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The general relativistic Dirac-Pauli particle: an underlying classical model

by

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ABSTRACT. — It is shown that the Dirac-Pauli particle classical model is characterized by a specific equation of state:

$$P_{\mu} \cdot P^{\mu} = m^2 + \frac{gq}{2} S_{\mu\nu} \cdot F^{\mu\nu}.$$

The Dixon-Souriau universal equations are satisfied.

INTRODUCTION

It is well established that one knows how to « prequantize » and then to quantize the space of motions of a free particle. (Kostant-Souriau-K. S. procedure: [3] [1] and references quoted therein). We recall that for a free relativistic particle with spin, the space of motions is an 8-dimensional symplectic manifold in one-to-one correspondence with an orbit of the coadjoint representation of the restricted Poincaré group. One is led in the spin 1/2 case to the free Dirac equation.

How shall we tackle the problem when the particle interacts with an external field? The K. S. theory yields partial answers in several (non relativistic) cases, e. g.: the harmonic oscillator [12] [13], the Kepler problem [14], thanks to the « pairing » technique [13] [15] [16] which makes it possible to work out the corresponding Schrödinger equation.

Everything is quite different when Lagrangian foliations do not exist, as in the case of spinning particles. (No real polarizations on the sphere S^2). It is then necessary to consider complex polarizations which demand a

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special treatment which raises difficulties we cannot get rid of up to now.

Nevertheless the correspondence principle is a clue for the existence of a classical symplectic and prequantum model for a spinning particle in an external electromagnetic and/or gravitational field. Numerous models with various specific properties can be found in the literature [5] [9] [11]. Which ones should properly describe the characteristic features of the electron or a particle with anomalous magnetic moment?

The Bargmann-Michel-Telegdi equations give the behaviour of a polarized particle with small $g - 2$, but only in the case of a weak and constant electromagnetic field (additional remarks and criticisms of Souriau can be found in the reference [1]).

Now we have at our disposal a whole class of classical spinning particle models (interaction with an external electromagnetic field F and general relativistic treatment) labelled by one arbitrary function f such that $P_\mu \cdot P^\mu = f(S_{\mu\nu} F^{\mu\nu})$ is the *equation of state* of the particle. P is the linear momentum whose gauge invariant definition can be obtained from a geometric theory (*Hyperspace* in Souriau [1]) which also yields in a natural way the electric charge q , the spin tensor S , the electromagnetic moment tensor, and some universal equations of motion for these quantities. This theory is reviewed in § I.

Furthermore it has been shown [1] [7] that a natural symplectic structure can be given on the set of solutions of these universal equations, provided that reasonable extra-conditions are imposed, namely: monolocality and vanishing electric moment.

The Dirac equation (minimal coupling) or the Dirac-Pauli equation (if one cares about the $g - 2$ factor) seems to be the relevant ones to describe electron physics. We are thus faced with the puzzling problem of the dequantization of the Dirac equation. We mean to say that we must pick out of the class of previously introduced models, *the* underlying classical model for a Dirac or Dirac-Pauli particle.

The aim of Dixon [5] was to look for « physical » variables associated (up to a term of the 1st order in F) to the Dirac equation. He could not avoid several conceptual difficulties, but he presented the normal coupling term ($g = 2$ in the formula below) up to a term of 2nd order in the electric charge. We will show that Dixon's result (extended to the Dirac-Pauli case) is *exact* for any electromagnetic field.

Our point of view is quite different. We compare the classical and quantum conserved quantities computed with the help of the hyperspace techniques (no gauge problems). This seems to be the best way to bridge the gap between the two descriptions of physics. One must stress that first integrals of the universal equations can be defined independently of any equation of state (this is the strong point of the theory) for particles, when there exist symmetries of the external fields. Their quantum analogues are set up via the stress-energy tensor and the electromagnetic current of

the Dirac-Pauli equation. The Volkov-Chakrabarti solution is used to carry out actual computations of the conserved quantities in the case of a constant and nilpotent ($F^3 = 0$) electromagnetic field in special relativity. The dipole coupling term is calculated with the help of a Casimir number of the invariance group of the electromagnetic field which has already been investigated by Bacry, Combe and Richard [6]. We find that:

$$P_\mu \cdot P^\mu = m^2 + \frac{gq}{2} S_{\mu\nu} \cdot F^{\mu\nu},$$

where m is the bare mass and g the gyromagnetic factor, and thus conjecture that geometric quantization of the symplectic structure of such a particle model should provide us again with the Dirac-Pauli equation.

I. A CLASS OF CLASSICAL MODELS OF SPINNING PARTICLES INTERACTING WITH THE ELECTROMAGNETIC FIELD. GENERAL INTRODUCTION

We wish to present in this section a fairly sketchy — although necessary — introduction to the hyperspace geometry of particle physics in order to introduce general conservation laws. The basic reference is [1].

1) The Principle of General Relativity. Electrodynamics

i) Hyperspace.

Let V_4 be the space-time manifold and $\pi(V_4)$ the set of all C^∞ differentiable fields of symmetric 2-covariant tensors g with signature $(- - - +)$ and 1-forms A over V_4 : $X \mapsto \begin{pmatrix} g_x \\ A_x \end{pmatrix}$. X denotes a point of V_4 , $T_X V_4$ the tangent space at X . Let G_I be the group of diffeomorphisms with compact support in V_4 , G_{II} be the additive group of differentiable real functions with compact support in V_4 . One can define in a natural way the action of $G_I \times G_{II}$ over the so called *space of potentials* $\pi(V_4)$:

$$(1.1) \quad \pi(V_4) \ni \begin{pmatrix} g \\ A \end{pmatrix} \mapsto \begin{pmatrix} \#(a)(g) \\ \#(a)(A) + \nabla b \end{pmatrix}; \quad a \in G_I; b \in G_{II}$$

$\#$ denotes the composition of the pull back and of the inversion operation: $[a \mapsto a^{-1}]$. ∇ is the exterior derivative.

(1.1) gives rise to a structure of *semi-direct product* of groups on the set $G_I \times G_{II}$. ($G = G_I \otimes G_{II}$).

The principle of general relativity (G. R.) — roughly speaking: invariance of physics under a smooth change of coordinates — and the gauge invariance of the 2nd kind for electrodynamics make it reasonable to assume that the relevant space for a further description of dynamics should be

the quotient: $\pi(V_4)/G$ called the *Hyperspace* $\mathcal{H}(V_4)$ (whose putative manifold structure is not needed for our purpose).

Let $[\eta \mapsto \xi]$ be the projection $\pi(V_4) \rightarrow \mathcal{H}(V_4)$.

Matter and electricity is then introduced as an element μ of the cotangent space $T_\xi^ \mathcal{H}(V_4)$.*

The hyperspatial duality is lifted to $\pi(V_4)$ in the sense that there exists at each $X \in V_4$ a symmetric contravariant 2-tensor T and a vector J such that the expression:

$$(1.2) \quad \int_{V_4} \left\{ \frac{1}{2} T^{\alpha\beta} \delta g_{\alpha\beta} + J^\alpha \delta A_\alpha \right\} \text{vol}$$

characterizes a completely continuous functional Φ ⁽²⁾ on $\pi(V_4)$ which will be assumed to be *baselike* (i. e. vanishing on a vertical vector). δ denotes an arbitrary derivation, vol the Riemannian volume element. In other words, given over V_4 any vector field f and any function φ with compact supports ((f, φ) denotes an element of the Lie algebra \mathcal{G} of the gauge group G), it is easy to write down the infinitesimal version of the latter assumption ⁽³⁾:

$$(1.3) \quad \Phi \left(\begin{matrix} [f, g] \\ [f, A] + \nabla \varphi \end{matrix} \right) = 0$$

It is clear that the following definition of μ makes sense

$$(1.4) \quad \mu(\delta\xi) = \Phi(\delta\eta)$$

ξ denotes the orbit of $\eta \in \pi(V_4)$ under G , whereas:

$$(1.5) \quad \text{Ker}([\delta\eta \mapsto \delta\xi]) = T_\eta(\text{orbit of } \eta).$$

Let us recall that the usual *Einstein-Maxwell identities* ⁽⁴⁾ are easily derived from (1.3), namely:

$$(1.6) \quad \text{div } [T] + F(J) = 0$$

$$(1.7) \quad \text{div } [J] = 0$$

where $F = \nabla A$, is the electromagnetic field (E. M. F.) with components

⁽²⁾ $\Phi(\delta\eta) = \int_{V_4} \left\{ \frac{1}{2} T(\delta g) + J(\delta A) \right\} \text{vol}; \eta = \begin{pmatrix} g \\ A \end{pmatrix} \in \pi(V_4); \delta\eta \in T_\eta \pi(V_4)$.

⁽³⁾ $[f, \]$ is the Lie derivative with respect to the vector field f . In a chart:

$$[f, g]_{\mu\nu} = \hat{\partial}_\mu \bar{\nabla}_\nu + \hat{\partial}_\nu \bar{\nabla}_\mu \quad (V = f(X); \bar{\nabla} = g(V))$$

and

$$[f, A]_X + \nabla[\varphi(X)] = \nabla[A_X \cdot V + \varphi(X)] + \nabla A_X(V).$$

The hat « \wedge » denotes the covariant derivative.

⁽⁴⁾ In a chart: $\begin{cases} \hat{\partial}_\mu T_{\nu\rho} g^{\mu\nu} + F_{\mu\rho} J^\mu = 0 \\ \hat{\partial}_\mu J^\mu = 0. \end{cases}$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Whence the physical interpretation of the stress-energy tensor for T and of the current density for J.

ii) Particles.

A particle (Einstein-Maxwell field equations are to be ignored for passive matter) is introduced as a 1st order distribution of matter and electricity condensed on a curve of V_4 : the world line of the particle. The general result reads:

THEOREM [J]. — Let Λ be a 1-dimensional closed submanifold of V_4 . Let Φ be a 1st order base like distribution with support Λ on $\pi(V_4)$, then:

i) at each $X \in \Lambda$ there exist a scalar q , a vector P , two 2-forms of V_4 : S and M (all of them uniquely defined) such that:

$$(1.8) \quad \left. \begin{aligned} 1] \quad \Phi(\delta\eta) = \int_\Lambda \left\{ \frac{1}{2} \delta g_{\mu\nu} [P^\mu dX^\nu + M^{\mu\rho} F_\rho^\nu dt] + q \delta A_\mu dX^\mu \right. \\ \left. + \frac{1}{2} S^{\mu\nu} dX^\rho \hat{\partial}_\mu \delta g_{\nu\rho} + M^{\mu\nu} \hat{\partial}_\mu \delta A_\nu dt \right\} \end{aligned} \right\}$$

where t denotes some parameter of Λ .

ii) In addition, the theory yields ten universal equations (⁵)

$$2] \quad \frac{\hat{d}P}{dt} = qF \cdot \frac{dX}{dt} - \frac{1}{2} R(S) \cdot \frac{dX}{dt} - \frac{1}{2} \text{Tr}(M \hat{\partial}_\rho F) \overline{\mathcal{R}}^{-1}$$

$$3] \quad \frac{\hat{d}S}{dt} = P \cdot \frac{\overline{dX}}{dt} - \frac{dX}{dt} \cdot \bar{P} - [M \cdot F - F \cdot M]$$

with

$$4] \quad q = \text{const.}$$

The interpretation of the afore mentioned physical quantities readily follows:

- $\left\{ \begin{aligned} q &= \text{the electric charge} \\ P &= \text{the linear momentum (gauge invariant definition!)} \\ S &= \text{the spin tensor} \\ M &= \text{the electromagnetic moment tensor.} \end{aligned} \right.$

(⁵) One can write equivalently:

$$\frac{\hat{d}P^\sigma}{dt} = qF_\mu^\sigma \frac{dX^\mu}{dt} - \frac{1}{2} R_{\mu\nu,\rho}^\sigma S^{\mu\nu} \frac{dX^\rho}{dt} + \frac{1}{2} M^{\mu\nu} \hat{\partial}_\rho F_{\mu\nu} g^{\rho\sigma}$$

$$\frac{\hat{d}S^{\mu\nu}}{dt} = P^\mu \frac{dX^\nu}{dt} - P^\nu \frac{dX^\mu}{dt} - [M^{\mu\rho} F_\rho^\nu - M^{\nu\rho} F_\rho^\mu]$$

R is the Riemannian curvature of V_4 . \mathcal{R} denotes a natural basis at X. The bar « — » stands for the transposition with respect to the metric g .

II) Conservation Laws

i) *The general procedure.*

The key point of the present formulation lies in the fact that general laws of conservation can be exhibited despite the indeterministic set of equations of motion ((1.8); 2], 3]) in the case of symmetric external fields, i. e. when there exists a non trivial isotropy group $\tilde{G}_\eta^{(6)}$ of a given field of potentials $\eta \in \pi(V_4)$.

Let $Z = (f, \varphi) \in \tilde{\mathcal{G}}$, then $Z \in \tilde{\mathcal{G}}_\eta$ (the Lie algebra of \tilde{G}_η) iff $Z_{\pi(V_4)}(\eta) = 0$ where $[Z \mapsto Z_{\pi(V_4)}]$ denotes the infinitesimal action of $\tilde{\mathcal{G}}$ on $\pi(V_4)$ (1.3):

$$(1.9) \quad [Z = (f, \varphi) \in \tilde{\mathcal{G}}_\eta] \Leftrightarrow [[f, g] = 0; [f, A] + \nabla\varphi = 0].$$

So f must be a Killing vector field of the metric g , which leaves the electromagnetic field invariant (note that the Lie derivative commutes with ∇ and remember: $\nabla^2 = 0$) i. e.: $[f, F] = 0$.

Suppose that V_4 can be endowed with nice global topological properties; i. e. that past and future may be globally well defined. Assume furthermore that matter is bounded in space. Then choose a real function λ on V_4 ranging from 0 (past) to 1 (future) and define $(f, \varphi)_\lambda = (\lambda f, \lambda\varphi)$ where (f, φ) is a solution of (1.9) and $[\lambda f](X) = \lambda(X)f(X) \forall X \in V_4$.

(1.10) PROPOSITION. — The quantity $\Phi(\delta_\lambda\eta)^{(7)}$ is independent of the choice of λ . It depends upon the distribution of matter and of the generator $(f, \varphi) \in \tilde{\mathcal{G}}_\eta$. One may arbitrarily shrink the time interval in which λ increases and claim that the latter quantity is a constant of motion of the matter, provided that it does not change under a shift of the little time interval.

For a proof, see: ([I] (61)(62)(63)).

(1.11) COROLLARY. — Let $V = f(X) \in T_X V_4$; $(f, \varphi) \in \tilde{\mathcal{G}}_\eta$, then the flux $\int_{C_3} \text{vol}(T.V + [A.V + \varphi(X)]J)$ through any arbitrary space-like 3-chain C_3 is a constant of motion. T and J are respectively the stress energy tensor and the E. M. current of the matter.

Proof. — Let $(f, \varphi)_\lambda(X) = (V\alpha, \alpha\varphi(X))$; $\alpha = \lambda(X)$. So

$$(1.3) \quad \Phi(\delta_\lambda\eta) = \left\{ \left\{ \text{Tr} \left(T \cdot \frac{\hat{\partial}}{\partial X} [V\alpha] \right) + \nabla[(A.V + \varphi(X))\alpha] \cdot J + F(V\alpha)(J) \right\} \text{vol} \right.$$

(6) As a subgroup of the semi-direct product \tilde{G} of the group of diffeomorphisms and the additive group of real C^∞ functions of V_4 (no compact support!), $G(1.1)$ is an invariant subgroup of \tilde{G} .

(7)
$$\delta_\lambda\eta = \begin{pmatrix} [\lambda f, g] \\ [\lambda f, A] + \nabla[\lambda\varphi] \end{pmatrix}$$

Use

$$\text{div} [T \cdot V] = \text{div} [T] \cdot V + \text{Tr} \left(T \cdot \frac{\partial V}{\partial X} \right); \text{div} [\alpha V] = \alpha \text{div} (V) + \nabla \alpha \cdot V;$$

(1.7), and get:

$$\Phi(\delta_\lambda \eta) = \int \left\{ \text{div} [\alpha(T \cdot V + [A \cdot V + \varphi(X)]J)] - \alpha[\text{div} [T] + F(J)] \cdot V \right\} \text{vol}.$$

Put $\alpha = 1$ (1.10) and use (1.6) to complete the proof.

ii) Conservation laws for particles.

(1.12)
PROPOSITION. — Given $(f, \varphi) \in \mathcal{G}_n$ (solution of (1.19)), the quantity:

$$[\bar{P} + qA] \cdot V + q\varphi(X) + \frac{1}{2} \text{Tr} \left(S \cdot \frac{\partial V}{\partial X} \right)$$

is a 1st integral of the system: ((1.8) 2]; 3])

for a proof: [I] (66).

Remark. — ($f = 0$; $\varphi = 1$) is obviously a solution of (1.9). So (1.11) and (1.12) assume that $\int_{C_3} \text{vol} (J)$ and q correspond to each other.

III) Conservation Laws in the Specific Case of the Constant Electromagnetic Field

Given a constant E. M. F.: $[X \mapsto F]$ over E_4 (the Minkowski flat space-time), one wishes to look for infinitesimal isometries f of E_4 such that:

$$(1.13) \quad [f, A] + \nabla \varphi = 0 \quad (\text{for some function } \varphi),$$

i. e. an element f of a Lie subalgebra of the Poincaré Lie algebra whose action on E_4 is [3]:

$$(1.14) \quad f(X) = \Lambda \cdot X + \Gamma; \quad X \in E_4; \quad \Gamma \in E_4 \text{ (translations)}$$

$$\Lambda \in L(E_4); \quad \Lambda + \bar{\Lambda} = 0 \quad (\text{infinitesimal Lorentz rotations}).$$

PROPOSITION. — Let $[X \mapsto F]$ be a constant E. M. F. over E_4 with potential $[X \mapsto A]$; let f be an infinitesimal isometry of E_4 , i. e. a pair (Λ, Γ) (1.14), then the general solution (f, φ) of (1.13) is of the form:

$$(1.15) \quad \left\{ \begin{array}{l} \Lambda = aF + b*(F); \quad a, b \in \mathbb{R}; \quad \Gamma \in E_4^{(8)} \\ \varphi(X) = -A \cdot (\Lambda \cdot X + \Gamma) + F(X) \left(\Gamma + \frac{1}{2} \Lambda \cdot X \right) + \text{const.} \end{array} \right.$$

⁽⁸⁾ The skew-symmetric 2-form F is identified with a linear anti-hermitian operator by:

$$F(dX)(\delta X) = d\bar{X} \cdot F \cdot \delta X \quad \forall dX, \delta X \in E_4.$$

Proof. — $[f, A]_X = \nabla[A \cdot f(X)] + F(f(X))$ (Cartan formula [3]). (1.13) reads:

$$\begin{aligned} d[A \cdot f(X) + \varphi(X)] &= F(dX)(f(X)) \quad \forall dX \in T_X E_4 = E_4 \\ &= F(dX)(\Lambda \cdot X + \Gamma) \quad (1.14) \\ &= d[F(X)(\Gamma)] - \frac{1}{2} [F(\Lambda \cdot X)(dX) - F(dX)(\Lambda \cdot X)] \quad (F = \text{const.}) \\ &= d[F(X)(\Gamma)] + \frac{1}{2} [\bar{X} \cdot \Lambda \cdot F \cdot dX + d\bar{X} F \cdot \Lambda \cdot X] \\ &= d \left[F(X) \left(\Gamma + \frac{1}{2} \Lambda \cdot X \right) \right] + \frac{1}{2} \bar{X} [\Lambda \cdot F - F \cdot \Lambda] dX \end{aligned}$$

Furthermore (1.13) yields $[f, F] = 0$ (Poincaré lemma) which implies:

$$\Lambda \cdot F - F \cdot \Lambda = 0$$

the general solution of which reads [6]

$$\Lambda = aF + b * (F); \quad a, b \in \mathbb{R}.$$

where $*(F)$ denotes the usual adjoint of F [3]

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Thus: $F, *(F)$ together with the translations span a 6-dimensional Lie subalgebra of the Poincaré Lie algebra.

It is then straightforward to exhibit the 6 constants of motion associated to a constant E. M. F. via (1.12).

We have:

$$(1.16) \quad f(X) = aF \cdot X + b * (F) \cdot X + \Gamma$$

Let us define the quantities $\pi \in E_4; \gamma, \gamma^* \in \mathbb{R}$ by:

$$[\bar{P} + qA] \cdot f(X) + \frac{1}{2} \text{Tr} \left(S \cdot \frac{\partial f(X)}{\partial X} \right) + q\varphi(X) = \bar{\pi} \cdot \Gamma + a\gamma + b\gamma^*$$

then:

$$(1.17) \quad \pi = P - qF \cdot X \quad (\text{the linear momentum})$$

$$(1.18) \quad \gamma = \left[\bar{P} + \frac{q}{2} \bar{X} \cdot F \right] \cdot F \cdot X + \frac{1}{2} \text{Tr} (S \cdot F)$$

$$(1.19) \quad \gamma^* = \left[\bar{P} + \frac{q}{2} \bar{X} \cdot F \right] \cdot * (F) \cdot X + \frac{1}{2} \text{Tr} (S \cdot * (F))$$

are 1st integrals of motion associated to the isotropy subgroup (1.9).

They can be alternatively interpreted as the *momenta* of that subgroup [3].

IV) A Model of Spinning Particle in an External Gravitational and Electromagnetic Field

A particle is called *elementary* iff it can be given a complete description through the sole set of the physical parameters q, P, S, M introduced in (1.8).

In order to set up a complete deterministic system of equations of motion, one must restrict the host of different equations of state investigated in the literature. It has been assumed [1] [5] [7] [8] [9] that physically relevant additional assumptions should be:

(1.20) $P \in \text{Ker}(S)$ (monolocality)

(1.21) M parallel to S (no electric dipole moment)

(1.20) and (1.21) together with (1.8) yield:

$$-\frac{1}{2} \text{Tr}(S^2) = s^2 = \text{const.}$$

(s denotes the scalar spin: constant of the system) whence the:

PROPOSITION. — Let f be an arbitrary real positive function (at least C^1). It can be associated an elementary particle model in which the relations:

i) $P \cdot P = f(\alpha)$ ($\alpha = -\text{Tr}(S \cdot F) = S_{\mu\nu} F^{\mu\nu}$)

ii) $S \cdot P = 0$

hold. The magnetic moment operator reads:

$$M = \left[\frac{f'(\alpha)}{f(\alpha)} P \cdot \frac{dX}{dt} \right] \cdot S = \frac{\mu(\alpha)}{s} \cdot S$$

(definition of the scalar magnetic moment $\mu(\alpha)$).

(1.22) Given initial data obeying: i), ii), the motion is determined by the universal equations ((1.8): 2]; 3]) and the additional equation for the velocity:

1) $\frac{dX}{dt}$ parallel to:

$$\omega P + S \left[\frac{1}{2} R(S) \cdot P + (q - f'(\alpha)) F \cdot P - \frac{1}{2} f'(\alpha) \text{Tr}(S \hat{\partial}_\rho F)^\rho \mathcal{R}^{-1} \right]$$

2) $\omega = f(\alpha) + \frac{1}{2} q\alpha + \frac{1}{4} R(S)(S)$

For a proof: [1] [7].

M is a linear density of the world line Λ with values in the set of skew-symmetric-2-tensors of V_4 . One can identify M and the magnetic moment tensor if t is required to be the proper time of the particle whenever Λ is timelike.

Note that in the case of a weak E. M. F., the gyromagnetic Landé factor is defined by:

(1.23) $g = \frac{2m\mu(0)}{qs}$

$$(1.24) \quad m = \sqrt{f(0)} \quad (\text{the bare mass})$$

$$(1.25) \quad \mu(0) = \frac{s}{m} f'(0)$$

In the case of a constant E. M. F. in special relativity, one can easily check that the spin-E. M. F. coupling term α ((1.22)(i)) is *constant*.

II. THE DIRAC-PAULI EQUATION

This section is devoted to the explicit calculation of the conserved quantities deduced from the Dirac-Pauli (D. P.) equation in the case of a constant E. M. F. Since we are going to deal — within the classical field framework — with densities (over spacetime), the hyperspatial machinery previously scribed will be useful to work out the stress energy tensor T and the current density J of a D. P. particle in an arbitrary gravitational and E. M. field. The specific case of a constant E. M. F. we are interested in, will readily follow; as for the constants of motion deduced from the general theorem (1.11), the exact solution of the D. P. equation [4] which stems from the Volkov solution [10] of the Dirac equation will be needed to achieve their actual computation in the case of a nilpotent (crossed) E. M. F. The function f (1.22) which characterizes the E. M. interaction is then explicitly stated for a D. P. particle.

The geometric meaning of the so called anomalous magnetic moment is also clearly analysed.

Let us first introduce the main results of spinor-calculus on a Riemannian manifold.

I) Spinors

Let (V_4, g) be the Riemannian space time manifold with connexion Γ . We recall that the spinor space $\varepsilon \cong \mathbb{C}^4$ can be given a real hermitian structure (we denote the transposition by a bar —: $\varepsilon \ni \psi \mapsto \bar{\psi} \in \varepsilon^*$) with signature $(++--)$. In a local chart the Dirac operators γ_μ turn out to be hermitian ($\gamma_\mu = \bar{\gamma}_\mu \quad \forall \mu = 1, 2, 3, 4$) and satisfy the anticommutation relations:

$$(2.1) \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \mathbb{1}_\varepsilon$$

Suppose (as will be assumed later on) that there exists over V_4 a global C^∞ field of orthonormal frames, then there exists a mapping $\lambda: T_X V_4 \mapsto \mathcal{G}_{\text{cliff}}$ (the Lie algebra of the Clifford group) characterized by:

$$(2.2) \quad \lambda_\mu = \frac{1}{8} [\gamma^\nu, \partial_\mu \gamma_\nu - \Gamma_{\mu\nu}^\rho \gamma_\rho]$$

(the brackets denote the commutator; $\gamma^\nu = g^{\mu\nu} \gamma_\mu$).

Define then the covariant derivative of a spinor $\psi \in \varepsilon$ by

$$(2.3) \quad \hat{\partial}_\mu \psi = \partial_\mu \psi + \lambda_\mu \cdot \psi$$

Let us exhibit some useful formulae which will be involved in the proof of (2.9).

$$(2.4) \quad \lambda_\mu \cdot \gamma^\mu + \gamma^\mu \cdot \lambda_\mu = -\frac{1}{4} \gamma^\mu [\partial_\mu \gamma_\nu - \partial_\nu \gamma_\mu] \gamma^\nu$$

$$(2.5) \quad \gamma^\mu \lambda_\mu - \lambda_\mu \cdot \gamma^\mu = \frac{1}{u} \partial_\mu [u \gamma^\mu] = \partial_\mu \gamma^\mu + \Gamma_{\mu\rho}^\mu \gamma^\rho$$

($u =$ the Riemannian density $= \sqrt{|\det(g)|}$).

II) The General Expression of the Stress Energy Tensor and the Current Density of a D. P. Particle in an E. M. and Gravitational Field

The D. P. field equation:

$$(2.6) \quad \gamma^\mu \hat{\partial}_\mu \psi i - \gamma^\mu \psi A_\mu q - \psi m - \gamma^\mu \gamma^\nu \psi i k F_{\mu\nu} = 0$$

(k is a real constant which takes into account the anomalous magnetic moment) is deduced from the well known variational problem:

$$\delta \int_{C_4} l \text{ vol} = 0$$

$\forall [X \mapsto \delta\psi]$ with compact support contained in the 4-chain C_4 , where the Lagrangian l is given by

$$(2.7) \quad l = \text{Re}(\bar{\psi}[\gamma^\mu \hat{\partial}_\mu \psi i - \gamma^\mu \psi A_\mu q - \psi m - \gamma^\mu \gamma^\nu \psi i k F_{\mu\nu}])$$

q, m are respectively the *electric charge* and the (proper) *mass* of the particle.

In order to calculate T and J , let us assume that the 1-form of $T^*\mathcal{H}(V_4)$ associated ⁽⁹⁾ to the functional (1.3) is « exact »:

$$(2.8) \quad \int_{C_4} \left\{ \frac{1}{2} T(\delta g) + J(\delta A) \right\} \text{ vol} + \delta \int_{C_4} l \text{ vol} = 0$$

$\forall [X \mapsto \delta g]; \forall [X \mapsto \delta A]$ with compact support in C_4 . Note that the only constraints entering (2.8) are given by (2.1).

⁽⁹⁾ Let $\zeta \in T^*\mathcal{H}(V_4)$ and define the 1-form ϖ of $T^*\mathcal{H}(V_4)$ by:

$$\varpi(\delta\zeta) = \Phi(\delta\eta)$$

So (2.8) will provide us with a Lagrangian section of $T^*\mathcal{H}(V_4)$.

One can now state the following:

PROPOSITION. — *Given the Lagrangian density (2.7) for the Dirac-Pauli particle, the stress energy tensor T and the current density J are respectively:*

$$\begin{aligned}
 (2.9) \quad i) \quad T_{\alpha\beta} &= \frac{1}{2} \operatorname{Re} (\bar{\psi} [\gamma_\alpha \hat{\partial}_\beta + \gamma_\beta \hat{\partial}_\alpha] \psi i) - \frac{1}{2} \bar{\psi} [\gamma_\alpha A_\beta + \gamma_\beta A_\alpha] \psi q \\
 &+ \frac{k}{2} \{ \operatorname{Re} (\bar{\psi} [\gamma_\alpha, \gamma^\rho] \psi i) F_{\rho\beta} + \operatorname{Re} (\bar{\psi} [\gamma_\beta, \gamma^\rho] \psi i) F_{\rho\alpha} \} \\
 &= \operatorname{Re} (\bar{\psi} \gamma_\alpha \hat{\partial}_\beta \psi i) - \bar{\psi} \gamma_\alpha \psi A_\beta q - \frac{1}{4} \operatorname{div} [S]_{\alpha\beta} \\
 &+ k \operatorname{Re} (\bar{\psi} [\gamma_\alpha, \gamma^\rho] \psi i) F_{\rho\beta}.
 \end{aligned}$$

where:

a) $S_{\alpha\beta}^\rho = \operatorname{Re} (\bar{\psi} \gamma_\alpha \gamma^\rho \gamma_\beta \psi i)$ with the following algebraic properties:

b) $S_{\alpha\beta}^\rho + S_{\beta\alpha}^\rho = 0$

c) $g^{\mu\rho} g_{\lambda\nu} S_{\alpha\rho}^\lambda + S_{\alpha\nu}^\mu = 0$

ii) $J \cdot V = \bar{\psi} \gamma(V) \psi q + \operatorname{div} [\Theta] \cdot V \quad \forall V \in T_X V_4$

where:

a) $\Theta_{\alpha\beta} = 2k \operatorname{Re} (\bar{\psi} \gamma_\alpha \gamma_\beta \psi i)$

b) $\Theta_{\alpha\beta} + \Theta_{\beta\alpha} = 0.$

Proof. — Compute, compute, compute!

A tedious calculation shows — as expected — that the identities (1.6) and (1.7) are satisfied.

We would like to emphasize at this stage that we get rid of the controversy of the symmetry of T (here $T_{\alpha\beta} = T_{\beta\alpha}$, even in the case of a spinning particle) arising from the use of the Euler-Lagrange formalism. Proposition (2.9) is a generalization of parts of [2].

III) Conserved Quantities in the Case of a Constant E. M. F. in flat spacetime E_4

The computation of the 6 above mentioned (1.17) (1.18) (1.19) first integrals $\{\pi, \gamma, \gamma^*\}$ is performed by means of the corollary (1.11) and the expression (2.9) of T and J.

Let $N \in E_4$ define the pointwise orthogonal direction with respect to the spacelike « hypersurface » C_3 involved in (1.11) ⁽¹⁰⁾.

⁽¹⁰⁾ Since the corollary (1.11) allows any arbitrary 3-hypersurface C_3 , let us choose for the sake of simplicity $C_3 = \{X \in E_4 / NX = 0 / N = \text{const.} \in E_4 / NN \geq 0\}$.

One gets using (1.15) and (1.16):

$$(2.10) \quad \bar{\pi} \cdot \Gamma = \int_{C_3} \{ T(N)(\Gamma) - [\bar{N} \cdot J][\overline{F \cdot X}] \cdot \Gamma \} \text{vol}_3$$

$$(2.11) \quad \gamma = \int_{C_3} \left\{ T(N)(F \cdot X) - [\bar{N}J] \frac{[\overline{F \cdot X}] \cdot F \cdot X}{2} \right\} \text{vol}_3$$

$$(2.12) \quad \gamma^* = \int_{C_3} \left\{ T(N)(*(F) \cdot X) - [\bar{N}J] \frac{[\overline{F \cdot X}] \cdot *(F) \cdot X}{2} \right\} \text{vol}_3$$

Let us define a new operator Ω by:

$$(2.13) \quad \Omega_{\beta}^{\alpha} = S_{\rho\beta}^{\alpha} N^{\rho} = \text{Re} (\bar{\psi} \gamma(N) \gamma^{\alpha} \gamma_{\beta} \psi i).$$

Obviously:

$$(2.14) \quad \Omega_{\alpha\beta} + \Omega_{\beta\alpha} = 0$$

and

$$(2.15) \quad N \in \text{Ker} (\Omega) \quad (\text{Rank} (\Omega) = 2)$$

Ω will be, in the sequel, easily related to the spin operator by a mere comparison of the quantum conserved quantities with the classical ones. Thus:

$$\bar{\pi}_{\alpha} = \int_{C_3} \left\{ \text{Re} (\bar{\psi} \gamma(N) [\partial_{\alpha} \psi i - \psi (A_{\alpha} + [\overline{F \cdot X}]_{\alpha} q)]) - \frac{1}{4} \text{div} [\Omega]_{\alpha} + 2k (Z^{\rho} F_{\rho\alpha} - [\overline{F \cdot X}]_{\alpha} \text{div} [Z]) \right\} \text{vol}_3$$

with: $Z^{\rho} = \text{Re} (\bar{\psi} \gamma(N) \gamma^{\rho} \psi i)$

Discarding in view of Stokes theorem quantities which vanish on the boundary of C_3 , we are left with:

$$(2.16) \quad \bar{\pi}_{\alpha} = \int_{C_3} \text{Re} (\bar{\psi} \gamma(N) [\partial_{\alpha} \psi i - \psi (A_{\alpha} + [\overline{F \cdot X}]_{\alpha} q)]) \text{vol}_3$$

$$(2.17) \quad \gamma = \int_{C_3} \left\{ \text{Re} (\bar{\psi} \gamma(N) [\partial_{\alpha} \psi i - \psi A_{\alpha} q]) [\overline{F \cdot X}]^{\alpha} - \bar{\psi} \gamma(N) \psi \frac{[\overline{F \cdot X}] F \cdot X}{2} + \frac{1}{4} \text{Tr} (\Omega \cdot F) \right\} \text{vol}_3$$

$$(2.18) \quad \gamma^* = \int_{C_3} \left\{ \text{Re} (\bar{\psi} \gamma(N) [\partial_{\alpha} \psi i - \psi A_{\alpha} q]) [* (F) \cdot X]^{\alpha} - \bar{\psi} \gamma(N) \psi \frac{[\overline{F \cdot X}] * (F) \cdot X}{2} + \frac{1}{4} \text{Tr} (\Omega * (F)) \right\} \text{vol}_3$$

Compare with (1.17), (1.18), (1.19).

One must now insist on the fact that the latter conserved quantities have the same formal expression as those associated with the Dirac equation ⁽¹⁾.

⁽¹⁾ We thus prove a result quoted in [6]. Our point of view is however different.

The information on the anomalous magnetic moment lies only in the solution ψ of the D. P. equation.

Finally, the computation of (2.16), (2.17), (2.18), will be carried out in a rather special case with the aid of the following exact solution [4] of the D. P. equation coupled with a constant nilpotent E. M. F.

Let \mathbf{B}, \mathbf{K} be constant vectors of E_4 , and $A = \mathbf{B}\bar{\mathbf{K}}X$ be a potential of the E. M. F.: $F = \frac{\partial \bar{A}}{\partial X} - \frac{\partial A}{\partial X} = \mathbf{K}\mathbf{B} - \mathbf{B}\bar{\mathbf{K}}$. The wave vector \mathbf{K} will be null ($\bar{\mathbf{K}} \cdot \mathbf{K} = 0$) for obvious physical reasons. F turns out to be singular. Furthermore if $\bar{\mathbf{B}}\mathbf{K} = 0$ (as will be assumed later on), $\text{Tr}(F^2) = 0$. Hence the electric and magnetic fields have the same intensity and are orthogonal to each other (briefly $F^3 = 0$).

The Maxwell equations together with the gauge condition $\text{div}[A] = 0$ are satisfied.

One can easily check that in that case:

$$(2.19) \quad \psi = \exp\left(\frac{q\bar{\mathbf{K}} \cdot X}{4\bar{\mathbf{K}} \cdot \mathbf{I}} \gamma_\mu \gamma_\nu F^{\mu\nu} - i\sigma\right) [\gamma(\mathbf{I}) + m]\gamma(\mathbf{K}) \exp(2k\bar{\mathbf{K}} \cdot X \gamma(\mathbf{B}))\xi$$

$$\text{together with: } \sigma = \bar{\mathbf{I}} \cdot X + q \frac{\bar{\mathbf{I}} \cdot \mathbf{B}}{2\bar{\mathbf{K}} \cdot \mathbf{I}} [\bar{\mathbf{K}} \cdot X]^2 - q^2 \frac{\bar{\mathbf{B}} \cdot \mathbf{B}}{6\bar{\mathbf{K}} \cdot \mathbf{I}} [\bar{\mathbf{K}} \cdot X]^3$$

(\mathbf{I} is a constant vector of E_4 , whilst ξ is a constant spinor of ϵ) is a solution of (2.6) whenever:

$$(2.20) \quad \bar{\mathbf{I}} \cdot \mathbf{I} = m^2 \quad (1^2)$$

Put judiciously $\mathbf{N} = \mathbf{K}$ in the definition of C_3 .

Let us now come back to the definition (1.12) of the electric charge:

$$(2.21) \quad q = \int_{C_3} \text{vol}(\mathbf{J})$$

But:

$$\int_{C_3} \text{vol}(\mathbf{J}) = \int_{C_3} \{ \bar{\psi}\gamma(\mathbf{K})\psi q + \text{div}[\Theta \cdot \mathbf{K}] \} \text{vol}_3 = \int_{C_3} \bar{\psi}\gamma(\mathbf{K})\psi q \text{vol}_3$$

since the term under the divergence vanishes on the boundary of C_3 ($\text{div}[\Theta \cdot \mathbf{K}] = \text{div}_3[\Theta \cdot \mathbf{K}]$), whence (2.19) yields (1³)

$$(2.22) \quad \int_{C_3} \bar{\psi}\gamma(\mathbf{K})\psi q \text{vol}_3 = 4[\bar{\mathbf{K}} \cdot \mathbf{I}]^2 \bar{\xi}\gamma(\mathbf{K})\xi q \int_{C_3} \text{vol}_3$$

(1²) \mathbf{I} is timelike, and, in addition, future-pointing for convenience.

(1³) $\exp\left(\frac{q\bar{\mathbf{K}} \cdot X}{4\bar{\mathbf{K}} \cdot \mathbf{I}} \gamma_\mu \gamma_\nu F^{\mu\nu}\right) = \mathbb{1}_\epsilon + \frac{q\bar{\mathbf{K}} \cdot X}{2\bar{\mathbf{K}} \cdot \mathbf{I}} \gamma(\mathbf{K})\gamma(\mathbf{B})$

and provides us with an infinite charge! As a matter of fact, the solution (2.19) of the D. P. equation unfortunately does not lie in the Hilbert space of square integrable functions as ⁽¹⁴⁾

$$(2.23) \quad \bar{\psi}\psi = 4m[\bar{K} \cdot I]\bar{\xi}\gamma(K)\xi = \text{const.}$$

Let us — as a new procedure — replace the integrations (2.16), (2.17), (2.18), (2.22) by mean values over a compact subset Δ_3 of C_3 . Since the integrand is a constant (e. g.: $4[\bar{K} \cdot I]^2\bar{\xi}\gamma(K)\xi q$ in (2.22)), the final result turns out to be independent of the chosen Δ_3 . In order to recover the electric charge, we deal with the following normalization:

$$(2.24) \quad 4[\bar{K} \cdot I]^2\bar{\xi}\gamma(K)\xi \int_{\Delta_3} \text{vol}_3 = 1$$

Furthermore ⁽¹⁵⁾ $F \cdot X = K[\bar{B}X]$ is null whilst

$$\text{Tr} (\Omega \cdot F) = \text{Re} (\bar{\psi}\gamma(K)\gamma_\mu\gamma_\nu\psi i)F^{\mu\nu}$$

vanishes ($\bar{K}K = 0$), whence (2.16), (2.17) yield:

$$(2.25) \quad \begin{aligned} \pi &= I + 4k [\bar{I} \cdot K]K \int_{\Delta_3} \text{Re} (\bar{\psi}\gamma(K)\gamma(B)\xi i)\text{vol}_3 \\ &\quad - K \int_{\Delta_3} [\bar{B} \cdot X]\bar{\psi}\gamma(K)\psi q \text{vol}_3 \\ &= I + 4k [\bar{I} \cdot K]^2 \text{Re} (\bar{\xi}\gamma_\mu\gamma_\nu\xi i)F^{\mu\nu}K \int_{\Delta_3} \text{vol}_3 - Kaq \end{aligned}$$

(where $a = \int_{\Delta_3} [\bar{B} \cdot X]\bar{\psi}\gamma(K)\psi \text{vol}_3$ is a term one needs not to worry about, since our final result will be independent of it), and:

$$(2.26) \quad \begin{cases} \gamma = a\bar{I} \cdot K \end{cases}$$

$$(2.27) \quad \begin{cases} \pi = I + k \text{Re} (\bar{\psi}\gamma_\mu\gamma_\nu\psi i) |_{\bar{K} \cdot X = 0} F^{\mu\nu}K \int_{\Delta_3} \text{vol}_3 - Kaq \end{cases}$$

By a suitable rescaling of ξ , ψ may be normalised to unity, i. e.:

$$(2.28) \quad \bar{\psi}\psi = 4m[\bar{I} \cdot K]\bar{\xi}\gamma(K)\xi = 1$$

So (2.27) reads now:

$$(2.29) \quad \pi = I + 2k \left[\frac{1}{2} \text{Re} (\bar{\psi}\gamma_\mu\gamma_\nu\psi i) \right]_{\bar{K} \cdot X = 0} F^{\mu\nu} \frac{m}{\bar{I} \cdot K} \cdot K - Kaq$$

The expression of γ^* (2.18), slightly more intricate, will not be presented here, for all the physics we are interested in lies in (2.26) and (2.29).

⁽¹⁴⁾ $\gamma(K) \cdot \exp(2k \bar{K} \cdot X\gamma(B)) = \exp(-2k \bar{K} \cdot X\gamma(B)) \cdot \gamma(K)$

⁽¹⁵⁾ Remember that now $\bar{K}X = 0$.

Notice then that the function f (introduced in (1.22)) which labels the elementary particle model in an external E. M. F. can be reached within the following element $[\sigma]$ of the enveloping algebra (Casimir) of the momenta previously computed in the case of a constant E. M. F. (1.27), (1.28):

$$(2.30) \quad \bar{\pi} \cdot \pi + 2q\gamma = f(\alpha) - q\alpha$$

say.

The results (2.26), (2.29) yield the quantum analogue:

$$(2.31) \quad \bar{\pi} \cdot \pi + 2q\gamma = \bar{I} \cdot I + 4km \left[\frac{1}{2} \operatorname{Re} (\bar{\psi} \gamma_\mu \gamma_\nu \psi i) \right] \Big|_{\mathbb{K}, \mathbf{x}=0} F^{\mu\nu}$$

For a D. P. particle, the function f giving rise to a classical (non quantum) particle model turns out to be ⁽¹⁶⁾ affine:

$$(2.32) \quad f(\alpha) = m^2 + [q - 4km]\alpha \quad (m = \text{const.} = \text{the bare mass})$$

This result is in full accordance with the conclusions of [5] in the case $k = 0$ (Dirac equation) and with the formula: (193) in [I].

IV) Conclusion

One can now give a dynamical interpretation of (2.32). The velocity and the linear momentum \mathbf{P} of a D. P. particle are *almost parallel* (up to a factor of $0(k)$) as long as the E. M. F. is constant. We mean to say that if we define: t — the proper time in the weak field limit — by:

$$(2.33) \quad \mathbf{P} \cdot \frac{d\mathbf{X}}{dt} = \sqrt{f(\alpha)}$$

then (1.22):

$$(2.34) \quad \frac{d\mathbf{X}}{dt} = \frac{1}{\sqrt{f(\alpha)}} \left[\mathbf{P} + \left(\frac{q - f'(\alpha)}{f(\alpha) + q \frac{\alpha}{2}} \right) \mathbf{S} \cdot \mathbf{F} \cdot \mathbf{P} \right]$$

with f given by (2.32).

The velocity and the linear momentum remain *always parallel* in the case of the Dirac electron in a constant E. M. F., and that characterizes in our sense the normal dipole coupling.

⁽¹⁶⁾ The term: $-\frac{1}{2} \operatorname{Re} (\bar{\psi} \gamma_\mu \gamma_\nu \psi i) F^{\mu\nu}$ can be interpreted as the (quantum) term of spin-E. M. F. interaction α , as $S_{\mu\nu} = -\frac{1}{2} \operatorname{Re} (\bar{\psi} \gamma_\mu \gamma_\nu \psi i)$ satisfying the algebraic relations:

$$S_{\mu\nu} S^{\mu\nu} = \frac{1}{2}; \quad S_{\mu\nu} + S_{\nu\mu} = 0;$$

rank (S) = 2 deduced from the fact that $\bar{\psi} \psi = 1; \bar{\psi} \gamma_5 \psi = 0$ ($\gamma_5 + \gamma_1 \gamma_2 \gamma_3 \gamma_4 = 0$) can be interpreted as the quantum spin tensor [I] [3] [7].

What about the magnetic moment? We get from (1.22), (2.32) and (2.33):

$$(2.35) \quad \mu(\alpha) = \frac{[q - 4km]s}{\sqrt{m^2 + [q - 4km]\alpha}} \quad \left(s = \frac{1}{2} \right)$$

As for the Landé factor (1.23):

$$(2.36) \quad g = \frac{2m}{qs} \mu(0) = 2 \left[1 - \frac{4km}{q} \right]$$

or:

$$(2.37) \quad k = -\frac{q}{4m} \left[\frac{g}{2} - 1 \right]$$

The function (2.32) can then be written ⁽¹⁷⁾:

$$(2.38) \quad f(\alpha) = m^2 + \frac{gq}{2} \alpha$$

and provides us with a *D. P. particle classical model*.

For a Dirac electron, the magnetic moment $\frac{qs}{\sqrt{m^2 + q\alpha}}$ differs from the careless $\frac{qs}{m}$ given in the literature (which can be recovered in the weak field limit).

We wish at last to point out the coherence between the universal model investigated in Section I and the specific features of the D. P. and Dirac equations scribed in Section II.

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