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## NEW OPERATORS ON JET SPACES

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**Résumé :** Nous introduisons des opérations et des techniques fonctionnelles nouvelles dans l'étude des espaces de jets d'un espace fibré. Ces méthodes permettent d'obtenir en particulier une caractérisation canonique des transformations infinitésimales d'ordre quelconque et des prolongements de champs des vecteurs. Les expressions locales explicites des formules intrinsèques sont aussi présentées. Ces résultats peuvent être utilisés dans le cadre du calcul des variations d'ordre supérieur.

**Summary :** New operators and functorial techniques are introduced on the  $k$ -jet spaces of a fibered space, which lead to interesting characterizations of the infinitesimal contact transformations at any order. Canonical prolongations at all orders of vector fields are obtained as an application useful in the calculus of variations. Global results as well as explicit local formulas are given.

### INTRODUCTION

There is growing interest in jet spaces with respect to Mathematical-Physics, for they allow a structural and unifying analysis of differential equations. Jet spaces may be considered to be a natural framework for higher order field theories, in particular for the calculus of variations. We might also expect that they will play a role in the quantization via deformation. So we are concerned with all the prolongation techniques which naturally involve higher order jet spaces.

In this paper we introduce some new operators and techniques on jet spaces, and obtain results which are useful, for instance, in the calculus of variations [8].

We will start with a systematic recall of functorial techniques -widely used in the paper-, giving intrinsic as well explicit local formulas. In particular we will make a broad analysis of affine structures involved in jet spaces (for the classical results see, for instance, [3]).

Then the contact form  $\vartheta$  at any order will be investigated with respect to different possible global definitions (see also [12] and [4]) and to the functorial invariance properties. We will consider the kernel  $\Delta$  of  $\vartheta$  at any order, which is a non involutive distribution. For some purposes it is more important than  $\vartheta$  itself.

Then we will introduce a new operator  $r$  at any order, which will allow the exchange between jet and tangent spaces and maps and which will be shown to have analogies and relations with  $\vartheta$ . We will investigate its functorial invariance properties and also the close connection with different affine structures.

At this point we will be in the position to introduce the set of infinitesimal contact transformations of any order, which are composed of the vector fields which preserve  $\Delta$ . In this way we may avoid non-essential connections, which may be involved with the vector valued form  $\vartheta$  (see, for instance, [2]). Moreover we can give several new characterizations of this set by means of the operator  $r$  and also explicit local formulas thus obtained.

For useful application -for instance in the calculus of variations [8]- we will show a canonical prolongation at all orders of any vector field, by using the jet functor and  $r$ . This new approach may also be applied to non-projectable vector fields and moreover will naturally produce explicit local formulas.

## 1. - THE JET FUNCTOR

We start with the basic notations on jet spaces and we give explicit formulas with respect to the functorial techniques which will be widely used in the following. All spaces and maps will be  $C^\infty$ .

1.1 - Henceforth  $p : E \rightarrow M$  is a fibered space (i.e. a surjective submersion), with

$$m \equiv \dim M \quad \text{and} \quad m + l \equiv \dim E.$$

The standard chart of  $E$  is denoted by  $(x^\lambda, y^i)$ , with  $1 \leq \lambda \leq m$ ,  $1 \leq i \leq l$ .

In some cases we will deal with a further fibered space  $q : F \rightarrow N$ , with  $n \equiv \dim N$ , whose standard chart is denoted by  $(z^\alpha, w^j)$ .

1.2 -  $J^k E$  is the  $k$ -jet space and  $J^0 E \equiv E$ , with  $0 \leq k$ . It can be viewed naturally as a fibered space and a bundle

$$p^k : J^k E \rightarrow M \quad \text{and} \quad p_h^k : J^k E \rightarrow J^h E,$$

respectively, with  $0 \leq h < k$ . Moreover

$$p_h^k \circ p_l^h = p_l^k \quad \text{and} \quad p_h^k \circ p^h = p^k.$$

The standard chart of  $J^k E$  is denoted by  $(x^\lambda, y_\Lambda^i)$ , with  $0 \leq |\Lambda| \leq k$ , where  $\Lambda \equiv (\Lambda_1, \dots, \Lambda_m)$  is a multi-index and  $|\Lambda| \equiv \Lambda_1 + \dots + \Lambda_m$ . We put

$$0 \equiv (0, \dots, 0), \quad \Lambda + \lambda \equiv (\Lambda_1, \dots, \Lambda_\lambda + 1, \dots, \Lambda_m)$$

$$y^i \equiv y_0^i, \quad y_\lambda^i \equiv y_{0+\lambda}^i, \quad y_{\lambda\mu}^i \equiv y_{0+\lambda+\mu}^i, \dots$$

If  $s : M \rightarrow E$  is a (local) section, then  $j^k s : M \rightarrow J^k E$  is its  $k$ -lift, whose expression is  $(x^\lambda, y_\Lambda^i) \circ j^k s = (x^\lambda, \partial_\Lambda s^i)$ , where

$$s^i \equiv y^i \circ s \quad \text{and} \quad \partial_\Lambda s^i \equiv \partial_1^{\Lambda_1} \dots \partial_m^{\Lambda_m} s^i.$$

Moreover we have  $p_h^k \circ j^k s = j^h s$ .

**1.3 - PROPOSITION.** *Let  $H : E \rightarrow F$  be a morphism over the diffeomorphism  $h : M \rightarrow N$ . Then there is a unique morphism  $J^k H : J^k E \rightarrow J^k F$  over  $h : M \rightarrow N$ , such that, for each (local) section  $s : M \rightarrow E$ , the following diagram commutes*

$$\begin{array}{ccc}
 J^k E & \xrightarrow{J^k H} & J^k F \\
 \uparrow j^k s & & \uparrow j^k(H*s) \\
 M & \xrightarrow{h} & N
 \end{array}$$

where  $H*s$  is the section  $H*s \equiv H \circ s \circ h^{-1} : N \rightarrow F$ .

Its expression is given by

$$z^\alpha \circ J^k H = h^\alpha,$$

$$w_A^j \circ J^k H = \partial_\Lambda \partial_I H^j y_{M(1,1)}^1 \dots y_{M(1,I_1)}^1 \dots y_{M(l,1)}^l \dots y_{M(l,I_l)}^l$$

$$\partial_{B(1,1)} g^1 \circ h \dots \partial_{B(1,b(1))} g^1 \circ h \dots \partial_{B(m,1)} g^m \circ h \dots \partial_{B(m,b(m))} g^m \circ h,$$

where  $h^\alpha \equiv z^\alpha \circ h$ ,  $H^j \equiv w^j \circ H$ ,  $g^\lambda \equiv x^\lambda \circ h^{-1}$ ,

$\Lambda, M^{(1,1)}, \dots, M^{(l,I_l)}$  are multi-indices in  $(1, \dots, m)$   
 $B^{(1,1)}, \dots, B^{(m,b(m))}$  " " "  $(1, \dots, n)$   
 $I$  is a multi-index in  $(1, \dots, l)$

and where the summation is extended to all the multi-indices whith the conditions

$$b(\mu) = \Lambda_\mu + M_\mu^{(1,1)} + \dots + M_\mu^{(l,I_l)}, \quad 1 \leq \mu \leq m,$$

$$A = B^{(1,1)} + \dots + B^{(m,b(m))}$$

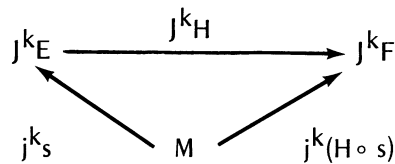
$$I_h > 0 \Rightarrow |M^{(h,1)}|, \dots, |M^{(h,I_h)}| > 0, \quad 1 \leq h \leq l,$$

$$b(\mu) > 0 \Rightarrow |B^{(\mu,1)}|, \dots, |B^{(\mu,b(\mu))}| > 0.$$

*Proof.* It will follow by induction on  $|A|$  such that  $1 \leq |A| \leq k$  (after a long calculation!).

If  $H$  is over  $M$ , then we have a simpler expression.

**COROLLARY.** Let  $H : E \rightarrow F$  be a morphism over  $M \equiv N$ . Then there is a unique morphism  $J^k H : J^k E \rightarrow J^k F$  over  $M$ , such that, for each (local) section  $s : M \rightarrow E$ , the following diagram commutes



Its expression is

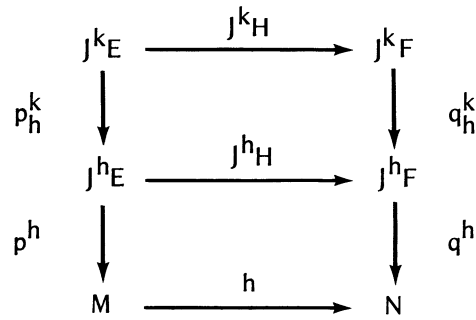
$$(x^\lambda, w^j_A) \circ J^k H = (x^\lambda, \partial_\Lambda \partial_I H^j y^1_{M^{(1,1)}} \dots y^l_{M^{(l,I_l)}}),$$

where the summation is extended to all the multi-indices with the conditions

$$\Lambda + M^{(1,1)} + \dots + M^{(l,I_l)} = A,$$

$$I_h > 0 \Rightarrow |M^{(h,1)}|, \dots, |M^{(h,I_h)}| > 0.$$

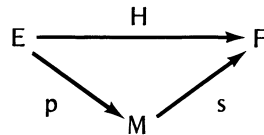
1.4 - Coming back to the general case, we can see that the following diagram commutes, for  $0 \leq h \leq k$ ,



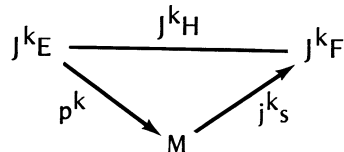
1.5 - Moreover, from the uniqueness of  $J^k H$ , it follows that  $J^k$  is a co-variant functor in the category of fibered spaces, namely

$$J^k H \circ J^k H' = J^k(H \circ H') \quad \text{and} \quad J^k \text{id}_E = \text{id}_{J^k E}.$$

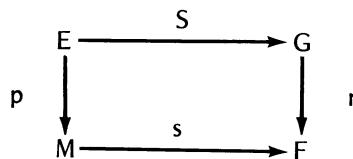
1.6 - Sometimes, the following remark, which is a simple consequence of the local expression, will be useful. Let  $N = M$  and let  $s : M \rightarrow F$  be a section such that the following diagram commutes



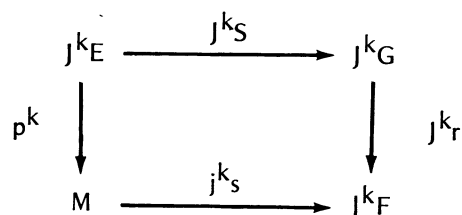
Then the following diagram commutes



A further consequence is the following. Let  $r : G \rightarrow F$  be a further fibered space, so that  $q \circ r : G \rightarrow M$  is also a fibered space. Let  $S : E \rightarrow G$  be a morphism over the section  $s : M \rightarrow F$ , so that the following diagram commutes



Then the following diagram commutes (putting  $H \equiv r \circ S$  and taking all the  $J^k$  over  $M$ )



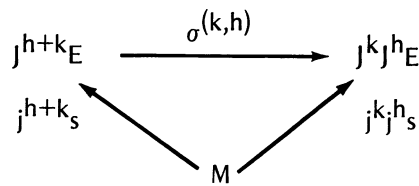
1.7 - If we consider the fibered space  $p^h : J^h E \rightarrow M$ , with  $0 \leq h$ , then we have the two bundle structures, for  $0 \leq k$ ,

$$(p^h)_0^k : J^k J^h E \rightarrow J^h E \quad \text{and} \quad J^k p_0^h : J^k J^h E \rightarrow J^k E,$$

whose expressions are

$$(x^\lambda, y_\Lambda^i) \circ (p^h)_0^k = (x^\lambda, y_{\Lambda 0}^i) \quad \text{and} \quad (x^\lambda, y_M^i) \circ J^k p_0^h = (x^\lambda, y_{0\Lambda}^i).$$

1.8 - There is a unique morphism  $\sigma^{(k,h)} : J^{h+k} E \rightarrow J^k J^h E$  over  $J^h E$ , such that the following diagram commutes, for each (local) section  $s : M \rightarrow E$ ,



Moreover  $\sigma^{(k,h)}$  is an embedding. By identification of  $J^{h+k} E$  with its image, we have

$$J^{h+k} E = \overset{\text{arc}}{\underbrace{J^{k'} J^{h'} E}_{h'+k'=h+k}}$$

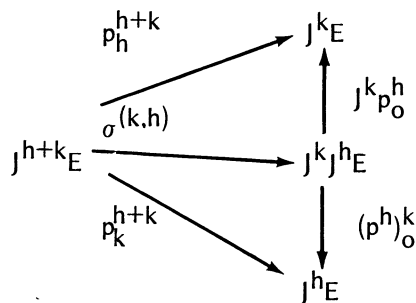
The expression of  $\sigma^{(k,h)}$  is

$$(x^\lambda, y_{\Lambda M}^i) \circ \sigma^{(k,h)} = (x^\lambda, y_{\Lambda+M}^i).$$

The subbundle  $J^{h+k} E \hookrightarrow J^k J^h E$  is locally characterized by

$$y_{\Lambda M}^i = y_{\Lambda' M'}^i \iff \Lambda + M = \Lambda' + M'.$$

Moreover the following diagram commutes



1.9 - We will often use the canonical isomorphism

$$J^k(E \times_M F) \rightarrow J^k E \times_M J^k F.$$

2. - THE TANGENT FUNCTOR AND JET SPACES

T denotes the covariant tangent functor in the category of manifolds and V denotes the covariant vertical functor in the category of fibered spaces.

2.1 - We will be concerned with the tangent bundles of M and of  $J^k E$

$$\pi : TM \rightarrow M \quad \text{and} \quad \pi_k : TJ^k E \rightarrow J^k E$$

and with the bundles tangent to  $p^k : J^k E \rightarrow M$  and to  $p_h^k : J^k E \rightarrow J^h E$

$$Tp^k : TJ^k E \rightarrow TM \quad \text{and} \quad Tp_h^k : TJ^k E \rightarrow TJ^h E.$$

The standard charts of TM and of  $TJ^k E$  are, respectively

$$(x^\lambda, \dot{x}^\lambda) \quad \text{and} \quad (x^\lambda, y_\Lambda^i; \dot{x}^\lambda, \dot{y}_\Lambda^i)$$

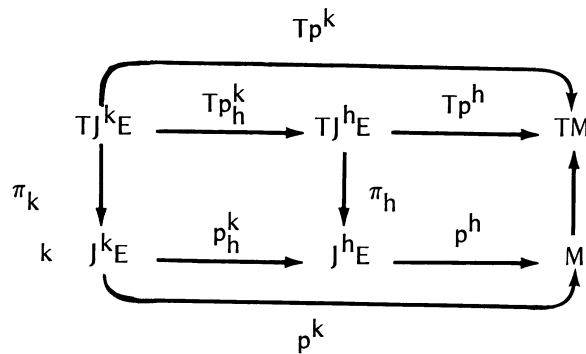
and we have the expressions

$$(x^\lambda) \circ \pi \equiv x^\lambda, \quad (x^\lambda, y_\Lambda^i) \circ \pi_k \equiv (x^\lambda, y_\Lambda^i),$$

$$(x^\lambda, \dot{x}^\lambda) \circ Tp^k \equiv (x^\lambda, \dot{x}^\lambda), \quad (x^\lambda, y_M^i; \dot{x}^\lambda, \dot{y}_M^i) \circ Tp_h^k \equiv (x^\lambda, y_M^i; \dot{x}^\lambda, \dot{y}_M^i),$$

with  $0 \leq |\Lambda| \leq k, \quad 0 \leq |M| \leq h.$

The following diagram commutes



and  $Tp_h^k$  and  $Tp^k$  are linear morphisms.

2.2 - The vertical spaces of  $p^k : J^k E \rightarrow M$  and of  $p_h^k : J^k E \rightarrow J^h E$  are the linear sub-bundles of  $\pi_k : TJ^k E \rightarrow J^k E$



$$VJ^{kE} \equiv \ker Tp^k \quad \text{and} \quad V_h J^{kE} \equiv \ker Tp_h^k,$$

which are locally characterized by

$$\dot{x}^\lambda = 0 \quad \text{and} \quad \dot{x}^\lambda = \dot{y}_M^i = 0, \quad \text{with} \quad 0 \leq |\Lambda| \leq h.$$

And so naturally we have the following sequence of inclusions

$$V_{k-1} J^{kE} \hookrightarrow \dots \hookrightarrow V_0 J^{kE} \hookrightarrow VJ^{kE} \hookrightarrow TJ^{kE}.$$

The transverse spaces of  $p^k : J^{kE} \rightarrow M$  and of  $p_h^k : J^{kE} \rightarrow J^{hE}$  are the pull-back bundles of  $\pi : TM \rightarrow M$  and of  $\pi_h : TJ^{hE} \rightarrow J^{hE}$  with respect to  $p^k : J^{kE} \rightarrow M$  and to  $p_h^k : J^{kE} \rightarrow J^{hE}$ , respectively,

$$HJ^{kE} \equiv J^{kE} \times_M TM \quad \text{and} \quad H_h J^{kE} \equiv J^{kE} \times_{J^{hE}} TJ^{hE},$$

whose standard charts are

$$(x^\lambda, y_\Lambda^i; \dot{x}^\lambda) \quad \text{and} \quad (x^\lambda, y_\Lambda^i; \dot{x}^\lambda, \dot{y}_M^i)$$

with  $0 \leq |\Lambda| \leq k$ ,  $0 \leq |\Lambda| \leq h$ .

And so naturally we have the following sequence of projections

$$TJ^{kE} \rightarrow H_{k-1} J^{kE} \rightarrow \dots \rightarrow H_0 J^{kE} \rightarrow HJ^{kE}.$$

Moreover one has the following exact sequences

$$0 \rightarrow VJ^{kE} \rightarrow TJ^{kE} \rightarrow HJ^{kE} \rightarrow 0$$

$$0 \rightarrow V_h J^{kE} \rightarrow TJ^{kE} \rightarrow H_h J^{kE} \rightarrow 0.$$

2.3 - Moreover we will be concerned with the  $k$ -jet spaces of the fibered spaces  $\pi : TM \rightarrow M$  and  $p^h \circ \pi_h : TJ^{hE} \rightarrow M$

$$\pi^k : J^k TM \rightarrow M \quad \text{and} \quad (p^h \circ \pi_h)^k : J^k TJ^{hE} \rightarrow M,$$

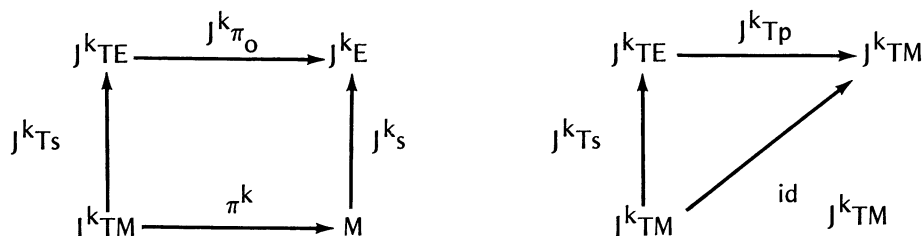
with  $0 \leq h, k$ , whose standard charts are

$$(x^\lambda, \dot{x}_\Lambda^\lambda) \quad \text{and} \quad (x^\lambda, y_{M\Lambda}^i; \dot{x}_\Lambda^\lambda, \dot{y}_{M\Lambda}^i)$$

with  $0 \leq |\Lambda| \leq k$ ,  $0 \leq |\Lambda| \leq h$ .

In (5.2) we will use the following result.

2.4 - LEMMA. *Let  $s : M \rightarrow E$  be a (local) section. Then the following diagrams commute*



*Proof.* The first one follows from (1.6). The second one follows from the functorial properties of  $J^k$  and  $T$ .

### 3. - AFFINE STRUCTURES ON JET SPACES

3.0 - Let  $a : A \rightarrow M$  be an affine bundle (see also [3] ). Then we denote by  $\bar{a} : \bar{A} \rightarrow M$  its vector bundle. We note that  $VA = A \times_M \bar{A}$ .

Let  $b : B \rightarrow N$  be a further affine bundle and  $F : A \rightarrow B$  be an affine morphism over  $f : M \rightarrow N$ . Then we denote by  $DF : A \rightarrow \text{Hom}(\bar{A}, \bar{B})$  the «fibre derivative» of  $F$ .

3.1 - In order to exploit further affine structures coming from a given one, we need the following lemmas.

a) LEMMA. *Let  $a : A \rightarrow M$  be an affine bundle. Then  $a^k : J^k A \rightarrow M$  is naturally an affine bundle, whose vector bundle is  $\bar{a}^k : J^k \bar{A} \rightarrow M$ .*

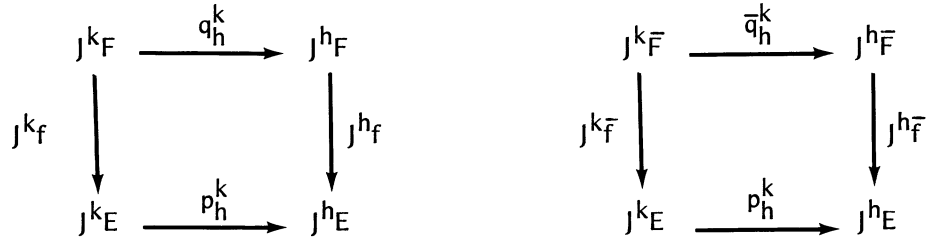
*Proof.* The covariant functor  $J^k$  prolongs naturally the translation  $+ : A \times_M \bar{A} \rightarrow A$  and its properties to  $\overset{k}{+} \equiv J^k + : J^k \bar{A} \times_M J^k A \rightarrow J^k A$ , so that, for each (local) section  $s : M \rightarrow A$  and  $v : M \rightarrow \bar{A}$ , we have

$$j_s^k + j_v^k = j_{(s+v)}^k \quad \text{and} \quad \overset{k}{0} = j_0^k.$$

LEMMA. *Let  $f : F \rightarrow E$  be a linear (affine) bundle (whose vector bundle is  $\bar{f} : \bar{F} \rightarrow E$ ), so that  $q \equiv f \circ p : F \rightarrow M$  ( $\bar{q} \equiv \bar{f} \circ p : \bar{F} \rightarrow M$ ) is a fibered space,  $f$  and  $J^k f : J^k F \rightarrow J^k E$  ( $\bar{f}$  and  $J^k \bar{f} : J^k \bar{F} \rightarrow J^k E$ ) are morphisms over  $M$ .*

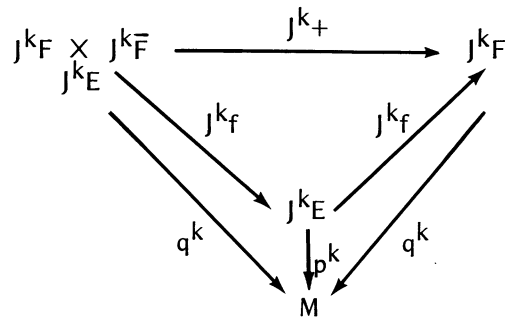
b) Then  $J^{kf} : J^{kF} \rightarrow J^{kE}$  is a linear (affine) bundle (whose vector bundle is  $J^{k\bar{F}} : J^{k\bar{F}} \rightarrow J^{kE}$ ).

c) Then  $q_h^k : J^{kF} \rightarrow J^{hF}$  is a linear (affine) morphism over  $p_h^k : J^{kE} \rightarrow J^{hE}$  (whose fibre derivative is  $\bar{q}_h^k : J^{k\bar{F}} \rightarrow J^{h\bar{F}}$ ), so that the following diagrams commute



d) Then  $(J^{kf}, q_h^k) : J^{kF} \rightarrow J^{kE} \times_{J^{hE}} J^{hF}$  is an affine bundle.

*Proof.* b) It follows, in the same way as (a), by taking into account the commutative diagram



and sections  $(s,v) : M \rightarrow F \times_M F$  such that  $f \circ s = \bar{f} \circ v : M \rightarrow E$ .

c) It follows from  $q_0^k \circ (j_s^k + j_v^k) = q_0^k \circ j_{(s+v)}^k = s + v$ .

d) The fibre of  $J^{kF}$  over  $(\sigma, y) \in J^{kE} \times_{J^{hE}} J^{hF}$  is the affine subspace of the fibre of  $J^{kF}$  over  $\sigma \in J^{kE}$  which is projected onto  $y \in J^{hF}$ .

3.2 - We will be concerned with different structures of the space  $TJ^{kE}$ . Besides the vector bundle

$$\pi^k : TJ^{kE} \rightarrow J^{kE}$$

we will consider the affine bundle

$$\omega_k \equiv (\pi_k, T p^k) : TJ^{kE} \rightarrow HJ^{kE} \equiv J^{kE} \times_M TM,$$

whose vector bundle is (up to an obvious pull-back)  $VJ^kE$  and the affine bundle

$$\mu_{(k,h)} \equiv (\pi_k, T\rho_h^k) : TJ^kE \rightarrow H_h J^kE \equiv J^kE \times_{J^hE} TJ^hE,$$

whose vector bundle is (up to an obvious pull-back)  $V_h J^kE$ .

3.3 - We have a basic affine structure on jet spaces. Namely

$$p_{k-1}^k : J^kE \rightarrow J^{k-1}E$$

is an affine bundle, whose vector bundle is (up to an obvious pull-back)

$$\overline{J^kE} = \underset{k}{V} T^*M \otimes VE$$

where  $\underset{k}{V}$  is the symmetrized tensor product.

Then (see 3.0) we have

$$V_{k-1} J^kE = J^kE \times_E (\underset{k}{V} T^*M \otimes VE).$$

3.4 - Moreover, the previous lemmas allow us to discover further linear and affine structures, which will be useful in the following.

PROPOSITION. a)  $J^k\pi_0 : J^kTE \rightarrow J^kE$  is a vector bundle.

b)  $J^k\pi_0 : J^kVE \rightarrow J^kE$  is a linear sub-bundle of the previous bundle.

c)  $(J^k\pi_0, (\rho \circ \pi_0)^k) : J^kTE \rightarrow J^kE \times_E TE$  is an affine bundle.

d)  $(J^k\pi_0, (\pi_0)^k_{k-1}) : J^kTE \rightarrow J^kE \times_{J^{k-1}E} J^{k-1}TE$  is an affine bundle, whose vector bundle is (up to an obvious pull-back)  $\underset{k}{V} T^*M \rightarrow TE$ .

e)  $(J^k\pi_0, J^kTp) : J^kTE \rightarrow J^kE \times_M J^kTM$  is an affine bundle, whose vector bundle is (up to an obvious pull-back)  $J^kVE$ .

*Proof.* a) It follows from (3.1b), taking into account the fact that  $\pi_0 : TE \rightarrow E$  is a vector bundle.

b) It follows from (3.1b) and (3.4a), by taking into account the fact that  $VE$  is a subbundle of  $TE$  over  $E$ .

c) It follows from (3.1d), by taking into account the fact that  $\pi_0 : TE \rightarrow E$  is a vector bundle.

d) It follows from (3.1d), by taking into account the fact that  $\pi_0 : TE \rightarrow E$  is a vector bundle, and using (3.3).

e) It follows from (3.1b), by taking into account the fact that (see 3.2)  $\mu_0 : TE \rightarrow E \times_M TM$  is an affine bundle.

4. - THE FIRST FUNDAMENTAL STRUCTURE ON JET SPACES

4.1 - The exact sequence (2.2)

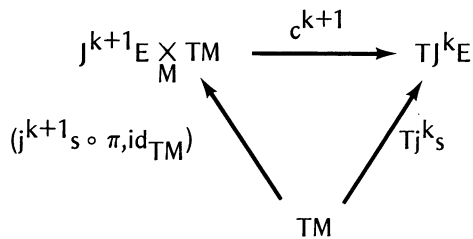
$$0 \rightarrow VJ^k E \rightarrow TJ^k E \rightarrow HJ^k E \rightarrow 0$$

has not in general a canonical splitting, but its pull-back over  $J^{k+1} E$  splits.

PROPOSITION. *There is a unique morphism*

$$c^{k+1} : p^{k+1} * TM \rightarrow TJ^k E$$

over  $p_k^{k+1} : J^{k+1} E \rightarrow J^k E$ , such that, for each (local) section  $s : M \rightarrow E$ , the following diagram commutes



Moreover  $c^{k+1}$  is a linear morphism, with respect to the fibres, which induces a splitting of the previous sequence over  $J^{k+1} E$ .

Its expression is

$$(x^\lambda, y_M^i ; \dot{x}^\lambda, \dot{y}_M^i) \circ c^{k+1} = (x^\lambda, y_M^i ; \dot{x}^\lambda, y_{M+\mu}^i \dot{x}^\mu) , \quad 0 \leq |\mu| \leq k.$$

*Proof. Uniqueness.* Let  $(\sigma, u) \in J^{k+1} E \times_M TM$  be a point over  $x \in M$ . Let  $s : M \rightarrow E$  be one of the (local) sections such that  $j^{k+1} s(x) = \sigma$ . Then we have  $c^{k+1}(\sigma, u) = (x^\lambda, y_M^i ; \dot{x}^\lambda, \dot{y}_M^i) \circ Tj^k s(u) = (x^\lambda, \partial_M s^i ; \dot{x}^\lambda, \partial_{M+\mu} s^i \dot{x}^\mu)(u)$ .

*Existence.* The previous formula does not depend on the choice of  $s$ .

The complementary linear epimorphism is denoted by

$$\bar{\nu}^{k+1} : p_k^{k+1} * TJ^k E \rightarrow VJ^k E.$$

Its expression is

$$(x^\lambda, y_M^i; \dot{y}_M^i) \circ \bar{\vartheta}^{k+1} = (x^\lambda, y_M^i; \dot{y}_M^i - y_{M+\mu}^i \dot{x}^\mu), \quad o \leq |M| \leq k.$$

Moreover, taking into account the projection  $(\pi_{k+1}, \text{Tp}_k^{k+1}) : \text{TJ}^{k+1}E \rightarrow p_k^{k+1} * \text{TJ}^kE$ , we can also characterize  $\bar{\vartheta}^{k+1}$  by means of the vector valued form

$$\vartheta^{k+1} : \text{J}^{k+1}E \rightarrow \text{T}^* \text{J}^{k+1}E \otimes \text{VJ}^kE.$$

Its expression is

$$\vartheta^{k+1} = (dy_M^i - y_{M+\mu}^i dx^\mu) \otimes \partial_i, \quad o \leq |M| \leq k.$$

We call  $c^{k+1}$  (or, equivalently,  $\bar{\vartheta}^{k+1}$ , or  $\vartheta^{k+1}$ ) the FIRST FUNDAMENTAL STRUCTURE of order  $k+1$  on jet spaces.

4.3 -  $c^{k+1}$  and  $c^{h+1}$ , with  $o \leq h < k$ , are related by the following commutative diagram, which follows immediately from the local expressions,

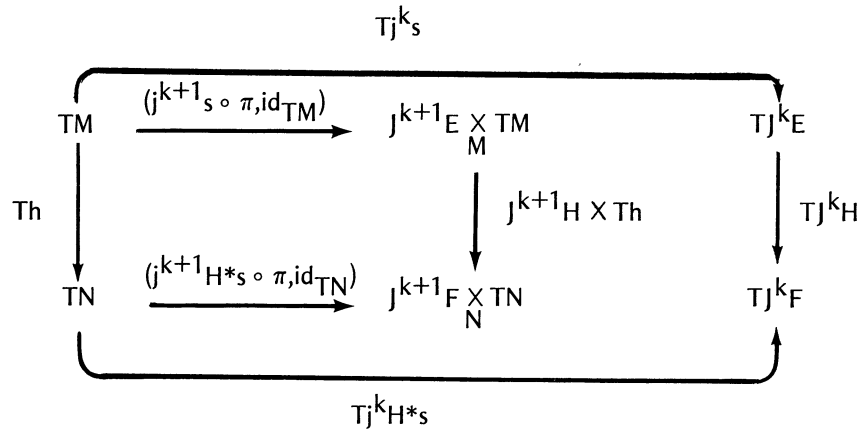
$$\begin{array}{ccc} p^{k+1} * \text{TM} & \xrightarrow{c^{k+1}} & \text{TJ}^kE \\ \downarrow p_{h+1}^{k+1} \times \text{id}_{\text{TM}} & & \downarrow \text{Tp}_h^k \\ p^{h+1} * \text{TM} & \xrightarrow{c^{h+1}} & \text{TJ}^hE \end{array}$$

4.4 - The first fundamental structure is functorially invariant

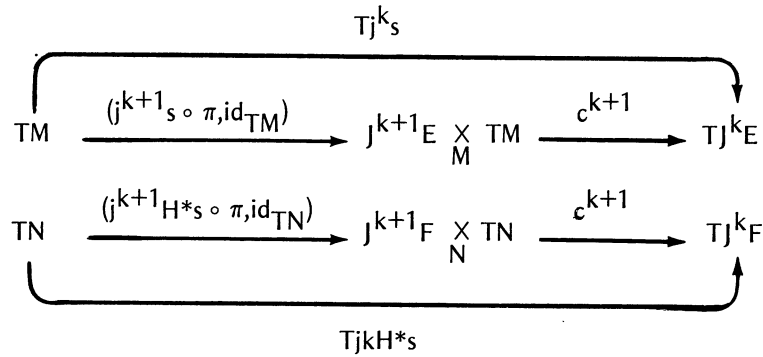
PROPOSITION. Let  $H : E \rightarrow F$  be a morphism over the diffeomorphism  $h : M \rightarrow N$ . Then the following diagram commutes

$$\begin{array}{ccc} p^{k+1} * \text{TM} & \xrightarrow{c^{k+1}} & \text{TJ}^kE \\ \downarrow J^{k+1}H \times \text{Th} & & \downarrow \text{TJ}^kH \\ q^{k+1} * \text{TN} & \xrightarrow{c^{k+1}} & \text{TJ}^kF \end{array}$$

*Proof.* It follows by taking into account the diagram



which commutes by the properties of the covariant functors  $J^k$  and  $T$ , and the diagrams



which commute by the definition of  $c^{k+1}$ , where  $s : M \rightarrow E$  is a (local) section and  $H^*s \equiv H \circ s \circ h^{-1} : N \rightarrow F$ .

4.5 - The form  $\vartheta^{k+1}$  determines a differential system  $\Delta^{k+1} \equiv \ker \vartheta^{k+1} \hookrightarrow Tj^{k+1}E$  or, equivalently,

$$(\Delta^{k+1})^\perp \equiv (\ker \vartheta^{k+1})^\perp \hookrightarrow T^*j^{k+1}E.$$

Locally,  $\Delta^{k+1}$  is generated by the vector fields  $\partial_\mu + y_{M+\mu}^i \partial_i^M$  and  $\partial_i^\Lambda$  and  $(\Delta^{k+1})^\perp$  is generated by the forms

$$\alpha_M^i \equiv dy_M^i - y_{M+\mu}^i dx^\mu, \quad 0 \leq |M| \leq k, \quad |\Lambda| = k+1.$$

$\Delta^{k+1}$  has constant rank and is not involutive. In fact we can easily prove that the exterior product of  $d\alpha_M^i$  times all the  $\alpha$  is different from 0.

Moreover we note that  $(d\alpha_M^i)^m \wedge \alpha_M^i = 0$ . Hence  $\alpha_M^i$  can be viewed as a «contact

form». More precisely, in the particular case of  $E \equiv M \times \mathbb{R}$ , we have  $JE = \mathbb{R} \times T^*M$  and  $\vartheta^1$  is the veritable contact form induced by the Liouville form. For this reason, we call  $\Delta^{k+1}$  the CONTACT SYSTEM of degree  $k+1$ .

4.6 - The fundamental form characterizes the sections of  $p^{k+1} : J^{k+1}E \rightarrow M$  which are the lifts of their projections on  $E$ .

PROPOSITION. Let  $S : M \rightarrow J^{k+1}E$  be a section. Then the following conditions are equivalent

- a)  $S = j^{k+1}(p^{k+1} \circ S)$
- b)  $TS : TM \rightarrow \Delta^{k+1}$ .

*Proof.* It follows from the local expression by induction on  $|M|$  such that  $0 \leq |M| \leq k+1$ .

4.7 - PROBLEM. Is there a way to get a splitting of the sequence

$$0 \rightarrow V_h J^k E \rightarrow T J^k E \rightarrow H_h J^k E \rightarrow 0$$

up to a suitable pull-back ?

### 5. - THE SECOND FUNDAMENTAL STRUCTURE ON JET SPACES

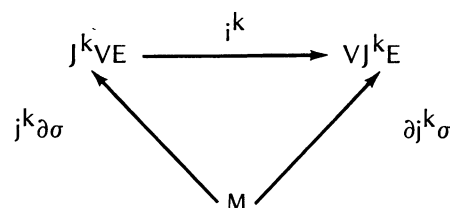
We can introduce a new fundamental map, which accounts for the exchange between  $J^k, j^k$  and  $T$ , in terms of a morphism between the spaces  $J^k T E$  and  $T J^k E$ .

5.1 - First we recall the following map (see also [3]).

LEMMA. *There is a unique morphism*

$$i^k : J^k V E \rightarrow V J^k E$$

over  $J^k E$ , such that, for each 1-parameter family of (local) sections  $\sigma : \mathbb{R} \times M \rightarrow E$ , the following diagram commutes





where  $\partial$  is the «variational derivative», i.e. the derivative with respect to the parameter evaluated at  $o \in \mathbb{R}$ . Moreover  $i^k$  is linear over  $J^k E$  (see 3.4b) and is an isomorphism. Its expression is

$$(x^\lambda, y_\Lambda^i; \dot{y}_\Lambda^i) \circ i^k = (x^\lambda, y_\Lambda^i; \dot{y}_\Lambda^i).$$

*Proof.* We have

$$(x^\lambda, y_\Lambda^i; \dot{y}_\Lambda^i) \circ j^k \partial \sigma = (x^\lambda, \partial_\Lambda \sigma^i_o; \partial_\Lambda \partial \sigma^i) = (x^\lambda, \partial_\Lambda \sigma^i_o; \partial \partial_\Lambda \sigma^i) = (x^\lambda, y_\Lambda^i; \dot{y}_\Lambda^i) \circ \partial j^k \sigma.$$

5.2 - PROPOSITION. There is a unique affine morphism (see 3.4d and 3.2)

$$r^k : J^k TE \rightarrow TJ^k E$$

over

$$\text{id}_{J^k E} \times \pi^k : J^k E \times_M J^k TM \rightarrow J^k E \times_M TM$$

such that

- a) for each (local) section  $s : M \rightarrow E$ , the following diagram commutes

$$\begin{array}{ccc} J^k TE & \xrightarrow{r^k} & TJ^k E \\ J^{kT}s \uparrow & & \uparrow Tj^k s \\ J^k TM & \xrightarrow{(\pi^k)_o} & TM \end{array}$$

- b)  $Dr^k = i^k$  (see 3.0).

Moreover  $r^k$  is a surjective map and its local expression is

$$(x^\lambda, y_\Lambda^i; \dot{x}^\lambda, \dot{y}_\Lambda^i) \circ r^k = (x^\lambda, y_\Lambda^i; \dot{x}^\lambda, \dot{y}_\Lambda^i - y_{\phi+\mu}^i \dot{x}^\mu_\psi),$$

with  $0 \leq |\Lambda| \leq k$  and where the summation is extended to all the multi-indices such that  $\phi + \psi = \Lambda$ ,  $|\psi| > 0$ .

*Proof. Uniqueness.* Let  $(\sigma, u^k) \in J^k E \times_M J^k TM$  be a point which is projected on  $(\sigma, u) \in J^k E \times_M TM$  by the map  $\text{id}_{J^k E} \times \pi^k$  and let  $x \in M$  be the common basis of  $\sigma, u^k, u$ . Let  $s : M \rightarrow E$  be one of the (local) sections such that  $j^k s(x) = \sigma$ . Then (see 2.4)  $a \equiv j^{kT}s(u^k) \in J^k TE$  and  $b \equiv Tj^k s(u) \in TJ^k E$  are, respectively, points of the affine fibres of

$$(J^k \pi_o, J^k T_p) : J^k TE \rightarrow J^k E \times_M J^k TM \text{ over } (\sigma, u^k)$$

and of

$$(\pi_k, T_p^k) : TJ^k E \rightarrow J^k E \times_M TM \text{ over } (\sigma, u).$$

Hence there is a unique affine morphism  $r^k$  between these affine fibres, such that  $r^k(a) = b$  and  $Dr^k = i^k$ .

*Existence.* The local expression shows that this map  $r^k$  does not depend on the choice of  $s$ .

We call  $r^k$  the SECOND FUNDAMENTAL STRUCTURE of order  $k$  on jet spaces. The local expression of  $r^k$  shows the further properties.

5.3 - PROPOSITION.

- a)  $r^k$  is a linear morphism (see 3.4a) over  $J^k E$ .
- b)  $r^k$  is an affine morphism (see 3.4c) over  $J^k E \times_E TE$ .

5.4 - We can view  $i^k$  as the restriction of  $r^k$ .

PROPOSITION. The following diagram commutes (see 5.1 and 5.4b)

$$\begin{array}{ccc} J^k TE & \xrightarrow{r^k} & TJ^k E \\ \uparrow J & & \uparrow J \\ J^k VE & \xrightarrow{i^k} & VJ^k E \end{array}$$

5.5 -  $r^k$  and  $r^h$ , with  $0 \leq h < k$ , are related by the following commutative diagram

$$\begin{array}{ccc} J^k TE & \xrightarrow{r^k} & TJ^k E \\ \downarrow (p \circ \pi_o)_h^k & & \downarrow T_p^k \\ J^h TE & \xrightarrow{r^h} & TJ^h E \end{array}$$

5.6 - We have a relation between  $r^k$  and  $\vartheta^1$ .

PROPOSITION.  $r^k$  is an affine morphism (see 3.4d) over

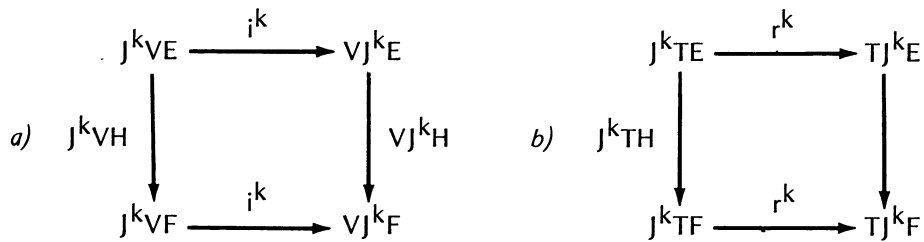
$$\text{id}_{J^k E} \times r^{k-1} : J^k E \times_{J^{k-1} E} J^{k-1} TE \rightarrow J^k E \times_{J^{k-1} E} TJ^{k-1} E$$

whose fibre derivative is (up to an obvious pull-back)

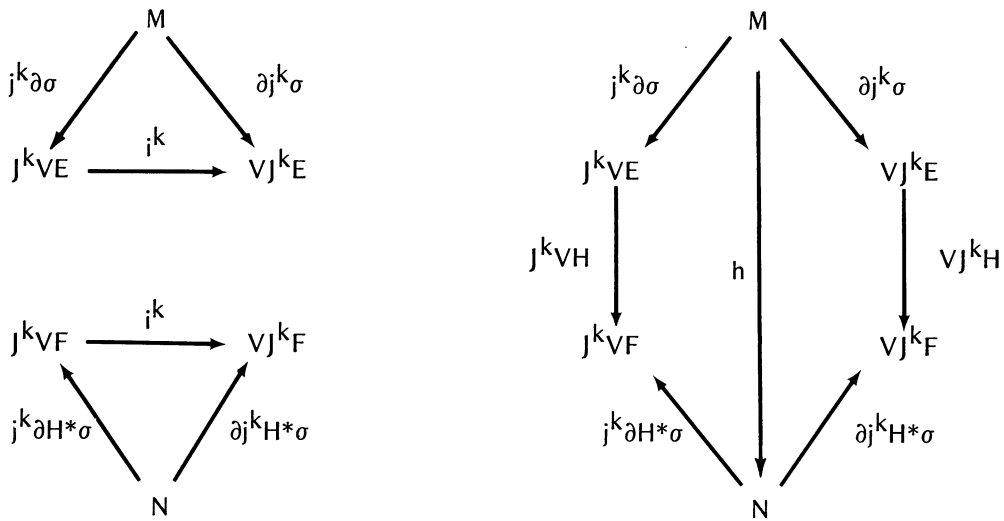
$$\text{id}_{\mathbb{V}T^*M} \otimes \bar{\partial}^1 : \mathbb{V}T^*M \otimes TE \rightarrow \mathbb{V}T^*M \otimes VE.$$

5.7 - The second fundamental structure is functionally invariant.

PROPOSITION. Let  $H : E \rightarrow F$  be a morphism over the diffeomorphism  $h : M \rightarrow N$ . Then the following diagrams commute

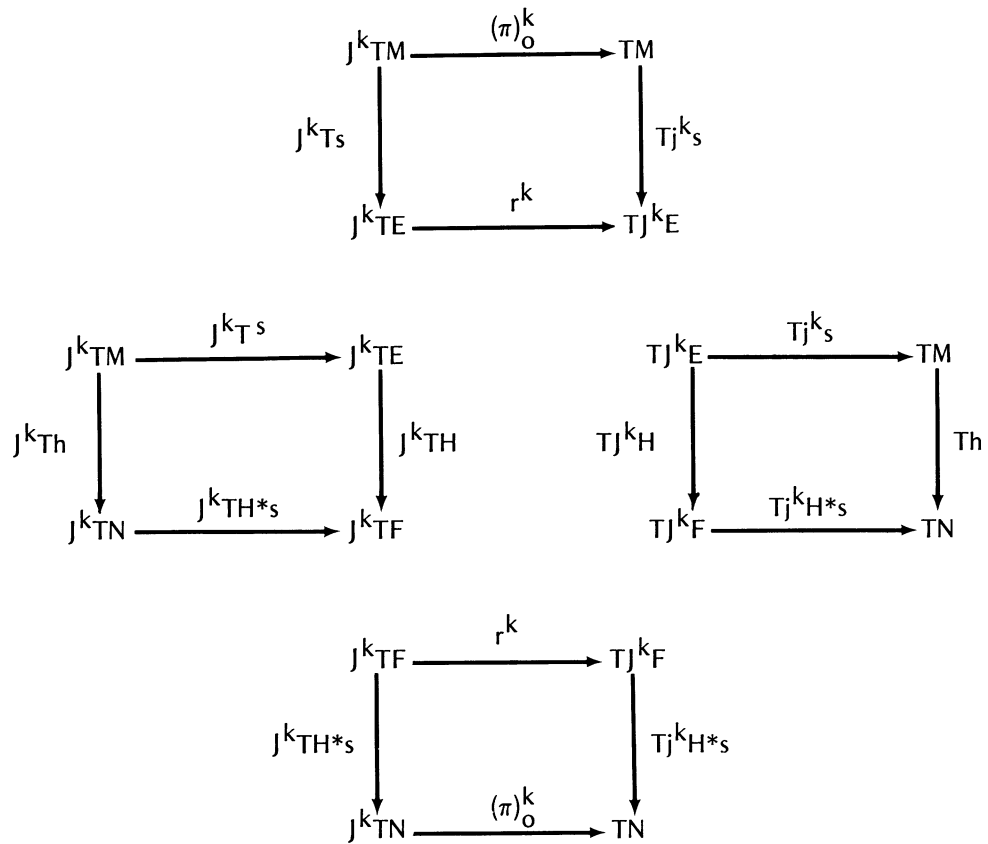


Proof. a) Let  $a \in J^kVE$  be a point which is projected on  $x \in M$ . Then there is a 1-parameter family of (local) sections  $\sigma : \mathbb{R} \times M \rightarrow E$ , such that  $j^k\partial\sigma(x) = a$ . Then the proof follows from the commutative diagrams

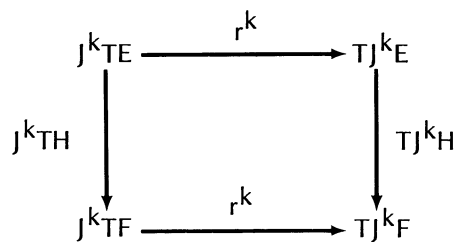


b) Let  $(\sigma, u) \in J^kE \times_M J^kTM$  be a point which is projected on  $x \in M$  and let  $s : M \rightarrow E$  be a (local) section, such that  $j^ks(x) = \sigma$ .

Then, from the commutative diagrams



it follows that the following diagram commutes with respect to the point  $J^{kTs}(u) \in J^{kTE}$  which is on the fibre  $(\sigma, u) \in J^k_E \times_M J^{kTM}$



Moreover, the diagram (a) can be viewed as the fibre derivative of (b). Then (b) commutes for each point of the fibre  $(\sigma, u) \in J^k_E \times_M J^{kTM}$ .

5.8 -  $r^k$  allows the exchange between  $\partial$  and  $J^k$ . This result will be utilized in (7.5).

PROPOSITION. Let  $H : \mathbb{R} \times E \rightarrow E$  be a 1-parameter family of morphisms over the 1-parameter family of diffeomorphisms  $h : \mathbb{R} \times M \rightarrow M$ , such that  $H_0 = \text{id}_E$  (hence  $h_0 = \text{id}_M$ ). Then

$$\partial J^k H = r^k \circ J^k H : J^k E \rightarrow T J^k E.$$

*Proof.* We have (see 1.2)

$$\text{a) } x^\alpha \circ J^k H = h^\alpha = x^\alpha$$

$$\text{b) } y_A^i \circ \partial J^k H = \partial_j H_0^i y_M^j (\partial_1 h_0^1)^{b(1)} \dots (\partial_m h_0^m)^{b(m)} = y_A^i$$

for

$$H_0^i = y^i, \quad h_0^\alpha = x^\alpha,$$

$$B(\alpha, 1) = \dots = B(\alpha, M_\alpha) = 0 + \alpha,$$

$$B(1, 1) + \dots + B(m, b(m)) = A, \quad \text{hence } M = A;$$

$$\text{c) } \dot{x}^\alpha \circ \partial J^k H = \partial h^\alpha \equiv u^\alpha;$$

$$\text{d) } \dot{y}_A^i \circ \partial J^k H =$$

$$\begin{aligned} &= \partial_\Lambda \partial_I \partial H^i y_M^1(1,1) \dots y_M^l(l, I_l) (\partial_1 h_0^1)^{b(1)} \dots (\partial_m h_0^m)^{b(m)} - \\ &- \partial_j H_0^i y_m^j (\partial_1 h_0^1)^{b(1)} \dots \partial_{B(\mu, 1)} \partial h^\mu (\partial_\mu h_0^\mu)^{b(\mu)-1} \dots (\partial_m h_0^m)^{b(m)} \\ &= \partial_\Lambda \partial_I u^i y_M^1(1,1) \dots y_M^l(l, I_l) - y_{N+\mu}^i \partial_B u^\mu \end{aligned}$$

with

$$\Lambda + M(1, 1) + \dots + M^{(l, I_l)} = A,$$

$$I_h > 0 \Rightarrow |M(h, 1)|, \dots, |M^{(h, I_h)}| > 0,$$

for

$$b(\mu) = \Lambda_\mu + M_\mu(1, 1) + \dots + M_\mu^{(l, I_l)},$$

$$B(\mu, 1) + \dots + B(\mu, b(\mu)) = 0 + b(\mu),$$

hence

$$\Lambda + M(1, 1) + \dots + M^{(l, I_l)} = A,$$

and with  $N + B = A$ ,

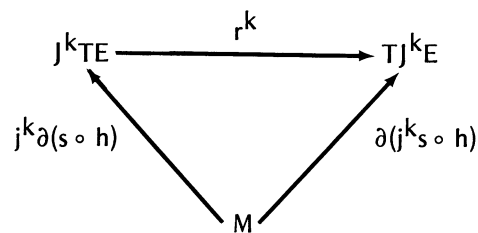
for  $b(\lambda) = M_\lambda$ ,

$$A_1 = M_1 + B_1, \dots, A_\mu = B_\mu + M_{\mu-1}, \dots, A_m = M_m + B_\mu,$$

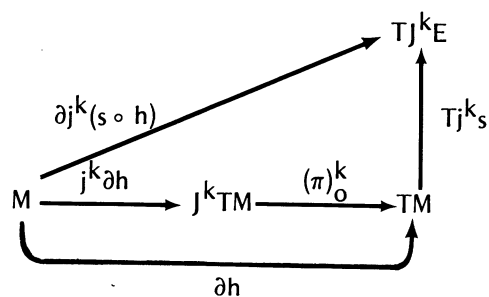
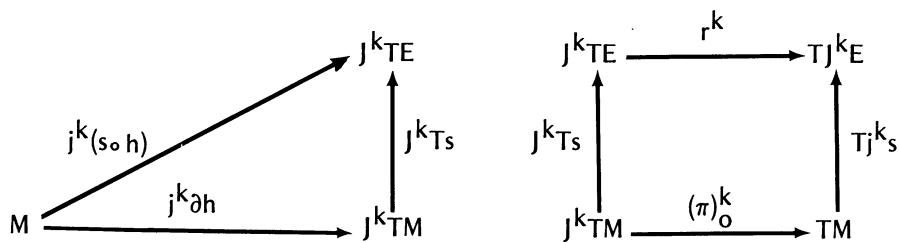
hence  $N + B = A$ , where  $M = N + \mu$ .

5.9 -  $r^k$  also allows the exchange between  $\partial$  and  $j^k$ .

**PROPOSITION.** Let  $h : \mathbb{R} \times M \rightarrow M$  be a 1-parameter family of maps such that  $h_0 = \text{id}_M$ . Then, for each (local) section  $s : M \rightarrow E$ , the following diagram commutes



*Proof.* It follows from the following commutative diagrams



5.10 - By replacing  $p : E \rightarrow M$  with  $p^h : J^h E \rightarrow M$  in the definition of  $r^k$ , we obtain the map

$$r^{(k,h)} : J^k T J^h E \rightarrow T J^k J^h E \quad 0 \leq h, k.$$

Its expression is

$$(x^\lambda, y_{\Lambda M}^i; \dot{x}^\lambda, \dot{y}_{\Lambda M}^i) \circ r^{(k,h)} = (x^\lambda, y_{\Lambda M}^i; \dot{x}^\lambda, \dot{y}_{\Lambda M}^i - y_{\Lambda}^i \phi + \mu \dot{x}^\mu_\psi),$$

with

$$0 \leq |\Lambda| \leq h, \quad 0 \leq |M| \leq k, \quad \phi + \psi = M, \quad |\psi| > 0.$$

In particular we have  $r^{(k,0)} = r^k$ . We may also note that the image of  $r^{(k,h)}$  is larger than the subspace

$$T J^{h+k} E \hookrightarrow T J^k J^h E.$$

## 6. - INFINITESIMAL CONTACT TRANSFORMATIONS ON JET SPACES

We can introduce the infinitesimal contact transformations of any order, by using both the first and the second fundamental structures.

6.1 - DEFINITION. An *INFINITESIMAL CONTACT TRANSFORMATION* is a vector field

$$u : J^{k+1} E \rightarrow T J^{k+1} E$$

such that

$$L_u \Delta^{k+1} \subset \Delta^{k+1}.$$

Of course, if  $u, v$  are i.c.t., then  $[u, v]$  is an i.c.t.

PROPOSITION. Let  $u : J^{k+1} E \rightarrow T J^{k+1} E$  be a vector field, whose expression is

$$u = u^\lambda \partial_\lambda + u_\Lambda^i \partial_i^\Lambda + u_\theta^i \partial_i^\theta, \quad \text{with } 0 \leq |\Lambda| \leq k, \quad |\theta| = k+1.$$

Then the following conditions are equivalent :

a)  $u$  is an i.c.t.

b)  $u_{\Lambda+\lambda}^i = (\partial_\lambda u_\Lambda^i + \partial_j^M u_\Lambda^i y_{M+\lambda}^j) - y_{\Lambda+\mu}^i (\partial_\lambda u^\mu + \partial_j^M u^\mu y_{M+\lambda}^j)$

$$\partial_j^\theta u_\Lambda^i = \gamma_{\Lambda+\lambda}^i \partial_j^\theta u^\lambda, \quad \text{with} \quad 0 \leq |\lambda| \leq k.$$

*Proof.* It follows from the expression of  $L_u v$ , with  $v : J^{k+1}E \rightarrow \Delta^{k+1}$ .

By extension we may naturally consider each vector field  $u : E \rightarrow TE$  as an i.c.t. of order 0.

6.2 - We can characterize the i.c.t. by means of  $r^k$ ,  $r^{(1,k-1)}$  and  $r^{(1,k)}$ .

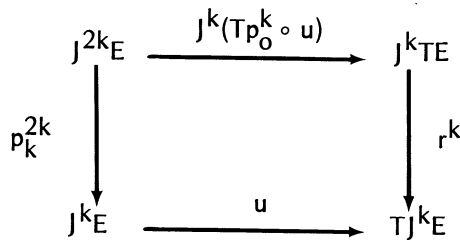
LEMMA. Let  $u : J^k E \rightarrow TJ^k E$  be a vector field. Then

$$r^{(1,k-1)} \circ J^1(T\rho_{k-1}^k \circ u) = TJ^1 \rho_{k-1}^k \circ r^{(1,k)} \circ J^1 u : J^{k+1}E \rightarrow TJ^1 J^{k-1}E.$$

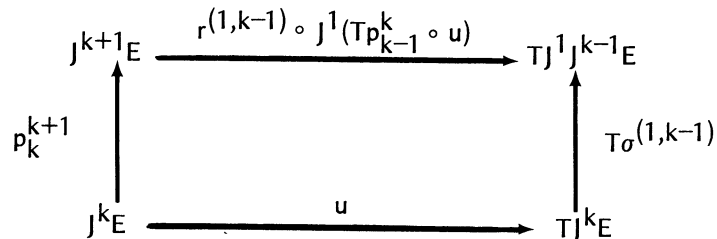
*Proof.* It follows from the local expression.

PROPOSITION. Let  $u : J^k E \rightarrow TJ^k E$  be a vector field. Then the following conditions are equivalent :

- a)  $u$  is an i.c.t.
- b) The following diagram commutes



- c) The following diagram commutes



- d) The following diagram commutes



$$\begin{array}{ccc}
 J^{k+1}E & \xrightarrow{TJ^1 p_{k-1}^k \circ r^{(1,k)} \circ J^1 u} & TJ^1 J^{k-1}E \\
 \uparrow p_k^{k+1} & & \uparrow T_{\sigma}^{(1,k-1)} \\
 J^k E & \xrightarrow{u} & TJ^k E
 \end{array}$$

*Proof.* a)  $\iff$  b). It follows, after a long calculation, by induction on  $|\Lambda|$  such that  $0 \leq |\Lambda| \leq k$ , by the formula

$$\begin{aligned}
 \dot{y}_{\Lambda}^i \circ r^k \circ J^k(Tp_0^k \circ u) &= \partial_A \partial_I u^i \dots y_{\theta_{1+\psi}^h}^h(h,1;1) \dots y_{\theta_{l_k+\psi}^h}^h(h,l_k;I(h,l_k)) \dots - \\
 &- y_{B+\mu}^i \partial_C \partial_J u^\mu \dots y_{\theta_{1+\phi}^h}^h(h,1;1) \dots y_{\theta_{l_k+\phi}^h}^h(h,l_k;J(h,l_k))
 \end{aligned}$$

with

$$0 \leq |\Lambda| \leq k, \quad 0 \leq |\theta^a| \leq k, \quad 1 \leq h \leq l$$

$$A + \psi^{(1,1;1)} + \dots + \phi^{(l,l_k;I(l,l_k))} = B + C + \phi^{(1,1;1)} + \dots + \phi^{(l,l_k;J(l,l_k))} = \Lambda$$

$$I(a,b) > 0 \Rightarrow |\psi(a,b;1)|, \dots, |\psi(a,b;I(a,b))| > 0$$

$$J(a,b) > 0 \Rightarrow |\phi(a,b;1)|, \dots, |\phi(a,b;J(a,b))| > 0$$

$$|C + \phi^{(h,1;1)} + \dots + \phi^{(h,l_k;J(h,l_k))}| > 0$$

a)  $\iff$  c). It follows from the formula

$$(\dot{y}_{\Sigma}^i, \dot{y}_{\Sigma\mu}^i) \circ r^{(1,k-1)} \circ J^1(Tp_{k-1}^k \circ u) = (u_{\Sigma}^i, \partial_{\mu} u_{\Sigma}^i + \partial_j^M u_{\Sigma}^i y_{M+\mu}^j - y_{\Sigma+\lambda}^i (\partial_{\mu} u^{\lambda} + \partial_j^M u^{\lambda} y_{M+\mu}^j))$$

with

$$0 \leq |\Sigma| \leq k-1.$$

c)  $\iff$  d). It follows from the previous lemma.

7. - PROLONGATION OF VECTOR FIELDS ON JET SPACES

7.1 - We recall the following

DEFINITION. Let  $u^k : J^k E \rightarrow TJ^k E$  be a vector field. Then we say that  $u^k$  is projectable on  $u : M \rightarrow TM$ , or on  $u^h : J^h E \rightarrow TJ^h E$ , with  $0 \leq h < k$ , if the following diagrams commute, respectively,



7.2 - THEOREM. Let  $u^0 : E \rightarrow TE$  be a vector field. Then there is a unique i.c.t.  $u^k : J^k E \rightarrow TJ^k E$ , which is projectable on  $u^0$ . Namely, we have

$$u^k = r^k \circ j^k u^0.$$

The expression of

$$u^k = u^\lambda \partial_\lambda + u^i_\Lambda \partial_i^\Lambda, \quad 0 \leq |\Lambda| \leq k,$$

is given by

$$u^i_\Lambda = \partial_\phi \partial_I u^i u^1_\psi(1,1) \dots y^1_\psi(1, I_1) \dots y^l_\psi(l,1) \dots y^l_\psi(l, I_l) -$$

$$- y^i_{M+\mu} \partial_\theta \partial_J u^\mu y^1_\Sigma(1,1) \dots y^1_\Sigma(1, J_1) \dots y^l_\Sigma(l,1) \dots y^l_\Sigma(l, J_l)$$

with

$$\phi + \psi(1,1) + \dots + \psi(l, I_l) = M + \theta + \Sigma(1,1) + \dots + \Sigma(l, J_l) = \Lambda$$

$$I_h > 0 \Rightarrow |\psi^{(h,1)}|, \dots, |\psi^{(h, I_h)}| > 0$$

$$J_h > 0 \Rightarrow |\Sigma^{(h,1)}|, \dots, |\Sigma^{(h, J_h)}| > 0$$

$$|M| < |\Lambda|.$$

*Proof. Uniqueness.* If  $u^k$  is an i.c.t. which is projectable on  $u^0$ , then it is determined by  $u^0$ , taking into account its inductive expression (or more precisely 6.2b).

*Existence.*  $r^k \circ J^k u^0$  is an i.c.t. for (6.2b).

We call

$$u^k \equiv r^k \cdot u^0 \equiv r^k \circ J^k u^0$$

the PROLONGATION of order  $k$  of  $u^0$ .

7.3 - There is a natural relation among the prolongations of  $u^0$  at different orders.

PROPOSITION. Let  $u^0 : E \rightarrow TE$  be a vector field. Then  $u^k$  is projectable on  $u^h$ , for  $0 \leq h < k$ .

*Proof.* It will result from the local expressions of  $u^k$  and  $u^h$ .

7.4 - PROPOSITION. The map  $u^0 \mapsto u^k$  is a homomorphism of Lie algebras, i.e.

$$r^k \cdot [u^0, v^0] = [r^k \cdot u^0, r^k v^0].$$

*Proof.* It will result, after a long calculation, by induction on  $|\Lambda|$  such that  $0 \leq |\Lambda| \leq k$ , taking into account the local expression.

7.5 - We find an equivalent prolongation in the particular case of projectable vector fields. First we recall the following obvious

LEMMA. Let  $u^0 : E \rightarrow TE$  be a vector field. Then the following conditions are equivalent.

a) The flow of  $u^0$   $H : \mathbb{R} \times E \rightarrow E$  is a 1-parameter group of (local) isomorphisms over the 1-parameter group of (local) diffeomorphisms  $h : \mathbb{R} \times M \rightarrow M$ .

b)  $u^0$  is projectable on  $u : M \rightarrow TM$ .

Moreover, in such a case, we have

$$u^0 = \partial H \qquad u = \partial h.$$

In particular, the following conditions are equivalent.

a') The flow of  $u^0$   $H : \mathbb{R} \times E \rightarrow E$  is a 1-parameter group of (local) isomorphisms over  $M$ .

b')  $u^0$  is vertical.

PROPOSITION. Let  $u^0 : E \rightarrow TE$  be a vector field projectable on  $u : M \rightarrow TM$  and let  $H : \mathbb{R} \times E \rightarrow E$  be its flow over  $h : \mathbb{R} \times M \rightarrow M$ . Then we have

$$u^k = \partial J^k H.$$

*Proof.* It results from (5.8).

7.6 - We could try to prolong any vector field  $u^h : J^h E \rightarrow TJ^h E$  to  $u^k : J^k E \rightarrow TJ^k E$ , with  $0 \leq h < k$ , by means of  $r^{(k-h,h)}$ , but we find that

$$u^k \equiv r^{(k-h,h)} \circ J^{k-h} u^h : J^k E \rightarrow TJ^{k-h} J^h E$$

generally does not take its values in

$$TJ^k E \hookrightarrow TJ^{k-h} J^h E.$$

We can find the local necessary and sufficient condition for it to hold. In particular we have the following condition.

PROPOSITION. Let  $u^h : J^h E \rightarrow TJ^h E$  be a vector field. If  $r^{(k-h,h)} \circ J^{k-h} u^h$  is factorizable through a vector field  $u^k : J^k E \rightarrow TJ^k E$ , i.e. if the following diagram commutes

$$\begin{array}{ccc}
 J^{k-h} TJ^h E & \xrightarrow{r^{(k-h,h)}} & TJ^{k-h} J^h E \\
 \uparrow J^{k-h} u^h & & \uparrow T\sigma^{(k-h,h)} \\
 J^k E & \xrightarrow{u^k} & TJ^k E
 \end{array}$$

then  $u^h$  is an i.c.t.

*Proof.* It results from the local expression.

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