}{}_i \qquad i = 1, \ldots, n .$$

Let us denote the associative algebra generated by $\bar{U}_n$ and the monomials $x^i x^j, \ldots, x^k$ by $\bar{U}_{n+1}$. Clearly $\bar{U}_{n+1}$ contains U_{n+1} as well as $\bar{U}_n$.

We denote the GL(n + 1) matrix by $\hat{a}$;

viz.
$$\hat{a} = \begin{pmatrix} a^1{}_1 & a^1{}_2 & \ldots & a^1_{n+1} \\ a^2{}_1 & a^2{}_2 & \ldots & a^2{}_{n+1} \\ a_1^{n+1} & a_2^{n+1} & \ldots & a^{n+1}{}_{n+1} \end{pmatrix}$$

As for GL(n) the centre Z_{n+1} of U_{n+1} is generated by the fundamental invariants $\hat{\sigma}_k = \operatorname{tr}(\hat{a}^k)$ and $\hat{a}$ satisfies a polynomial identity over Z_{n+1} which can also be written in split form

$$\prod_{r=1}^{n+1} (\hat{a} - \hat{b}_r) = 0 .$$

The roots $\hat{b}_r$ of this identity are related to the GL(n + 1) invariants $\hat{\sigma}_k$ by equations analogous to the equations (2) for GL(n).

In most applications we shall regard GL(n) as embedded in GL(n + 1), so it is convenient to extend $\bar{Z}_n = F(b_1, \ldots, b_n)$ to include the GL(n + 1) roots $\hat{b}_1, \ldots, \hat{b}_{n+1}$. Since the $\hat{b}_r$ are GL(n + 1) invariants they must also be GL(n) invariants. In particular $[\sigma_k, \hat{b}_r] = 0$, and since the b_r are well defined functions of the GL(n) invariants σ_k we must have $[b_r, \hat{b}_k] = 0$. So we extend $\bar{Z}_n$ to

$$Z = F(b_1, \ldots, b_n, \hat{b}_1, \ldots, \hat{b}_{n+1}) .$$

Every element of Z is a GL(n) invariant and includes the GL(n + 1) invariants $\hat{\sigma}_r$. Thus Z contains the centre Z_{n+1} of the universal enveloping algebra U_{n+1} of GL(n + 1). Since $a^{n+1}{}_{n+1} = \hat{\sigma}_1 - \sigma_1$, we see that $a^{n+1}{}_{n+1}$ is an element of Z, and it can be shown inductively that $p(\hat{a})^{n+1}{}_{n+1}$ belongs to Z for every $p(x)$ in Z[x]. Moreover if $p(x)$ and $h(x)$ belong to Z[x], then

$$\left[p(\hat{a})^{n+1}{}_{n+1}, h(\hat{a})^{n+1}{}_{n+1}\right] = 0 .$$

If we wish to work over Z it is not sufficient to work in U_{n+1} and we must consider an extended enveloping algebra of U_{n+1}. We now define

$$U = \bar{Z}_{n+1} \otimes_{Z_{n+1}} \bar{U}_{n+1} ,$$

where

$$\bar{Z}_{n+1} = F(\hat{b}_1, \ldots, \hat{b}_{n+1}) .$$

In the light of our previous remarks we see that U contains $\bar{U}_n$, $\bar{U}_{n+1}$ and also Z.

From now on we shall work in U so that we may write both the GL(n) and GL(n + 1) identities in their split forms.

3. SHIFT OPERATORS

There are two obvious properties of vector operators ψ:

i) $$\sum_{r=1}^{n}[b_r, \psi^i] = [\sigma_1, \psi^i] = \psi^i$$

ii) $$\sum_{r=1}^{n}[(b_r - n + r)(b_r + 1 - r), \psi^i] = [\sigma_2, \psi^i] = (2a - n)^i{}_j\psi^j .$$

Following Green [2] [3] a vector operator ψ can be decomposed into a sum of component vector operators

$$\psi^i = \sum_{r=1}^{n} \psi_r{}^i$$

where each ψ_r satisfies

$$b_k\psi_r{}^i = \psi_r{}^i(b_k + \delta_{rk})$$

or

$$[b_k, \psi_r{}^i] = \delta_{kr}\psi_r{}^i . \tag{5}$$

From this we obtain

$$[b_k, \psi^i] = \sum_{r=1}^{n}[b_k, \psi_r{}^i] = \psi_k{}^i . \tag{6}$$

From equation (6) it is easy to see that if ψ is a vector operator then so is each of its component operators ψ_r. Moreover equation (5) implies that $\psi_r{}^i$ increases the eigenvalue β_r of b_r in an irreducible representation of GL(n) by one unit, leaving the other eigenvalues β_k unchanged.

Hence the tensor

$$T^{ij} = \psi_r{}^i\psi_r{}^j$$

increases the eigenvalue β_r of b_r by two units while the other β_k remain unaltered. Since this is a well known property of symmetric tensors only (see [5]) it follows that

$$[\psi_r{}^i, \psi_r{}^j] = 0 .$$

From equation (4) it is easily verified (see [2]) that if ψ is a vector operator then

$$(a - b_r)^i{}_j\psi_r{}^j = 0$$

or

$$a^i{}_j\psi_r{}^j = b_r\psi_r{}^i . \tag{7}$$

LEMMA 1. — If $p(x) \in Z[x]$, then

$$p(a)^i{}_j\psi_r{}^j = p(b_r)\psi_r{}^i . \tag{8}$$

Proof. — The proof holds by induction, the result being obvious for $p(x) = x$ from equation (7).

By summing equation (8) over r we obtain

COROLLARY. — If ψ is a vector operator and $p(x) \in Z[x]$, then

$$p(a)^i{}_j\psi^j = \sum_{r=1}^{n} p(b_r)\psi_r{}^i . \tag{9}$$

Using lemma 1 we may now derive the explicit form of the components ψ_r of ψ. Substituting

$$p(x) = \prod_{l \neq k}(x - b_l)$$

into equation (9) gives

$$\prod_{l \neq k}(a - b_l)^i{}_j\psi^j = \sum_{r=1}^{n} \prod_{l \neq k}(b_r - b_l)\psi_r{}^i$$
$$= \prod_{l \neq k}(b_k - b_l)\psi_k{}^i .$$

Since the roots b_r are all distinct (see [1]) we obtain

$$\psi_k{}^i = \prod_{l \neq k}(b_k - b_l)^{-1}(a - b_l)^i{}_j\psi^j .$$

This equation agrees with the result given in Green [2]. Let us write

$$(f_k)^i{}_j = \prod_{l \neq k}(b_k - b_l)^{-1}(a - b_l)^i{}_j \tag{10}$$

so that

$$\psi_k{}^i = f_k{}^i{}_j\psi^j . \tag{11}$$

LEMMA 2. — If $p(x) \in Z[x]$, then

$$p(a)^i{}_j = \sum_{r=1}^{n} p(b_r) f_r{}^i{}_j . \tag{12}$$

Proof. — Let ψ be an arbitrary vector operator. In view of equation (9) we have

$$p(a)^i{}_j\psi^j = \sum_{r=1}^{n} p(b_r) f_r{}^i{}_j\psi^j$$

or

$$\left[p(a)^i{}_j - \sum_{r=1}^{n} p(b_r) f_r{}^i{}_j\right]\psi^j = 0\,.$$

Since ψ is an arbitrary vector operator this implies

$$p(a)^i{}_j = \sum_{r=1}^{n} p(b_r) f_r{}^i{}_j$$

as required.

COROLLARY. — If $p(x) \in Z[x]$, then

$$\operatorname{tr}\,[p(a)] = \sum_{k=1}^{n} p(b_k) \operatorname{tr}\,(f_k)\,.$$

This last corollary together with equation (11) both appear in Green [2] [3] but in our case we treat the b_r as operators rather than as scalars (which can only apply when working in a particular irreducible representation of GL(n)).

Some important properties of the operators $f_r{}^i{}_j$ are given in the following lemma.

LEMMA 3. — The operators $f_r{}^i{}_j$ satisfy the following two conditions:

i)
$$f_r{}^i{}_j f_k{}^j{}_l = \delta_{rk} f_k{}^i{}_l$$

ii)
$$\sum_{r=1}^{n} f_r{}^i{}_j = \delta^i{}_j\,.$$

Proof. — Property *ii)* follows immediately from lemma 2 if we substitute $p(x) = 1$. So it just remains to prove *i)*.

Let ψ be an arbitrary vector operator. Then

$$\begin{aligned} f_r{}^i{}_j \psi^j &= \psi_r{}^i \\ &= [b_r, \psi^i]. \end{aligned}$$

Since ψ_r is itself a vector operator we obtain

$$\begin{aligned} f_r{}^i{}_j f_k{}^j{}_l \psi^l &= f_r{}^i{}_j \psi_k{}^j \\ &= [b_r, \psi_k{}^i] \\ &= \delta_{rk} \psi_k{}^i \\ &= \delta_{rk} f_k{}^i{}_j \psi^j\,. \end{aligned}$$

Thus $(f_r{}^i{}_j f_k{}^j{}_l - \delta_{rk} f_k{}^i{}_l)\psi^l = 0$ and since ψ was arbitrary, the result follows.

As for vector operators a contragredient vector operator ϕ may be decomposed into components ϕ_r which satisfy

$$\begin{aligned} &i) && [b_k, \phi_{r_i}] = -\delta_{kr}\phi_{r_i} \\ &ii) && [b_k, \phi_i] = -\phi_{k_i}. \end{aligned} \tag{14}$$

Similarly the components ϕ_r satisfy the equation $\phi_{r_i}(a - b_r)^i{}_j = 0$ or $\phi_{r_i}a^i{}_j = \phi_{r_j}b_r$.

Using an argument analogous to that for vector operators we obtain the following results:

LEMMA 4. — If $p(x) \in Z[x]$ and ϕ is a contragredient vector operator, then

$$\phi_{r_i}p(a)^i{}_j = \phi_{r_j}p(b_r).$$

COROLLARY. — If p and ϕ satisfy the conditions of the lemma then

$$\phi_i p(a)^i{}_j = \sum_{r=1}^{n} \phi_{r_j}p(b_r). \tag{15}$$

If we substitute

$$p(x) = \prod_{k \neq r}(x - b_k)$$

into (15) we obtain

$$\phi_{r_j} = \phi_i f_r{}^i{}_j. \tag{16}$$

The Adjoint Equation

Following Green [2] we define the adjoint $\bar{a}$ of the GL(n) matrix a by $\bar{a}_j{}^i = -a^i{}_j$. From equation (7) we obtain

$$\begin{aligned} \psi_r{}^j\bar{a}_j{}^i &= (n - b_r)\psi_r{}^i \\ &= \psi_r{}^i(n - b_r - 1). \end{aligned}$$

Hence $\psi_r{}^j(\bar{a} + b_r - n + 1)_j{}^i = 0$ and $\bar{a}$ satisfies the polynomial equation

$$\prod_{r=1}^{n}(\bar{a} + b_r - n + 1) = 0.$$

As for the matrix a we obtain the following results

LEMMA 5. — Let $p(x) \in Z[x]$ and ψ, ϕ be arbitrary vector and contragredient vector operators respectively. Then

$$\psi_r{}^j p(\bar{a})_j{}^i = \psi_r{}^i p(n - b_r - 1)$$

and

$$p(\bar{a})_j{}^i\phi_{r_i} = p(n - b_r - 1)\phi_{r_j}.$$

COROLLARY. — With $p(x)$, ψ and ϕ as above we have

$$\psi^j p(\bar{a})_j{}^i = \sum_{r=1}^{n} \psi_r{}^i p(n - b_r - 1)$$

$$p(\bar{a})_j{}^i \phi_i = \sum_{r=1}^{n} p(n - b_r - 1)\phi_{r_j}. \tag{17}$$

If we substitute

$$p(x) = \prod_{r \neq k} (x + b_r + 1 - n)$$

into equation (17) we obtain

$$\begin{aligned} \psi_r{}^i &= \psi^j g_{r_j}{}^i \\ \phi_{r_i} &= g_{r_i}{}^j \phi_j \end{aligned} \tag{18}$$

where

$$g_{r_j}{}^i = \prod_{k \neq r} (\bar{a} + b_k + 1 - n)_j{}^i (b_k - b_r)^{-1}. \tag{19}$$

LEMMA 6. — The g_r satisfy the following two conditions:

i) $$g_{r_i}{}^j g_{k_j}{}^l = \delta_{rk} g_{k_i}{}^l$$

ii) $$\sum_{r=1}^{n} g_{r_i}{}^j = \delta^j{}_i.$$

Proof. — Analogous to the proof of lemma 3.

Finally, combining equations (17) and (18) we obtain

LEMMA 7. — If $p(x) \in Z[x]$, then

$$p(\bar{a})_j{}^i = \sum_{r=1}^{n} p(n - b_r - 1) g_{r_j}{}^i.$$

4. INVERSES AND DETERMINANTS

In this section we show that several results which are well known to hold for numerical matrices also hold for the GL(n) matrix a.

We have already remarked that a satisfies a polynomial identity $m(a) = 0$, where

$$m(x) = \prod_{r=1}^{n} (x - b_r)$$

and the roots b_r are all distinct. When such a situation occurs in the classical theory of numerical matrices $m(x)$ would be the minimum polynomial. We shall now prove that this result also holds for the GL(n) matrix a.

Consider the vector operator ψ with components $\psi^i = a^i{}_{n+1}$; $i=1, \ldots, n$. Then each component $\psi_r{}^i$ is a well defined element of U. For each r it is easily shown that there exists representations of GL(n + 1) (and GL(n)) on which $\psi_r{}^i$ does not vanish, so we say $\psi_r{}^i$ is a non zero element of U. To say that χ is a zero element of U means that χ must vanish on all representations of GL(n + 1) and GL(n).

From these remarks we see that the component operators ψ_r of the vector operator ψ considered above form a Z-linearly independent set; that is if

$$\sum_{r=1}^{n} \gamma_r \psi_r{}^i = 0 , \qquad \gamma_r \in Z ,$$

then

$$\gamma_r = 0 \qquad r = 1, \ldots, n .$$

Since

$$\sum_{r=1}^{n} \gamma_r \psi_r{}^i = 0$$

implies

$$\begin{aligned} 0 &= \left[b_k, \sum_{r=1}^{n} \gamma_r \psi_r{}^i \right] \\ &= \sum_{r=1}^{n} \gamma_r [b_k, \psi_r{}^i] \\ & \gamma_k \psi_k{}^i . \end{aligned}$$

Since $\psi_k{}^i$ is non zero this implies $\gamma_k = 0$.

THEOREM 8. — The characteristic polynomial $m(x)$ of GL(n) is its minimum polynomial. That is if $p(x) \in Z[x]$ and $p(a) = 0$ then $m(x)$ divides $p(x)$.

Proof. — Suppose $p(x) \in Z[x]$ and $p(a) = 0$. Let ψ be the vector operator with components $\psi^i = a^i{}_{n+1}$. Then

$$\begin{aligned} 0 &= p(a)^i{}_j \psi^j \\ &= \sum_{r=1}^{n} p(b_r) \psi_r{}^i . \end{aligned}$$

Since the ψ_r form a linearly independent set this implies $p(b_r) = 0$ ($r = 1, \ldots, n$). Hence $x - b_r$ divides $p(x)$ for each r and since the b_r are all distinct $m(x)$ must divide $p(x)$.

This theorem shows that $m(x)$ is the minimum polynomial for GL(n) in its universal enveloping algebra. However it need not be the minimum polynomial when we consider certain irreducible representations of GL(n). This is because although the $\psi_r{}^i$ are non zero elements of U there may exist representations of GL(n) on which some of the ψ_r vanish. In such a situation (see Green [2]) we may drop a factor corresponding to each zero ψ_r from the characteristic identity and obtain a reduced identity. Since the remaining ψ_k are non zero this reduced identity will be the minimum polynomial of GL(n) on this particular representation.

We now show that the matrix a of GL(n) has a two sided inverse with eigenvalues $(b_r)^{-1}$ in agreement with the classical theory of numerical matrices.

If $p(x) \in Z[x]$ we may define an inverse for the matrix $p(a)$ by

$$p^{-1}(a)^i{}_j = \sum_{r=1}^{n} p(b_r)^{-1} f_r{}^i{}_j \tag{20}$$

where $[p(b_r)]^{-1}$ may be interpreted as that operator whose eigenvalue on an irreducible representation of GL(n) is $p(\beta_r)^{-1}$ where β_r is the eigenvalue of the operator b_r.

Clearly the inverse of the matrix $p(a)$ will not exist if $p(b_r) = 0$ (i. e. if b_r is a root of $p(x)$); e. g. $(a - b_r)$ and $f_r(a)$ do not have inverses. (The f_r may be regarded as an orthogonal set of idempotent matrix operators. For such operators it is well known that an inverse does not exist).

In the case where b_r is not a root of $p(x)$ the inverse $p^{-1}(a)$ of $p(a)$ only exists on irreducible representations of GL(n) where the eigenvalues $p(\beta_r)$ of the operators $p(b_r)$ are non zero. On representations where the eigenvalue $p(\beta_r)$ of some $p(b_r)$ is zero we may redefine our GL(n) generators to be $a^i{}_j + c\delta^i{}_j$ (for a suitable choice of constant c) to ensure that the inverse $p^{-1}(a)$ exists.

Equation (20) allows us to define the matrix $h(a)$ for every $h(x) \in Z(x)$ (the field of quotients for $Z[x]$) by setting

$$h(a)^i{}_j = \sum_{r=1}^{n} h(b_r) f_r{}^i{}_j .$$

In fact we may define any well defined function of a in this way. In particular we define the inverse of the GL(n) matrix a to be

$$(a^{-1})^i{}_j = \sum_{r=1}^{n} (b_r)^{-1} f_r{}^i{}_j . \tag{21}$$

Then a^{-1} satisfies $a^i{}_j(a^{-1})^j{}_k = (a^{-1})^i{}_j a^j{}_k = \delta^i{}_k$ and we see that the matrix a not only has an adjoint $\bar{a}$ but an inverse a^{-1} as well.

The inverse matrix a^{-1} will not be defined on representations of GL(n) where the eigenvalue β_r of some b_r is zero. However on finite dimensional irreducible representations of GL(n) with weights $(\lambda_1. \dots, \lambda_n)$ the eigenvalues of the b_r are $\lambda_r + n - r$ where the λ_r are integers satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n \geqslant 0$. Hence $\lambda_r + n - r = 0$ implies $n - r \leqslant 0$ which can only occur when $r = n$ and $\lambda_n = 0$. Thus $(a^{-1})^i{}_j$ is not defined on irreducible representations where $\lambda_n = 0$ but will be defined on all other finite dimensional irreducible representations of GL(n).

In the classical theory of numerical matrices it is well known that the determinant of a matrix is simply the product of its eigenvalues. Extending this idea we define the determinant of the GL(n) matrix a by

$$\det a = \prod_{r=1}^{n} b_r$$

and if $h(x) \in Z[x]$ we define

$$\det [h(a)] = \prod_{r=1}^{n} h(b_r) . \tag{22}$$

In view of our previous remarks we see that the inverse of the matrix $h(a)$ exists if and only if its determinant is non zero. Also if $h(x)$, $g(x) \in Z(x)$ then equation (22) implies

$$\det [g(a)h(a)] = \det [g(a)] \det [h(a)]$$

and the product rule for determinants is satisfied.

The determinant of the matrix a is of particular interest since it is the constant term (up to a factor ± 1) which appears in the characteristic polynomial $m(x)$. Since the coefficients of $m(x)$ lie in the centre Z_n of U_n we see that det a is a polynomial (over the underlying field F) in the GL(n) invariants $\sigma_1, \dots, \sigma_n$.

For example the constant term in the GL(2) characteristic identity is

$$\det a = \frac{1}{2}\left[\sigma_1{}^2 + \sigma_1 - \sigma_2\right] .$$

Although our definition of determinant is in agreement with the classical case, it should be noted that unlike a numerical matrix the trace of the matrix a (i. e. σ_1) is not the sum of the eigenvalues b_r of a, but is the sum of the operators $b_r + r - n$.

It has been suggested by Lehrer-ilamed (private communication) that we may take the determinant of the GL(n) matrix a to be the symmetrized expression obtained by evaluating the determinant of a as though it were a numerical matrix and then symmetrizing the resultant expression.

For example we may define the determinant of the GL(2) matrix

$$a = \begin{pmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & a^2{}_2 \end{pmatrix}$$

by

$$\det a = a^1{}_1 a^2{}_2 - \frac{1}{2}(a^2{}_1 a^1{}_2 + a^1{}_2 a^2{}_1).$$

But

$$a^1{}_1 a^2{}_2 = \frac{1}{2}\left[(a^1{}_1 + a^2{}_2) - (a^1{}_1)^2 - (a^2{}_2)^2\right]$$

which gives

$$\det a = \frac{1}{2}\left[\sigma_1{}^2 - \sigma_2\right].$$

Thus the determinant of a as defined by ilamed differs from our definition by a factor $\frac{1}{2}\sigma_1$ for the case $n = 2$. In view of our definition of inverse however, definition (22) for determinants is more applicable for our purposes.

5. INVERSE CHARACTERISTIC IDENTITY

From equation (21) we have

$$(a^{-1})^i{}_j f_k{}^j{}_l = \sum_{r=1}^{n} (b_r)^{-1} f_r{}^i{}_j f_k{}^j{}_l$$
$$= (b_k)^{-1} f_k{}^i{}_l .$$

Hence

$$\prod_{r=1}^{n} [a^{-1} - (b_r)^{-1}]^i{}_j = \prod_{r=1}^{n} [a^{-1} - (b_r)^{-1}]^i{}_l \delta^l{}_j$$
$$= \sum_{k=1}^{n} \prod_{r=1}^{n} [a^{-1} - (b_r)^{-1}]^i{}_l f_k{}^l{}_j$$
$$\sum_{k=1}^{n} \prod_{r=1}^{n} [(b_k)^{-1} - (b_r)^{-1}] f_k{}^i{}_j$$
$$= 0 .$$

Therefore a^{-1} satisfies the polynomial identity $m^{(-)}(a^{-1}) = 0$, where

$$m^{(-)}(x) = \prod_{r=1}^{n} [x - (b_r)^{-1}] .$$

We call $m^{(-)}(x)$ the inverse polynomial of GL(n). As for the matrix a we can show that $m^{(-)}(x)$ is the minimum polynomial for the matrix a^{-1}. From equation (21) we obtain the following results

LEMMA 9. — Let $p(x) \in Z[x]$. Then

$$P(a^{-1})^i{}_j = \sum_{r=1}^{n} P[(b_r)^{-1}] f_r{}^i{}_j . \qquad (23)$$

COROLLARY.

$$\mathrm{Tr}\, P(a^{-1}) = \sum_{k=1}^{n} P[(b_k)^{-1}]\, \mathrm{tr}\, (f_k) .$$

This last result allows us to calculate the inverse invariants

$$\sigma_r{}^{(-)} = \mathrm{tr}\, [(a^{-1})^r] .$$

Substituting

$$p(x) = \prod_{k \neq r} ((x) - (b_k)^{-1})$$

into (23) gives

$$\prod_{k \neq r} [a^{-1} - (b_k)^{-1}]^i{}_j = \sum_{l=1}^{n} \prod_{k \neq r} [(b_l)^{-1} - (b_k)^{-1}] f_l{}^i{}_j$$

or

$$f_r{}^i{}_j = \prod_{k \neq r} [(b_r)^{-1} - (b_k)^{-1}]^{-1} [a^{-1} - (b_k)^{-1}]^i{}_j . \qquad (24)$$

Inverse of Adjoint

As for the GL(n) matrix a we may define for each $p(x)$ in $Z[x]$

$$p^{-1}(\bar{a})_j{}^i = \sum_{r=1}^{n} g_{rj}{}^i [P(n - b_r - 1)]^{-1} .$$

In particular we may define the inverse adjoint

$$(\bar{a}^{-1})_j{}^i = \sum_{r=1}^{n} (n - b_r - 1)^{-1} g_{rj}{}^i .$$

LEMMA 10. — If $p(x) \in Z[x]$, then

$$p(\bar{a}^{-1})_j{}^i = \sum_{r=1}^{n} p[(n - b_r - 1)^{-1}] g_{rj}{}^i .$$

Proof. — Analogous to lemma 10. Substituting

$$p(x) = \prod_{k \neq r} [x - (n - b_k - 1)^{-1}]$$

gives

$$g_{r_j}{}^i = \prod_{k \neq r} [(n - b_r - 1)^{-1} - (n - b_k - 1)^{-1}]^{-1} [\bar{a}^{-1} - (n - b_k - 1)^{-1}]_j^i.$$

6. INVERSE LIE ALGEBRA AND ITS REPRESENTATIONS

We shall now show that the entries $(a^{-1})^i{}_j$ of the inverse GL(n) matrix a^{-1} form a Lie algebra. Now

$$\begin{aligned}[a^i{}_l, (a^{-1})^k{}_m] &= \sum_{r=1}^{n} (b_r)^{-1} [a^i{}_l, f_r{}^k{}_m] \\ &= \sum_{r=1}^{n} (b_r)^{-1} (\delta^k{}_l f_r{}^i{}_m - \delta^i{}_m f_r{}^k{}_l) \\ &= \delta^k{}_l (a^{-1})^i{}_m - \delta^i{}_m (a^{-1})^k{}_l .\end{aligned}$$

But

$$\begin{aligned}0 &= [\delta^i{}_j, (a^{-1})^k{}_m] \\ &= a^i{}_l [(a^{-1})^l{}_j, (a^{-1})^k{}_m] + [a^i{}_l, (a^{-1})^k{}_m](a^{-1})^l{}_j .\end{aligned}$$

Hence

$$\begin{aligned}a^i{}_l [(a^{-1})^l{}_j, (a^{-1})^k{}_m] &= - [a^i{}_l, (a^{-1})^k{}_m](a^{-1})^l{}_j \\ &= \delta^i{}_m (a^{-2})^k{}_j - (a^{-1})^i{}_m (a^{-1})^k{}_j\end{aligned}$$

where

$$(a^{-2})^k{}_j = (a^{-1})^k{}_i (a^{-1})^i{}_j .$$

Multiplying by $(a^{-1})^p{}_i$ and summing over i gives

$$[(a^{-1})^p{}_j, (a^{-1})^k{}_m] = (a^{-1})^p{}_m (a^{-2})^k{}_j - (a^{-2})^p{}_m (a^{-1})^k{}_j . \tag{25}$$

More generally we can show by induction that if $p(x) \in Z[x]$ then

$$[(a^{-1})^p{}_j, P(a^{-1})^k{}_m] = (a^{-1})^p{}_m P(a^{-1})^k{}_l (a^{-1})^l{}_j - (a^{-1})^p{}_i P(a^{-1})^i{}_m (a^{-1})^k{}_j .$$

From equation (25) we see that the $(a^{-1})^i{}_j$ generate an infinite dimensional Lie algebra, denoted $GL^{(-)}(N)$, which we call the inverse GL(N) Lie algebra.

We may now define the inverse Lie algebra of GL(n) as the Lie algebra with generators $b^i{}_j$ which satisfy the commutation relations (25) and the condition

$$a^i{}_j b^j{}_k = b^i{}_j a^j{}_k = \delta^i{}_k .$$

Since

$$(a^{-1})^i{}_j = \sum_{r=1}^{n} (b_r)^{-1} f_r{}^i{}_j$$

where

$$f_r{}^i{}_j = \prod_{k \neq r} (b_r - b_k)^{-1} (a - b_k)^i{}_j$$

we see that every representation of GL(n) is also a representation of $\mathrm{GL}^{(-)}(n)$. Conversely we have also shown that

$$f_r{}^i{}_j = \prod_{k \neq r} [(b_r)^{-1} - (b_k)^{-1}]^{-1} [a^{-1} - (b_k)^{-1}]^i{}_j$$

and since

$$a^i{}_j = \sum_{r=1} b_r f_r{}^i{}_j$$

we see that every representation of $\mathrm{GL}^{(-)}(n)$ is a representation of GL(n).

Thus the representations of GL(n) are representations of $\mathrm{GL}^{(-)}(n)$ and conversely. Hence we just need to investigate the representations of GL(n).

However we have already remarked that $(a^{-1})^i{}_j$ will not be defined on irreducible representations of GL(n) with weights $(\lambda_1, \ldots, \lambda_n)$ where $\lambda_n = 0$. Henceforth we only consider finite dimensional irreducible representations of GL(n) with weights $(\lambda_1, \ldots, \lambda_n)$ satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 1$.

Let v_0 be a maximal weight vector of such a GL(n) representation. Since

$$f_r{}^i{}_j v_0 = 0 \qquad \text{for} \quad j > i \text{ (see [1])}$$

we have

$$\delta^i{}_k v_0 = (a^{-1})^i{}_j a^j{}_k v_0 = \sum_{j \geqslant k} (a^{-1})^i{}_j a^j{}_k v_0 .$$

$$= \sum_{j \geqslant k} a^j{}_k (a^{-1})^i{}_j v_0 - \delta^i{}_k \sum_{j \geqslant k} (a^{-1})^j{}_j v_0 + \sum_{j \geqslant k} \delta^j{}_j (a^{-1})^i{}_k v_0 .$$

Putting $i = k$ we get

$$v_0 = \sum_{j \geqslant i} a^j{}_i (a^{-1})^i{}_j v_0 - \sum_{j \geqslant i} (a^{-1})^j{}_j v_0 + (n + 1 - i)(a^{-1})^i{}_i v_0$$

$$= (b_i + 1)(a^{-1})^i{}_i v_0 - \sum_{j \geqslant i} (a^{-1})^j{}_j v_0 .$$

Hence for $i = 1$, we obtain

$$v_0 = (b_1 + 1)(a^{-1})^1{}_1 v_0 - \sigma_1{}^{(-)} v_0$$

where $\sigma_1^{(-)}$ is the inverse invariant

$$\sigma_1^{(-)} = \operatorname{tr}(a^{-1}) = \sum_{j=1}^{n} (a^{-1})^j{}_j .$$

Therefore

$$(a^{-1})^1{}_1 v_0 = (\sigma_1^{(-)} + 1)(b_1 + 1)^{-1} v_0 .$$

For $i = 2$, we have

$$(b_2 + 1)(a^{-1})^2{}_2 v_0 - [\sigma_1^{(-)} - (a^{-1})^1{}_1] v_0 = v_0$$

which gives

$$(a^{-1})^2{}_2 v_0 = (\sigma_1^{(-)} + 1) b_1 [(b_1 + 1)(b_2 + 1)]^{-1} v_0 .$$

By recursion it is easily verified that

$$(a^{-1})^r{}_r v_0 = (\sigma_1^{(-)} + 1)(b_r + 1)^{-1} \prod_{k<r} b_k (b_k + 1)^{-1} v_0 .$$

Summing on r gives

$$\sigma_1^{(-)} v_0 = (\sigma_1^{(-)} + 1) \sum_{r=1}^{n} (b_r + 1)^{-1} \prod_{k<r} b_k (b_k + 1)^{-1} v_0 .$$

Solving this equation for $\sigma_1^{(-)}$ we obtain

$$\sigma_1^{(-)} = \left[1 - \sum_{r=1}^{n} (b_r)^{-1} \prod_{k \leqslant r} b_k (b_k + 1)^{-1} \right]^{-1} \sum_{r=1}^{n} (b_r)^{-1} \prod_{k \leqslant r} b_k (b_k + 1)^{-1} . \quad (26)$$

Therefore

$$(a^{-1})^r{}_r v_0 = \Lambda_r v_0$$
$$(a^{-1})^i{}_j v_0 = 0 \qquad j > i$$

where

$$\Lambda_r = \left[1 - \sum_{k=1}^{n} (b_k)^{-1} \prod_{l \leqslant k} b_l (b_l + 1)^{-1} \right]^{-1} (b_r)^{-1} \prod_{k \leqslant r} b_k (b_k + 1)^{-1} . \quad (27)$$

We may label the representations of $GL^{(-)}(n)$ by the eigenvalues of the inverse invariants $\Lambda_1, \ldots, \Lambda_n$. Formula (27) shows the relation between the labelling operators for $GL(n)$ and those for $GL^{(-)}(n)$.

7. VECTOR OPERATORS

In this section we study vector operators in more detail and obtain conditions under which the components ψ^i of a vector operator ψ commute.

Firstly we note that if ψ is a vector operator then each of its component operators $\psi_r = [b_r, \psi]$ is also a vector operator. Hence it follows that any Z-linear combination of the ψ_r is also a vector operator. Thus if $p(x)$ belongs to Z[x] then $p(a)\psi$ is a vector operator since

$$p(a)^i{}_j\psi^j = \sum_{r=1}^{n} p(b_r)\psi_r{}^i,$$

and

$$p(b_r) \in \mathrm{Z}.$$

Suppose now that ψ is a vector operator whose components ψ^i commute. Then

$$\begin{aligned}[\psi_r{}^i, \psi^k] &= [[b_r, \psi^i], \psi^k] \\ &= [[b_r, \psi^k], \psi^i] \qquad \text{(Jacobi identity)} \\ &= [\psi_r{}^k, \psi^i].\end{aligned}$$

Thus $[\psi_r{}^i, \psi^k]$ is symmetric with respect to i and k and it follows that if $l \neq r$ then $[\psi_r{}^i, \psi_l{}^k]$ is symmetric with respect to i and k and if $r = l$ the commutator is zero.

Hence we have proved that if ψ is a vector operator with commuting components then $[\psi_r{}^i, \psi_k{}^j]$ is symmetric with respect to i and j for $r, k = 1, \ldots, n$. Conversely suppose ψ is a vector operator such that $[\psi_r{}^i, \psi_k{}^j]$ is symmetric with respect to i and j for $r, k = 1, \ldots, n$. Then

$$[\psi^i, \psi^j] = \sum_{r,k=1}^{n} [\psi_r{}^i, \psi_k{}^j]$$

is also symmetric with respect to i and j. However $[\psi^i, \psi^j] = \psi^i\psi^j - \psi^j\psi^i$ is obviously antisymmetric with respect to i and j so it follows that

$$[\psi^i, \psi^j] = 0.$$

Hence we have proved the following result:

LEMMA 11. — Let ψ be a vector operator. Then the components ψ^i of ψ commute if and only if $[\psi_r{}^i, \psi_k{}^j]$ is symmetric with respect to i and j for $r, k = 1, \ldots, n$.

From now on we shall only consider vector operators ψ with components ψ^i which lie in U. Recall that $\bar{\mathrm{Z}}_{n+1} = \mathrm{F}(\hat{b}_1, \ldots, \hat{b}_{n+1})$ and every element of $\bar{\mathrm{Z}}_{n+1}$ is a GL(n + 1) invariant and therefore commutes with all the elements of U.

Let us define

$$\mathrm{Y}_r = \mathrm{F}(\hat{b}_1, \ldots, \hat{b}_{n+1}; b_r) \qquad r = 1, \ldots, n.$$

Then if $\gamma_r \in \mathrm{Y}_r$ it follows that if ψ is a vector operator then $[\gamma_r, \psi_k{}^i] = 0$ for $r \neq k$.

Now let ψ be a vector operator whose components ψ^i commute and let

$$\chi^i = \sum_{r=1}^{n} \gamma_r \psi_r{}^i, \qquad \gamma_k \in Y_k$$

be a Z-linear combination of the vector operators $\psi_r{}^i$ where the coefficient γ_r belongs to Y_r. Then $\chi_r{}^i = \gamma_r \psi_r{}^i$ and

$$\begin{aligned}[\chi_r{}^i, \chi_k{}^j] &= \gamma_r \gamma_k [\psi_r{}^i, \psi_k{}^j], \qquad r \neq k \\ &= 0, \qquad r = k.\end{aligned}$$

Since ψ is a vector operator, lemma (11) implies that $[\psi_r{}^i, \psi_k{}^j]$ and hence $[\chi_r{}^i, \chi_k{}^j]$ is symmetric with respect to i and j. Therefore χ is a vector operator with commuting components.

In particular if $p(x) \in \bar{Z}_{n+1}[x]$ then $p(a)\psi$ is a vector operator with commuting components since

$$p(a)^i{}_j \psi^j = \sum_{\gamma=1}^{n} p(b_r) \psi_r{}^i$$

and

$$p(b_r) \in Y_r.$$

The most obvious example of a vector operator with components that commute is the vector operator ψ with components $\psi^i = a^i{}_{n+1}$. From our previous remarks any Z-linear combination of the component operators $\psi_r = f_r \psi$ is also a vector operator and if the coefficient of each ψ_k lies in Y_k then the vector operator has commuting components. We shall now show that every vector operator in U must be a Z-linear combination of the vector operators ψ_r.

LEMMA 12. — Let a be the matrix of GL(n). Then for every $p(x)$ in Z[x] there exists $h(x)$ in Z[x] such that $p(\hat{a})^i{}_{n+1} = h(a)^i{}_j a^j{}_{n+1}$ and $p(\hat{a})^i{}_{n+1}$ is a vector operator.

Proof. — We first prove the result for $p(x) = x^m$ by induction on m, the result being obvious for $m \leqslant 1$.

$$\begin{aligned}(\hat{a}^{m+1})^i{}_{n+1} &= \sum_{k=1}^{n+1} \hat{a}^i{}_k (\hat{a}^m)^k{}_{n+1} \\ &= \sum_{k=1}^{n} a^i{}_k (\hat{a}^m)^k{}_{n+1} + a^i{}_{n+1} (\hat{a})^{n+1}{}_{n+1}.\end{aligned}$$

By the induction hypothesis there exists $h(x)$ in Z[x] such that

$$(\hat{a}^m)^k{}_{n+1} = h(a)^k{}_j a^j{}_{n+1}.$$

Therefore

$$(\hat{a}^{m+1})^i{}_{n+1} = a^i{}_k h(a)^k{}_j + (\hat{a}^m)^{n+1}{}_{n+1} a^i{}_{n+1} + [a^i{}_{n+1}, (\hat{a}^m)^{n+1}{}_{n+1}] = g(a)^i{}_j a^j{}_{n+1}$$

where

$$g(x) = (1 + x)h(x) + (\hat{a}^m)^{n+1}{}_{n+1} \in Z[x] .$$

Hence the result holds by induction for $p(x) = x^m$. Since every polynomial in $Z[x]$ is a Z-linear combination of the x^m the result follows.

We shall later derive the explicit form of $h(x)$ in terms of $p(x)$.

Now let ψ be an arbitrary vector operator with components ψ^i which lie in U. Since ψ^i depends only on the index i, there must be no other free indices (so all remaining indices are summed over). Hence, using the $GL(n + 1)$ commutation relations, we see that ψ^i must be of the form $p(\hat{a})^i{}_{n+1}$ for some $p(x)$ in $Z[x]$. Therefore there exists $h(x)$ in $Z[x]$ such that

$$\psi^i = p(\hat{a})^i{}_{n+1} = h(a)^i{}_j a^j{}_{n+1} = \sum_{\gamma=1}^{n} h(b_r) f_r{}^i{}_j a^j{}_{n+1}$$

and ψ is hence a Z-linear combination of the operators ψ_r with components $\psi_r{}^i = f_r{}^i{}_j a^j{}_{n+1}$.

We have therefore proved the following theorem:

THEOREM 13. — Let ϕ be a $GL(n)$ vector operator with components $\phi^i \in U$. Then ϕ is a Z-linear combination of the vector operators $\psi_r{}^i = f_r{}^i{}_j a^j{}_{n+1}$ and there exists $h(x)$ in $Z[x]$ such that

$$\phi^i = h(a)^i{}_j a^j{}_{n+1} .$$

If moreover

$$\phi^i = \sum_{r=1}^{n} \gamma_r \psi_r{}^i \qquad \text{where} \quad \gamma_k \in Y_k$$

then the components ϕ^i of ϕ commute.

DEFINITION. — From now on by a $GL(n)$ vector operator we mean a vector operator whose components lie in U. In view of the previous theorem we see that all of our vector operators must be expressible as a Z-linear combination of the vector operators

$$\psi_r{}^i = f_r{}^i{}_j a^j{}_{n+1} .$$

Let us denote the space of $GL(n)$ vector operators by W_n. Then W_n can be regarded as an n dimensional vector space over Z with basis bectors ψ_r.

With this interpretation we see that the matrix a of GL(n) is a diagonal matrix on W_n with the b_r's appearing along the diagonal:

viz.
$$a = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & b_n \end{pmatrix}.$$

The b_r may be interpreted as distinct eigenvalues of a with corresponding eigenvectors ψ_r. As in the classical theory we see that every vector in W_n can be written as a Z-linear combination of the eigenvectors ψ_r. Also in accordance with the classical theory the inverse a^{-1} of a is a diagonal matrix with the $(b_r)^{-1}$ appearing along the diagonal. Since

$$f_k\psi_r = \delta_{kr}\psi_r(f_k{}^i{}_j\psi_r{}^j = \delta_{kr}\psi_r{}^i)$$

we may interpret f_k as an idempotent matrix over W_n with 1 in the (k, k) position and zeros elsewhere.

DEFINITION. — Let J be the ideal in Z[x] consisting of all polynomials $p(x)$ in Z[x] such that $p(a) = 0$. In view of theorem (8), J consists of all polynomials in Z[x] divisible by the GL(n) minimum polynomial $m(x)$. Thus J is the two sided ideal in Z[x] generated by $m(x)$.

If $p(x)$ belongs to Z[x] then denote by $\bar{p}(x)$ the image of $p(x)$ under the canonical map
$$\pi : Z[x] \to Z[x] \mid J .$$

In view of lemma (2) we see that every element of Z[x] | J is a Z-linear combination of the $\bar{f}_r(x)$; i. e. if $\bar{p}(x) \in Z[x] \mid J$ then

$$\bar{p}(x) = \sum_{r=1}^{n} p(b_r)\bar{f}_r(x) .$$

The polynomials $\bar{f}_r(x)$ form an orthogonal set of idempotents in the algebra Z[x] | J which add up to give the identity:

$$\bar{f}_r(x)\bar{f}_k(x) = \delta_{rk}\bar{f}_k(x)$$

and

$$\sum_{r=1}^{n} \bar{f}_r(x) = 1 .$$

This gives a decomposition of Z[x] | J into a direct sum of irreducible left ideals:
$$Z[x] \mid J = Z\bar{f}_1(x) \oplus Z\bar{f}_2(x) \oplus \dots \oplus Z\bar{f}_n(x) .$$

Thus Z[x] | J is a semi simple associative algebra which is Z isomorphic to W_n (under the mapping $\bar{f}_r(x) \to \psi_r$).

8. DETERMINATION OF THE GL($n+1$) PROJECTION OPERATORS

As an application of our previous results we shall determine the GL(n + 1) operators

$$F_r{}^i{}_j = \prod_{k \neq r} (\hat{b}_r - \hat{b}_k)^{-1} (\hat{a} - \hat{b}_k)^i{}_j$$

which are analogues to the GL(n + 1) operators $f_r{}^i{}_j$.

Following Green [3] we introduce two sets of operators

$$u_k{}^i = F_k{}^i{}_{n+1} c_k{}^{-1}$$
$$v_{k_i} = c_k{}^{-1} F_k{}^{n+1}{}_i$$

where

$$c_k = F_k{}^{n+1}{}_{n+1} \,.$$

From the GL(n + 1) characteristic identity we have

$$\sum_{j=1}^{n+1} \hat{a}^i{}_j u_k{}^j = \hat{b}_k u_k{}^i$$

so that

$$\sum_{j=1}^{n} \hat{a}^i{}_j u_k{}^j + a^i{}_{n+1} = \hat{b}_k u_k{}^i \,.$$

Therefore

$$a^i{}_{n+1} = (\hat{b}_k - a)^i{}_j u_k{}^j \,.$$

Similarly we can show that

$$a^{n+1}{}_i = v_{k_j} (\hat{b}_k - a)^j{}_i \,.$$

Using our inverse polynomials we may invert these formulae to give

$$u_k{}^i = (\hat{b}_k - a)^{-1\,i}{}_j a^j{}_{n+1}$$
$$v_{k_i} = a^{n+1}{}_j (\hat{b}_k - a)^{-1\,j}{}_i \qquad (28)$$

where

$$(\hat{b}_k - a)^{-1\,i}{}_j = \sum_{r=1}^{n} (\hat{b}_k - b_r)^{-1} f_r{}^i{}_j \,.$$

The operator $(\hat{b}_k - b_r)^{-1}$ is a mixed GL(n) invariant and we may interpret it as that operator whose eigenvalue is $(\mu_k + 1 + r - \lambda_r - k)^{-1}$ on an irreducible representation of GL(n) with highest weight $(\lambda_1, \ldots, \lambda_n)$ contained in an irreducible representation of GL(n + 1) with highest weight $(\mu_1, \ldots, \mu_{n+1})$.

Using equation (28) we may now write

$$\begin{aligned} F_{k\ n+1}^{\ i} &= (\hat{b}_k - a)^{-1i}{}_j a^j{}_{n+1} c_k \\ F_k^{\ n+1}{}_i &= c_k a^{n+1}{}_j (\hat{b}_k - a)^{-1j}{}_i . \end{aligned} \tag{29}$$

But

$$F_{k\ j}^{\ i} = \delta^i{}_j c_k - [a^{n+1}{}_j, F_{k\ n+1}^{\ i}] \qquad i, j = 1, \ldots, n,$$

and using

$$[a^{n+1}{}_j, (\hat{b}_k - \hat{a})^i{}_l F_{k\ n+1}^{\ l}] = 0$$

we obtain

$$[a^{n+1}{}_j, F_{k\ n+1}^{\ i}] = \delta^i{}_j c_k - (\hat{b}_k - a)^{-1i}{}_p a^p{}_{n+1} F_k^{\ n+1}{}_j .$$

Hence we get

$$F_{k\ j}^{\ i} = (\hat{b}_k - a)^{-1i}{}_p a^p{}_{n+1} c_k a^{n+1}{}_q (\hat{b}_k - a)^{-1q}{}_j \qquad i, j = 1, \ldots, n. \tag{30}$$

Substituting for u_k and v_k using equation (29) we obtain

$$F_{k\ j}^{\ i} = u_k^{\ i} c_k v_{k_j} \qquad i, j = 1, \ldots, n+1. \tag{31}$$

As for GL(n) we have, for all $p(x)$ in Z[x],

$$\begin{aligned} p(\hat{a})^i{}_j &= \sum_{r=1}^{n+1} p(\hat{b}_r) F_{r\ j}^{\ i} \\ &= \sum_{r=1}^{n+1} p(\hat{b}_r) u_r^{\ i} c_r v_{r_j} . \end{aligned} \tag{32}$$

Equation (32) is in agreement with the result obtained by Green [3], and, as Green points out, can be regarded as analogous to a well known result for numerical matrices.

However we now have a little more information. From equation (28) we see that $u_k^{\ i}$ is a Z-linear combination of the vector operators $\psi_r^{\ i} = f_{r\ j}^{\ i} a^j{}_{n+1}$ and the coefficient of ψ_r is $(\hat{b}_k - b_r)^{-1}$ which lies in Y_r. Therefore $u_k^{\ i}$ is a vector operator with commuting components; i. e. $[u_k^{\ i}, u_k^{\ j}] = 0$.

Now from equation (28) we can write

$$\begin{aligned} u_k^{\ i} &= \sum_{r=1}^{n} (\hat{b}_k - b_r)^{-1} f_{r\ j}^{\ i} a^j{}_{n+1} \\ &= \sum_{r=1}^{n} f_{r\ j}^{\ i} a^j{}_{n+1} (\hat{b}_k - b_r - 1)^{-1} . \end{aligned} \tag{33}$$

This last result implies that although the components $u_k^{\ i}$ are well defined elements of U they may not always be defined on certain representations of GL(n) and GL($n+1$).

Indeed suppose $V_\mu(n+1)$ is a finite dimensional irreducible representation of GL($n+1$) with highest weight $(\mu_1, \ldots, \mu_{n+1})$. Then $V_\mu(n+1)$ can be decomposed into irreducible representations $V_\lambda(n)$ of GL(n) with highest weights $(\lambda_1, \ldots, \lambda_n)$ which satisfy

$$\mu_i \geqslant \lambda_i \geqslant \mu_{i+1} \qquad \text{(see [6])}.$$

Hence on the space $V_\lambda(n) \subseteq V_\mu(n+1) u_k{}^i$ takes the form

$$u_k{}^i = \sum_{r=1}^{n} f_r{}^i{}_j a^j{}_{n+1}(\mu_k - \lambda_r + r - k)^{-1}.$$

Thus $u_k{}^i$ will not be defined on subspaces where $\mu_k - \lambda_r + r - k = 0$ for some r. In particular the irreducible GL(n) representation $V_\mu(n)$ with highest weight $(\mu_1, \ldots, \mu_n)$ occurs in $V_\mu(n+1)$. Hence, on this space we have

$$u_k{}^i = \sum_{r=1}^{n} f_r{}^i{}_j a^j{}_{n+1}(\mu_k - \mu_r + r - k)^{-1}$$

and $u_k{}^i$ is not defined. An analogous statement also holds for the v_{k_i}.

Using equation (29) we have

$$\begin{aligned} F^i_{kn+1} &= (\hat{b}_k - a)^{-1i}{}_j a^j{}_{n+1} c_k \\ &= c_k(\hat{b}_k - a)^{-1i}{}_j a^j{}_{n+1} + (\hat{b}_k - a)^{-1i}{}_j F_k{}^j{}_{n+1}. \end{aligned}$$

Hence

$$[1 - (\hat{b}_k - a)^{-1}]^i{}_j F_k{}^j{}_{n+1} = c_k(\hat{b}_k - a)^{-1i}{}_j a^j{}_{n+1},$$

which gives

$$\begin{aligned} F_k{}^i{}_{n+1} &= c_k[1 - (\hat{b}_k - a)^{-1}]^{-1}(\hat{b}_k - a)^{-1i}{}_j a^j{}_{n+1} \\ &= c_k(\hat{b}_k - a - 1)^{-1i}{}_j a^j{}_{n+1}. \end{aligned} \tag{34}$$

Therefore, if $p(x) \in Z[x]$ then

$$\begin{aligned} p(a)^i{}_{n+1} &= \sum_{k=1}^{n+1} p(\hat{b}_k) F_k{}^i{}_{n+1} \\ &= \sum_{k=1}^{n+1} c_k p(\hat{b}_k)(\hat{b}_k - a - 1)^{i1i}{}_j a^j{}_{n+1} \\ &= \sum_{k=1}^{n+1} \sum_{r=1}^{n} c_k p(\hat{b}_k)(\hat{b}_k - b_r - 1)^{-1} f_r{}^i{}_j a^j{}_{n+1}. \end{aligned} \tag{35}$$

Hence we have proved that

$$p(\hat{a})^i{}_{n+1} = h(a)^i{}_j a^j{}_{n+1}$$

where

$$h(x) = \sum_{k=1}^{n+1} \sum_{r=1}^{n} c_k p(\hat{b}_k)(\hat{b}_k - b_r - 1)^{-1} f_r(x).$$

This result offers an alternative proof of lemma 12.

If we substitute $p(x) = 1$ into equation (35) we obtain

$$\sum_{r=1}^{n} \sum_{k=1}^{n+1} c_k(\hat{b}_k - b_r - 1)^{-1} \psi_r{}^i = 0.$$

Since the $\psi_r{}^i$ form a Z-linearly independent set this implies

$$\sum_{k=1}^{n+1} c_k(\hat{b}_k - b_r - 1)^{-1} = 0, \qquad r = 1, \ldots, n. \tag{36}$$

Equation (36) together with the condition

$$\delta^{n+1}{}_{n+1} = \sum_{k=1}^{n+1} \mathrm{F}_k{}^{n+1}{}_{n+1} \quad \text{or} \quad 1 = \sum_{k=1}^{n+1} c_k$$

may in fact be used to define the GL(n) invariants c_k. We may write these equations in matrix form

$$\sum_{k=1}^{n+1} \gamma_{rk} c_k = \delta_{rn+1}$$

where

$$\gamma_{rk} = (\hat{b}_k - b_r - 1)^{-1} \qquad r = 1, \ldots, n$$
$$\gamma_{n+1k} = 1.$$

Since the c_l, b_r and $\hat{b}_k$ commute these equations are easily solved using matrix methods and yield the solution

$$c_k = \prod_{p \neq k} (\hat{b}_k - \hat{b}_p)^{-1} \prod_{r=1}^{n} (\hat{b}_k - b_r - 1). \tag{37}$$

Therefore if $p(x) \in Z[x]$ then from equation (37) and the relation

$$p(\hat{a})^{n+1}{}_{n+1} = \sum_{k=1}^{n+1} p(\hat{b}_k) c_k$$

we see that we may express $p(\hat{a})^{n+1}{}_{n+1}$ as a rational function of the b_r and $\hat{b}_k$.

We have already shown that every vector operator of the form $p(\hat{a})^i{}_{n+1}$ where $p(x) \in Z[x]$ can be expressed in the form $h(a)^i{}_j a^j{}_{n+1}$ for some $h(x)$ in $Z[x]$. We shall now show that every vector operator of the form $h(a)^i{}_j a^j{}_{n+1}$ where $h(x) \in Z[x]$ can be expressed in the form $p(a)^i{}_{n+1}$ for $p(x)$ in $Z[x]$.

Let us introduce a set of polynomials

$$P_l(x) = d_l(x - b_l - 1)^{-1} \qquad l = 1, \dots, n \tag{38}$$

where

$$d_l = \prod_{p=1}^{n+1} (\hat{b}_p - b_l - 1) \prod_{r \neq l} (b_r - b_l)^{-1}$$

and

$$(x - b_l - 1)^{-1} = \sum_{k=1}^{n+1} (\hat{b}_k - b_l - 1)^{-1} F_k(x).$$

Then it can be shown that

$$p_l(\hat{a})^{n+1}{}_{n+1} = 0$$

and

$$\sum_{k=1}^{n+1} c_k P_l(\hat{b}_k)(\hat{b}_k - b_r - 1)^{-1} = \delta_{rl}.$$

In fact the $P_l(\hat{b}_k)$ may be regarded as solutions to these equations. Using equation (35) it follows that

$$\psi_l{}^i = p_l(\hat{a})^i{}_{n+1}$$

and hence if $p(x) \in Z[x]$ then

$$\begin{aligned} p(a)^i{}_j a^j{}_{n+1} &= \sum_{r=1}^{n} p(b_r)\psi_r{}^i \\ &= \sum_{r=1}^{n} p(b_r)p_r(\hat{a})^i{}_{n+1} \\ &= \sum_{k=1}^{n+1}\sum_{r=1}^{n} d_r p(b_r)(\hat{b}_k - b_r - 1)^{-1} F_k{}^i{}_{n+1} \end{aligned} \tag{39}$$

Therefore we have proved that

$$p(a)^i{}_j a^j{}_{n+1} = h(\hat{a})^i{}_{n+1}$$

where

$$h(x) = \sum_{k=1}^{n+1} \sum_{r=1}^{n} d_r(b_r)(\hat{b}_k - b_r - 1)^{-1} F_k(x) .$$

From equation (31) it may be of interest to compute the commutation relations between the u_k, v_r and c_m, since then it would be possible to compute the commutation relations between the ${F_r}^i{}_j$.

i) Since $[u_r{}^i, c_k] = [{F_r}^i{}_{n+1}, c_k]c_r{}^{-1}$ we may substitute for ${F_r}^i{}_{n+1}$ using equation (34) and obtain

$$[u_r{}^i, c_k] = c_r(\hat{b}_r - a - 1)^{-1i}{}_j u_k{}^j c_k c_r{}^{-1} .$$

Similarly one can show

$$[v_{r_i}, c_k] = c_r{}^{-1} c_k v_{k_j}(\hat{b}_r - a - 1)^{-1j}{}_i c_r .$$

ii) From the GL($n+1$) characteristic identity we can write

$$a^{n+1}{}_{n+1} = v_{r_i}(a - \hat{b}_r)^i{}_j u_k{}^j + \hat{b}_k .$$

Now

$$\begin{aligned} a^i{}_j - \delta^i{}_j a^{n+1}{}_{n+1} &= [a^i{}_{n+1}, a^{n+1}{}_j] \\ &= [(\hat{b}_k - a)^i u_k, v_{r_p}(\hat{b}_r - a)^p{}_j] . \end{aligned}$$

Using the fact that $v_r . u_k = \delta_{rk} c_k{}^{-1} - 1$ (easily verified using the GL($n+1$) identity) and substituting for $a^{n+1}{}_{n+1}$ we get

$$[v_{k_j}, u_r{}^i] = \delta_{rk} c_k{}^{-1}(\hat{b}_k - a)^{-1i}{}_j .$$

iii) It is easily shown that

$$[c_r, a^i{}_{n+1}] = -{F_r}^i{}_{n+1}$$

from which it follows that

$$[c_r{}^{-1}, a^i{}_{n+1}] = c_r{}^{-1} u_r{}^i .$$

Also

$$\begin{aligned} [(\hat{b}_k - a)^i{}_j u_k{}^j, u_r{}^m] &= [a^i{}_{n+1}, u_r{}^m] \\ &= -u_r{}^m u_r{}^i . \end{aligned}$$

Therefore

$$\begin{aligned} -u_r{}^m u_r{}^i &= [(\hat{b}_k - a)^i{}_j u_k{}^j, u_r{}^m] \\ &= (\hat{b}_k - a)^i{}_j [u_k{}^j, u_r{}^m] - \delta^m{}_j u_r{}^i u_k{}^j . \end{aligned}$$

Thus

$$(\hat{b}_k - a)^i{}_j [u_k{}^j, u_r{}^m] = u_r{}^i(u_k{}^m - u_r{}^m)$$

which gives

$$[u_r{}^i, u_r{}^m] = (\hat{b}_k - a)^{-1i}{}_j u_r{}^j(u_k{}^m - u_r{}^m) .$$

In a similar way we may evaluate the commutator between two v_r and thus compute the commutation relations of the F_k.

The evaluation of the above commutators illustrates how inverse polynomials in a may be used to derive commutation relations which would otherwise be difficult to obtain.

9. YOUNG PROJECTION OPERATORS

We conclude by showing how the characteristic identity can be applied to obtain an explicit expression for the Young projection operators in terms of the GL(n) invariants $\sigma_1, \ldots, \sigma_n$.

We apply the characteristic identity in its trace form

$$\sum_{r=0}^{n} S_r \sigma_{n-r} = 0 \tag{40}$$

where

$$S_r = (-1)^r \sum_{1 \leqslant i < j < \ldots < k \leqslant n} b_i b_j \ldots b_k \qquad r = 2, \ldots, n-1$$

the sum being taken over all sets of r integers $i, j, \ldots, k$ such that $1 \leqslant i < j < \ldots < k \leqslant n$ and

$$S_0 = 1$$

$$S_n = n(-1)^n b_1 \ldots b_n = n(-1)^n \det a .$$

We call equation (40) the GL(n) scalar identity. In its split form the GL(n) characteristic polynomial is

$$m(x) = \prod_{r=1}^{n} (x - b_r)$$

and this can be written

$$m(x) = \sum_{r=0}^{n-1} S_r X^{n-1} + (-1)^n \det a .$$

Since the coefficients of $m(x)$ belong to $Z_n = F[\sigma_1, \ldots, \sigma_n]$ we see that each of the coefficients S_r appearing in equation (40) are polynomial functions (over F) of the σ_k. On an irreducible finite dimensional representation of GL(n) with highest weight $(\lambda_1, \ldots, \lambda_n)$ the b_r take constant values $b_r = \lambda_r + n - r$. However, as we have already mentioned, on some irreducible representations certain factors from the characteristic identity may be omitted and we obtain a reduced equation. Hence the scalar identity must also reduce on such representations.

We write the scalar identity on the irreducible representation with highest weight $(\lambda_1, \ldots, \lambda_n)$ in the form

$$\sum_{r=0}^{n} S_r(\lambda_1, \ldots, \lambda_n) \sigma_{n-r} = 0 ,$$

where it is understood that the $S_r(\lambda_1, \ldots, \lambda_n)$ take the values corresponding to the trace of the reduced identity on this particular irreducible representation.

For example the reduced identity on the $(1, 0, \ldots, 0)$ representation is $(a - n)a = 0$ (see [2]). Therefore we have

$$S_{n-2}(1, 0, \ldots, 0) = 1$$
$$S_{n-1}(1, 0, \ldots, 0) = -n$$
$$S_r(1, 0, \ldots, 0) = 0 \qquad r \neq n-1, n-2$$

and the reduced scalar identity is $\sigma_2 - n\sigma_1 = 0$.

Now let us consider an n-dimensional vector space V over F and denote by V^r the tensor product space $V \oplus V \oplus \ldots \oplus V$ (r times). Every rth rank tensor in V^r can be written as a sum of tensors corresponding to each partition $[\lambda]$ of the rth order Young diagram.

We represent each such partition by a tuple $(\lambda_1, \ldots, \lambda_n)$ where λ_k denotes the number of boxes in the kth row of the Young diagram. The λ_i must be integers which satisfy $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_n = r$. Thus, in the case where $r < n$, we see that $\lambda_k = 0$ for $k > r$.

Hence we have a decomposition

$$V^r = \oplus \sum_{[\lambda]} V(\lambda) \tag{41}$$

where the sum is taken over all tuples $(\lambda_1, \ldots, \lambda_n)$ satisfying

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$$

and $\lambda_1 + \ldots + \lambda_n = r$ and where $V(\lambda_1, \ldots, \lambda_n)$ denotes the space of rth rank tensors of symmetry type corresponding to the partition $(\lambda_1, \ldots, \lambda_n)$ of the Young diagram.

It is well known that an n dimensional vector space V affords the irreducible $(1, 0, \ldots, 0)$ representation of $GL(n)$ and each of the spaces $V(\lambda_1, \ldots, \lambda_n)$ will give the irreducible representation of $GL(n)$ with highest weight $(\lambda_1, \ldots, \lambda_n)$. Hence on the space $V(\lambda_1, \ldots, \lambda_n)$ we obtain the reduced scalar identity

$$\sum_{r=0}^{n} S_r(\lambda_1, \ldots, \lambda_n)\sigma_{n-r} = 0 .$$

From the decomposition (41) we see that the generalized scalar identity

$$\prod_{[\lambda]} \left[\sum_{r=0}^{n} S_r(\lambda)\sigma_{n-r} \right] = 0$$

must be satisfied on the entire space V^r.

Corresponding to each partition $(m_1, \ldots, m_n)$ of the rth order Young diagram let us define an operator

$$P(m) = \prod_{[\lambda]} \left[\sum_{r=0}^{n} S_r(\lambda)\sigma_{n-r}(m) \right]^{-1} \left[\sum_{r=0}^{n} S_r(\lambda)\sigma_{n-r} \right]$$

where the product is over all partitions of the rth order Young diagram excluding the partition $(m_1, \ldots, m_n)$ and $\sigma_{n-r}(m)$ is the eigenvalue of the invariant σ_{n-r} on the irreducible $(m_1, \ldots, m_n)$ representation of $GL(n)$;

i. e.
$$\sigma_{n-r}(m) = \sum_{j=1}^{n} (b_j)^{n-r} \prod_{k \neq j} (b_j - b_k)^{-1}(b_j - b_k - 1)$$

where

$$b_j = m_j + n - j \qquad \text{(see [3])}.$$

The operator $P(m)$ corresponds to the left hand side of equation (42) (up to a normalization factor) with the term corresponding to the partition $(m_1, \ldots, m_n)$ omitted. Hence $P(m)$ will vanish on each of the spaces $V(\lambda_1, \ldots, \lambda_n)$ except the space $V(m_1, \ldots, m_n)$ on which it takes the value 1. Therefore $P(m)$ will project the space V^r onto the subspace of rth rank tensors of symmetry type $(m_1, \ldots, m_n)$.

Suppose T is an arbitrary tensor in V^r, and let

$$T = \sum_{[\lambda]} T_\lambda \tag{43}$$

be the decomposition of T into tensors of the various symmetries. Then application of the operator $P(m)$ gives

$$P(m)T = T_{m_1 \ldots m_n}.$$

Hence

$$P(m)^2 = P(m) \qquad (\text{on } V^r)$$

and we have a resolution of the identity on the space V^r.

$$1 = \sum_{[\lambda]} P(\lambda). \tag{44}$$

Therefore the operators $P(m)$ correspond to the Young projection operators of order r.

As an example consider the case $r = 2$. Then V^2 is the direct sum of the $(2, 0, \ldots, 0)$ (symmetric) and $(1, 1, 0, \ldots, 0)$ (antisymmetric) representations of $GL(n)$. The $GL(n)$ invariant σ_1 takes the constant value 2 on the space V^2.

The characteristic identity of GL(n) on the (2, 0, ..., 0) representation is $(a - n - 1)a = 0$, and the scalar identity is therefore $\sigma_2 - 2(n + 1) = 0$ (which therefore shows that $\sigma_2(2, 0, \ldots, 0) = 2(n+1)$). On the (1, 1, 0, ..., 0) representation the GL(n) identity is $(a - n + 1)a = 0$ and the scalar identity is $\sigma_2 - 2(n - 1) = 0$ (and hence

$$\sigma_2(1, 1, 0, \ldots, 0) = 2(n - 1))$$

Therefore the generalized scalar identity on V^2 is

$$(\sigma_2 - 2(n + 1)(\sigma_2 - 2(n - 1)) = 0$$

and the Young projection operators are:

$$\begin{aligned} P(2, 0, \ldots, 0) &= [2(n + 1) - 2(n - 1)]^{-1}[\sigma_2 - 2(n - 1)] \\ &= \frac{1}{4}(\sigma_2 - 2n + 2) \end{aligned}$$

and

$$\begin{aligned} P(1, 1, 0, \ldots, 0) &= [2(n - 1) - 2(n + 1)]^{-1}[\sigma_2 - 2(n + 1)] \\ &= -\frac{1}{4}(\sigma_2 - 2n - 2). \end{aligned}$$

This method for the construction of Young projection operators can also be applied to various other situations. For instancc it is well known (see [6]) that a finite dimensional irreducible representation $V_\mu(n + 1)$ of GL($n + 1$) can be decomposed into a direct sum of irreducible representations $V_\lambda(n)$ of GL(n) with weights $(\lambda_1, \ldots, \lambda_n)$ satisfying $\mu_i \geqslant \lambda_i \geqslant \mu_{i+1}$. Since the characteristic identity of the GL(n) matrix a on each of the spaces $V_\lambda(n)$ is known (see Green [2]) we may multiply all of these identities together to obtain a generalized characteristic identity satisfied on the entire space $V_\mu(n + 1)$. By taking the trace of this identity and omitting the factor corresponding to the space $V_\lambda(n)$ we obtain projection operators P_λ which project $V_\mu(n + 1)$ onto $V_\lambda(n)$.

Although in this paper we have restricted ourselves to GL(n) many of our previous results also hold in an analogous way for the subgroups Sp(n) and O(n). Green [2] has determined the characteristic identities of Sp(n) and O(n) on finite dimensional irreducible representations. Hence we may apply the methods just described to the decomposition of a finite dimensional irreducible representation of GL(n) into irreducible representations of SO(n).

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