ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 6^e série, tome 14, n° 2 (2005), p. 185-200

<http://www.numdam.org/item?id=AFST_2005_6_14_2_185_0>

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Annales de la Faculté des Sciences de Toulouse Vol. XIV, n° 2, 2005
pp. 185–200

Geometry of superficial elements (*)

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ABSTRACT. — In this paper, we study superficial elements of an ideal with respect to a module from a geometrical point of view, using blowing-ups. The notion of weak transform is particularly relevant to this study. We use this viewpoint to get another natural characterization of these elements, which turns out to be equivalent to one given by D. Kirby in algebraic terms. We also indicate how the same result may be algebraically derived from a more recent theorem of Flenner and Vogel.

RÉSUMÉ. — Dans ce travail, nous étudions les éléments superficiels d'un idéal relativement à un module d'un point de vue géométrique, en terme d'éclatements. Ils sont caractérisés par une propriété de leur transformée faible sur cet éclatement. Nous donnons alors comme application directe une autre caractérisation de ces éléments superficiels qui est le pendant géométrique d'un théorème de D. Kirby. On indique aussi comment le même résultat peut aussi être obtenu algébriquement à partir d'un théorème de Flenner et Vogel.

1. Introduction

The notion of superficial element was introduced by P. Samuel in [Sa] for the study of multiplicities of primary ideals in local rings.

As is well-known and recalled in § 2, superficial elements are not only well-behaved with respect to multiplicities but also with respect to Hilbert polynomials. More basically, superficial elements can be defined for *any* ideal simply by a property of *the kernel* of the multiplication by the class of this element in the associated graded ring (resp. module) and an at first sight

(*) Reçu le 23 octobre 2003, accepté le 31 mars 2004
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surprising fact (yielding the afore-mentioned nice behaviour w.r.t. numerical invariants) is that this property forces the *cokernel* of the same multiplication map to be very nice.

This property is part of a characterization of superficial elements given by D. Kirby, recalled in thm. 2.3.

The purpose of this paper is to give a more picturesque approach to superficial elements using projective geometry and blowing-up schemes.

It is natural to translate the condition to be superficial as a property on the Proj on the associated graded ring. However, one gets further understanding if one embeds this Proj as the exceptional divisor of the blow-up scheme.

A first characterization of superficial elements on the blow-up is given at the end of § 3 and is expressed in term of *the weak transform* of these elements (cf. prop. 3.6).

In § 4, we compare this weak transform to the more usual *strict* (or *proper* transform) giving a criterion of equality.

Section 5 contains our main result: since superficiality is a condition of regularity on the exceptional divisor of the blow-up, i.e. on a Cartier subscheme, it may be used to define a two-terms regular sequence, which, once inverted, give two non-trivial regularity conditions, the second precisely connecting the weak and strict transforms (thm. 5.3).

In Section 6 we go back to Kirby's formulation showing that the two conditions found in our theorem 5.3 exactly give the two Kirby's conditions.

Eventually, we mention in the last section how Kirby's result may be also derived from a more general result by Flenner and Vogel which makes clear the connection between kernel and cokernel of certain maps between associated graded rings.

We end this introduction by setting-up hypotheses and notation valid throughout this paper:

Setup – Let R always stand for a noetherian ring, I an ideal of R and M a finitely generated R-module. It is standard to consider the following graded rings and modules (cf. e.g. [Ei] Chap. 5 for the ring and module structure):

$$G(R) := G_I(R) := \bigoplus_{n\in\mathbb{N}} I^n/I^{n+1}, \tag{1.1}$$

$$G(M) := G_I(M) := \bigoplus_{n\in\mathbb{N}} I^n M/I^{n+1}M, \tag{1.2}$$

associated to the filtration of R by the $(I^n)_{n\in\mathbb{N}}$.

We will also denote G_n for the component of degree n of a graded module G, and write $G(s)$ for the graded module with "shifted" gradation $[G(s)]_n := G_{n+s}$.

2. Superficial elements after Samuel and Kirby

Multiplication by $\bar{f}$ – For an element $f \in I^s \setminus I^{s+1}$ one considers its so-called initial form $\bar{f}$ which is its class in I^s/I^{s+1}.

Multiplication by $\bar{f}$ defines a map of degree zero between graded modules:

$$m_{\bar{f}} : G(M) \to G(M)(s). \tag{2.1}$$

Being interested in the properties of this map, one studies the associated exact sequence:

$$0 \to \ker m_{\bar{f}} \to G(m) \to G(M)(s) \to G(m)(s)/\bar{f}.G(m) \to 0. \tag{2.2}$$

Considering also the graded module $G(M/fM)$ defined as in (1.2) replacing M by M/fM, it is easy to check that, denoting by $\mathrm{in}_I(fM)$ the graded submodule of $G(M)$ whose n-th component is:

$$[\mathrm{in}_I(fM)]_n := (I^n M \cap fM)/I^{n+1}M, \tag{2.3}$$

one has the isomorphism:

$$G(M/fM) \cong G(M)/\mathrm{in}_I(f.M).$$

In particular, one has a natural surjection:

$$G(M)/\bar{f}G(M) \to G(M/fM), \tag{2.4}$$

corresponding to the obvious inclusion $\bar{f}.G(M) \subset \mathrm{in}_I(fM)$.

Introduction of superficial elements – As he was interested in properties of Hilbert functions (see below), Samuel only considered properties valid for components of large enough degree and introduced (cf. [Sa] p. 182, the following equivalent formulation is the one in [AC] VIII § 7 No. 5 p. 79):

DEFINITION 2.1. — *An element $f \in I^s \setminus I^{s+1}$ is said to be* superficial *(of degree s) for M with respect to I if, and only if, there is an integer n_0 such that the multiplication map $m_{\bar{f}}$ defined in (2.1) is injective from $G(M)_{n-s} \to G(M)_n$ for all $n \geqslant n_0$.*

A somewhat surprising property of superficial elements is the following:

PROPOSITION 2.2. — *Let R be a noetherian ring and M a finitely generated R-module. Let I be an ideal of R.*

If $f \in I^s \setminus I^{s+1}$ is superficial for M w.r.t. I then the epimorphism (2.4) is an isomorphism in degree $n \geqslant n_0$, and the following sequence:

$$0 \to G(M)(-s) \xrightarrow{m_{\bar{f}}} G(M) \to G(M/fM) \to 0, \tag{2.5}$$

is exact between homogeneous components of degree $n \geqslant n_0$.[1]

From the view-point of comparison of graded properties of M and M/fM, this miracle says that *injectivity of $m_{\bar{f}}$ is enough to get "the right cokernel".*

Application to Hilbert functions – If one is interested (as Samuel originally was) by properties of Hilbert functions, one takes R to be a noetherian local ring and I an ideal such that the length $l(M/IM)$ is finite, then one defines the Hilbert function of $G(M)$ as:

$$H_{I,M}(n) := l(I^n M/I^{n+1}M),$$

and the exact sequence (2.5) gives the nice relationship for all $n \geqslant n_0$:

$$H_{I,M/fM}(n) = H_{I,M}(n) - H_{I,M}(n-s),$$

which allows to compute this Hilbert function by induction on dimension.

Characterization of superficial elements by prop. 2.2

In fact, prop. 2.2 is part of a full characterization of superficial elements by D. Kirby (in [Ki] thm. 3) which we now state[2]. We use the following standard notation (cf. e.g. [Ei] § 3.6): denote by $(0 :_M I) := \{m \in M, mI = 0\}$ and $\Gamma_I(M) := (0 :_M I^\infty) := \cup_{n\in\mathbb{N}}(0 :_M I^n)$.

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(1) the n_0 is intended to be the same as in def. 2.1

(2) In loc. cit. it was formulated only for $M = R$ but the generalization to any R-module is straightforward.

THEOREM 2.3. — *Let R be a noetherian ring, I an ideal contained in the Jacobson radical of R (cf. [Ei] 4.8), and M a finitely generated R-module. An $f \in I^s \setminus I^{s+1}$ is superficial for M w.r.t. I if and only if both of the following conditions hold:*

(i) *multiplication by f from $M/\Gamma_I(M) \to M/\Gamma_I(M)$ is injective,*

(ii) *the map in (2.4): $G(M)/\bar{f}G(M) \to G(M/fM)$ is an isomorphism between components of degree $n \geqslant n_0$.*

3. Superficial elements, definition on the blowup

Before shifting to the language of projective geometry, we remind the reader about another piece of commutative algebra:

Associated primes – From the properties relating associated primes and non zero divisors (cf. [Ei] thm. 3.1) we immediately get (cf. [AC] VIII p. 79):

Remark 3.1. — With the same notation as in def. 2.1 and considering the set $\mathrm{Ass}(G(M))$ of homogeneous prime ideals in $G(R)$ associated to $G(M)$ (cf. [Ei] Chap. 3) the condition f *is superficial* is equivalent to the condition that $\bar{f}$ *does not belong to the* $p_i \in \mathrm{Ass}(G(M))$ *such that* p_i *does not contain* $G(R)_1 = I/I^2$.

Projective Formulation – Both def. 2.1 and rem. 3.1 are more naturally formulated using projective geometry.

Denote $\mathcal{E}_I := \mathrm{Proj}(G(R))$, then the $G(R)$-graded module $G(M)$ defines a (sheaf of) $\mathcal{O}_{\mathcal{E}_I}$-module $\mathcal{G}(\mathcal{M}) := \widetilde{G(M)}$ using the standard functor $\widetilde{\ }$ (cf. e.g. [Ii] § 3.4), which also gives a map:

$$\widetilde{m_{\bar{f}}} : \mathcal{G}(\mathcal{M}) \to \mathcal{G}(\mathcal{M})(s), \tag{3.1}$$

associated to the map $m_{\bar{f}}$ defined in (2.1).

A basic fact about this functor $\widetilde{\ }$ is (cf. e.g. [Ii] § 7.1 or [EGA] II § 2.7):

LEMMA 3.2. — *If R is a noetherian ring and M is a finitely generated module, then the injectivity of the map $m_{\bar{f}} : G(M)_n \to G(M)_{n+s}$ for n large enough is equivalent to the injectivity of $\widetilde{m_{\bar{f}}} : \mathcal{G}(\mathcal{M}) \to \mathcal{G}(\mathcal{M})(s)$.*

In the same spirit, the subset $\mathrm{Ass}(\mathcal{G}(\mathcal{M}))$ of points in $\mathcal{E}_I$ associated to the $\mathcal{O}_{\mathcal{E}_I}$-module $\mathcal{G}(\mathcal{M})$ (cf. [EGA] IV 3.1) precisely corresponds to the elements of $\mathrm{Ass}(G(M))$ fulfilling the condition in rem. 3.1 (called *projectively relevant*).

Finally, consider the subscheme of $\mathcal{E}_I$ defined by the homogeneous ideal $\bar{f}.G(R)$ that we denote by $(f)^{\#}_{\mathcal{E}_I}$ (this notation is to be explained below). With the foregoing remarks, we reformulate def. 2.1 and rem. 3.1 as follows:

LEMMA 3.3. — *Let R be a noetherian ring, I an ideal of R and M a finitely generated R-module. Let $f \in I^s \setminus I^{s+1}$. Then f is superficial for M w.r.t. I if, and only if, one of the following two equivalent conditions is satisfied:*

(i) $\widetilde{m_{\bar{f}}} : \mathcal{G}(\mathcal{M}) \to \mathcal{G}(\mathcal{M})(s)$ *is injective,*

(ii) $\mathrm{Supp}(f)^{\#}_{\mathcal{E}_I} \cap \mathrm{Ass}(\mathcal{G}(\mathcal{M})) = \emptyset$,

where Supp *denotes the underlying set of the scheme $(f)^{\#}_{\mathcal{E}_I}$.*

Embedding $\mathcal{E}_I$ on the blowup S_I – To get a better geometric understanding of both the map $\widetilde{m_{\bar{f}}}$ and the scheme $(f)^{\#}_{\mathcal{E}_I}$ in lem. 3.3, we may embed the scheme $\mathcal{E}_I$ in the blowup scheme $S_I := \mathrm{Proj}\, B(R)$, where:

$$B(R) := R[It] = \bigoplus_{n\in\mathbb{N}} I^n t^n,$$

is graded by the powers of t, with the convention $I^0 = R$.

For a subscheme of a projective scheme, it is possible to consider either its global "homogeneous" equations (i.e. the homogeneous ideal defining it) or its local ones in affine charts. As far as affine charts are concerned, fixing a basis $(h_0, \ldots, h_r)$ of the ideal I of R, one defines an affine open covering of S_I by the $U_i := D_+(h_i t) \cong \mathrm{Spec} R[I/h_i]$.

Denote also $b_I : S_I \to \mathrm{Spec} R$ the blowup morphism induced by the inclusion of R in $B(R)$.

Then $\mathcal{E}_I$ is the preimage $(b_I)^{-1}(V(I))$ and hence globally defined by the homogeneous ideal $I.B(R)$ in S_I and locally by the equation h_i in each U_i for $i = 0, \ldots, r$.

Total and weak transforms – Taking an $f \in I$, one defines its *total transform* $(b_I)^*(f)$ as the preimage $(b_I)^{-1}(V(f))$ hence globally defined by the homogeneous principal ideal $f.B(R)$ and locally also by f in each chart U_i.

Before going further, we put the emphasis on the following piece of terminology that will be of some importance later:

DEFINITION 3.4. — *i) If X is a scheme and Y is a subscheme of X, we say that Y is* locally principal *if there is a covering of X by affine open subsets $U_i = \mathrm{Spec}A_i$ such that $Y \cap U_i$ is defined by a principal ideal $(f_i) \subset A_i$.*

ii) We say that Y is a Cartier subscheme *if it fulfills condition i) and further the $f_i \in A_i$ are all non zero divisors.*

Making no special assumption on $f \in R$, we see that $(b_I)^*(f)$ is simply a locally principal subscheme of S_I whereas $\mathcal{E}_I$ is a Cartier subscheme. This is enough for the following:

DEFINITION 3.5. — *We define the* weak transform $(f)^\#$ *of an element $f \in I^s \setminus I^{s+1}$ on the blowup S_I of I in* $\mathrm{Spec}R$*, as the subscheme:*

$$(f)^\# := (b_I)^*(f) - s\,\mathcal{E}_I,$$

where the $-$ sign means that one takes locally the quotient of the equations in each affine chart.[3]

Remark that this weak transform $(f)^\#$ is also globally defined by the homogeneous ideal $ft^s B(R)$ of $B(R)$.

The scheme $(f)^\#_{\mathcal{E}_I}$ is the pull-back of the weak transform – Considering the homogeneous equation of $(f)^\#$ and the morphism (of degree zero) of graded rings $B(R) \to G(R) = B(R)/IB(R)$, that sends ft^s on $\bar{f} \in I^s/I^{s+1}$, one may define the intersection scheme $(f)^\# \cap \mathcal{E}_I$[4] as the subscheme of $\mathcal{E}_I$ defined by the homogeneous ideal $\bar{f}G(R)$.

Hence, $\mathcal{E}_I \cap (f)^\#$ is nothing but the scheme $(f)^\#_{\mathcal{E}_I}$ of lem. 3.3, so that (ii) of this lemma may be re-phrased as in the following:

PROPOSITION 3.6. — *Under the same notation as in def. 2.1, $f \in I^s \setminus I^{s+1}$ is superficial for M with respect to I if, and only if, on the blowup S_I of* $\mathrm{Spec}R$ *along I, denoting $\mathcal{G}(\mathcal{M}) = \widetilde{G(M)}$, and $(f)^\#$ the weak transform defined in 3.5, we have:*

$$\mathrm{Supp}((f)^\#) \cap \mathrm{Ass}(\mathcal{G}(\mathcal{M})) = \emptyset \tag{3.2}$$

where Supp *denotes the locus on S_I defined by the weak transform $(f)^\#$ and* Ass *the (finite) set of associated points of $\mathcal{G}(\mathcal{M})$ (included in $\mathcal{E}_I \subset S_I$).*

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(3) The terminology *weak transform* may be found in [Hk] p. 142. I chose the $^\#$ sign to mean that this weak transform can be bigger than the *strict* transform of (f).

(4) also called the pull-back on $\mathcal{E}_I$ of the locally principal subscheme $(f)^\#$

In the special case $M = R$, (3.2) reads:

$$\mathrm{Supp}((f)^{\#}) \cap \mathrm{Ass}(\mathcal{E}_I) = \varnothing \tag{3.3}$$

4. Weak transform vs. strict transform

In def. 3.5, we introduced the *weak transform* $(f)^{\#}$ of an element $f \in I^s \setminus I^{s+1}$ on the blowup S_I of the ideal I in $\mathrm{Spec}R$.

Let us recall the classical definition of strict (or proper) transform of a subscheme (cf. e.g. [E-H] p. 168). Recall first that the scheme-theoretic closure of a subscheme is by definition the smallest closed subscheme containing it.

DEFINITION 4.1. — *Let $Y \subset \mathrm{Spec}R$ a closed subscheme and $b_I : S_I \to S$ the blowup of I in $S = \mathrm{Spec}R$. The strict transform Y' of Y by b_I is by definition the scheme-theoretic closure of the preimage of $Y \setminus V(I)$ by b_I, which we denote by:*

$$Y' := \overline{(b_I)^{-1}(Y \setminus V(I))}^{sch}.$$

Because of the uniqueness of the structure of subscheme on an open subset, this subscheme is also the scheme-theoretic closure of $(b_I)^{-1}(Y) \setminus (b_I)^{-1}(V(I))$ $= (b_I)^{-1}(Y) \setminus \mathcal{E}_I$. For the same reason the scheme-structure of $V(I)$ is not to be taken into account in this definition, but simply its support.

In the particular case of $Y = (f) := \mathrm{Spec}R/(f)$, its strict transform is:

$$(f)' := \overline{(f)^* \setminus \mathcal{E}_I}^{sch},$$

but since, with $(f)^{\#}$ the weak transform of def. 3.5, we trivially have $(f)^* \setminus \mathcal{E}_I = (f)^{\#} \setminus \mathcal{E}_I$, one may just as well say:

$$(f)' = \overline{(f)^{\#} \setminus \mathcal{E}_I}^{sch}.$$

This gives in particular the inclusion of schemes $(f)' \subset (f)^{\#}$ (whence the $^{\#}$ notation cf. footnote 3), and the condition of equality is given by the following elementary lemma:

LEMMA 4.2. — *With the same notation as in § 3 on the blowup, the weak transform $(f)^{\#}$ is equal to the strict transform $(f)'$ if, and only if, the support of the exceptional divisor $\mathcal{E}_I$ does not contain any associated point to the scheme $(f)^{\#}$, what we denote by:*

$$\mathrm{Supp}(\mathcal{E}_I) \cap \mathrm{Ass}(f)^{\#} = \varnothing \tag{4.1}$$

Proof. — This is a simple application of the theory of primary decomposition: the inclusion of closed subschemes $(f)' \subset (f)^{\#}$ is proper if, and only if, one has a decomposition:

$$(f)^{\#} = (f)' \cup X_I,$$

into closed subschemes, where X_I contains a point associated to $(f)^{\#}$ (non trivial decomposition), which must be in the support of $\mathcal{E}_I$ since $(f)'$ and $(f)^{\#}$ coincide outside $\mathcal{E}_I$. □

Because of the similarity of conditions in prop. 3.6 (3.3) and in lem. 4.2, up to the permutation of the roles of Supp and Ass, we will investigate the precise connection between these two conditions in the next section (see the conclusive remark 5.4).

Case of modules – Let M be a (finitely generated) R-module and denote by $\mathcal{B}(\mathcal{M})$ the $\mathcal{O}_{S_I}$-module defined on the blowup S_I of I in Spec$R$ by the graded module:

$$B(M) = \bigoplus_{n \in \mathbb{N}} I^n M t^n.$$

Then, for $f \in I$ one may define the "restrictions" of $\mathcal{B}(\mathcal{M})$ to the strict transforms $(f)'$ and weak transform $(f)^{\#}$ by:

$$\mathcal{B}(\mathcal{M})_{(f)'} := \mathcal{B}(\mathcal{M}) \otimes_{\mathcal{O}_{S_I}} \mathcal{O}_{(f)'}, \tag{4.2}$$

(resp. $\mathcal{B}(\mathcal{M})_{(f)^{\#}}$ by tensor product with $\mathcal{O}_{(f)^{\#}}$).

In this context, the more general form of lemma 4.2 (with the same proof, i.e. primary decomposition for modules) is:

LEMMA 4.3. — *The two $\mathcal{O}_{S_I}$-modules $\mathcal{B}(\mathcal{M})_{(f)'}$ and $\mathcal{B}(\mathcal{M})_{(f)^{\#}}$ defined in (4.2) are isomorphic if, and only if,*

$$\mathrm{Supp}(\mathcal{E}_I) \cap \mathrm{Ass}(\mathcal{B}(\mathcal{M})_{(f)^{\#}}) = \emptyset.$$

5. Superficiality and regularity

We first recall standard constructions in projective geometry (cf. [EGA] II § 2.6 or [Ii] § 7.1):

Morphism α – Let $A := \oplus_{n \in \mathbb{N}} A_n$ be a graded ring and $M = \oplus_{n \in \mathbb{Z}} M_n$ a graded A-module. Then one defines for all $n \in \mathbb{Z}$ a morphism (of A_0-modules):

$$\alpha_n : M_n \to \Gamma(\mathrm{Proj}\, A, \tilde{M}(n)), \tag{5.1}$$

by, locally in each chart $D_+(f_i)$ of $\operatorname{Proj} A$, sending $m \in M_n$ simply to $m/1 \in \Gamma(D_+(f_i), \tilde{M}(n))$.

Multiplication by sections of $\mathcal{O}_X(s)$ – Take again A to be a graded ring, $X = \operatorname{Proj} A$ and $\sigma \in \Gamma(X, \mathcal{O}_X(s))$. Such a global section defines for any $\mathcal{O}_X$-module $\mathcal{M}$, a morphism of multiplication by σ:

$$m_\sigma : \mathcal{M} \to \mathcal{M}(s). \tag{5.2}$$

Explicity, assume for simplicity that $A = A_0[A_1]$ so that charts $D_+(f_i)$ with $f_i \in A_1$ cover X. Then, in each $U_i := D_+(f_i)$, $\sigma = a_i f_i^s$ with $a_i \in \mathcal{O}_X(U_i)$.

Remark 5.1. — In particular, since multiplication by f_i^s is certainly injective from $\mathcal{M}(U_i) \to \mathcal{M}(s)(U_i)$ the injectivity of m_σ is equivalent to the elements $a_i \in \mathcal{O}_X(U_i)$ being $\mathcal{M}(U_i)$ regular, with the usual terminology of a regular element for a module (cf. e.g. [Ei] Chap. 17).

If this is the case we will say that the section $\sigma \in \Gamma(X, \mathcal{O}_X(s))$ is $\mathcal{M}$-regular.

Application to $X = \operatorname{Proj}(G(R))$ – Now we return to the setting of lemma 3.3: for $G(R) = \oplus_{n\in\mathbb{N}} I^n/I^{n+1}$, $\mathcal{E}_I := \operatorname{Proj} G(R)$ and $\bar{f} \in G(R)_s$ one gets from (5.1) a global section $\alpha_s(\bar{f}) \in \Gamma(\mathcal{O}_{\mathcal{E}_I}, \mathcal{O}_{\mathcal{E}_I}(s))$ and it is direct from the definitions that the morphism $\widetilde{m_f} : \mathcal{G}(\mathcal{M}) \to \mathcal{G}(\mathcal{M})(s)$ considered in the cited lemma, coincides with the multiplication by $\alpha_s(\bar{f})$ in the sense of (5.2).

With the terminology of rem. 5.1, we get the following avatar of lemma 3.3:

LEMMA 5.2. — *In the same setting as in lemma 3.3 and using the foregoing definitions: f is superficial for M w.r.t. I if, and only if, the section $\alpha_s(\bar{f})$ is $\mathcal{G}(\mathcal{M})$-regular.*

Application to $X = S_I = \operatorname{Proj} B(R)$ – As we did in § 3, we now shift from $\mathcal{E}_I$ to the larger space $S_I = \operatorname{Proj}(\oplus_{n\in\mathbb{N}} I^n t^n)$ of blowup of I in $\operatorname{Spec} R$.

The morphism α defined in (5.1) above also applies to get a global section $\alpha(ft^s) \in \Gamma(S_I, \mathcal{O}_{S_I}(s))$. Recall that we denote $\mathcal{B}(\mathcal{M})$ for the $\mathcal{O}_{S_I}$-module associated to $B(M) := \oplus_{n\in\mathbb{N}} I^n M t^n$.

Using all this, our main result boils down to the following permutation of elements in a regular sequence:

THEOREM 5.3. — *Let R be a noetherian ring, I an ideal of R with basis $(h_0, \dots, h_r)$. Denote $U_i = D_+(h_i t)$ the corresponding affine open subsets in the blowup space $S_I = \mathrm{Proj}(B(R))$. Let M be a finitely generated R-module and $\mathcal{B}(\mathcal{M})$ the corresponding $\mathcal{O}_{S_I}$-module as defined above.*

Then for an element $f \in I^s \setminus I^{s+1}$, the following four conditions are equivalent:

a) *f is superficial for M w.r.t. I (cf. lem. 5.2, or lem. 3.3),*

b) *$\forall\, i = 0, \dots, r, \forall x \in S_I, (h_i, f/h_i^s)$ is a $\mathcal{B}(\mathcal{M})_x$-regular sequence (where "$_x$" denotes the localization at x),*

c) *$\forall\, i = 0, \dots, r, \forall x \in S_I, (f/h_i^s, h_i)$ is a $\mathcal{B}(\mathcal{M})_x$-regular sequence,*

d) *the following two conditions are satisfied:*

(i) *$\alpha_s(ft^s)$ is $\mathcal{B}(\mathcal{M})$-regular (definition in rem. 5.1),*

(ii) *$\mathrm{Supp}(\mathcal{E}_I) \cap \mathrm{Ass}(\mathcal{B}(\mathcal{M})_{(f)^\#}) = \emptyset$, where $(f)^\#$ is the weak transform introduced in 4.1 and we consider the "restriction" modules in the sense of (4.2).*

Proof. — Let us fix an $i \in \{0, \dots, r\}$, and work in the corresponding open subset $U_i \cong \mathrm{Spec} R[I/h_i]$ on the blowup.

• (a)⇔(b): since $\mathcal{B}(\mathcal{M})(U_i) = M[I/h_i]$, h_i is regular for any R-module M, and hence (b) reduces to the condition: f/h_i^s is regular for $\mathcal{B}(\mathcal{M})(U_i)/h_i\mathcal{B}(\mathcal{M})(U_i) = \mathcal{G}(\mathcal{M})(U_i)$ at all $x \in U_i$. But, considering the corresponding classes $\bar{f}$ and $\bar{h_i}$ in $G(R)_s$ and $G(R)_1$ respectively, this is the same as $\bar{f}/\bar{h_i}^s$ being regular for $\mathcal{G}(\mathcal{M})(U_i)$.

We now recognize the local equation of $\alpha_s(\bar{f})$ and hence the condition $\alpha_s(f)$ is $\mathcal{G}(\mathcal{M})$-regular of lemma 5.2, whence (b)⇔(a).

• (b)⇔(c): standard permutation property for regular sequences over a local ring, cf. e.g. [Ei] cor. 17.2.

• (c)⇔(d): in the sequence $(f/h_i^s, h_i)$ in U_i, the first element defines the equation of $\alpha_s(ft^s)$ whence (i) in d), and the condition for h_i to be regular with respect to the quotient $\mathcal{B}(\mathcal{M})(U_i)$ by f/h_i^s translates as (ii) in d), since $(f)^\#$ is the subscheme defined by $\alpha_s(ft^s)$ (using always the same relation between associated points and zero divisors). □

In the case $M = R$ we get the better sounding formulation:

Remark 5.4. — *In the special case $M = R$ in the foregoing theorem, we get that*

(a) f is superficial w.r.t. I

if, and only if,

(d) (i) the weak transform $(f)^{\#}$ is a Cartier subscheme of S_I (cf. def. 3.4), and

(ii) $\mathrm{Supp}(\mathcal{E}_I) \cap \mathrm{Ass}((f)^{\#}) = \emptyset$, which is exactly the condition (4.1) in lem. 4.2 so that the weak transform equals the strict transform.

6. Back to Kirby's theorem

We claim now that thm. 2.3 is easily recovered from our thm. 5.3 condition (d). Precisely conditions (i) (resp. (ii)) correspond in both results, as we now check:

LEMMA 6.1. — *The following two conditions are equivalent:*

(*) Condition (i) in thm. 2.3, i.e. multiplication by f: $M/\Gamma_I(M) \to M/\Gamma_I(M)$ is injective,

(**) Condition (i) in thm. 5.3 (d), i.e. $\alpha_s(ft^s)$ is $\mathcal{B}(\mathcal{M})$-regular on the blowup scheme S_I.

Proof. — Condition (**) is equivalent to the multiplication by ft^s: $I^nMt^n \to I^{n+s}Mt^{n+s}$ being injective for n large, and since t is certainly a non-zero divisor, we may just as well consider multiplication by f: $I^nM \to I^nM$.

Hence to prove the lemma, one has to check, for n large, the following:

$$\mathrm{Ass}_R(I^nM) = \mathrm{Ass}(M/\Gamma_I(M)). \tag{6.1}$$

To prove (6.1), one may reduce to the case $\Gamma_I(M) = 0$.

Indeed, by noetherian condition $\Gamma_I(M) = (0 :_M I^{n_1})$ for a certain n_1 and hence $I^{n_1}.M = I^{n_1}.(M/\Gamma_I(M))$. So replacing M by $M/\Gamma_I(M)$ and taking $n \geqslant n_1$, we may assume $\Gamma_I(M) = 0$.

We now prove $\mathrm{Ass}(I^nM) = \mathrm{Ass}(M)$ for all n in the case $\Gamma_I(M) = 0$.

From the inclusion $I^n M \subset M$, one always has $\mathrm{Ass}(I^n M) \subset \mathrm{Ass}(M)$. Conversely, if $p \in \mathrm{Ass}(M)$, one may localize at p and denote M and R for M_p and R_p. Then $p = \mathrm{Ann}(m)$ is equivalent to $p.m = 0$, since p the maximal ideal.

Then, as $\Gamma_I(M) = 0$ we have for any fixed n, $I^n.m \neq 0$ i.e. there is an $i \in I^n$ such that $i.x \neq 0$. Hence, $i.p.x = 0$ gives $p = \mathrm{Ann}(ix)$ i.e. $p \in \mathrm{Ass}(I^n.M)$. □

LEMMA 6.2. — *The following two conditions are equivalent:*

(∗) *Condition* (ii) *in Kirby's thm. 2.3 i.e. the epimorphism:*

$G(M)/\bar{f}.G(M) \to G(M/f.M)$ *is an isomorphism between components of large degree,*

(∗∗) *Condition* (ii) *in our thm. 5.3 (d):* $\mathrm{Ass}(\mathcal{B}(\mathcal{M})_{(f)^\#}) \cap \mathrm{Supp}(\mathcal{E}_I) = \emptyset$.

Proof. — From lemma 4.3, we know that condition (∗∗) is exactly the condition for the isomorphism of the two $\mathcal{O}_{S_I}$-modules defined by the restriction $\mathcal{B}(\mathcal{M})_{(f)^\#}$ and $\mathcal{B}(\mathcal{M})_{(f)'}$ of $\mathcal{B}(\mathcal{M})$ to the weak and strict transform respectively.

But since these two modules are clearly isomorphic at each point outside $\mathcal{E}_I$ it is equivalent to check that their pull-back on $\mathcal{E}_I$ are actually isomorphic.

These pull-back (taking $\otimes\mathcal{E}_I$) are exactly $\widetilde{G(M)/\bar{f}.G(M)}$ for the weak transform and $G(\widetilde{M/fM})$ for the strict transform.

This is equivalent to condition (∗) from the standard lemma 3.2. □

7. More reasons for the miracle: a theorem of Flenner-Vogel

The miracle referred to in the title is the one mentioned after prop. 2.2.

As H. Flenner explained it to us, one may also get Kirby's characterization (thm. 2.3) from the following more precise result in [F-V]; recall first that for an R-module M one defines the *cycle* $Z(M)$ associated to M by:

$$Z(M) = \sum_{p \in \mathrm{Min}(M)} l(M_p)[p],$$

where $\mathrm{Min}(M)$ is the set of minimal associated primes to M.

THEOREM 7.1. — *Let R be a noetherian ring, I an ideal of R and*

$$0 \to M_0 \to M_1 \to M_2 \to 0,$$

an exact sequence of finitely generated R-modules.

Considering the associated graded modules, one gets a complex $G_I(M_0) \to G_I(M_1) \to G_I(M_2)$ in which the last map is still onto and one has the equality of cycles:

$$Z\big(\ker(G_I(M_0) \to G_I(M_1))\big) = Z\big(\ker(G_I(M_1)/G_I(M_0) \to G_I(M_2))\big),$$

where $G_I(M_1)/G_I(M_0)$ stands for $G_I(M_1)/\mathrm{Im}(G_I(M_0) \to G_I(M_1))$.

Remark 7.2. — For a map R-modules $\varphi : M \to N$, the associated map of graded $G_I(R)$-modules is understood to be the map from $I^n M/I^{n+1}M \to I^{n+\sigma}N/I^{n+\sigma+1}N$, where σ is defined by the condition that $\mathrm{Im}\varphi$ is in $I^\sigma N$ and not in $I^{\sigma+1}N$.

For the existence of such a σ, and hence in the theorem above, one has to assume that Krull's intersection theorem (cf. e.g. [Ei] 5.4) applies. In the following, we will assume that I is in the Jacobson radical of R (which is also the hypothesis of thm. 2.3).

For example, the inclusion map $f.M \to M$ for an f in I^s and not in I^{s+1}, actually gives a map $G_I(f.M)(-s) \to G_I(M)$, whose image is exactly $\bar{f}.G_I(M)(-s)$, where $\bar{f}$ is the class of f in I^s/I^{s+1}. However, following [F-V], we will omit the shift sign $(-s)$ in what follows.

Considering now an $f \in I^s \setminus I^{s+1}$, multiplication by f in M gives an exact sequence:

$$0 \to \mathrm{Ann}_M(f) \to M \to M \to M/fM \to 0, \tag{7.1}$$

that one has to split, in order to apply thm. 7.1, into the two sequences:

$$0 \to \mathrm{Ann}_M(f) \to M \to f.M \to 0, \tag{7.2}$$

$$0 \to f.M \to M \to M/f.M \to 0. \tag{7.3}$$

Taking G_I in (7.1) gives a complex, hence multiplication by $\bar{f} \in I^s/I^{s+1}$ from $G_I(M)$ to $G_I(M)$ may be decomposed as the following chain:

$$G_I(M) \xrightarrow{1} G_I(M)/G_I(\mathrm{Ann}_M f) \xrightarrow{2} G_I(f.M) \xrightarrow{3} G_I(M). \tag{7.4}$$

For simplicity, we will denote $\mathrm{Ann}_M(f)$ with $\mathrm{Ann}f$.

By def. 2.1, f is superficial if the kernel of the total map in (7.4) vanishes in high degree, which is equivalent to the same property for the kernels of the three maps $1, 2, 3$ above.

For the map 1, this means that $\mathrm{Im}\big(G_I(\mathrm{Ann}f) \to G_I(M)\big)$ vanishes in high degree (the convention for notation is as on the last line of thm. 7.1).

For the map 2, this means that $\ker\big(G_I(\mathrm{Ann}f) \to G_I(M)\big)$ vanishes in high degree, applying thm. 7.1 to (7.2).

For the map 3, this means that $\ker\big(G_I(M)/G_I(f.M) \to G_I(M/f.M)\big)$ vanishes in high degree, applying thm. 7.1 to (7.3).

The condition for the third map is explicitly condition (ii) in Kirby's thm, because $\bar{f}.G_I(M)$ is the image of the map $G_I(f.M) \to G_I(M)$ (cf. rem. 7.2).

The condition for the first two maps is equivalent to $G_I(\mathrm{Ann}f)$ vanishing in high degree. As put in the following remark, this is equivalent to condition (i) in Kirby's thm. 2.3. This ends the proof of Kirby's theorem.

Remark 7.3. — Kirby's condition (i) that multiplication by f is injective inside $M/\Gamma_I(M)$ is equivalent to $G_I(\mathrm{Ann}f)$ vanishing in high degree.

Indeed, recall that $\Gamma_I(M) = (0 :_M I^k)$ for any large enough k. Hence, the condition of injectivity in $M/\Gamma_I(M)$ means that $\mathrm{Ann}f \subset \Gamma_I(M)$ i.e. that $I^k.\mathrm{Ann}f = 0$ for k large enough.

On the other hand, $[G_I(\mathrm{Ann}f)]_k = 0$ also means $I^k.\mathrm{Ann}f = 0$ by Nakayama's lemma (cf. [Ei] Cor. 4.8), whence the remark.

Acknowledgment. — The paper is a souped-up version of the first chapter of the author's Ph.D. thesis under Lê Dũng Tráng (cf. [Bo]). The motivation for such a geometric study was mainly to complete the point of view in [B-L], where in spite of the title, we focused not on superficial elements but on so-called v-superficial elements of ideals (developed in the following chapters of the thesis). I would like to thank Mark Spivakovsky for his remarks as a referee for my dissertation and H. Flenner for his suggestions relative to the last parts of this work. I also thank the referee for his remarks.

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