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ON CONDITION (a_f) OF A STRATIFIED MAPPING

by Satoshi KOIKE

In [3], D.J.A. Trotman showed that Whitney's condition (a) on the pair of adjacent strata is equivalent to condition (a^s) which has more obvious geometric content. These conditions can be generalized to the conditions of the kernel of the mapping called a stratified mapping. The generalization of condition (a) is condition (a_f) which is well-known in the stratification theory. On the other hand, we shall call the generalization of condition (a^s) condition (a_f^s). Then, we have already known that (a_f) implies (a_f^s) from the proof of Lemma 11.4 in J.N. Mather [2] (or Lemma (2.4) of Chapter II in [1]). In this paper, we show that (a_f) is equivalent to (a_f^s). In § 2 we prove this result, and in § 3 we give the illustrative example of the fact.

1. Definitions and the result.

Let X, Y be disjoint C^1 submanifolds of $\mathbf{R}^n$, and let y_0 be a point in $Y \cap \overline{X}$. We say the pair (X, Y) satisfies *Whitney's condition* (a) at y_0 if for any sequence of points $\{x_i\}$ in X tending to y_0 such that the tangent space $T_{x_i}X$ tends to τ, we have $T_{y_0}Y \subset \tau$. As stated above, this condition is equivalent to the following condition; (a^s): For any local C^1 retraction at y_0, $\pi_Y : \mathbf{R}^n \longrightarrow Y$, there exists a neighborhood W of y_0 in $\mathbf{R}^n$ such that $\pi_{Y|W \cap X}$ is a submersion.

Let $f: A \longrightarrow \mathbf{R}^p$ be a smooth mapping defined in a neighborhood A of $X \cup Y$ in $\mathbf{R}^n$. Suppose that the restricted mappings

$f|_X : X \longrightarrow \mathbf{R}^p$ and $f|_Y : Y \longrightarrow \mathbf{R}^p$ are of constant ranks. Then we say the pair (X, Y) satisfies *condition* (a_f) at y_0 if for any sequence of points $\{x_i\}$ in X tending to y_0 such that the plane $\ker d(f|_X)_{x_i}$ tends to κ, we have $\ker d(f|_Y)_{y_0} \subset \kappa$, where $\ker d(f|_X)_x$ denotes the kernel of the differential of $f|_X$ at x.

Let U, V be C^1 submanifolds of $\mathbf{R}^p$ such that $U \cap V = \phi$ or $U = V$. Further, suppose that $f(X)$, $f(Y)$ are contained in U, V respectively, and that $f|_X : X \longrightarrow U$ and $f|_Y : Y \longrightarrow V$ are submersions. Then we call this mapping f a *stratified mapping.* From now, we shall think of a stratified mapping.

We say that a local C^1 retraction at y_0, $\pi_Y : \mathbf{R}^n \longrightarrow Y$, and a local C^1 retraction at $f(y_0)$, $\pi_V : \mathbf{R}^p \longrightarrow V$, satisfy the commutation relation (CRf) if it holds that $f \circ \pi_Y = \pi_V \circ f$ in a neighborhood of y_0.

Remark 1. – For a stratified mapping, the following facts hold.

1) For any local C^1 retraction at $f(y_0)$, $\pi_V : \mathbf{R}^p \longrightarrow V$, there exists a local C^1 retraction at y_0, $\pi_Y : \mathbf{R}^n \longrightarrow Y$ such that they satisfy (CRf). Consider the mapping $\pi_V \circ f$ in a neighborhood of y_0. Since $\pi_V \circ f|_Y : Y \longrightarrow V$ is a submersion, there exists a local C^1 retraction at y_0, $\pi_Y : \mathbf{R}^n \longrightarrow Y$, such that $\pi_V \circ f \circ \pi_Y = \pi_V \circ f$. Thus, we see that they satisfy (CRf) in a neighborhood of y_0.

2) On the other hand, it is not true that for any local C^1 retraction at y_0, there exists a local C^1 retraction at $f(y_0)$ such that they satisfy (CRf): See the example in 3.

Here we introduce the next condition;

(a_f^s): For any local C^1 retraction at y_0, $\pi_Y : \mathbf{R}^n \longrightarrow Y$, and local C^1 retraction at $f(y_0)$, $\pi_V : \mathbf{R}^p \longrightarrow V$, satisfying (CRf), there exists a neighborhood W of y_0 in $\mathbf{R}^n$ such that for any $x \in W \cap X$,

$$d(\pi_{YX})_x : \ker d(f|_X)_x \longrightarrow \ker d(f|_Y)_y$$

is onto, where $\pi_{YX} = \pi_Y|_X$ and $y = \pi_Y(x)$.

THEOREM. – *For a stratified mapping,* (a_f) *is equivalent to* (a_f^s).

Remark 2. – Theorem A in [3] is the case where $U = V = \{f(y_0)\}$ in the above theorem. Because, in that case, the kernel is the tangent space, and (CRf) is satisfied for any local C^1 retraction at y_0, $\pi_Y : \mathbf{R}^n \longrightarrow Y$.

2. Proof of the theorem.

Let f be a stratified mapping i.e.

$$f|_X : X \longrightarrow U \quad \text{and} \quad f|_Y : Y \longrightarrow V$$

are submersions. We introduce the condition of "transverse foliation" defined locally in a neighborhood of y_0 in $\mathbf{R}^n$;

$(\mathcal{H}^1)$: For any local C^1 foliation $\mathcal{F}$ which is transversal to the fiber of $f|_Y$ at y_0, and whose leaves are unions of fibers of a local C^1 retraction π_Y satisfying the relation (CRf), there exists a neighborhood W of y_0 in $\mathbf{R}^n$ such that $\mathcal{F}$ is transversal to the fibers of $f|_X$ in W.

LEMMA. – *(a_f^s) is equivalent to $(\mathcal{H}^1)$.*

Proof. – As it is trivial that (a_f^s) implies $(\mathcal{H}^1)$, we shall show that $(\mathcal{H}^1)$ implies (a_f^s). Consider a local C^1 retraction at y_0, $\pi_Y : \mathbf{R}^n \longrightarrow Y$, and a local C^1 retraction at $f(y_0)$, $\pi_V : \mathbf{R}^p \longrightarrow V$, which satisfy (CRf) in a neighborhood of y_0. Let N_{y_0} denote the normal space of $\ker d(f|_Y)_{y_0}$ in $T_{y_0}Y$. Then, there exist a neighborhood W_1 of y_0 in Y, and a local C^1 foliation $\widetilde{\mathcal{F}}$ of W_1 such that $N_{y_0} = T_{y_0}\widetilde{F}_{y_0}$, where $\widetilde{F}_{y_0}$ denotes the leaf of $\widetilde{\mathcal{F}}$ which contains y_0. Shrinking the neighborhood W_1 if necessary, $\mathcal{F} \equiv \{\{v \in \mathbf{R}^n \mid \pi_Y(v) \in \widetilde{F}\}\}_{\widetilde{F}\in\widetilde{\mathcal{F}}}$ is a local C^1 foliation of $\mathbf{R}^n$ in a neighborhood of y_0. From the construction, we have

$$T_{y_0}F_{y_0} \oplus \ker d(f|_Y)_{y_0} = T_{y_0}\mathbf{R}^n,$$

where F is a leaf of $\mathcal{F}$. Since $f|_Y : Y \longrightarrow V$ is a submersion, $\ker d(f|_Y)_y$ is continuous in the Grassman manifold of

$$\dim \ker(f|_Y)_{y_0}\text{-planes in } n\text{-space.}$$

Further, $(\mathcal{H}^1)$ holds from the assumption. Therefore, there exists a neighborhood W_2 of y_0 in $\mathbf{R}^n$ such that for any $y \in W_2$,

$$T_yF_y \oplus \ker d(f|_Y)_y = T_y\mathbf{R}^n \tag{2.1}$$

and for any $x \in W_2 \cap X$,

$$T_xF_x + \ker d(f|_X)_x = T_x\mathbf{R}^n. \tag{2.2}$$

From the relation (CRf), we have

$$d(\pi_{YX})_x : \ker d(f|_X)_x \longrightarrow \ker d(f|_Y)_y \qquad (2.3)$$

in a neighborhood of y_0. By (2.1), (2.2), and (2.3), we see that the differential mapping (2.3) is onto near y_0.

Remark 3. – From the proof of Theorem A in [3], we can see easily that (a_f) is equivalent to the following condition;

($\mathfrak{F}^1$): For any C^1 foliation $\mathfrak{F}$ which is transversal to the fiber of $f|_Y$ at y_0, there exists a neighborhood W of y_0 in $\mathbf{R}^n$ such that $\mathfrak{F}$ is transversal to the fibers of $f|_X$ in W.

PROPERTY 1. – *Let* $\pi_V : \mathbf{R}^p \longrightarrow V$ *be a local* C^1 *retraction at* $f(y_0)$.

1) *There exists a neighborhood* W *of* $f(y_0)$ *in* V *such that* $\{\pi_V \circ f)^{-1}(w)\}_{w \in W}$ *is a* C^1 *foliation of codimension* V *in a sufficiently small neighborhood of* y_0 *in* $\mathbf{R}^n$.

Since $\pi_Y \circ f|_Y : Y \longrightarrow V$ is a submersion, $\pi_V \circ f$ has a maximal rank (of dimension V) at $y_0 \in \mathbf{R}^n$. Thus (1) follows.

2) *If a point* q *in* $\mathbf{R}^n$ *is contained in* $(\pi_V \circ f)^{-1}(w)$, *then we have* $\ker df \subset T_q(\pi_V \circ f)^{-1}(w)$.

It is clear from the fact that $T_q(\pi_V \circ f)^{-1}(w) = \ker d(\pi_V \circ f)_q$.

PROPERTY 2. – *Let* $\pi_Y : \mathbf{R}^n \longrightarrow Y$ *be a local* C^1 *retraction at* y_0, *and* $\pi_V : \mathbf{R}^p \longrightarrow V$ *be a local* C^1 *retraction at* $f(y_0)$. *If any fiber of* π_Y *is contained in some fiber of* $\pi_V \circ f$ *in a neighborhood of* y_0, *then* (CRf) *holds.*

It is trivial.

From Lemma, it is sufficient to show that ($\mathcal{H}^1$) implies (a_f). We suppose that the pair (X, Y) does not satisfy condition (a_f) at y_0. Then, there exists a sequence of points $\{x_i\}$ in X tending to y_0 with $\lim_i \ker d(f|_X)_{x_i} = K$, such that $K \not\supset \ker d(f|_Y)_{y_0}$. Thus, there exists a vector $k \in \ker d(f|_Y)_{y_0}$ such that $k \notin K$. By the similar way as the proof of Theorem A in [3], we can construct a C^1 foliation $\mathfrak{F}$ of codimension 1 such that $T_{y_0}F_{y_0} \not\ni k$ and $T_{x_i}F_{x_i} \supset \ker d(f|_X)_{x_i}$ i.e. $\mathfrak{F}$ is transversal to the fiber of $f|_Y$ at y_0, and $\mathfrak{F}$ is not transversal to the fiber of $f|_X$ at x_i.

We take a local C^1 retraction at $f(y_0)$, $\pi_V : \mathbf{R}^n \longrightarrow V$, arbitarily. From Property 1 (2), we have

$$k \in \ker d(f|_Y)_{y_0} \subset \ker df_{y_0} \subset T_{y_0}(\pi_V \circ f)^{-1}(w_0),$$

where $w_0 = f(y_0)$. Therefore, the local foliations $\{(\pi_V \circ f)^{-1}(w)\}_{w \in W}$ and $\mathcal{F}$ are transversal near y_0. Thus,

$$\{(\pi_V \circ f)^{-1}(w) \cap F\}_{\substack{w \in W \\ F \in \mathcal{F}}} \tag{2.4}$$

is a C^1 foliation in a neighborhood of y_0 in $\mathbf{R}^n$.

PROPERTY 3. – $(\pi_V \circ f)^{-1}(w_0) \cap F_{y_0}$ *is transversal to* Y *at* y_0. Since $\pi_V \circ f|_Y : Y \longrightarrow V$ is a submersion and

$$T_{y_0}(\pi_V \circ f)^{-1}(w_0) = \ker d(\pi_V \circ f)_{y_0},$$

we have

$$T_{y_0}Y + T_{y_0}(\pi_V \circ f)^{-1}(w_0) = T_{y_0}\mathbf{R}^n. \tag{2.5}$$

As $(\pi_V \circ f)^{-1}(w_0)$ is transversal to F_{y_0} *at* y_0, we have

$$T_{y_0}((\pi_V \circ f)^{-1}(w_0) \cap F_{y_0}) = T_{y_0}(\pi_V \circ f)^{-1}(w_0) \cap T_{y_0}F_{y_0}.$$

Further, the vector k is not an element of $T_{y_0}F_{y_0}$. Therefore, we have

$$T_{y_0}((\pi_V \circ f)^{-1}(w_0) \cap F_{y_0}) + \langle k \rangle = \ker d(\pi_V \circ f)_{y_0} \tag{2.6}$$

where $\langle k \rangle$ denotes the subvector space spanned by the vector k of $T_{y_0}\mathbf{R}^n$. From (2.5), (2.6), and the fact that $k \in \ker d(f|_Y)_{y_0} \subset T_{y_0}Y$, we see that $T_{y_0}((\pi_V \circ f)^{-1}(w_0) \cap F_{y_0}) + T_{y_0}Y = T_{y_0}\mathbf{R}^n$.

By using Property 3, we can construct a local C^1 retraction at y_0, $\pi_Y : \mathbf{R}^n \longrightarrow Y$, along leaves of the local foliation (2.4). Then, these local retractions π_Y and π_V satisfy (CRf) in a neighborhood of y_0 in $\mathbf{R}^n$, from Property 2. Further, from the construction it is clear that each leaf of $\mathcal{F}$ is a union of fibers of π_Y. Thus, $(\mathcal{H}^1)$ does not hold. This completes the proof of the theorem.

3. An Example.

In this section, we give an example which illustrates the proof of the theorem, and demonstrates Remark 1 (2).

Let $f = (f_1, f_2) : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ be a mapping defined by

$$f(x, y, z) = (x, y^4 + 2y^2z^2).$$

We take $X = \{y \neq 0\}$ and $Y = \{y = 0\}$ as disjoint submanifolds in $\mathbf{R}^3$, and take $U = \{y \neq 0\}$ and $V = \{y = 0\}$ as disjoint submanifolds in $\mathbf{R}^2$. Then, restricted mappings $f|_X : X \longrightarrow U$ and $f|_Y : Y \longrightarrow V$ are submersions i.e. f is a stratified mapping.

Put $S = \{p = (x, y, z) \in X \mid y = z\}$. For any point $p \in S$, we have

$$\text{grad}\,(f_{1|X})_p = (1, 0, 0) \text{ and } \text{grad}\,(f_{2|X})_p = (0, 8y^3, 4y^3).$$

Therefore, we have $\ker d(f|_X)_p = \langle(0, 1, -2)\rangle$. We take a sequence of points $\{p_i\}$ in S tending to $0 = (0, 0, 0) \in Y$. We have

$$\lim_i \ker d(f|_X)_{p_i} = \langle(0, 1, -2)\rangle.$$

On the other hand, $\ker d(f|_Y)_0 = \langle(0, 0, 1)\rangle$. Therefore, (X, Y) does not satisfy condition (a_f) at 0.

In this case, (X, Y) does not satisfy condition (a_f^s) at 0 as a matter of course. For example, we take the canonical projection over Y as a local retraction at $f(0)$, $\pi_V : \mathbf{R}^2 \longrightarrow V$. Then, the foliation whose leaves are fibers of $\pi_V \circ f$ is

$$\mathcal{F}_1 = \{\{(x, y, z) \in \mathbf{R}^3 \mid x = k_1\}\}_{k_1 \in \mathbf{R}}.$$

Further, we consider the foliation

$$\mathcal{F}_2 = \{\{(x, y, z) \in \mathbf{R}^3 \mid z + 2y = k_2\}\}_{k_2 \in \mathbf{R}},$$

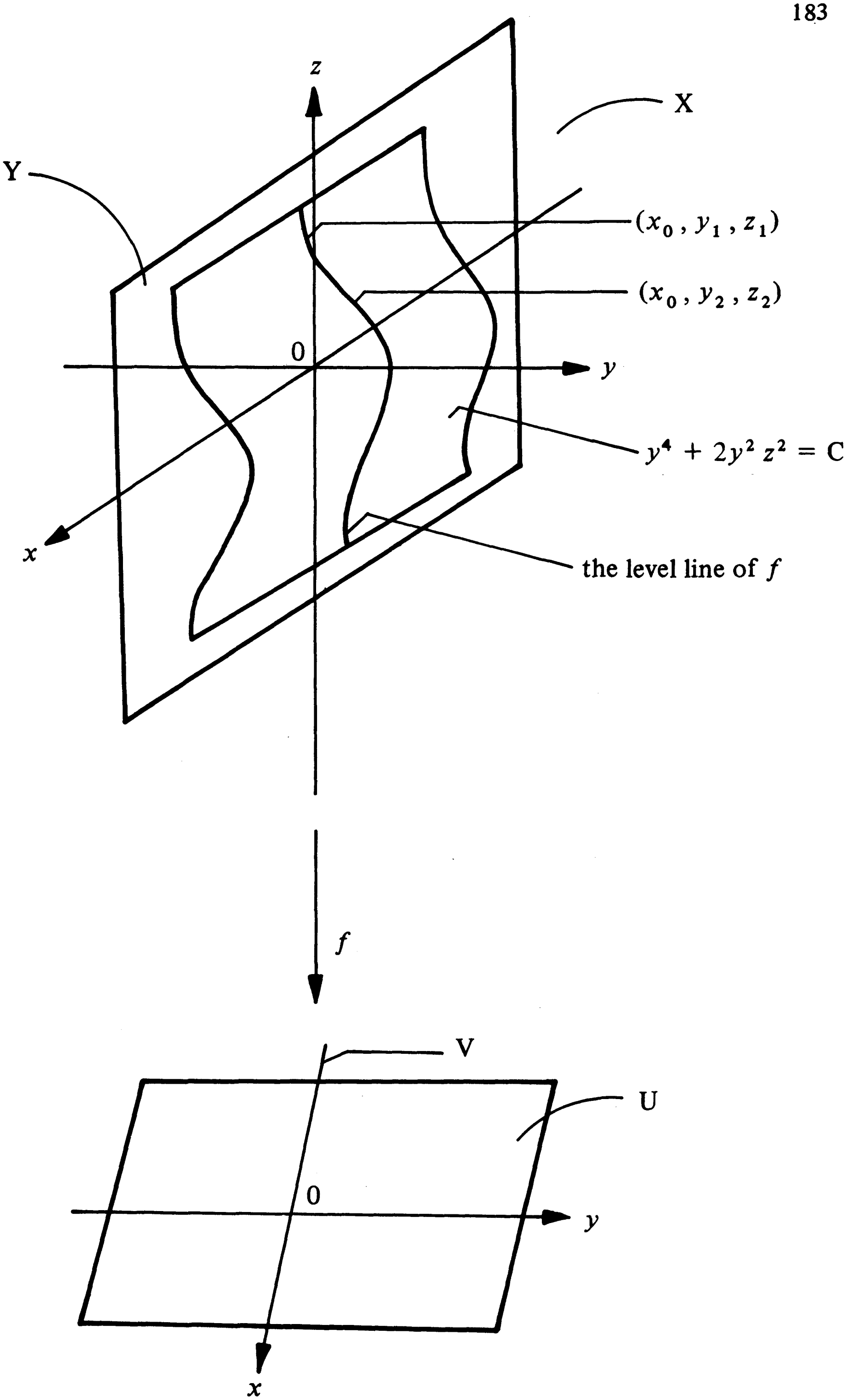
which is transversal to the fiber of $f|_Y$ at 0, and is not transversal to the fibers of $f|_X$ in S. As $\mathcal{F}_1$ and $\mathcal{F}_2$ are transversal, the intersection of $\mathcal{F}_1$ and $\mathcal{F}_2$ becomes a foliation of $\mathbf{R}^3$,

$$\{\{(x, y, z) \in \mathbf{R}^3 \mid x = k_1, z + 2y = k_2\}\}_{\substack{k_1 \in \mathbf{R} \\ k_2 \in \mathbf{R}}}.$$

It is clear that the leaves of this foliation induce a retraction π_Y which does not admit condition (a_f^s) at $0 \in \mathbf{R}^3$.

Nextly, we show that this example demonstrates Remark 1 (2). We consider a retraction at $0 \in \mathbf{R}^3$, $\pi_Y(x, y, z) = (x + yz^2, 0, z)$. Then, we have $f \circ \pi_Y(x, y, z) = (x + yz^2, 0)$. Let (x_0, y_1, z_1), (x_0, y_2, z_2) be points in X such that $0 < y_1 < y_2$ and

$$y_1^4 + 2y_1^2z_1^2 = y_2^4 + 2y_2^2z_2^2 = C > 0.$$


z
X
Y
(x_0, y_1, z_1)
(x_0, y_2, z_2)
0
y
$y^4 + 2y^2z^2 = C$
x
the level line of f
f
V
U
0
y
x

From the fact that $0 < y_1 < y_2$, we have

$$x_0 + y_1 z_1^2 = x_0 + \frac{C - y_1^4}{2y_1} \neq x_0 + \frac{C - y_2^4}{2y_2} = x_0 + y_2 z_2^2$$

i.e. $$f \circ \pi_Y(x_0, y_1, z_1) \neq f \circ \pi_Y(x_0, y_2, z_2).$$

On the other hand, $f(x_0, y_1, z_1) = f(x_0, y_2, z_2)$. Therefore, there does not exist a local C^1 retraction at $0 \in \mathbf{R}^2$, $\pi_V : \mathbf{R}^2 \longrightarrow V$, such that they satisfy (CRf).

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