

ANNALES DE L'INSTITUT FOURIER

K. NOMIZU

K. YANO

**On infinitesimal transformations preserving the
curvature tensor field and its covariant differentials**

Annales de l'institut Fourier, tome 14, n° 2 (1964), p. 227-236

[<http://www.numdam.org/item?id=AIF_1964__14_2_227_0>](http://www.numdam.org/item?id=AIF_1964__14_2_227_0)

© Annales de l'institut Fourier, 1964, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

ON INFINITESIMAL TRANSFORMATIONS PRESERVING THE CURVATURE TENSOR FIELD AND ITS COVARIANT DIFFERENTIALS

by Katsumi NOMIZU and Kentaro YANO ⁽¹⁾

We shall say that a transformation ϕ of a Riemannian manifold M is *strongly curvature-preserving* if it preserves the curvature tensor field R and all its successive covariant differentials $\nabla^m R$. Similarly, an infinitesimal transformation X on M is strongly curvature-preserving if

$$L_X(\nabla^m R) = 0, \quad m = 0, 1, 2, \dots,$$

where L_X denotes Lie differentiation with respect to X and $\nabla^0 R = R$.

Of course, an affine transformation or an infinitesimal affine transformation is strongly curvature-preserving. In the present note, we shall prove the converse in the following form. Recall that an infinitesimal transformation X is conformal, homothetic, or Killing according as $L_X g = fg$ (f : function), $L_X g = cg$ (c : constant), or $L_X g = 0$, respectively, where g denotes the metric tensor.

THEOREM 1 ⁽²⁾. — *Let M be an irreducible analytic Riemannian manifold of dimension ≥ 2 . Then a strongly curvature-preserving infinitesimal transformation is necessarily homothetic. If M is furthermore complete, then X is Killing.*

⁽¹⁾ Both authors are being partially supported by an NSF Grant No. 24026.

⁽²⁾ We have since extended theorem 1 to the case of a global transformation; this result will appear elsewhere.

Note that the additional assertion is a consequence of a result of Kobayashi [2]. The proof of Theorem 1 will depend on the following results.

THEOREM 2. — *Let M be an irreducible Riemannian manifold of dimension > 2 . An infinitesimal conformal transformation X is homothetic if $L_X R = 0$.*

THEOREM 3. — *Let M be an irreducible analytic Riemannian manifold of dimension 2. An infinitesimal transformation X is homothetic if $L_X R = 0$ and $L_X(\nabla R) = 0$.*

The proof of Theorem 2 makes use of a result of Guillemin and Sternberg [1] on the prolongations of the conformal algebra.

Finally, we shall prove the following generalization of Theorem 1.

THEOREM 4. — *Let M be a connected, complete and analytic Riemannian manifold which has no Euclidean part (i.e., the restricted homogeneous holonomy group Ψ^0 has no non-zero fixed vector). Then any strongly curvature-preserving infinitesimal transformation X is a Killing vector field.*

1. Preliminaries.

For an arbitrary infinitesimal transformation X on M , we shall define a tensor field K of type $(1, 2)$ which measures the deviation of X from being affine; X is affine if and only if $K = 0$. For any vector field Y , consider the derivation

$$(1) \quad K(Y) = [L_X, \nabla_Y] - \nabla_{[X, Y]}$$

of the algebra of tensor fields. It is easy to verify that $K(Y)$ is actually a tensor field of type $(1, 1)$ and that $K(fY) = fK(Y)$ for any differentiable function f . This means that K is a tensor field of type $(1, 2)$ which associates to a vector field Y the tensor field $K(Y)$ of type $(1, 1)$.

Using the formula $L_X = A_X + \nabla_X$, where A_X is the tensor field of type $(1, 1)$ defined by $A_X Y = -\nabla_Y X$ (cf. [3], p. 235), we may express $K(Y)$ as follows:

$$(2) \quad K(Y) = R(X, Y) - \nabla_Y(A_X).$$

In fact, we have

$$\begin{aligned} K(Y) &= [A_X + \nabla_X, \nabla_Y] - \nabla_{[X, Y]} \\ &= [A_X, \nabla_Y] + [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \\ &= -\nabla_Y(A_X) + R(X, Y). \end{aligned}$$

We now prove

LEMMA 1. — *The tensor field K corresponding to a vector field X has the following properties :*

- 1) $K(Y)Z = K(Z)Y$ for any vector fields Y and Z ;
- 2) $(\nabla_U K)(Y)Z = (\nabla_U K)(Z)Y$ for any vector fields Y, Z , and U ;
- 3) If $L_X R = 0$, then $(\nabla_Y K)(Z) = (\nabla_Z K)(Y)$ for any vector fields Y and Z ;

- 4) If X is conformal: $L_X g = fg$, then

$$(3) \quad K(Y)g = -\alpha(Y)g$$

for any vector field Y , where $\alpha = df$.

- 5) If X is conformal, then, for the form α in 4), we have

$$(\nabla_U K)(Y)g = -(\nabla_U \alpha)(Y)g$$

for any vector fields Y and U .

Proof. — 1) By using (2), we have

$$\begin{aligned} K(Y)Z &= R(X, Y)Z - [\nabla_Y(A_X)]Z \\ &= R(X, Y)Z - \nabla_Y(A_X Z) + A_X(\nabla_Y Z) \end{aligned}$$

and hence

$$K(Y)Z = R(X, Y)Z + \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X$$

by definition of A_X . Thus alternating with respect to Y and Z , we have

$$\begin{aligned} &K(Y)Z - K(Z)Y \\ &= R(X, Y)Z - R(X, Z)Y + ([\nabla_Y, \nabla_Z] - \nabla_{[Y, Z]})X = 0 \end{aligned}$$

by virtue of Bianchi's identity :

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$$

and the definition of the curvature tensor :

$$[\nabla_Y, \nabla_Z] - \nabla_{[Y, Z]} = R(Y, Z).$$

2) We take ∇_u of 1) and obtain

$$\begin{aligned} (\nabla_u K)(Y)Z + K(\nabla_u Y)Z + K(Y)\nabla_u Z \\ = (\nabla_u K)(Z)Y + K(\nabla_u Z)Y + K(Z)\nabla_u Y, \end{aligned}$$

from which, using 1) again, we find

$$(\nabla_u K)(Y)Z = (\nabla_u K)(Z)Y.$$

3) By using (2), we have

$$\begin{aligned} (\nabla_Y K)(Z) &= \nabla_Y(K(Z)) - K(\nabla_Y Z) \\ &= (\nabla_Y R)(X, Z) + R(\nabla_Y X, Z) + R(X, \nabla_Y Z) - \nabla_Y \nabla_Z(A_X) \\ &\quad - R(X, \nabla_Y Z) - \nabla_{\nabla_Y Z}(A_X) \end{aligned}$$

or

$$(\nabla_Y K)(Z) = (\nabla_Y R)(X, Z) - R(A_X Y, Z) - (\nabla_Y \nabla_Z - \nabla_{\nabla_Y Z})(A_X).$$

Alternating with respect to Y and Z , we find

$$\begin{aligned} (\nabla_Y K)(Z) - (\nabla_Z K)(Y) \\ &= (\nabla_Y R)(X, Z) - (\nabla_Z R)(X, Y) - R(A_X Y, Z) + R(A_X Z, Y) \\ &\quad - ([\nabla_Y, \nabla_Z] - \nabla_{[Y, Z]})(A_X) \\ &= (\nabla_X R)(Y, Z) - R(A_X Y, Z) - R(Y, A_X Z) - R(Y, Z)A_X \\ &= [(\nabla_X + A_X)R](Y, Z) = (L_X R)(Y, Z) = 0, \end{aligned}$$

by virtue of Bianchi's identity :

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0$$

and the assumption $L_X R = 0$.

4) By definition of $K(Y)$, we have

$$K(Y) = L_X \nabla_Y - \nabla_Y L_X - \nabla_{[X, Y]}.$$

Applying this derivation to g , we find

$$K(Y)g = -\nabla_Y L_X g.$$

Thus if $L_X = fg$, then we have

$$K(Y)g = -\alpha(Y)g,$$

where $\alpha = df$.

5) Taking ∇_u of the equation in 4), we have

$$(\nabla_u K)(Y)g + K(\nabla_u Y)g = -(\nabla_u \alpha)(Y)g - \alpha(\nabla_u Y)g,$$

which implies

$$(\nabla_v K)(Y)g = -(\nabla_v \alpha)(Y)g,$$

since $K(\nabla_v Y)g = -\alpha(\nabla_v Y)g$ by 4).

We shall now interpret Lemma 1 above in terms of the prolongations of the conformal algebra [1]. By the conformal algebra over an n -dimensional real vector space V with inner product, we mean the following. Let $\text{co}(V)$ be the set of all linear endomorphisms A of V such that

$$(AX, Y) + (X, AY) = c(X, Y)$$

for all X, Y in V , where c is a constant which depends on A . With respect to the usual bracket $[A, B] = AB - BA$, $\text{co}(V)$ forms a Lie algebra.

Suppose X is conformal. Property 4) means that for any Y in the tangent space $T_x(M)$ at a point $x \in M$, the endomorphism $K(Y)$ is in the conformal algebra $\text{co}(x)$ over $T_x(M)$, of course, with respect to the metric g_x . Property 1) means that the linear mapping $K: Y \in T_x(M) \rightarrow K(Y) \in \text{co}(x)$ is an element of the first prolongation $\text{co}(x)^{(1)}$. Property 5) means that for any $U \in T_x(M)$, the endomorphism $(\nabla_U K)(Y)$ belongs to $\text{co}(x)$ for any $Y \in T_x(M)$. Property 2) means that the linear mapping $\nabla_U K: Y \in T_x(M) \rightarrow (\nabla_U K)(Y) \in \text{co}(x)$ is an element of $\text{co}(x)^{(1)}$. Now assume that $L_X R = 0$. Property 3) means that the linear mapping $\nabla K: U \in T_x(M) \rightarrow \nabla_U K \in \text{co}(x)^{(1)}$ is actually an element of the second prolongation $\text{co}(x)^{(2)}$. It is known [1], however, that $\text{co}(x)^{(2)} = 0$ when $\dim M > 2$. Thus we arrive at the following consequence of the lemma above:

If X is conformal and $L_X R = 0$, then the corresponding tensor field K satisfies $\nabla K = 0$.

2. Proof of Theorem 2.

From the preceding interpretation of the Lemma, we see that $\nabla K = 0$. Let γ be the 1-form defined by $\gamma(Y) = \text{trace of } K(Y)$. We have then $\nabla \gamma = 0$. Since M is irreducible, we have $\gamma = 0$, that is, $\text{trace } K(Y) = 0$ for any Y . Since $K(Y)$ is in $\text{co}(x)$, it follows that $K(Y)$ is skew-symmetric. In equation (3), we have $K(Y)g = -\alpha(Y)g = 0$ for any Y , which means that $\alpha = 0$. Since $\alpha = df$ in the proof of equation (3), we see that f is a constant, that is X is homothetic.

3. Proof of Theorem 3.

In a two-dimensional irreducible Riemannian manifold, the Ricci tensor S has the form

$$S = \lambda g,$$

where λ is a function which is not identically zero. From this we have

$$\nabla_Y S = (Y\lambda)g$$

for any vector Y .

If the infinitesimal transformation X satisfies $L_X R = 0$ and $L_X(\nabla R) = 0$, then it satisfies $L_X S = 0$ and $L_X(\nabla S) = 0$. From $S = \lambda g$ and $L_X S = 0$, we obtain

$$(4) \quad (X\lambda)g + \lambda(L_X g) = 0.$$

From $\nabla_Y S = (Y\lambda)g$ and $L_X(\nabla S) = 0$, we obtain

$$0 = L_X \nabla_Y S - \nabla_{[X, Y]} S = (XY\lambda)g + (Y\lambda)L_X g - ([X, Y]\lambda)g \\ = (YX\lambda)g + (Y\lambda)L_X g,$$

that is,

$$(5) \quad (YX\lambda)g + (Y\lambda)(L_X g) = 0.$$

Taking ∇_Y of (4) and taking (5) into account, we get

$$\lambda \nabla_Y (L_X g) = 0.$$

Since our manifold is real analytic, the set of zero points of λ is nowhere dense. Hence we have

$$\nabla L_X g = 0.$$

Since the manifold is irreducible, we get

$$L_X g = cg,$$

where c is a constant.

4. Proof of Theorem 1.

Since M is an analytic Riemannian manifold, the holonomy algebra h_x (Lie algebra of the restricted holonomy group at x) is generated by all endomorphisms of the form

$$R(Y, Z), (\nabla_U R)(Y, Z), \dots, (\nabla^m R)(Y, Z; U_1; \dots; U_m), \dots,$$

where $Y, Z, U_1, \dots, U_m$ are arbitrary vectors at x

(cf. [3, p. 152]). From the assumption $L_X(\nabla^m R) = 0$, it follows that $A_X(\nabla^m R) = -\nabla_X(\nabla^m R)$. It is easy to see that

$$[A_X, (\nabla^m R)(Y, Z; U_1; \dots; U_m)] \in h_x$$

and hence

$$[A_X, h_x] \subset h_x.$$

The tensor $L_X g = A_X g$ at x is then invariant by h_x . In fact, for any $B \in h_x$, we have

$$B(A_X g) = A_X(Bg) + [A_X, B]g = 0,$$

since B and $[A_X, B]$ are skew-symmetric as elements in h_x . Since h_x is irreducible, $A_X g$ at x is a scalar multiple of the tensor g_x . This being the case at every point x of M , we have $A_X g = fg$, that is, $L_X g = fg$, where f is a function. This means that X is conformal.

Thus, if the dimension of $M > 2$, then Theorem 2 implies that X is homothetic.

If the dimension of M is 2, then Theorem 1 is as special case of Theorem 3.

5. Proof of Theorem 4.

We may assume that M is simply connected. Let $M = M_1 \times \dots \times M_k$ be the de Rham decomposition, where $M_1, \dots, M_k$ are irreducible, complete and analytic Riemannian manifolds. We shall show that the vector field X decomposes naturally, that is, there exists a strongly curvature-preserving infinitesimal transformation X_i on M_i , $1 \leq i \leq k$, such that

$$X_{(x_1, \dots, x_k)} = (X_1)_{x_1} + \dots + (X_k)_{x_k}$$

for any point $x = (x_1, \dots, x_k) \in M_1 \times \dots \times M_k$. Once this is shown, we see that X_i is Killing on M_i by Theorem 1 and hence X is Killing on M .

In order to prove a natural decomposition of X , we proceed as follows. Let $(T_1), \dots, (T_k)$ be the parallel distributions corresponding to the de Rham decomposition $M_1 \times \dots \times M_k$.

LEMMA 2. — $L_X(T_i) \subset (T_i)$ for each i , in the sense that if Y is a vector field belonging to the distribution (T_i) , then

$$L_X(Y) = [X, Y]$$

belongs to (T_i) .

Proof. — Since $L_X = \nabla_X + A_X$ and since $\nabla_X(T_i) \subset (T_i)$ because (T_i) is parallel, it is sufficient to show that $A_X(T_i) \subset (T_i)$. Let x be an arbitrary point. In the proof of Theorem 1, we have seen that $(A_X)_x$ lies in the normalizer of the holonomy algebra h_x . Thus the 1-parameter group of linear transformations $\exp tA_X$ of $T_x(M)$ lies in the normalizer of the holonomy group Ψ_x . It follows that, for each t , $(\exp tA_X) \cdot (T_i)_x$ coincides with some $(T_j)_x$ by virtue of the uniqueness of the de Rham decomposition

$$T_x(M) = (T_1)_x + \cdots + (T_k)_x$$

(cf. Theorem 5.4, (4), p. 185, and Lemma, p. 186, in [3]). By continuity, we see that $(\exp tA_X) \cdot (T_i)_x = (T_i)_x$ for every t . This implies $A_X(T_i)_x \subset (T_i)_x$.

LEMMA 3. — *Let Δ be a differentiable distribution on a differentiable manifold M . If a vector field X on M satisfies $L_X(\Delta) \subset \Delta$, then a local 1-parameter group φ_t of local transformations generated by X preserves the distribution.*

Proof. — Let $Y_1, \dots, Y_r$ be a local basis for Δ in a neighborhood of x . It is sufficient to show that $(\varphi_t \cdot (Y_i))_x$ belongs to Δ_x for every t . We recall the formula

$$\frac{d(\varphi_t \cdot Y_i)_x}{dt} = -(\varphi_t \cdot [X, Y_i])_x$$

(Corollary 1.10, p. 16, [3]).

Since $[X, Y_i]$ belongs to Δ , we have

$$[X, Y_i] = \sum_{j=1}^r f_{ij} Y_j,$$

where f_{ij} are certain functions. Therefore

$$\begin{aligned} \frac{d(\varphi_t Y_i)_x}{dt} &= -\left(\varphi_t \cdot \left(\sum_{j=1}^r f_{ij} Y_j\right)\right)_x \\ &= -\sum_{j=1}^r (f_{ij} \circ \varphi_t^{-1}) \cdot (\varphi_t Y_j)_x. \end{aligned}$$

If we denote $(\varphi_t Y_i)_x$ by $Y_i(t)$, then the functions $Y_i(t)$ with

values in $T_x(M)$ satisfy a system of differential equations

$$(6) \quad \frac{dY_i(t)}{dt} = \sum_{j=1}^r g_{ij}(t) Y_j(t),$$

where $g_{ij}(t) = -f_{ij}(\varphi_t^{-1}(x))$. The initial conditions are $Y_i(0) = (Y_i)_x$. It follows that $Y_i(t)$ has to be a linear combination

$$Y_i(t) = \sum_{j=1}^r F_{ij}(t) (Y_j)_x$$

of the vectors $(Y_1)_x, \dots, (Y_r)_x$, that is, $Y_i(t) \in \Delta_x$. $F(t) = [F_{ij}(t)]$ is the matrix function which is a unique solution of

$$\frac{dF}{dt} = G(t)F(t)$$

with initial condition $F(0) = [\delta_{ij}]$. The existence of such a solution is a special case of Lemma, p. 69, [3].) This proves Lemma 3.

Now we can prove that X decomposes naturally. Let φ_t be a local 1-parameter group of local transformations generated by X in a neighborhood of a point x . By Lemma 2,

$$L_X(T_i) \subset (T_i).$$

By Lemma 3, φ_t preserves each distribution (T_i) and hence its maximal integral manifold. It follows, by an argument similar to the proof of Theorem 3.5, p. 240, in [3], that there exists, for each t a local transformation $\varphi_t^{(i)}$ of M_i such that

$$\varphi_t(x_1, \dots, x_k) = (\varphi_t^{(1)}(x_1), \dots, \varphi_t^{(k)}(x_k)).$$

Each $\varphi_t^{(i)}$ is a local 1-parameter group and defines a vector field X_i on M_i . It is clear that $X = X_1 + \dots + X_k$. Since the curvature tensor R and its successive covariant differentials $\nabla^m R$ decompose naturally, it is obvious that each X_i is strongly curvature-preserving on M_i .

BIBLIOGRAPHY

- [1] V. W. GUILLEMIN and S. STERNBERG, An algebraic model of transitive differential geometry, *Bull. Amer. Math. Soc.*, 70 (1964), 16-47.
- [2] S. KOBAYASHI, A theorem on the affine transformation group of a Riemannian Manifold, *Nagoya Math. J.*, 9 (1955), 39-41.
- [3] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry*, Vol. I, Interscience Tracts No. 15, John Wiley et Sons, New York, 1963.

Manuscrit reçu en juin 1964

Katsumi NOMIZU,
Department of Mathematics,
Brown University,
Providence 12, R.I. (U.S.A.).

Kentaro YANO
Department of Mathematics,
Tokyo Institute of Technology,
Ohokayama, Meguro,
Tokyo (Japon).
