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Section A :
Physique théorique.

Einstein's causality as a consequence of equal-time commutation relations in some soluble models

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ABSTRACT. — The evolution of the commutation (anticommutation) relations given at a fixed time is shown to give rise to space like (anti) commutation properties for approximate fields governed by a differential equation.

RÉSUMÉ. — Dans des modèles de champs approchés caractérisés par des équations différentielles, l'évolution des relations d'(anti) commutation assignées à un temps fixé donne lieu à des propriétés d'(anti) commutation de genre espace pour les champs.

Equal-time commutation (anticommutation) relations cannot be formulated in the most general setting of quantum field theory; yet once they can, one expects, at least in a relativistic theory, the *causal* commutation (anticommutation) rules to be valid. In the present paper we shall prove this guess on some models, where the fields are replaced by some approaching quantities. To clarify the scope of our approximation, we observe that quantum field theory differs from classical theories in two main aspects, first, the fact that the relevant quantities are (non commuting) operators, secondly, that the dependence of such operators on the space-time variables is not function-like, but rather distribution-like. The operator character of the field is related to the quantum aspect of the theory, whereas the distribution character, related roughly speaking to the field aspect, is forced by the quantum character itself if ones requires the theory under consideration to satisfy Wightman's axioms [1]. However, if we relax some of the axioms, we may consider " fields " which are non commuting operators, and as such *bona fide* quantum objects, yet whose space-time dependence is function-like (typically, one may think of such objects as obtained from a Wightman field by regularization, i. e., by convolution with a smooth function). For such quantities one may try to formulate and solve the " field equations " as suggested by the standard treatment of field theory.

Furthermore, if the " true " (Wightman) fields under consideration are known to allow for sharp time restrictions (as operator-valued distributions), space-variables regularization is enough, and formulating a Cauchy problem for such regularized quantities may be given a sound interpretation. In particular, one may assume the Cauchy data at a given time, say, $t = 0$, to satisfy some relation expressing the vanishing of the (anti) commutators taken at different (space) points : the corresponding solutions will be expected to satisfy the (anti) commutativity at spacelike distances. This will in fact be shown to hold true in a version of Thirring's model, and in a model in a two-dimensional space time with a nonpolynomial interaction of the type $u^{2n+1}(u^{2n}+1)^{-1}$, or, more generally, of the type $u(u^{2n}+a)(u^{2n}+b)^{-1}$.

A. — MASSIVE THIRRING'S MODEL

The result is essentially contained in [2], where a massless model was considered. The proof being exactly the same, we shall only describe the setting of the model and state the results in a thorough way.

The equation for this model is

$$(1)\qquad \begin{cases} \dfrac{d}{dt}u(t) + \mathrm{L}\,u(t) + \mathrm{T}\,u(t) = 0, \\ u(0) = u_0, \end{cases}$$

where $t \to u(t)$ is a function from $\mathbf{R}_+$ to the space X :

$$\mathrm{X} = \mathrm{C}_0(\mathbf{R};\mathcal{L}(\mathcal{H}))\,\mathrm{C}_0(\mathbf{R};\mathcal{L}(\mathcal{H}))$$

of couples of continuous functions on $\mathbf{R}$, vanishing at infinity, taking their values in the space $\mathcal{L}(\mathcal{H})$ of linear continuous operators on the Hilbert space of physical states $\mathcal{H}$; the space X is a Banach space with the norm :

$$u \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to |u| = \sup_{x\in\mathbf{R}} |u_1(x)|_{\mathcal{L}(\mathcal{H})} + \sup_{x\in\mathbf{R}} |u_2(x)|_{\mathcal{L}(\mathcal{H})}.$$

In addition, L is the linear closed operator on X,

$$(2)\qquad \mathrm{L} = \begin{pmatrix} \dfrac{d}{dx} & 0 \\ 0 & -\dfrac{d}{dx} \end{pmatrix}$$

which generates a semigroup of linear contractions on X and T is the non linear operator defined by

$$T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = ig\begin{pmatrix} u_2^* \, u_2 \, u_1 \\ u_1^* \, u_1 \, u_2 \end{pmatrix} + im\begin{pmatrix} u_2 \\ u_1 \end{pmatrix},$$

where g, m are real numbers.

In view of the results of [3], [4], a unique solution of this equation is known to exist.

As to the anticummotation relations for the above introduced fields, we note that the fields under consideration being continuous functions, the standard (singular) equal time anticommutation relations cannot be imposed on the Cauchy data. One should rather require the more regular condition :

$$(3) \qquad [u_0 (x), u_a (y)]_+ = \delta_d (x - y),$$

where $\delta_d (s)$ is a selfadjoint operator valued symmetric function vanishing for $|s| > d$. Then one may show that (3) has a consequence

$$[(u(t))(x), (u(0))(x')]_+ = 0$$

whenever $|x - x'| - |t| > d$, that is, a property analogous to the local anticommutativity up to d [1].

B. — BOSE FIELD MODEL

We consider the model (in the two dimensional space-time) [2] :

$$(4) \qquad \square u - m^2 u + g \frac{u^{2n+1}}{1 + u^{2n}} = 0$$

which can be written in the form

$$(5) \quad \begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 0 & \mathrm{I} \\ \partial_{xx} & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -m^2 u_1 + g \dfrac{u_1^{2n+1}}{1 + u_1^{2n}} \end{pmatrix} = 0,$$

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(1) The reference [2] contains a slight mistake : in fact causality is shown to hold in the region $|x - x'| - |t| > d$ but not in the region $(x - x')^2 - t^2 > d^2$.

Notice further that, as explained in the remark at the end of the present paper, in view of the minimal structure put in the models we are dealing with, we are not able to prove a similar statement for the anticommutators taken at different times. Hence testing our results on the solutions of Thirring's model found by Dell'Antonio *et al.* [5] is impossible, as such solutions need *time* smearing in order the fields to be defined as operators.

(2) The same results are valid for a slightly more general type of interaction, namely

$$f(u) = gu\,(u^{2n} + a)\,(u^{2n} + b)^{-1} \qquad (a, b > 0)$$

which, if $a > b$, is not bounded (*cf.* [8]).

where u_1 is the u in (4). To the equation one should add the Cauchy data

$$(5') \qquad u_1(0) = u_{10}; \qquad u_2(0) = u_{20}.$$

This model can be treated on a similar footing, taking

$$X = H^1(\mathbf{R}; \mathcal{L}_{s.a.}(\mathcal{H})) \oplus L^2(\mathbf{R}; \mathcal{L}_{s.a.}(\mathcal{H})),$$

where $\mathcal{L}_{s.a.}(\mathcal{H})$ denotes the space of linear continuous selfadjoint operators on the Hilbert space $\mathcal{H}$.

The space X is turned into a Banach space by putting on it the norm

$$u \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to |u| = \left(\int |\partial_x u_1(x)|^2_{\mathcal{L}(\mathcal{H})} dx + \int |u(x)|^2_{\mathcal{L}(\mathcal{H})} dx \right)^{1/2}.$$

The problem (5), (5′) may therefore be written in the form (1) by setting

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad L = -\begin{pmatrix} 0 & I \\ \dfrac{d^2}{dx^2} & 0 \end{pmatrix}, \qquad u_0 = \begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix}$$

and T the non linear operator defined by

$$T u = T \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -m^2 u_1 + g \dfrac{u_1^{2n+1}}{1 + u_1^{2n}} \end{pmatrix}.$$

As an elementary calculation shows, L generates a semigroup $S_0(t)$ of class C_0 on X, namely

$$(S_0(t)\, u)(x) = \frac{1}{2} \begin{pmatrix} u_{10}(x+t) + u_{10}(x-t) + \displaystyle\int_{x-t}^{x+t} u_{20}(\xi)\, d\xi \\ u'_{10}(x+t) - u'_{10}(x-t) + u_{20}(xt+) + u_{20}(x-t) \end{pmatrix}.$$

Moreover, as shown in Appendix A, T is Lipschitz continuous on the whole of X, with a Lipschitz norm K depending only on the degree n of the interaction.

Therefore [6], a result analogous to that of [3] for Thirring's model can be established, namely the existence and uniqueness of the solution of (5), (5′).

Now, in order to quantize u, we require

$$(6) \qquad [(u_1(0))(x), (u_2(0))(x')] = i\, \delta_d(x - x'),$$

$$(7) \qquad [(u_1(0))(x), (u_1(0))(x')] = 0,$$

$$(8) \qquad [(u_2(0))(x), (u_2(0))(x')] = 0,$$

where $s \to \delta_d(s)$ is a symmetric $\mathcal{L}_{s.a.}(\mathcal{H})$-valued function vanishing for $|s| > d$.

Our claim is

$$[(u_1(t))(x), (u_1(0))(x')] = 0 \qquad \text{for} \quad |x - x'| - t > d \qquad (t \geq 0),$$

that is, Einstein's causality up to d.

An essential ingredient of the proof is the validity of the required property for the free equation (namely that with $T = 0$). This will be shown to be true in the Appendix B.

Another ingredient of the proof is the locality of T, namely the pointwise action of T on $(u(t))(.)$. Thew we proceed as follows :

(i) define, for any integer k, the " approaching solutions " u^k :

$$\text{(8)} \qquad \begin{cases} u^k(t) = S_0(t)\, u_0 + \int_0^t S_0(t-s)\, T\, u^{k-1}(s)\, ds, \\ u^0(t) = S_0(t)\, u_0. \end{cases}$$

Then for any t we have

$$\text{(9)} \qquad \lim_{k \to \infty} |\, u^k(t) - u(t)\,| = 0,$$

which holds true because, as a direct calculation shows,

$$|\, u^{k+1}(t) - u^k(t)\,| \leq \int_0^t |\, T\, u^k(s) - T\, u^{k-1}(s)\,|\, ds \leq \frac{(Kt)^k}{k\,!};$$

(ii) show that the required property is valid for any k, that is

$$\text{(10)} \qquad [(u_1^k(t))(x), (u_1^k(0))(x')] = 0 \qquad \text{if} \quad |x - x'| - |t| > d.$$

Putting $\hat{T}\, u_1 = -m^2 u_1 + g u_1^{2n+1} (1 + u_1^{2n})^{-1}$, we get from (3) :

$$\begin{aligned} \text{(11)} \quad (u_1(t))(x) &= (u_0(t))(x) \\ &+ \sum_{j=1}^{k} (2^{-j-1}) \int_0^{t_0} dt_1 \int_{x_0-(t_0-t_1)}^{x_0+(t_0-t_1)} dx_1\, \hat{T} \\ &\times \int_0^{t_1} dt_2 \int_{x_1-(t_1-t_2)}^{x_1+(t_1-t_2)} dx_2\, \hat{T} \ldots \\ &\times \int_0^{t_{j-1}} dt_j \int_{x_{j-1}-(t_{j-1}-t_j)}^{x_{j-1}+(t_{j-1}-t_j)} dx_j\, \hat{T} \\ &\times \left[u_{10}(x_j + t_j) + u_1(x_j - t_j) + \int_{x_j - t_j}^{x_j + t_j} dx_{j+1}\, u_{20}(x_{j+1}) \right], \end{aligned}$$

where we have put $x_0 = x$, $t_0 = t$.

In order to prove (10) we can, in putting (11) into the commutator, forget about $\hat{T}$: its pointwise action does not affect the commutation properties of u_1.

Then, keeping in mind the expression (11) for u_1^k, we see that the commutator in (10) vanishes if

$$x' \notin [x_j - t_j - d, x_j + t_j + d]$$

for all j from 1 to k.

Recalling the domain of integration of the variables x_j, t_j, as read off from (11) :

$$x_j \in [x_{j-1} - (t_{j-1} - t_j), x_{j-1} + (t_{j-1} - t_j)],$$
$$t_j \in [0, t_{j-1}],$$

we see that (10) vanishes if

$$x' \notin [x_{j-1} - t_{j-1} - d, x_{j-1} + t_{j-1} + d]$$

and so on, until we end up with

$$x' \notin [x - t - d, x + t + d]$$

that is,

$$|x - x'| - t > d.$$

Finally, combining (9) with (10) completes the proof.

Remark. — We note that, in order to generalize (10) to

$$(12) \quad [(u_1(t))(x), (u_1(t'))(x')] = 0 \qquad \text{if} \quad |x - x'| - |t - t'| > d,$$

one might be tempted to prove the time-translational invariance of the commutators like that in (11); however, this seems hard to be done, even if we could assume δ_d to be a complex-valued function, essentially because no hypothesis was made on the representation of the time translations by operators leaving a vector in $\mathcal{H}$ invariant (vacuum), as in the usual setting of Wightman's theory. Such invariance property holds nevertheless true for the free equation, in view of the particularly simple structure of the free solution.

As a further remark, it would be nice to compare the method used here to derive causal propagation with that devised by J. Glimm and A. Jaffe in the models they construct. However, our method is essentially based on differential equations, and no assumption on the Hamiltonian is made : the latter, on the contrary, plays a crucial role in Glimme and Jaffe's approach. On the relationship between the field equation approach and the Hamiltonian one, compare, e. g. [7].

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APPENDIX A

We prove the Lipschitz continuity of the operator T :

$$\mathrm{T}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -m^2 u_1 + g u_1^{2n+1} (1 + u_1^{2n})^{-1} \end{pmatrix}$$

on $\mathrm{H}^1 (\mathbf{R}; \mathcal{L}_{\text{s.a.}} (\mathcal{H})) \oplus \mathrm{L}^2 (\mathbf{R}; \mathcal{L}_{\text{s.a.}} (\mathcal{H}))$.

Let us first consider the function $u \to f(u) = \dfrac{u^{2n+1}}{1 + u^{2n}}$ defined on $\mathcal{L}_{\text{s.a.}} (\mathcal{H})$, and remark that, once f is proven to be Lipschitz continuous, it follows immediately that the function $\hat{f}$ is Lipschitz continuous from $\mathrm{L}^2 (\mathbf{R}; \mathcal{L}_{\text{s.a.}} (\mathcal{H}))$ to $\mathrm{L}^2 (\mathbf{R}; \mathcal{L}_{\text{s.a.}} (\mathcal{H}))$, where $\hat{f}$ is defined by

$$(\hat{f} u)(x) = f(u(x))$$

(almost everywhere in x). Therefore $\hat{f}$ is *a fortiori* Lipschitz continuous from $\mathrm{H}^1 (\mathbf{R}; \mathcal{L}_{\text{s.a.}} (\mathcal{H}))$ to $\mathrm{L}^2 (\mathbf{R}; \mathcal{L}_{\text{s.a.}} (\mathcal{H}))$, which entails the Lipschitz continuity of T.

Hence we may concentrate ourselves on the function f. To prove its Lipschitz continuity, we shall essentially use the fact that the resolvent set of u is the whole complex plane excepted the real line, and the resolvent set of u^2 is the whole complex plane excepted the positive half-axis, and whe shall repeatidly apply the identity of the resolvent. Let us first consider the case $n = 1$: for $u, v \in \mathcal{L}_{\text{s.a.}} (\mathcal{H})$,

$$\begin{aligned}
&\left| \frac{u^3}{1 + u^2} - \frac{v^3}{1 + v^2} \right| \\
&\quad = | u u^2 \mathrm{R}(-1, u^2) - v v^2 \mathrm{R}(-1, v^2) | \\
&\quad \leqq | u - v | + | u \mathrm{R}(i, u) \mathrm{R}(-i, u) - v \mathrm{R}(i, v) \mathrm{R}(-i, v) | \\
&\quad \leqq | u - v | + | u \mathrm{R}(i, u) \mathrm{R}(-i, u) - v \mathrm{R}(i, v) \mathrm{R}(-i, u) | \\
&\qquad + | v \mathrm{R}(i, v) \mathrm{R}(-i, u) - v \mathrm{R}(i, v) \mathrm{R}(-i, v) | \\
&\quad \leqq 2 | u - v | + | (\mathrm{R}(i, u) - \mathrm{R}(i, v)) \mathrm{R}(-i, u) | \\
&\qquad + | v \mathrm{R}(i, v) (\mathrm{R}(-i, u) - \mathrm{R}(-i, v)) | \\
&\quad \leqq 2 | u - v | + | (\mathrm{R}(i, u)(u - v) \mathrm{R}(i, v)) \mathrm{R}(-i, u) | \\
&\qquad + | \mathrm{R}(-i, u) - \mathrm{R}(-i, v) | \leqq 4 | u - v |.
\end{aligned}$$

The case n arbitrary can be treated in a similar way, by writing

$$(1 + u^{2n})^{-1} = \prod_{i=1}^{2n} (r_i + u)^{-1},$$

where r_i are the $2\,n^{\text{th}}$ roots of -1 (which are of course non-real). We omit the details of the calculations, which are lengthy but straightforward.

In the same way we may prove the Lipschitz continuity of

$$u \to f(u) = u \, \frac{u^{2n} + a}{u^{2n} + b} \qquad (a > b > 0),$$

that is, of an interaction which does not satisfy $|f(u)| \leq k\,|u|$ [8].

APPENDIX B

We prove the validity of the causal propagation property for the semigroup S_0 expressing the free evolution of our system. Recalling the explicit form of S_0 as given in the text, we have

$$\text{(B 1)} \qquad [(S_0(t)\,u)_1(x), (S_0(t')\,u)_1(x')]$$
$$= \begin{cases} \text{(i)} & + \dfrac{1}{4}\left[u_{10}(x-t), \displaystyle\int_{x'-t'}^{x'+t'} u_{20}(s)\,ds\right] \\ \text{(ii)} & + \dfrac{1}{4}\left[u_{10}(x+t), \displaystyle\int_{x'-t'}^{x'+t'} u_{20}(s)\,ds\right] \\ \text{(iii)} & + \dfrac{1}{4}\left[\displaystyle\int_{x-t}^{x+t} u_{20}(s)\,ds, u_{10}(x'-t')\right] \\ \text{(iv)} & + \dfrac{1}{4}\left[\displaystyle\int_{x-t}^{x+t} u_{20}(s)\,ds, u_{10}(x'+t')\right]. \end{cases}$$

This expression vanishes if

$$\text{(B 2)} \qquad \text{if} \quad \text{(i)} = \text{(iv)} = 0 \qquad \text{and if} \quad \text{(ii)} + \text{(iii)} = 0$$

or

$$\text{(B 3)} \qquad \text{if} \quad \text{(i)} + \text{(iv)} = 0 \qquad \text{and if} \quad \text{(ii)} = \text{(iii)} = 0.$$

We shall prove that these conditions are impled by

$$|x - x'| - |t - t'| > d.$$

Let us in fact consider the first alternative, (B 2) : here the first condition, " (i) = (iv) = 0 " is satisfied if and only if

$$x - t \notin [x' - t' - d, x' + t' + d]$$

and

$$x' + t' \notin [x - t - d, x + t + d],$$

that is, if and only if

(B 4) either $x - t < x' - t' - d$ or $x - t > x' + t' + d$

and

(B 5) either $x' + t' < x - t - d$ or $x' + t' > x + t + d$.

The first alternative of (B 4) and the second of (B 5) are both satisfied if

(B 6) $$x' - x - (t' - t) > d \quad \text{if} \quad t - t' \geqq 0$$

or if

(B 7) $$x' - x + (t - t') > d \quad \text{if} \quad t - t' \leqq 0.$$

The second condition of (B 9), namely (ii) + (iii) = 0 is satisfied if

$$\int_{x'-t'}^{x'+t'} \delta_d (x + t - s)\, ds - \int_{x-t}^{x+t} \delta_d (x' - t' - s)\, ds = 0.$$

The last expression, by a suitable change in the integration variables, reduces to

$$\int_{x'-x-(t'-t)}^{x'-x+(t'-t)} \delta_d (s)\, ds = 0,$$

which vanishes if

(B 8) $$\begin{cases} \text{either} \quad x' - x - (t' - t) > d \quad \text{or} \quad x' - x + (t' - t) < - d, \\ \text{when} \quad t' - t \geqq 0 \end{cases}$$

and if

(B 9) $$\begin{cases} \text{either} \quad x' - x + (t' - t) > d \quad \text{or} \quad x' - x - (t' - t) < - d, \\ \text{when} \quad t' - t \leqq 0. \end{cases}$$

Hence (B 8), respectively (B 9) do hold once (B 6), (B 7) do.
Thus (B 2) is satisfied if

$$x' - x - | t - t' | > d \quad \text{for} \quad x' - x > 0.$$

As to (B 3), a similar argument applies : in fact, suffice it to interchange x with x' and t with t'. Thus (B 3) is satisfied if

$$x - x' - | t - t' | > d \quad \text{for} \quad x - x' > 0.$$

Putting together the two alternatives (B 2) and (B 3) we end up with the desired condition

$$| x - x' | - | t - t' | > d$$

which proves in particular our claim as to the free equation.

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