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Weber's Antenna and the Radiation Sources

par

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ABSTRACT. — This paper contains a discussion of some questions concerning the relation of what is detected by the antenna of Weber to the sources of the gravitational radiation.

I

The antenna which is presently being used to search for gravitational radiation [*1*] is an elastic cylinder which is fixed to the earth with its axis in an east-west direction and so executes a complete rotation in a 24 (sidereal) hour period. This antenna is now detecting certain events which are probably pulses of gravitational radiation. This paper contains a discussion of some questions concerning the relation of what is detected by the antenna to the sources of the gravitational radiation.

We assume that the source of the radiation lies in a small region around the centre of the galaxy and therefore it has a well defined propagation vector $\vec{\xi}$. We choose a coordinate system so that $\vec{\xi}$ coincides with the unit vector Ox^1, while the axis of rotation of the earth lies in the plane Ox^1x^2 and forms the angle α with Ox^1 (fig. 1).

In the calculations which follow we shall idealize the cylinder as a pair of material points held together by an elastic force. In figure 1 we show the vector $\vec{\eta}$ joining these two points. Consider the world lines of these points. Let p^μ (greek indices take the values 0, 1, 2, 3) be the unit tangent to one of them and s the proper time. Let η^μ be the vector normal to p^μ joining the two world lines. We have the following identities in the rest frame of the cylinder, κ being the constant of gravitation:

$$(1) \qquad \begin{aligned} p^\mu &= (1, \vec{0}) + 0(\kappa), \\ \eta^\mu &= (0, \vec{\eta}) + 0(\kappa). \end{aligned}$$

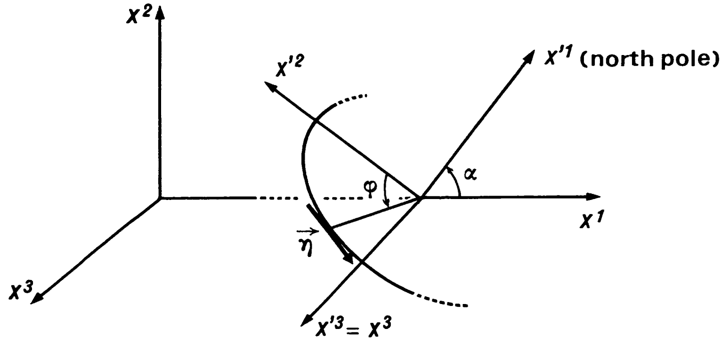

FIG. 1.

The two points are held together by an elastic force and their world-lines are not geodesics. Therefore the equation for geodesic deviation [2] must be modified [3] to read

$$(2) \qquad \frac{\delta^2 \eta^\mu}{\partial s^2} + R^\mu{}_{\nu\lambda\sigma} p^\nu \eta^\lambda p^\sigma = f^\mu,$$

where f^μ is the term which arises from the elastic forces binding the two points.

The elastic force is proportional to the deviation of η^μ from its mean value and consequently equation (2) is the equation of a forced elastic vibration with the term $R^\mu{}_{\nu\rho\sigma} p^\nu \eta^\rho p^\sigma$ playing the role of the external force. What is actually measured is the deviation of the proper length η from its mean value (i. e. the deformation of the cylinder). It is easily seen then that the essential part of equation (2) is obtained if we multiply it by η_μ/η. Therefore the external force is characterised by the scalar

$$(3) \qquad \Phi = \frac{1}{\eta} R_{\mu\nu\rho\sigma} \eta^\mu p^\nu \eta^\rho p^\sigma.$$

In Weber's approach a resonance effect is used and consequently this deformation will be related to the Fourier component of the external force corresponding to the resonant frequency ω. Therefore the apparatus will be measuring indirectly the Fourier component of the quantity (3). To avoid introducing more complicated notation we shall assume in future that the quantity (3) represents just this Fourier component.

To find the detailed expression for Φ we have to take into account that the apparatus rotates in the plane $Ox'^2x'^3$ (fig. 1) with the angular velocity $\omega_0 \ll \omega$; $\phi = \omega_0 t$. Introduce the unit vector l^μ parallel to η^μ. In the rest frame of the cylinder

$$l^\mu = (0, \vec{l}),$$

the vector $\vec{l}$ having in the frame (x^i) the components

$$(\sin\alpha \sin\phi, \ -\cos\alpha \sin\phi, \cos\phi).$$

Therefore:

$$\frac{1}{\eta}\Phi = \cos^2\alpha \sin^2\phi R_{0202} + \cos^2\phi R_{0303} - \sin^2\phi \cos\alpha R_{0203} + \sin^2\phi\,(\sin^2\alpha R_{0101} - \sin 2\alpha R_{0102}) + \sin\alpha \sin 2\phi R_{0103}.$$

It will be found in the next section that

$$R_{0101} = R_{0102} = R_{0103} = 0, \quad R_{0202} = -R_{0303},$$

and so we shall have finally:

$$\frac{1}{\eta}\Phi = (\cos^2\alpha \sin^2\phi - \cos^2\phi)R_{0202} - \cos\alpha \sin 2\phi R_{0203}. \tag{4}$$

This formula shows that Φ is periodic in ϕ with the period π.

We now shall try to express the quantities $R_{\mu\nu\rho\sigma}$ by the quantities which characterise the source of the radiation.

Far from its sources the field has the general form

$$g^{\mu\nu} = \eta^{\mu\nu} + \underset{1}{g}^{\mu\nu} + \dots$$

For our purpose it is sufficient to consider the term $\underset{1}{g}^{\mu\nu}$ which is the retarded solution of the linearized field equations in harmonic coordinates. We shall have

$$\underset{1}{g}^{\mu\nu} = \frac{1}{R}\Phi^{\mu\nu} + 0\left(\frac{1}{R^2}\right).$$

From the harmonic coordinate condition $g^{\mu\nu}{}_{,\nu} = 0$ we derive the relation

$$\Phi^{\mu\nu}\xi_\nu = 0,$$

where

$$(\dot{\ }) \equiv \frac{\partial}{\partial t}(\ ) \quad \text{and} \quad \xi_\nu = \frac{\partial(t - R)}{\partial x^\nu} = (1, -\vec{\xi}), \quad \xi^\nu = (1, \vec{\xi}).$$

Therefore

$$\Phi^{\mu\nu}\xi_\nu = a^\mu, \qquad \dot{a}^\mu = 0.$$

If we introduce

$$F_0{}^{\mu\nu} = \begin{pmatrix} a^0 + a^s\xi^s & a^i \\ a^i & 0 \end{pmatrix}, \quad F^{\mu\nu} = \Phi^{\mu\nu} - F_0{}^{\mu\nu},$$

we find

$$\text{(5)} \qquad \underset{1}{g}^{\mu\nu} = \frac{1}{R}(F^{\mu\nu} + F_0{}^{\mu\nu}) + 0\left(\frac{1}{R^2}\right);$$

$$\text{(5 a)} \qquad F^{\mu\nu}\xi_\nu = 0, \qquad \dot{F}_0{}^{\mu\nu} = 0.$$

Gravitational radiation is described by the term $F^{\mu\nu}$ alone. Because of the orthogonality relation (5 *a*) only the 6 components F^{ik} are independent.

For a calculation of an order of magnitude one has to remember that the most important contribution will come from the 4-pole term. In the case of a 4-pole radiation we have

$$F^{ik} = \frac{\kappa}{8\pi}\dddot{d}^{ik}, \qquad d^{ik} = \int \rho x^i x^k d^3x.$$

The quantity $\dot{F}^{\mu\nu}$ is identical with the $\Omega^{\mu\nu}$ of [*4*]. For $\Omega^{\mu\nu}$ we found in [*4*] the important reduced form

$$\omega^{ik} = \left(\zeta^i_m\zeta^k_n - \frac{1}{2}\zeta^{ik}\zeta_{mn}\right)\Omega^{mn};$$

$$\zeta^i_m = \delta^i_m + \xi^i\xi_m, \qquad \zeta^{im} = \zeta_{im} = \delta^{im} - \xi^i\xi^m,$$

which satisfies the relations

$$\omega^{ik}\xi^k = 0, \qquad \omega^{ii} = 0$$

(with $\omega^{i0} = \omega^{00} = 0$). Since $F^{\mu\nu}$ obeys also, as $\Omega^{\mu\nu}$, the orthogonality relation (5 *a*) we can reduce it in a similar way and introduce

$$\text{(6)} \qquad f^{ik} = \left(\zeta^i_m\zeta^k_n - \frac{1}{2}\zeta^i_k\zeta^m_n\right)F^{mn}$$

We have again

$$f^{ik}\xi^k = 0, \qquad f^{ii} = 0$$

(with $f^{i0} = f^{00} = 0$). It is easy to show that there is a coordinate transformation which transforms $F^{\mu\nu}$ into $f^{\mu\nu}$ ([1]).

In the case we are considering here we have

(7) $$\xi^i = (1, 0, 0).$$

The operator ζ^i_m is then

$$\zeta^i_m = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and equation (6) leads to the result

(8) $$2f^{ik} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & F^{22} - F^{33} & 2F^{23} \\ 0 & 2F^{23} & F^{33} - F^{22} \end{pmatrix}.$$

If we put $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ we find immediately for the $h_{\mu\nu}$ corresponding to (8):

(8 *a*) $$h_{22} = -h_{33} = -\frac{f^{22}}{R}, \quad h_{23} = -\frac{f^{23}}{R}, \quad \text{all other} \quad h_{\mu\nu} = 0.$$

Thus the $h_{\mu\nu}$ have the structure of a plane wave, as it should be. The more detailed formulae (8 *a*) have the advantage to relate the functions h_{22} and h_{23} to the quantities which characterize the source of the radiation.

The Riemann tensor is given in the first approximation by the formula

$$R_{\mu\nu\rho\sigma} = h_{\mu[\sigma,\rho]\nu} - h_{\nu[\sigma,\rho]\mu}.$$

Since we are interested only in the terms of the order 1/R we may write

$$h_{\mu\sigma,\rho\nu} = \ddot{h}_{\mu\sigma}\xi_\rho\xi_\nu,$$

and consequently

(9) $$R_{\mu\nu\rho\sigma} = 2\xi_{[\nu}\ddot{h}_{\mu][\rho}\xi_{\sigma]}.$$

Note that because of $h_{22} = -h_{33}$ we can write

$$\ddot{h}_{\mu\alpha} = 2A(a_\mu a_\alpha - b_\mu b_\alpha),$$

a_μ and b_μ being two space-like unit vectors orthogonal to each other

([1]) For example if $F^{\mu\nu}$ represents a 4-pole radiation only this transformation is:

$$x'^\mu = x^\mu + \varepsilon f^\mu; \quad \varepsilon f^i = \frac{1}{8R}(d^{ss} + d^{ls}\xi^l\xi^s)\xi^i - \frac{1}{2R}d^{is}\xi^s,$$

$$\varepsilon f^0 = -\frac{1}{8R}(d^{ss} + d^{sl}\xi^s\xi^l).$$

and to ξ_μ. Thus the tensor (9) has the general from of the Riemann tensor of type N, as it should be.

With $\xi_\nu = (1, -\xi^i)$, the ξ^i being given by (7), we get from (9):

$$(9\,a) \qquad R_{010i} = 0, \quad R_{0202} = -R_{0303} = \frac{1}{2}\ddot{h}_{22}, \quad R_{0203} = \frac{1}{2}\ddot{h}_{23}.$$

With (9 *a*) we get from (4):

$$(10) \quad \Phi = \frac{\eta}{2R}\{ -\dddot{f}^{22}(\cos^2\alpha \sin^2\phi - \cos^2\phi) + \dddot{f}^{23}\cos\alpha \sin 2\phi \}$$

When it is possible for the antenna to determine quantitatively the quantity Φ of an individual pulse of radiation we shall be able by using equation (10) to determine the quantities f^{22} and f^{23} of the corresponding source. If we suppose the values of α and R to be known (eg. source at the centre of the galaxy) it will be sufficient to use a second antenna whose position in the plane $Ox'^2x'^3$ (fig. 1) makes an angle $\phi_1 \neq 0, \pi$ with the position of the first one and to observe the same pulse of radiation with both antennas. Indeed we will mesure the quantity Φ_1 given by (10) with ϕ replaced by $\phi + \phi_1$ and so we shall have two equations linear in the unknowns $\dddot{f}^{22}$ and $\dddot{f}^{23}$.

Using a third antenna at a position making an angle $\phi_2 \neq \phi_1$ with the position of the first one we shall in principle be able to determine also the angle α. Indeed we shall then have a system of three equations linear in the quantities $\frac{\dddot{f}^{22}}{R}$. $\frac{\dddot{f}^{22}}{R}\cos^2\alpha$ and $\frac{\dddot{f}^{23}}{R}\cos\alpha$ from which we shall finally get $\frac{\dddot{f}^{22}}{R}$, $\frac{\dddot{f}^{23}}{R}$ and the angle α.

We are still far from such a possibility. What is possible to determine now is the distribution of the number of the events observed during a sufficiently long time interval on the position angle ϕ.

It is to be expected that a pulse of radiation will be observed by the antenna when the quantity Φ^2 is larger than a certain limiting value $(\Phi^2)_0$. This limiting value will depend on the detailed properties of the experimental arrangement. Instead of the rather laborious calculation of the number of events observed at the position angle ϕ with a value $\Phi^2 > (\Phi^1)_0$ we shall calculate the average value $\overline{\Phi^2}$ over a total number $N \gg 1$ of observed events as a function of the angle ϕ. It is reasonable to expect that the curve we shall obtain in this way will show the same characteristic trends as the curve representing the distribution of observed events over the angle ϕ.

We shall assume that in each event the pulse of radiation sweeps the antenna in a time interval short compared with the separation of two consecutive events. We shall also assume that the source of the radiation of each event lies at the same « point » (central part of the galaxy), i. e. that the value of α (and of R) is constant. Further we introduce the following two assumptions:

1) In the sources of the radiation we have for each event exactly the same phenomenon, when we refer it to an appropriate frame $\bar{x}^i$. Only the orientation of the frame $(\bar{x}^i)$ may be different in the different cases. This assumption contains a certain simplification. In fact not only the orientation but also a certain amplitude of the phenomenon may be varying. In our assumption we simply replace the eventually variable amplitude by its average value.

2) As for the orientation of the frame $(\bar{x}^i)$ we shall consider separately two different cases:

a) Complete isotropy in the 3-dimensional space.

b) Isotropy in a given plane $Ox''^2x''^3$ with the direction of the axis Ox''^1 given.

The second case is suggested by the observed flattening of the galaxy and corresponds to the limiting case of an infinitely thin galaxy. The combination of the results we shall obtain for *a*) and *b*) will allow a qualitative discussion of the actual situation.

We shall consider first the case characterised by hypothesis 2 *b*). Let the given direction of the axis Ox''^1 be determined by the angles λ and κ (fig. 2), OP being the projection of the direction of Ox''^1 on the

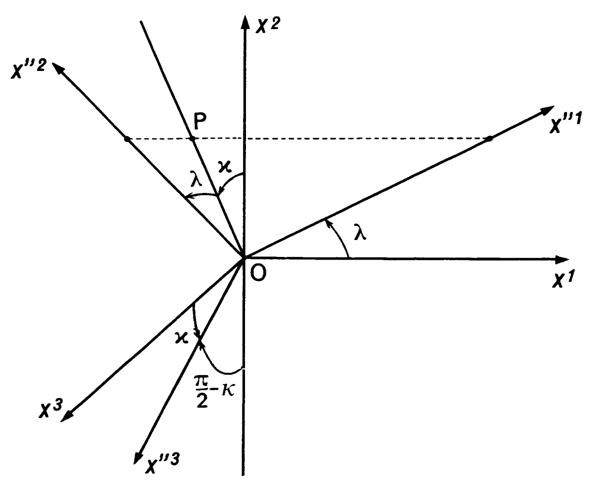

FIG. 2.

plane Ox^2x^3 and lying also in the plane $Ox''^1x''^2$. The relation between (x^i) and (x''^i) is:

$$(11)\qquad \begin{aligned} x^1 &= x''^1 \cos\lambda - x''^2 \sin\lambda, \\ x^2 &= x''^1 \sin\lambda \cos\kappa + x''^2 \cos\lambda \cos\kappa - x''^3 \sin\kappa, \\ x^3 &= x''^1 \sin\lambda \sin\kappa + x''^2 \cos\lambda \sin\kappa \end{aligned}$$

For the transformation of F^{ik} we have the formula

$$(12)\qquad F^{ik} = \frac{\partial x^i}{\partial x''^m}\frac{\partial x^k}{\partial x''^n} F''^{mn}$$

Using (11) and (12) we find for the quantities f^{22} and f^{33} given by (8) the following expressions:

$$(13)\qquad \begin{aligned} 2f^{22} &= \cos 2\kappa\,(\sin^2\lambda F''^{11} + \cos^2\lambda F''^{22} - F''^{33}) \\ &\quad + \sin 2\lambda \cos 2\kappa F''^{22} - 2 \sin 2\kappa\,(\sin\lambda F''^{13} + \cos\lambda F''^{13}), \\ 2f^{23} &= \sin 2\kappa\,(\sin^2\lambda F''^{11} + \cos^2\lambda F''^{22} - F''^{33}) \\ &\quad + \sin 2\lambda \sin 2\kappa F''^{12} + 2 \cos 2\kappa\,(\sin\lambda F''^{13} + \cos\lambda F''^{13}). \end{aligned}$$

It will be useful to introduce the quantities B and β defined by

$$(14)\qquad \cos^2\phi - \cos^2\alpha \sin^2\phi = B \sin\beta, \quad \cos\alpha \sin 2\phi = B \cos\beta.$$

With (13) and (14) we find from (10):

$$(15)\qquad \begin{aligned} \Phi = \frac{\eta B}{4R}\{ &\sin(2\kappa+\beta)(\sin^2\lambda \ddot{F}''^{11} + \cos^2\lambda \ddot{F}''^{22} - \ddot{F}''^{33}) \\ &+ \sin 2\lambda \sin(2\kappa+\beta)\ddot{F}''^{12} + 2\cos(2\kappa+\beta)(\sin\lambda \ddot{F}''^{13} + \cos\lambda \ddot{F}''^{23})\}. \end{aligned}$$

We note from (14) that B and β depend on α and ϕ only. For the quantity B we find from (14) the simple expression

$$(14\,a)\qquad B = 1 - \sin^2\alpha \sin^2\phi.$$

In the case which interests us the plane $Ox''^2x''^3$ is the galactic plane, i. e. Ox''^1 is the normal to this plane. As Ox^1 is the direction to the galactic centre we shall have

$$(16)\qquad \lambda = \frac{\pi}{2}.$$

To simplify further the expression (15) we shall introduce the following last assumption: the source of the radiation is undergoing a phenomenon which takes place entirely in the plane $Ox''^2x''^3$ i. e.:

$$(17)\qquad F''^{1i} = 0.$$

This assumption is compatible with the following two phenomena which constitute the simplest sources of radiation we could envisage: « collisions » (or gravitational deflections) of two objects in the galactic plane or linear explosions of one object also in the galactic plane. Without the assumption (17) certain expressions become longer but the general conclusions in which we are interested remain unchanged.

With (16) and (17) we get from (15)

$$\Phi = -\frac{\eta B}{4R}\ddot{F}''^{33}\sin(2\kappa+\beta). \tag{18}$$

In this formula κ represents the angle of the normal to the galactic plane with the axis Ox^2.

We now proceed to the calculation of the average of Φ^2 over all possible orientations of the frame $(\bar{x}^i)$. We shall have $x^1 = \bar{x}''^1$ and consequently $(\bar{x}^i)$ will be related to (x''^i) through a rotation by an angle μ in the plane $Ox''^2x''^3$. Therefore

$$(19\,a)\qquad x''^1 = \bar{x}^1,\quad x''^2 = \bar{x}^2\cos\mu + \bar{x}^3\sin\mu,\quad x''^3 = -\bar{x}^2\sin\mu + \bar{x}^3\cos\mu;$$

we have again

$$F''^{ik} = \frac{\partial x''^i}{\partial \bar{x}^m}\frac{\partial x''^k}{\partial \bar{x}^n}\bar{F}^{mn}$$

and so we find

$$F''^{33} = \bar{F}^{22}\sin^2\mu + \bar{F}^{33}\cos^2\mu - \bar{F}^{23}\sin 2\mu. \tag{19}$$

According to the hypothesis (2 *b*) we shall have

$$\overline{\Phi^2} = \int_0^{2\pi}\Phi^2\frac{d\mu}{2\pi}.$$

Using (19) we find finally

$$\overline{\Phi^2} = \frac{\eta^2B^2}{16R^2}\sin^2(2\kappa+\beta)\left\{\left(\frac{\ddot{\bar{F}}^{22}+\ddot{\bar{F}}^{33}}{2}\right)^2 + \frac{1}{2}\left(\frac{\ddot{\bar{F}}^{22}-\ddot{\bar{F}}^{33}}{2}\right)^2 + \frac{1}{2}(\ddot{\bar{F}}^{23})^2\right\}. \tag{20}$$

The right-hand side of this relation depends on the position angle ϕ of the antenna as the square of the quantity

$$B\sin(2\kappa+\beta) = \cos 2\kappa(\cos^2\phi - \cos^2\alpha\sin^2\phi) + \sin 2\kappa\cos\alpha\sin 2\phi.$$

This can be written also in the form

$$B\sin(2\kappa+\beta) = \frac{1}{2}\cos 2\kappa\sin^2\alpha + \frac{1+\cos^2\alpha}{2}\cos 2\phi\cos 2\kappa + \cos\alpha\sin 2\phi\sin 2\kappa$$

which shows that it is periodic in ϕ with the period π and it has in the interval $0 \leqslant \phi \leqslant \pi$ one maximum and one minimum. It is easily seen that the maximum and the minimum have different signs when $\cos \alpha \neq 0$ as is actually the case. Therefore the square of $B \sin (2\kappa + \beta)$ has in the interval $0 \leqslant \phi \leqslant \pi$ two maxima, the positions of which depend on the values of α and κ and are both different from $\phi = 0$ when $\cos \alpha \neq 0$.

The angle κ can be expressed with the help of the angle A of the equatorial and the galactic planes:

$$\cos \kappa = \frac{\cos A}{\sin \alpha}. \tag{21}$$

The case of the complete isotropy of the sources in the 3-dimensional space has to be treated with the help of the formula (15) corresponding to arbitrary values of λ and κ. We have now to average not only over the angle μ introduced by (19 *a*) but also over the angles λ and κ, in order to permit the axis Ox''^1 to take any orientation in the space. I. e. we have to multiply by $\frac{d\mu}{2\pi}\frac{d\kappa}{2\pi}\frac{\sin \lambda d\lambda}{2}$ and integrate over $0 \leqslant \mu,\ \kappa \leqslant 2\pi$ and $0 \leqslant \lambda \leqslant \pi$. Using again the simplifying assumption (17) we find finally:

$$\overline{\Phi^2} = \frac{\eta^2 B^2}{60R^2}\{(\ddot{F}^{22})^2 + (\ddot{F}^{33})^2 - \ddot{F}^{22}\ddot{F}^{33} + 3(\ddot{F}^{23})^2\}. \tag{22}$$

The right hand side of this relation depends on ϕ as the square of the quantity B and consequently it has in the interval $0 \leqslant \phi \leqslant \pi$ according to (14 *a*) one maximum only, for $\phi = 0$.

Our galaxy is not isotropic and the same will be true also for its central part. It follows that $\overline{\Phi^2}$ will not have exactly the form (22). To correct this formula qualitatively it will be sufficient to add to its right hand side the right hand side of (20) multiplied by a number $\varepsilon < 1$. If $\varepsilon \ll 1$ we shall again have for $\overline{\Phi^2}$ as a fonction of ϕ only the one maximum at $\phi = 0$ and this is seen to be the most plausible. However it cannot be excluded *a priori* that we should have $\varepsilon < 1$ but not $\varepsilon \ll 1$ and consequently the function $\overline{\Phi^2}$ might have three maxima.

II

In conclusion we shall define polarization states for gravitational waves and show that the number of maxima which $\overline{\Phi^2}$ possesses depends on the polarization of the waves at the moment of reception. If the waves are unpolarized or circularly polarized the function $\overline{\Phi^2}$ has one maximum; if the waves are linearly polarized $\overline{\Phi^2}$ has two maxima.

We assume that we are far enough away from the source of the radiation that the Riemann tensor can be considered to be of type N. Let ξ_μ be the principal null vector field. Then the Riemann tensor has the following representation (see for example [5]):

$$R_{\mu\nu\rho\sigma} = 4\xi_{[\mu}A_{\nu][\rho}\xi_{\sigma]}, \tag{23}$$

where $A_{\nu\rho}$ is of the form

$$A_{\nu\rho} = A(a_\nu a_\rho - b_\nu b_\rho). \tag{24}$$

The vectors a_μ and b_μ are two space-like unit vectors orthogonal to each other and to ξ_μ. They are not uniquely defined but admit transformations of the form

$$\begin{aligned} a_\mu &\to a_\mu + \lambda\xi_\mu, \\ b_\nu &\to b_\nu + \lambda'\xi_\nu. \end{aligned} \tag{25}$$

From formulae (23), (24) we see that $R_{\mu\nu\rho\sigma}$ is determined for example by the vector a_μ and the scalar A. Since a_μ is a unit vector orthogonal to ξ_μ it has only one essential component.

An analogous situation exists in classical electromagnetic theory. If the skew-symmetric Maxwell tensor $F_{\mu\nu}$ is of null type it possesses a representation of the form

$$F_{\mu\nu} = 2\xi_{[\mu}A_{\nu]} \tag{26}$$

where ξ_μ is the principal null vector field [6]. A_μ is a space-like vector orthogonal to ξ_μ and admits the transformations

$$A_\mu \to A_\mu + \lambda''\xi_\mu. \tag{27}$$

A_μ contains therefore only two physically significant components.

Let p_μ be the tangent vector to the world-line of the observer and eliminate the arbitrariness appearing in (27) by imposing the condition

$$A_\mu p^\mu = 0. \tag{28}$$

Then the polarization is defined by A_μ.

If the wave is coherent and monochromatic A_μ describes in general an ellipse in the 2-plane orthogonal to ξ_μ and p_μ. The superposition of two such waves is described by the vector sum of their corresponding vectors A_μ. Notice that because of (28) the state of polarization of a wave may depend on the observer.

We wish to describe a gravitational wave in a similar manner. Since polarization states are not observer independent, we shall in what follows carry out all calculations in the observer's rest frame and using a condition equivalent to (28) set

$$a^0 = b^0 = 0. \tag{29}$$

In the coordinate system of figure 1, we have

$$\vec{a} = (0, \cos\gamma, \sin\gamma),$$
$$\vec{b} = (0, \sin\gamma, -\cos\gamma).$$

The Riemann tensor is described by the two functions A and γ. A straightforward calculation yields from (3) the following formula for Φ^2:

$$\Phi^2 = A^2\eta^2 [\cos^2 2\gamma (\sin^2\phi \cos^2\alpha - \cos^2\phi)^2 + \sin^2 2\gamma \sin^2 2\phi \cos^2\alpha - 2\cos 2\gamma \sin 2\gamma \sin 2\phi \cos\alpha (\sin\phi \cos^2\alpha - \cos^2\phi]). \tag{30}$$

In contrast to the vector A_μ which is intrinsically defined in terms of $F_{\mu\nu}$ by the formula (8), the intrinsically defined vectors a_μ and b_μ appear quadratically in the expression (23) for the Riemann tensor and are not appropriate for studying the polarization states. We introduce therefore a 3-vector which completely describes the wave and which may be analyzed in terms of polarization states in the same way that the vector A_μ is analyzed in classical electromagnetic theory.

Define the vector $\vec{E}$ in the coordinate system of figure 1 by the formulae

$$\begin{aligned} E_1 &= 0, \\ E_2 &= R_{2020}, \\ E_3 &= R_{2030}. \end{aligned} \tag{31}$$

One easily sees that $\vec{E}$ is a vector with norm A and which forms an angle 2α with the x^2-axis. For the resonance frequency part of the incident wave, the vector $\vec{E}$ describes an ellipse in the plane normal to $\vec{\xi}$ exactly as in the electromagnetic case.

We shall now discuss the possible ways of averaging over A and γ in (30). If the wave is circularly polarized then A is constant and the average is taken over γ alone.

This yields immediately the following formula for $\overline{\Phi^2}$:

$$\overline{\Phi^2} = \frac{A^2\eta^2}{2} (\cos^2\phi + \sin^2\phi \cos^2\alpha) \tag{32}$$

If the wave is unpolarized, A and γ are statistically independent and the average in equation (30) may be taken over A and γ separately. This yields the same result as above. Formula (32) contains as a particular case formula (22).

If the wave is linearly polarized with a fixed direction then $\gamma = \gamma_0$ remains constant and the average is taken over A only. From (30) we find in this case

$$(33) \quad \overline{\Phi^2} = \overline{A^2}\eta^2 \left[\cos 2\gamma_0 (\sin^2 \phi \cos^2 \alpha - \cos^2 \phi) - \sin 2\gamma_0 \sin 2\phi \cos \alpha\right]^2.$$

A particular case of this formula is given by formula (20).

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